M.SC. MATHEMATICS

MAL-642

Differential Geometry

Directorate of Distance Education

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CONTENTS

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CHAPTER-1

CONCEPT OF A CURVE

Objectives: Students will learn about curve characteristics in space, curve parametric equations, curve vector representation, curve path, arc length, tangent to the curve, and inflexional tangent in this chapter. Throughout the booklet, the this chapter play vital role.

1.1 INTRODUCTION

The study of geometric figures using calculus methods is referred to as differential geometry. Curves and surfaces embedded in three-dimensional Euclidean space R^3 are investigated in detail in this chapter. Local characteristics are the qualities of the curves and surfaces that are dependent exclusively on points at a certain point on the figure. In the simple world, differential geometry is the study of local characteristics. Global characteristics are those that relate to the overall geometric shapes. Differential geometry in the large is the study of global characteristics, especially as they relate to local properties.

We know that the geometric character of the curves and the surfaces changes with time, and that this is accomplished by differential calculus. Differential geometry is divided into two branches: one studies the characteristics of curves and surfaces in the neighbourhood of a point, and the other studies the properties of curves and surfaces as a whole.

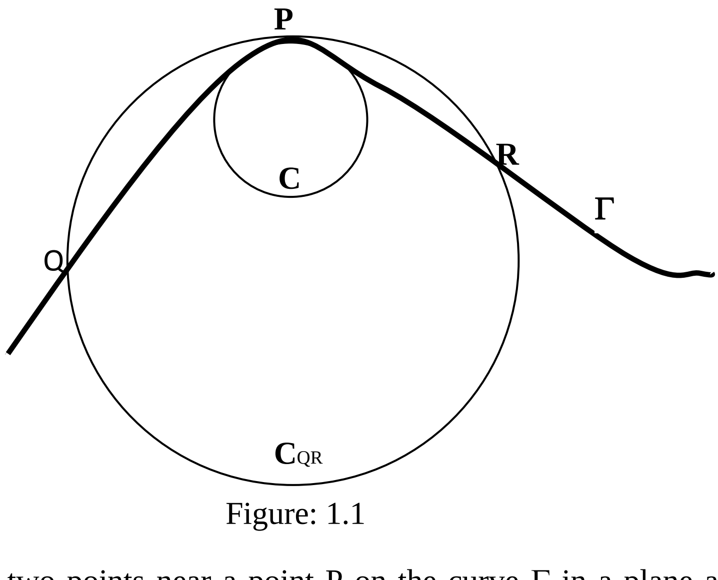


Figure: 1.1

Let Q and R be two points near a point P on the curve Γ in a plane and let C_{QR} be the circle through P,Q and R, as shown in the figure 1.1. Now consider the limiting position of the circles C_{QR} as

Q and R approach P. In general, the limiting point will be a circle C tangent to Γ at P. The radius of C is the radius of curvature of Γ at P. The radius of curvature is an example of a local property of the curve, for it depends only on the point on Γ near P.

The Moebius strips shown in figure 1.2 an example of a one-sided surface. One sidedness is an example of global property of a figure, for it depends on the nature of the entire surfaces. Observe that a small part of the surface surrounding an arbitrary point P is a regular two-sided surface, i.e. locally the Moebius strip is two-sided.

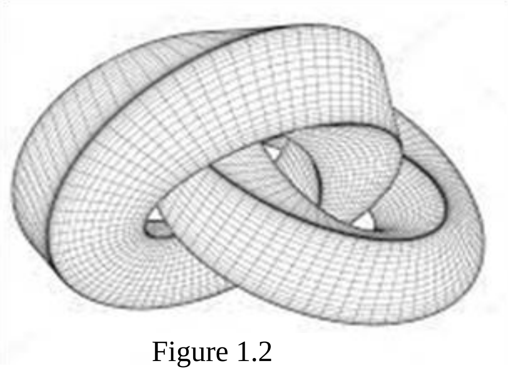


Figure 1.2

1.2 (i) SPACE CURVES

A curve in space can be described analytically by stating the equations of surfaces of which it is the intersection. Thus two surfaces represented by the equation of the form

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0 \tag{1.1}$$

Represent the curve of intersection of the surfaces.

For many practical purposes it is more convenient to describe a curve by parametric equations for the co-ordinates x, y, z.

(ii) TO FIND PARAMETRIC EQUATIONS FOR SPACE CURVES

To find the parametric equations for the curve, we eliminate x between equations (1.1) and get equation of the form $y = f_3(z)$, say; similarly eliminating y between equations (1.1), we get another equation of the form $x = f_4(z)$ say. Thus we have represented the co-ordinate x and y as some functions of z. Now if the co-ordinate z is a function of some parameter u say; then co-ordinates x and

y becomes functions of the same parameter u. Thus the co-ordinates of a point on the space curve can be represented by the following equations:

$$x = \phi_1(u), \quad y = \phi_2(u), \quad z = \phi_3(u) \tag{1.2}$$

Equations (1.2) are parametric equations of the curve in space where ϕ_1, ϕ_2, ϕ_3 are real valued functions of a single real parameter u ranging over the set of values $a \le u \le b$. In the light of the parametric representation of the space curves, we have the following definition.

Definition: A curve in the space is defined as the locus of a point whose Cartesian co-ordinates are the functions of a single variable parameter u, say.

(iii) VECTOR REPRESENTATION OF A CURVE

Definition: A curve in the space is the locus of a point whose position vector **r** relative to a fixed origin may be represented as a function of a single variable parameter u, say. It is represented as

$$\mathbf{r} = \mathbf{r}(u) \tag{1.3}$$

where **r** is a position vector of a current point on the curve. The vector representation (1.3) is equivalent to the Cartesian representation (1.2), since the Cartesian co-ordinates x, y, z of the point P(**r**) are called the component of **r**. Thus we have

$$\begin{aligned} \mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\ &= \phi_1(u)\,\mathbf{i} + \phi_2(u)\,\mathbf{j} + \phi_3(u)\,\mathbf{k} \end{aligned}$$

we shall often write this equation as

$$\mathbf{r} = (\phi_1(u), \phi_2(u), \phi_3(u)) \tag{1.4}$$

If the curve lies in a plane it is called plane curve, otherwise it is called skew, twisted or tortuous curve.

Note: A parametric representation of the curve gives sense of description of the curve also. The positive direction of the description of the curve is that in which the parameter u increases and the opposite direction is the negative direction.

(iv) FUNCTIONS OF CLASS r

A real valued function f is said to be of class **r** ($C^{\mathbf{r}}$-function), over a real interval ***I*** if it has $\mathbf{r}^{\text{th}}$ derivative at each point of ***I*** and this derivative is continuous on ***I***. Here **r** is a positive integer.

A function is said to be class ∞ or C^{∞} - function, if it is infinitely differentiable. If f is single valued and possesses continuous derivatives of all orders at each point of ***I***, then f is said to be analytic over ***I***, and f is called a function of class w or a C^{w} - function.

A vector valued function **R**=(X,Y,Z) is said to be of class **r** over a real interval ***I*** if it has **r**th derivative at each point of ***I*** and this derivative is continuous on ***I***; or if each of its components X, Y, Z are of class **r** .

The function **R** is said to be regular, if the derivative

$$\frac{d\mathbf{R}}{du} = \mathbf{R} \neq 0 \qquad \text{on the real interval } \boldsymbol{I}.$$

This is equivalent to saying that each of the components $\dfrac{dX}{du}, \dfrac{dY}{du}$ and $\dfrac{dZ}{du}$ never vanishes simultaneously on ***I***. The parameter u is called a regular parameter.

Note: In the rest of the book we shall always assume unless otherwise stated, that the equation (1.4) above always represents a regular analytic curve in terms of a regular parameter u .

Example: 1.1: The function $x = (u+1)i + (u^2+3)j$, $-\infty < u < \infty$ is a regular parametric representation, since $x' = i + 2uj$ is continuous and $x' \neq 0$ for all u. The image of the function is the parabola.

Example 1.2: The graph of the equation $r = 2\cos\theta - 1, 0 \le \theta \le 2\pi$, in polar coordinates. Polar and rectangular coordinates are related by the equations $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Upon substitution for r , we obtain the representation

$$x_1 = \cos\theta(2\cos\theta - 1)i + \sin\theta(2\cos\theta - 1)j$$

This representation is regular, since

$$x' = [-4\sin\theta\cos\theta + \sin\theta]i + [2\cos^2\theta - 2\sin^2\theta - \cos\theta]j$$

is continuous and it can be computed that $|x'| = \sqrt{5 - 4\cos\theta} \neq 0$ for all θ and hence $x' \neq 0$ for all θ.

Note: A regular parameter representation $x = x(u)$ on I can have multiple points, $u_1 \neq u_2$ in I for which $x(u_1) \neq x(u_2)$. However, locally this will not be the case.

1.3 PATH

A path of class **r** is a regular vector valued function of class **r** .

Equivalent Paths: Two paths R_1 and R_2 of the same class $\mathbf{r}$ on the real intervals $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ respectively are called equivalent if there exist a strictly increasing function θ of class $\mathbf{r}$,
which maps $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ and is such that $R_1 = R_2 \, o \, \theta$.

Change of Parameter: The function (or mapping) θ (as defined above) which relates two equivalent paths is called a change of parameter. This produces a change in the manner of description of the curve while sense remains the same.

Let the equation of the curve in parameter u be

$$\mathbf{r} = \mathbf{r}(u) \tag{1.5}$$

Consider the change in parameter $u = \theta(t)$, where $\theta(t)$ is a real single valued, analytic function of t defined on the same real interval. Now the equation (1.5) transform to

$$\mathbf{r} = \mathbf{R}(u) \tag{1.6}$$

Hence from equations (1.5) and (1.6), we get

$$\mathbf{R}(t) = \mathbf{r}(u)$$

$$\therefore \quad \frac{d\mathbf{R}}{dt} = \frac{dr}{du} \cdot \frac{du}{dt} \tag{1.7}$$

Since $\mathbf{r}(u)$ is regular, therefore $\frac{dr}{du} \neq 0$. Hence it follows from the equation (1.7) that

$$\frac{d\mathbf{R}}{dt} \neq 0, \qquad \text{iff} \quad \frac{du}{dt} \neq 0.$$

Thus t is also a regular parameter iff $\frac{du}{dt}$ is never zero.

Example 1.3: Consider the circular helix whose parametric equation is given by

$$\mathbf{r} = (a\cos u, a\sin u, cu) \qquad 0 \le u \le \pi \tag{1.8}$$

$$\therefore \qquad \frac{d\,\mathbf{r}}{du} = (-a\sin u, a\cos u, c).$$

We clearly see that $\frac{d\mathbf{r}}{du} \neq 0$ for any value of u in $0 \le u \le \pi$.

Hence u is a regular parameter. Let the change in parameter be $t = \tan\frac{u}{2}$.

$$i.e. \qquad u = 2\tan^{-1} t \tag{1.9}$$

$$\therefore \frac{du}{dt} = \frac{2}{1+t^2} \neq 0 \qquad \text{for } 0 \le t < \infty .$$

Hence **t** is also regular parameter defined by (1.9), in the interval $0 \le t < \infty$.

The equivalent representation of equation (1.8) in terms of parameter **t** is;

$$\mathbf{r} = \left(a.\frac{1-t^2}{1+t^2}, \frac{2at}{1+t^2}, 2c\tan^{-1} t \right), \quad 0 \le t < \infty .$$

1.4 ARC LENGTH

To find the arc length of a curve between two points.

Let us consider a curve C of class ≥ 1, and

$$\mathbf{r} = \mathbf{r}(u) \tag{1.10}$$

be the equation of the curve C. Suppose it is required to find the arc length between two points A and B on the curve (1.10) corresponding to the values a and b of the parameter u. [We shall find the length of the curve in the positive direction, as in relation 1.4]. Now corresponding to any subdivision Δ of the interval [a, b] by points

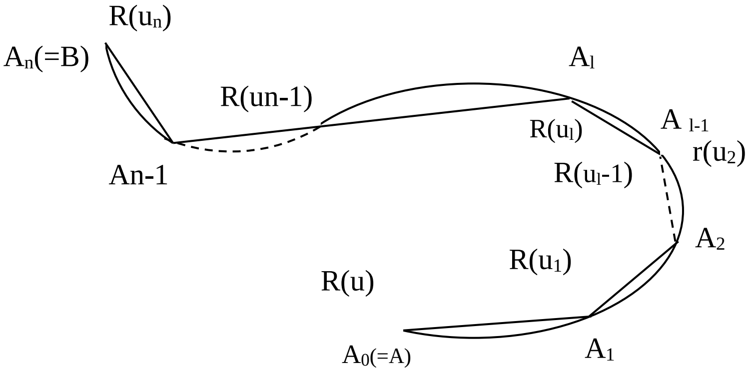
$$a = u_0 < u_1 < u_2 < u_3 < ...... < u_n = b$$

we have the length

$$L_\Delta = \sum_{l=1}^{n} A_{l-1}A_l = \sum \left| r(u_l - r(u_{l-1}) \right| \tag{1.11}$$

of the polygon inscribed to the arc by joining the successive points $A_0 (= A), A_1, ...A_n (= B)$

on it. If we increase the number of points of the subdivision, the length of the polygon will be increased (because the sum of the two sides of a triangle is greater than the third side). Therefore, **the length of the arc is defined to be the upper bound of** L_Δ take over all possible subdivisions of the interval [a, b]. Therefore we have from (1.11)



$$L_{\Delta} = \sum_{l=1}^{n} \left| \int_{u_{l-1}}^{u_l} \dot{\mathbf{r}}(u) du \right| \qquad \left[\because \left| \mathbf{r}(u_l) - \mathbf{r}(u_{l-1}) \right| = \int_{u_{l-1}}^{u_l} \dot{\mathbf{r}}(u)\, du \right] \tag{1.12}$$

The equation (1.12), using Schwartz inequality gives

$$L_{\Delta} \le \sum_{l=1}^{n} \int_{u_{l-1}}^{u_l} \left| \dot{\mathbf{r}}(u) \right| du = \int_a^b \left| \dot{\mathbf{r}}(u) \right| du \tag{1.13}$$

Now (1.13) shows the right hand member of (1.13) is finite and independent of Δ and hence upper bound of L_{Δ}. We shall now show that the upper bound of L_{Δ} is actually

$$\int_a^b \left| \dot{\mathbf{r}}(u) \right| du.$$

Let $s = s(u)$ denote the arc length from $u_0 (= a)$ to u *i.e.* $s(u) - s(u_0)$ where $a (= u_0) < u < b$ Therefore from equation (1.13), we have

$$s(u) - s(u_0) \le \int_{u_0}^{u} \left| \dot{\mathbf{r}}(u) \right| du \tag{1.14}$$

Also the definition of arc length implies that

$$\left| \mathbf{r}(u) - \mathbf{r}(u_0) \right| \le s(u) - s(u_0) \tag{1.15}$$

The equation (1.14) and (1.15) give

$$\left| \frac{\mathbf{r}(u) - \mathbf{r}(u_0)}{u - u_0} \right| \le \frac{s(u) - s(u_0)}{u - u_0} \le \frac{1}{u - u_0} \int_{u_0}^{u} \left| \dot{\mathbf{r}}(u) \right| du$$

Taking limit as $u \to u_0$, we get

$$|\dot{\mathbf{r}}(u)| \le \dot{\mathbf{s}}(u) \le |\dot{\mathbf{r}}(u)|$$

$$\therefore \qquad \dot{\mathbf{s}}(u) = |\dot{\mathbf{r}}(u)| \tag{1.16}$$

Since this is true for any value of u_0 in the range of u, Hence we have

$$s = s(u) = \int_a^u |\dot{\mathbf{r}}(u)\, du| \tag{1.17}$$

The formula (1.17) is used as formula to determine the arc length from a point a to any point u.

Formula (1.17) may also be rewritten as

$$s = s(u) = \int_a^u \sqrt{\{\dot{\mathbf{r}}^2(u)\}}\, du$$

Cartesian Equivalent: Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

So that $\dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$, $|\dot{\mathbf{r}}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

$$\therefore \qquad s = \int_a^u \sqrt{\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}\, du$$

Also the equation (1.16) may written as

$$s^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

In terms of differentials it gives

$$ds^2 = dx^2 + dy^2 + dz^2,$$

where ds is called the linear element of the curve C.

Note 1: Since $ds/du\,(=\dot{\mathbf{s}})$ will never vanish, we can use s as a new parameter. Changing the parameter from u to in the function $s = s(u)$, let $u = \theta(s)$, the parametric equation of curve with parameter s becomes $\mathbf{r} = \mathbf{r}\{\theta(s)\}$.

Note 2: We shall use dashes to denote differentiation with respect to arc length s and dots to denote differentiation with respect to any other parameter u. Thus we have

$$\frac{d\mathbf{r}}{ds} = \mathbf{r}', \frac{d^2\mathbf{r}}{ds^2} = \mathbf{r}'' \text{ etc.}$$

and

$$\frac{d\mathbf{r}}{du} = \dot{\mathbf{r}}, \frac{d^2\mathbf{r}}{du^2} = \ddot{\mathbf{r}} \text{ etc.}$$

Example 1.4: Find the length of circular helix.

$$\mathbf{r}(u) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + c\,u\,\mathbf{k}, \quad -\infty < u < \infty \text{ from } (a,0,0) \text{ to } (a,0,2\pi c).$$

Also obtain its equation in terms of parameter s.

Solution. Clearly the limits of u are from $cu = 0$ to $cu = 2\pi c$ *i.e.* from $u = 0$ to $u = 2\pi$.

The equation of circular helix is

$$\mathbf{r}(u) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + c\,u\,\mathbf{k}$$

$$\therefore \qquad \dot{\mathbf{r}} = \frac{d\mathbf{r}}{du} = -a\sin u\,\mathbf{i} + a\cos u\,\mathbf{j} + c\,\mathbf{k},$$

$$|\dot{\mathbf{r}}(u)| = (a^2\sin^2 u + a^2\cos^2 u + c^2)^{1/2} = (a^2 + c^2)^{1/2}$$

Therefore the length of the circular helix from $(a,0,0)$ to $(a,0,2\pi c)$ is

$$= \int_0^{2\pi} |\dot{\mathbf{r}}(u)|\,du = \int_0^{2\pi} \sqrt{a^2 + c^2}\,du$$

$$= (a^2 + c^2)[u]_0^{2\pi} = 2\pi\sqrt{(a^2 + c^2)}\,.$$

Again suppose s denotes the arc length from the point where $u = 0$ to any point u, we have

$$s = \int_0^u |\dot{\mathbf{r}}(u)|\,du$$

$$= \int_0^u \sqrt{a^2 + b^2}\,du = (a^2 + b^2)^{1/2}[u]_0^u = u(a^2 + b^2)^{1/2}$$

$$\therefore \qquad u = \frac{s}{(a^2 + b^2)^{1/2}}$$

Hence the given equation of the circular helix in terms of parameter s transforms to

$$\mathbf{r}(s) = a\cos\left\{\frac{s}{\sqrt{a^2 + c^2}}\right\}\mathbf{i} + a\sin\left\{\frac{s}{\sqrt{a^2 + c^2}}\right\}\mathbf{j} + \frac{cs}{\sqrt{a^2 + c^2}}\,\mathbf{k}$$

Example 1.5: Find the length of one complete turn of the circular helix

$$\mathbf{r}(u) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + c\,u\,\mathbf{k}, \quad -\infty < u < \infty.$$

Solution: The range of parameter u corresponding to one complete turn of the helix is

$$u_0 \le u \le u_0 + 2\pi.$$

Therefore the limit of u are from $u = u_0$ to $u = u_0 + 2\pi$.

From example (1) above, we have

$$|\dot{\mathbf{r}}(u)| = \sqrt{(a^2 + c^2)}$$

$\therefore$ Required arc length
$$= \int_{u_0}^{u_0+2\pi} |\dot{\mathbf{r}}(u)| du$$

$$= \int_{u_0}^{u_0+2\pi} \sqrt{a^2 + c^2}\, du$$

$$= \sqrt{a^2 + c^2}\,[u]_{u_0}^{u_0+2\pi}$$

$$= 2\pi\sqrt{a^2 + c^2}$$

Example 1.6: Find the length of the curve given as the intersection of the surfaces

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;\ x = \cosh\left(\frac{z}{a}\right) \text{ from the point } (a,0,0) \text{ to the point } (x, y, z).$$

Solution. The equation of the curve in the parametric form may be taken as

$$x = a\cosh u,\ y = b\sinh u,\ z = au$$

The position vector $\mathbf{r}$ of any point on the curve is given by

$$\mathbf{r}(u) = a\cosh u\ \mathbf{i} + b\sinh u\ \mathbf{j} + au\ \mathbf{k}$$

$$\dot{\mathbf{r}}(u) = a\sinh u\ \mathbf{i} + b\cosh u\ \mathbf{j} + a\ \mathbf{k}$$

$$|\dot{\mathbf{r}}(u)| = \{a^2\sinh^2 u + b^2\cosh^2 u + a^2\}^{1/2}$$

$$= \{a^2(1 + \sinh^2 u) + b^2\cosh^2 u\}^{1/2}$$

$$= \{a^2\cosh^2 u + b^2\cosh^2 u\}^{1/2}$$

$$= \sqrt{a^2 + b^2}\cosh u$$

Also limit of u are clearly from $u = 0$ to any point u .

$$\therefore \quad s = \int_0^u |\dot{\mathbf{r}}(u)| du$$

$$= \int_0^u \sqrt{a^2 + b^2} \cosh u \, du$$

$$= \sqrt{a^2 + b^2} \sinh u = \sqrt{a^2 + b^2}\, y / b$$

1.5 TANGENT LINE

Definition: The tangent line to a curve C at a point $P(u)$ of C is defined as the limiting position of a straight line L through $P(u)$ and neighboring point $Q(u+\delta u)$ on C as Q approaches P along the curve.

(i) To find the unit tangent vector to a curve: Consider two neighboring point $P(u)$ and $Q(u+\delta u)$ on C with position vectors $\mathbf{r}$ and $\mathbf{r} + \delta r$ respectively. We have

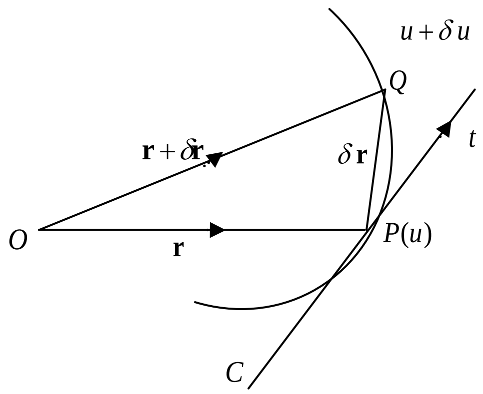
PQ=OQ-OP

Or $\delta \mathbf{r} = \mathbf{r}(u+\delta u) - \mathbf{r}u$

$$\frac{\delta \mathbf{r}}{\delta u} = \frac{\mathbf{r}(\mathrm{u}+\delta \mathrm{u}) - \mathbf{r}(\mathrm{u})}{\delta \mathrm{u}} \quad \ldots(1.18)$$

Now as $Q \to P, \delta u \to 0,$

$$\frac{\delta \mathbf{r}}{\delta u} \to \frac{d\mathbf{r}}{du} (= \dot{\mathbf{r}})$$



Therefore taking the limit $Q \to P$ of expression (1.18)

We get

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{du} = \lim_{\delta u \to 0} \frac{\mathbf{r}(u+\delta u) - \mathbf{r}(u)}{\delta u}$$

Thus we conclude that the vector $\mathbf{r}$ is parallel to the tangent line at P. The unit tangent vector is denoted by the symbol **t** and is, therefore, given by

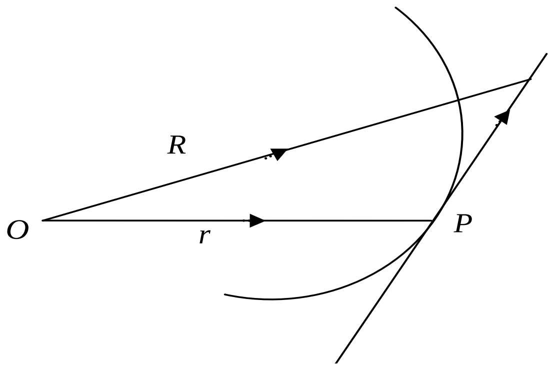
$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\dot{\mathbf{r}}}{\dot{\mathbf{s}}} \qquad \text{[Because, } \dot{\mathbf{s}} = |\dot{\mathbf{r}}|, \text{]} \qquad =$$

$$\frac{d\mathbf{r}}{du} / \frac{ds}{du} = \frac{d\mathbf{r}}{ds} = \mathbf{r}'.$$

Note that **t** always points in the direction of motion along the curve.

(ii) To find the equation of tangent line to a curve at a point: Suppose it is required to find the equation of the tangent line at a point $P(\mathbf{r})$ on the curve.

$$\mathbf{R} = \mathbf{r}(u) \qquad \ldots\ldots(1.19)$$



Consider a current point R with position $\mathbf{R}$ on the tangent line at P. We know that the vector $\dot{\mathbf{r}}$ is parallel to the tangent line at P.

Hence the vector equation of the tangent line at P is the vector sum of the position vector $\mathbf{r}$ of P and a vector in the direction of the tangent line. Hence the equation of tangent line in terms of parameter u is given by

$$\mathbf{R} = \mathbf{r} + w\mathbf{r} \qquad (1.20)$$

where w is a scalar parameter.

Again if instead of parameter u, we use parameter s (arc length), then since $\mathbf{t}$ is unit vector along the tangent at P, the equation of the tangent line at P is given by

$$\mathbf{R} = \mathbf{r} + \lambda\mathbf{t} \qquad (1.21)$$

$$\mathbf{R} = \mathbf{r} + \lambda\mathbf{r}' \quad [Because,\ \mathbf{t} = \mathbf{r}'] \qquad (1.22)$$

where, λ is a scalar parameter.

Cor.1. Tangent line in Cartesians: we may write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\ \dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$

and $R = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$

Substitute these values in eqn. (1.22) of tangent line, we get

$$X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + w(\dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k})$$

which gives on equating coefficients of **i**, **j**, **k** from both sides

$$X = x + w\dot{x},\ Y = y + w\dot{y},\ Z = z + w\dot{z}$$

i.e. $$\frac{X - x}{\dot{x}} = \frac{Y - y}{\dot{y}} = \frac{Z - z}{\dot{z}} = w$$

i.e $$\frac{X-x}{\dot{x}}=\frac{Y-y}{\dot{y}}=\frac{Z-z}{\dot{z}} \qquad (1.23)$$

This is required equation of tangent line at (x, y, z) and direction cosines of the tangent line are proportional to $(\dot{x}, \dot{y}, \dot{z})$.

If we use parameter s then substituting values in (1.21) and (1.22), we have

$$X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}+\lambda(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})$$

This gives

$$\frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'}=\lambda$$

This is the equation of tangent line and the quantities (x', y', z') are direction cosines of the tangent line.

Cor. 2. : If the equation of the curve is given as the intersection of two surfaces

$$F_1(x,y,z)=0 \textit{ and } F_2(x,y,z)=0$$

We have $$\frac{\partial F_1}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial F_1}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial F_1}{\partial z}.\frac{\partial z}{\partial u}=0$$

and $$\frac{\partial F_2}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial F_2}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial F_2}{\partial z}.\frac{\partial z}{\partial u}=0$$

or $$\dot{x}.\frac{\partial F_1}{\partial x}+\dot{y}.\frac{\partial F_1}{\partial y}+\dot{z}.\frac{\partial F_1}{\partial z}=0$$

and $$\dot{x}.\frac{\partial F_2}{\partial x}+\dot{y}.\frac{\partial F_2}{\partial y}+\dot{z}.\frac{\partial F_2}{\partial z}=0$$

Therefore, from last two relations for, $(\dot{x}, \dot{y}, \dot{z})$ we get

$$\frac{\dot{x}}{\frac{\partial F_1}{\partial y}.\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}.\frac{\partial F_2}{\partial y}}=\frac{\dot{y}}{\frac{\partial F_1}{\partial z}.\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}.\frac{\partial F_2}{\partial z}}=\frac{\dot{z}}{\frac{\partial F_1}{\partial x}.\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}.\frac{\partial F_2}{\partial x}} \qquad (1.24)$$

From which we obtain the direction cosines of the tangent. Substituting valuing of $(\dot{x}, \dot{y}, \dot{z})$ in equation (1.24) we shall get the equation of the tangent line at a point of the curve of intersection of the surface

$$F_1(x,y,z)=0 \textit{ and } F_2(x,y,z)=0$$

Example 1.7: Show that the tangent at any point of the curve whose equations are $x = 3u,\ y3u^2,\ z = 2u^3,$ makes a constant angle with the line $y = z - x = 0$.

Solution. The position vector $\mathbf{r}$ of any point of the curve is

$$\mathbf{r} = (3u, 3u^2, 2u^3)$$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{du} = (3, 6u, 6u^2)$$

$$\dot{\mathbf{s}} = |\dot{\mathbf{r}}| = (9 + 36u^2 + 36u^4)^{1/2} = (3 + 6u^2) = 3(1 + 2u^2)$$

The direction cosines of the tangent are given by

$$\frac{(3, 6u, 6u^2)}{3(1 + 2u^2)} \qquad i.e. \qquad \frac{(1, 2u, 2u^2)}{(1 + 2u^2)}$$

The equation of the given line is

$$y = z - x = 0 \qquad or \qquad \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

Its direction cosines are $(1/\sqrt{2}, 0, 1/\sqrt{2})$

$\therefore$ If θ is the required angel between the tangent and the given line

$$\cos\theta = \frac{1.1 + 2u.0 + 2u^2.1}{\sqrt{2}(1 + 2u^2)} = \frac{1 + 2u^2}{\sqrt{2}(1 + 2u^2)} = \frac{1}{\sqrt{2}}$$

Hence, θ is constant.

Example 1.8: Show that the tangent at a point of the curve of the intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and the confocal whose parameter is λ is given by

$$\frac{x(X - x)}{a^2(b^2 - c^2)(a^2 - \lambda)} = \frac{y(Y - y)}{b^2(c^2 - a^2)(b^2 - \lambda)} = \frac{z(Z - z)}{c^2(a^2 - b^2)(c^2 - \lambda)}$$

or

Find the tangent at a point of the curve of intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the confocal $\frac{x^2}{(a^2 - \lambda)} = \frac{y^2}{(b^2 - \lambda)} = \frac{z^2}{(c^2 - \lambda)} = 1$, where λ is a parameter.

Solution. The equation of the confocal to $F_1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ (1.25)

$$\text{is } F_{21} = \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} - 1 = 0 \tag{1.26}$$

Let the curve of the intersection of the surfaces (1.25) and (1.26) be $\mathbf{r}(u)$. Differentiating (1.25) and (1.26) with respect to (u), we have

$$\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} + \frac{z\dot{z}}{c^2} = 0, \qquad \frac{x\,x}{a^2 - \lambda} + \frac{y\dot{y}}{b^2 - \lambda} + \frac{z\dot{z}}{c^2 - \lambda} = 0$$

Solving for $(\dot{x}, \dot{y}, \dot{z})$ we have

$$\frac{\dot{x}}{\frac{yz}{b^2(c^2-\lambda)} - \frac{yz}{c^2(b^2-\lambda)}} = \frac{\dot{y}}{\frac{zx}{c^2(a^2-\lambda)} - \frac{zx}{a^2(c^2-\lambda)}} = \frac{\dot{z}}{\frac{xy}{a^2(b^2-\lambda)} - \frac{xy}{b^2(a^2-\lambda)}}$$

$$\text{or } \frac{\dot{x}}{\frac{yz\lambda(b^2-c^2)}{b^2c^2(c^2-\lambda)(b^2-\lambda)}} = \frac{\dot{y}}{\frac{zx\lambda(c^2-a^2)}{c^2a^2(a^2-\lambda)(c^2-\lambda)}} = \frac{\dot{z}}{\frac{xy\lambda(a^2-b^2)}{a^2b^2(b^2-\lambda)(a^2-\lambda)}}$$

$$\text{or, } \frac{\dot{x}}{a^2(b^2-c^2)(a^2-\lambda)/x} = \frac{\dot{y}}{b^2(c^2-a^2)(b^2-\lambda)/y} = \frac{\dot{z}}{c^2(a^2-b^2)(c^2-\lambda)/z}$$

The equation of required tangent is

$$x(X-x)/\dot{x} = y(Y-y)/\dot{y} = z(Z-z)/\dot{z}$$

$$i.e. \quad \frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)} = \frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)} = \frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}$$

Example 1.9:. Find the equation to the tangent at the point u on the circular helix.

$x = a\cos u,\ y = a\sin u,\ z = cu$

Solution. The vector equation of the helix is given by

$$\mathbf{r} = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + cu\,\mathbf{k}$$

$$\dot{\mathbf{r}} = -a\sin u\,\mathbf{i} + a\cos u\,\mathbf{j} + c\,\mathbf{k}$$

The equation of the tangent is

$$\mathbf{R} = \mathbf{r} + w\dot{\mathbf{r}}$$

or $\mathbf{R} = (a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + cu\,\mathbf{k}) + w(-a\sin u\,\mathbf{i} + a\cos u\,\mathbf{j} + c\,\mathbf{k})$

if $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$, then

$$X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k} = a(\cos u - w\sin u)\,\mathbf{i} + a(\sin u + w\cos u)\,\mathbf{j} + c(u+w)\,\mathbf{k}$$

which gives

$$\frac{X - a\cos u}{-a\sin u} = \frac{Y - a\sin u}{a\cos u} = \frac{Z - cu}{c}$$

It is the required equation of the tangent line.

Example 1.10: Prove the length of the curve

$$x = 2a\{\sin^{-1} t + t\sqrt{(1-t^2)}\},\ y = 2at^2,\ z = 4at$$

Between the points where $t = t_1$, and $t = t_2$ is $4\sqrt{2a(t_2 - t_1)}$. Show also that the curve is a helix drawn on a cylinder whose base is a cycloid and making an angle of 45^0 with the generators.

Solution. The positive vector $\mathbf{r}$ of any point o the curve is given by

$$\mathbf{r} = 2a(\sin^{-1} t + t\sqrt{(1-t^2)},\ y = 2at^2,\ z = 4at)$$

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \left(2a\left\{\frac{1}{\sqrt{(1-t^2)}} + \sqrt{(1-t^2)} - \frac{2t^2}{2\sqrt{(1-t^2)}}\right\}, 4at, 4a\right)$$

$$\therefore \qquad \dot{\mathbf{r}}.\dot{\mathbf{r}} = 4a^2\left\{\frac{1+t-t^2-t^2}{\sqrt{1-t^2}}\right\}^2 + 16a^2t^2 + 16a^2$$

$$= 16a^2(1-t^2) + 16a^2t^2 + 16a^2 = 32a^2$$

$\therefore$ The required length $s = \int_{t_1}^{t_2} 4\sqrt{2a}\,dt = 4\sqrt{2a}(t_2 - t_1)$

Again putting $t = \sin\theta$, in the equation of the curve.

$$x = 2a(\theta + \sin\theta\cos\theta),\ y = 2a\sin^2\theta,\ z = 4a\sin\theta$$

$$x = a(2\theta + \sin 2\theta),\ y = a(1 - \cos 2\theta),\ z = 4a\sin\theta$$

The first two equations clearly represent a cylinder whose base is cycloid and generators parallel to z-axis. Also the direction cosines of the tangent at the point θ are proportional to

$$dx/d\theta, dy/d\theta, dz/d\theta.$$

i.e. $a(2+2\cos 2\theta), a(2\sin 2\theta)$ *and* $4a\cos\theta$

i.e. $a2\cos^2\theta), \sin 2\theta, 2\cos\theta$

If ϕ is the angle between z-axis (a generator of the cylinder) and the tangent

$$\cos\phi = (0+0+1\times 2\cos\theta)/(4\cos^4\theta + 4\sin^2\theta\cos^2\theta + 4\cos^2\theta)^{1/2}$$

$$= 1/\sqrt{2} = \cos 45^0$$

Therefore $\phi = 45^0$, which was to be proved.

1.6 IMPLICIT REPRESENTATION OF CURVES

(x,y,z)

A curve in space can be determined as the intersection of two surfaces, i.e. as those points satisfying two relations of the form

$$F_1(x,y,z) = 0 \text{ and } F_2(x,y,z) = 0 \qquad (1.27)$$

If at a point (x,y,z) satisfying the above,

$$\det\begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix} \neq 0$$

then it follows from the implicit function that for some neighborhood of z, we can solve (1.27) for x and y as functions of z, obtaining a representation of the form

$$x=x(z),\ y=y(z),\ z=z \qquad (1.28)$$

with z itself the parameter. This defines at least locally a regular curve.

Example.1.11: The intersection of the two second degree surfaces $y - z^2 = 0$ and $zx - y^2 = 0$ is the third degree curve $x = t^3, y = t^2, z = t$ together with the x-axis, $x = t$,y=0, z=0. These are obtained as follows. For $z \neq 0$ we can solve the given relations for x and y in terms of z, obtaining

$$y = z^3, x = \frac{y^2}{z} = \frac{z^4}{z} = z^3,$$

$$\Rightarrow x = t^3, y = t^2, z = t \text{ or if we let z=t} \qquad (1.29)$$

If z=0, then y=z^2=0 and x can be arbitrary. This gives the x-axis, x=t, y=0, z=0. Observe that the point (0,0,0) is the intersection of the two curves.

1.7 CONTACT OF n^{th} ORDER OF A CURVE AND A SURFACE

Definition: If $P, P_1, P_2, ..., P_n$ points of a given curve lie on a given surface and $P, P_1, P_2, ..., P_n$ coincide with P, the curve and the surface are said to have the contact of n^{th} order at the point P.

(i) **To find the condition that a curve and a surface have a contact of n^{th} order.**

Let the equation to the curve C and the surface S be

$$r = [\phi_1(u), \phi_2(u), \phi_{31}(u),] \text{ and } f(x, y, z) = 0 \text{ respectively}$$

and let the class of curve C be sufficiently high. The value of 'u' which gives the point of intersection of the curve and the surface (i. e. the points common to C and S) are zeros of the functions

$$F(u) = \{\phi_1(u), \phi_2(u), \phi_{31}(u),\}$$

[Substituting the values of x, y, z (which are $x = \phi_1(u)$ etc. from the curve in the equation of surface)].

Let u_0 be such a zero, then expressing $F(u)$ by Taylor's theorem

$$F(u) = F(u - u_0 + u_0)$$

$$= F(h + u_0)$$

$$F(u_0) + hF'(u_0) + \frac{h^2}{2!}F''(u_0) + ... + \frac{h^n}{n!}F^n(u_0) + 0(h^{n+1}) \qquad (1.30)$$

where $h = u - u_0$

But $F(u_0) = 0$, therefore (1) reduces to

$$F(u) = hF'(u_0) + \frac{h^2}{2!}F''(u_0) + ... + \frac{h^n}{n!}F^n(u_0) + 0(h^{n+1})$$

Now u_0 is called a simple zero of $F(u)$ if $F'(u) \neq 0$ then in this case C and S are said to have a simple intersection at the point $\mathbf{r}(u_0)$.

There is a contact of first order at $u = u_0$, if $F'(u_0) = 0$ and $F''(u_0) \neq 0$ for $F(u)$ is of the second order of h, u_0 is a double zero of $F(u)$ and C and S have two point contact.

There is a contact of 2nd order at $u = u_0$, if $F'(u_0) = F''(u_0) = 0$ and $F'''(u_0) \neq 0$ for $F(u)$ is of 3rd order of h, u_0 is a triple zero of $F(u)$ and C and S have three point contact. Similarly there is a contact of nth order at $u = u_0$, if

$$F'(u_0) = F''(u_0) = ... = F^n(u_0) = 0 \text{ and } F^{n+1}(u_0) \neq 0 \qquad (1.31)$$

also then C and S are said to have (n+1) point contact.

(ii) INFLEXIONAL TANGENT

Definition: At a point P where $r'' = 0$, the tangent line is called inflexional and the point P is called the point of inflexion.

Alternative Definition: Let the equation to the line through a point (x_1, y_1, z_1) on a given surface be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = (u)$$

The inflexional tangent are the lines which have three point contact inside the given surface where $u = 0$.

Example 1.12: Find the plane that has three point contacts at the origin with the curve

$$x = u^4 - 1, y = u^3 - 1, z = u^2 - 1$$

Solution. Let the equation of the plane at the origin with the curve

$$lx + my + nz = 0 \qquad (1)$$

The equations of the given curve are

$$x = u^4 - 1, y = u^3 - 1, z = u^2 - 1 \qquad (2)$$

At the origin, $u^4 - 1 = 0, u^3 - 1 = 0, u^2 - 1 = 0$

Clearly $u = 1$ satisfies all of these three equations.

$\therefore$ at the origin, we have $u = 1$.

Now the points of intersections of the curve (2) and the surface (1) are given by the zeros of the function

$$F(u) = l(u^4 - 1) + m(u^3 - 1) + n(u^2 - 1)$$

or $$F(u) = lu^4 + mu^3 + nu^2 - l - m - n \qquad (3)$$

For three point contact, we should have $F'(u) = 0$

$$F''(u) = 0 \qquad \text{where } F'(u) = dF/du$$

$$\text{Now } F'(u) = 4lu^3 + 3mu^2 + 2nu = 0 \qquad (4)$$

$$\text{and } F''(u) = 12lu2 + 6mu + 2n = 0 \qquad (5)$$

At the origin i.e. at $u = 1$, the equation (4) and (5) becomes

$$4l + 3m + 2n = 0, \qquad 12l + 6m + 2n = 0$$

Solving, $m = -(8/3)l, n = 2l$

Putting values in (1), the equation of the required plane is given by

$$lx - (8/3)ly + 2l = 0 \qquad or \qquad 3x - 8y + 6z = 0.$$

Example 1.13: Find the lines that have four point contact at (0,0,1) with the surface

$$x^4 + 3xy + x^2 - y^2 - z^2 + 2yz - 3xy - 2y + 2z = 1$$

Solution. Any line through (0,0,1) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z-1}{n} = t \qquad \text{(say)} \qquad (1)$$

$\therefore \qquad x = lt, y = mt, z = 1 + nt$

$$\therefore \qquad F(t) = l^4t^4 3lmt^2(nt+1) + l^2t^2 - m^2t^2 - (nt+1)^2$$
$$+ 2mt(nt+1) - 3lmt^2 - 2mt + 2(nt+1) - 1 = 0$$

$$= l^4t^4 + 3lmnt^3 + t^2(l^2 - m^2 - n^2 + 2mn) = 0$$

For four point contact, we should have

$F'(u) = 0$, $F''(u) = 0$ and $F'''(u) = 0$ at (0,0,1) where $F'(u) = dF/du$, at the point (0,0,1) we clearly have $t = 0$. Now at $t = 0$, we have the following,

$$F'''(u) = 0 \quad \Rightarrow \quad lmn = 0 \qquad (2)$$

$$F''(u) = 0 \quad \Rightarrow \quad l^2 - m^2 - n^2 + 2mn = 0 \qquad (3)$$

Solving (2) and (3), we get

From (2) : $\qquad l = 0, \; m = 0 \; or \; n = 0$

If $l = 0$, from (3) $\qquad m = n \qquad (4)$

The required line is $\qquad x/0 = y = z - 1$

If $m = 0$, from (3) $\qquad l = \pm n \qquad (5)$

The required line is $x/1 = y/0 = (z-1)/\pm 1$

Lastly if $n = 0$, from (4) $l = \pm m$ (6)

The required line is $x/1 = y/\pm 1 = (z-1)/0$

Equation (4), (5) and (6) are the equations of required lines.

Example 1.14: Show that if the circle $lx + my + nz = 0,\ x^2 + ^2 + z^2 = 2cz$ has three point contact at the origin with the paraboloid

$$ax^2 + by^2 = 2z, \qquad then \quad c = (l^2 + m^2)/(bl^2 + am^2)$$

Solution. Let a point on the circle

$$ax^2 + by^2 = 2z, \qquad x^2 + y^2 + z^2 = 2cz$$

be denoted by $x = f_1(t), y = f_2(t),\ z = f_3(t)$ (1)

Substituting the values of x, y, z in $ax^2 + by^2 = 2z$ and differenting w.r.t. ‘t’, we get

$$ax\dot{x} + by\dot{y} = \dot{z}$$

$$a\dot{x}^2 + b\dot{y}^2 + ax\ddot{x} + by\ddot{y} = \ddot{z} \qquad (2)$$

where x, y, z are as in (1) and dots denote differentiation w.r.t. ‘t’.

Proceeding in a similar manner with the equation to the circle

$$l\dot{x} + m\dot{y} + n\dot{z} = 0 \qquad (3)$$

$$l\ddot{x} + m\ddot{y} + n\ddot{z} = 0$$

$$x\dot{x} + y\dot{y} + z\dot{z} = c\dot{z} \qquad (4)$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + x\ddot{x} + y\ddot{y} + z\ddot{z} = c\ddot{z} \qquad (5)$$

At the origin from (4), $c\ddot{z} = 0, i.e. \dot{z} = 0$ (6)

Hence from (3), $l\dot{x} + m\dot{y} = 0$ (7)

Also (3) and (7) at the origin reduce to

$$a\dot{x}^2 + b\dot{y}^2 = \dot{z}^2 \qquad (8)$$

$$\dot{x}^2 + \dot{y}^2 = c\dot{z}^2 \qquad (9)$$

Dividing (9) by (8), we get

$$c = \frac{\dot{x}^2 + \dot{y}^2}{a\dot{x}^2 + b\dot{y}^2} = \frac{\frac{m^2}{l^2}\dot{y}^2 + \dot{y}^2}{\frac{am^2}{l^2}\dot{y}^2 + b\dot{y}^2} \qquad \text{[from (9)]}$$

$$\frac{m^2 + l^2}{am^2 + bl^2}$$

Example 1.15: Determine a, h, b, so that the paraboloid $2z = ax^2 2hxy + by^2$ may have the closet possible contact at the origin with curve $x = t^3 - 2t^2 + 1;\ y = t^3 - 1;\ z = t^2 - 2t + 1$

Find also the order of contact.

Solution. Here

$$F(t) = 2(t^2 - 2t + 1) - a(t^3 - 2t^2 + 1)^2 - 2h(t^3 - 2t^2 + 1)(t^3 - 1) - b(t^3 - 1)^2$$

$$\frac{dF}{dt} = 4t - 4 - 2a(t^3 - 2t^2 + 1)^2(3t^2 - 4t) - 2h(6t^5 - 10t^4 + 4t) - 2b(t^3 - 1)3t^2$$

which is clearly zero at t=1 i.e. at the origin.

$$\frac{d^2F}{dt^2} = 4 - 2a(t^3 - 2t^2 + 1)(6t - 4) - 2a(3t^2 - 4t)^2$$
$$- 2h(30t^4 - 40t^3 + 4) - 6b.3t^4$$

Now for contact of 2nd order at the origin i.e. at t=1, we have

$$\frac{d^2F}{dt^2} = 4 - 2a + 12h - 18b = 0 \qquad (1)$$

$$\frac{d^3F}{dt^3} = -2a(t^3 - 2t^2 + 1)6 - 2a(6t - 4)(3t^2 - 4t) - 4a(3t^2 - 4t)(6t - 4)$$
$$- 2h(120t^3 - 120t^2) - 2b.(24t^3 - 6) - 72bt^3$$

Now for contact of 3rd order at the origin i.e. at t=1, we have

$$\frac{d^3F}{dt^3} = 4a + 8a - 108b = 0 \quad or 12a - 108b = 0 \qquad (2)$$

Again $\dfrac{d^4F}{dt^4} = -12a(3t^2 - 4t) - 2a(6t - 4)^2 - 2a(3t^2 - 4t)6$

$$- 4a(3t^2 - 4t)6 - 4a(6t - 4) - 2h(360t^2 - 240t) - 2b.(72t^2) - 216bt^2$$

Now for contact of 4th order at the origin i.e. at t=1, we have

$$\frac{d^4F}{dt^4} = 12a - 8a + 12a + 24a - 16a - 240h - 360b = 0$$

$$i.e. \quad a - 10h - 15b = 0 \qquad (3)$$

Now we have got three equations to find out a, h, b hence solving (1), (2) and (3), we have

$$a = \frac{5}{6},\ b = \frac{5}{54},\ h = -\frac{1}{18}$$

Also for these values $\frac{d^5F}{dt^5} \neq 0$

Hence when $\frac{a}{45} = \frac{b}{5} = \frac{h}{-3} = \frac{1}{54}$, there is a contact of 4th order at the origin.

Example 1.16: Find the inflexional tangent at (x_1, y_1, z_1) on the surface $y^2 z = 4ax$

Solution. The equation to a line through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = u \qquad \text{(say)} \qquad (1)$$

The inflexional tangents are the lines which have three-point contact inside the surface where $u = 0$.

From equation (1) substituting the values of (x, y, z) in the equation of surface $y^2 z = 4ax$ we get

$$F(u) = (mu + y_1)^2 (nu + z_1) - 4a(lu + x_1) = 0 \qquad (2)$$

For three point contact, we have

$$F'(u) = (mu + y_1)2m.(nu + z_1) + (mu + y_1)^2 n - 4al = 0 \qquad (3)$$

$$F''(u) = 2m^2 (nu + z_1) + 2mn(mu + y_1) + 2mn(mu + y_1) = 0 \qquad (4)$$

At $u = 0$, the above equations (2), (3) and (4) reduce to

$$y_1^2 z_1 - 4ax_1 = 0 \qquad (5)$$

$$2my_1 z_1 - ny_1^2 - 4al = 0 \qquad (6)$$

$$2m^2 z_1 - 2mny_1 + 2mny_1 = 0$$

or $\quad mz_1 + 2ny_1 = 0$

or $\quad n = -mz_1/(2y_1) \qquad (7)$

Using (6), (7) becomes

$$2my_1z_1 - \frac{mz_1}{2y_1}y_1^2 - 4al = 0 \qquad or \qquad l = \frac{3my_1z_1}{8a}$$

Substituting values of l and n in (4), we get

$$\frac{x-x_1}{\frac{3my_1z_1}{8a}} = \frac{y-y_1}{m} = \frac{z-z_1}{-\frac{mz_1}{2y_1}}$$

$$\frac{x-x_1}{\frac{3y_1^2z_1}{4a}} = \frac{y-y_1}{2y_1} = \frac{z-z_1}{-z_1}$$

or $\dfrac{x-x_1}{3x_1} = \dfrac{y-y_1}{2y_1} = \dfrac{z-z_1}{-z_1}$ which is the required equation of the inflexional tangent.

1.8 CHECK YOUR PROGRESS

SA1: Define a space curve and explain your definition by means of examples.

SA2: Deduce the formulae for the arc length of a curve between two points in vector form.

SA3: Find the length of one complete turn of the circular helix

SA3: Obtain the direction cosines of the tangent to the curve of intersection of the surfaces $F_1(x, y, z) = 0$ *and* $F_2(x, y, z) = 0$

SA4: Find the direction cosines and equation of tangents to the curve of intersection of the surfaces $F_1(x, y, z) = 0$ *and* $F_2(x, y, z) = 0$

SA5: Show that $\dfrac{d\vec{r}}{ds}$ is unit tangent vector to a space curve at the point $P(\vec{r})$.

SA6: Define the contact of n[th] order of a curve and surface and find the condition for this type of contact.

SA7: Show that a curve is a straight line if all tangent lines are parallel.

SA8: Find the intersection of the x_1x_2 plane and the tangent line to the curve $x = (1+t)e_1 - t^2e_2 + (1+t^3)e_3$ at t=1. Ans. (4/3, 1/3, 0)

SA9: Show that tangent vectors along the curve $x = ae_1 + bt^2 e_2 + t^3 e_3$ where $2b^2=3a$ make a constant angle with the vector $a = e_1 + e_3$.

SA10: Find the length of one complete turn on the circular helix

1.9 SELF ASSESSMENT TEST

i.) Find a representation of the intersection of the cylinders $z^2 = x$, $y^2 = 1 - x$ that does not involve radicals. **Hint:** $y^2 + z^2 = 1$

Ans: $x = \cos^2\theta$, $y = \sin\theta$, $z = \cos\theta$, $0 \le \theta \le 2\pi$.

ii) The conchold of Nicomedes in polar coordinates is $r = \dfrac{a}{\cos\theta} + c$, $a \ne 0, c \ne 0$ $-\pi \le \theta \le \pi$.

Sketch and find a representation in rectangular coordinates.

Ans: $x = a + c\cos\theta$, $y = a\tan\theta + c\sin\theta$.

iii) Show that the representation $x = it + j(t^2 + 2) + k(t^3 + t)$ is regular for all t and sketch the projections on the xz and xy planes.

iv) Show that $\theta = 3t^5 + 10t^3 + 15t + 1$ is an allowable change of parameter for all t.

v) Compute the length of the arc $x = e^t \cos t\, i + e^t \sin t\, j + e^t\, k$, $0 \le t \le \pi$.

Ans: $3(e^{\pi} - 1)$.

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CHAPTER-2

CURVES IN SPACE R^3(I)

Objectives: In continuation of first chapter, in this chapter students will learn about plane of curvature or Osculating plane, Osculating plane at the point of inflexion, Tangent plane at any point of the surface, normal plane, Principal normal, Binormal and their directions, Equations of principal normal, Binormal and their relationship, Curvature, Torsion of the curve and plane.

2.1 INTRODUCTION

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. In the first chapter we have studied different form of the curve and its representation, class, contact of nth order of a curve and surface, tangent line, and inflection tangent recall that the tangent line at a point on a curve can be defined as the limiting position of a line passing through two neighboring points on the curve as the two points approach the given point. In this way a line is obtained that in a sense best fits the curve at a point. Similarly, the osculating plane at a point can be defined as the limiting position of a plane passing through three neighbouring points on the curve as the points approach the given point. The tangent line and osculating plane are examples of geometric figures which have a certain order of contact with the curve.

Definition 2.1: Let C be a curve of class ≥ 2, consider two neighbouring points P and Q on C. Then the osculating plane of C at P is the limiting position of the plane which contains the tangent line at P and contains the point Q as $Q \to P$.

Definition 2.2: Let C be a curve of class ≥ 2, consider two neighbouring points P and Q on C. Then the osculating plane of C at P is the limiting position of the plane which contains the tangent line at P and is parallel to the tangent at Q as $Q \to P$.

Alternative Definition: If P, Q, R be three points on a curve , the limiting position of the plain PQR, when Q and R independently tend to P, is called the osculating plane at the point P. The definition implies that an osculating plane has a contact of 2nd order or three point contact.

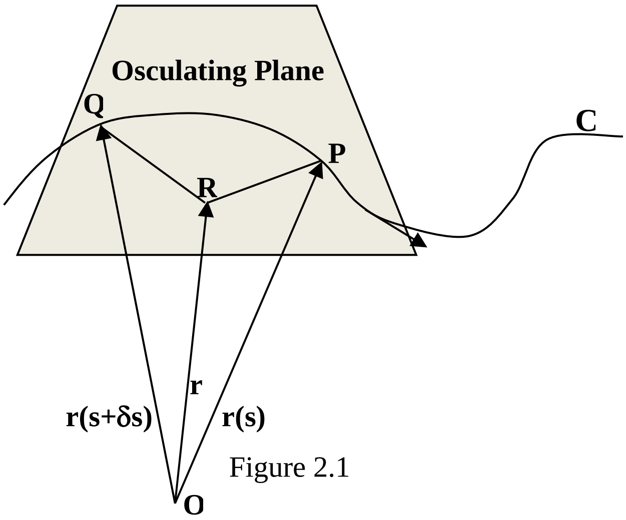

Figure 2.1

(A) To find the equation of the osculating plane (or Plane of curvature (figure 2.1)**):**

We shall find out the equation of the osculating plane separately by using definitions 1 and 2.

Using definition 1: Let the equation of the curve C be $\mathbf{r} = \mathbf{r}(s)$ where C is the class ≥ 2. Let $P(s)$ and $Q(s+\delta s)$ with position vectors respectively $\mathbf{r}(s)$ and $\mathbf{r}(s+\delta s)$ be two neighbouring points on the curve C where the arc length s is measured from some fixed on C. Let the position vector of a current point R, on the plane containing tangent line at P and containing the point Q be **R**. Now the vectors

$$\mathbf{PR} = \mathbf{R} - \mathbf{r}(s),\ \mathbf{t} = \mathbf{r}'(s)$$

and $$\mathbf{PQ} = \mathbf{r}(s+\delta s) - \mathbf{r}(s)$$

lie in the plane RPQ and therefore their scalar triple product must be zero, *i.e.* the equation of the plane RPQ is given by

$$[\mathbf{R} - \mathbf{r}(s), \mathbf{r}'(s), \mathbf{r}(s+\delta s) - \mathbf{r}(s)] = 0 \tag{2.1}$$

Now expanding $\mathbf{r}(s+\delta s)$ by Taylor's series in ascending powers of δs, we have

$$\mathbf{r}(s+\delta s) = \mathbf{r}(s) + \delta s\,\mathbf{r}'(s) + \frac{(\delta s)^2}{2!}\mathbf{r}''(s) 0\{(\delta s)^3\} \tag{2.2}$$

Equation (2.1) may be written as

$$\{\mathbf{R} - \mathbf{r}(s)\}\mathbf{r}'(s) \times \{\mathbf{r}(s+\delta s) - \mathbf{r}(s)\} = 0$$

or $$\{\mathbf{R}-\mathbf{r}(s)\}\mathbf{r}'(s)\times\left[\delta s\,\mathbf{r}'(s)+\frac{(\delta s)^2}{2!}\mathbf{r}''(s)+0\{(\delta s)^3\}\right]=0 \qquad \text{using (2.2)}$$

or $$\{\mathbf{R}-\mathbf{r}(s)\}\mathbf{r}'(s)\times\left[\frac{(\delta s)^2}{2!}\mathbf{r}''(s)+0\{(\delta s)^3\}\right]=0 \quad [\because \quad r'(s)\times r'(s)=0\,]$$

or $$\{\mathbf{R}-\mathbf{r}(s)\}\mathbf{r}'(s)\times[\mathbf{r}''(s)+0(\delta s)]=0$$

Hence the limiting position of the plane PQR as $Q\to P$

i.e. as $\delta s\to 0$ is

$$\{\mathbf{R}-\mathbf{r}(s)\}.\mathbf{r}'(s)\times\{\mathbf{r}''(s)\}=0$$

i.e. $$[\mathbf{R}-\mathbf{r}(s),\mathbf{r}'(s),\mathbf{r}''(s)]=0 \tag{2.3}$$

This is the equation of the osculating plane in parameter s at the point P on C. If the arc length s be measured from P, then at P, $s=0$ and equation (3) of the osculating plane becomes

$$[\mathbf{R}-\mathbf{r}(0),\mathbf{r}'(0),\mathbf{r}''(0)]=0 \tag{2.3$'$}$$

By Definition 2 : Here we shall find the equation of osculating plane in general parameter u.

Let $P(u)$ and $Q(u+\delta u)$ be two neighbouring points on the curve C. The tangent at these two points is parallel to the vectors $\dot{\mathbf{r}}(u)$ and $\dot{\mathbf{r}}(u+\delta u)$ respectively.

Therefore the plane through the tangents at $P(u)$ and $Q(u+\delta u)$ is perpendicular to the vector $\dot{\mathbf{r}}(u)\times\dot{\mathbf{r}}(u+\delta u)$ i.e. to the vector $\dot{\mathbf{r}}(u)\times[\dot{\mathbf{r}}(u+\delta u)-\dot{\mathbf{r}}(u)]$ $[\because \quad \dot{\mathbf{r}}\times\dot{\mathbf{r}}=0\,]$

i.e. $$\dot{\mathbf{r}}(u)\times\frac{\dot{\mathbf{r}}(u+\delta u)-\dot{\mathbf{r}}(u)}{\delta u}$$

As $Q\to P$, $\delta u\to 0$ in the limit the osculating plane is perpendicular to the vector

$$\dot{\mathbf{r}}(u)\times\ddot{\mathbf{r}}(u)=0\,.$$

If R be the position vector of any current point on the osculating plane, the equation of the osculating plane may be written as

$$(\mathbf{R}-\mathbf{r})\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=0$$

$$[\mathbf{R}-\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}]=0 \tag{2.4}$$

Corollary. In Cartesian: Let

$$\mathbf{R} = \xi\,\mathbf{i} + \eta\,\mathbf{j} + \zeta\,\mathbf{k}, \qquad \text{and} \qquad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Substituting these values in (4) the equation of the osculating plane is given by

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Remark 1: The definitions 1 and 2 of the osculating plane are equivalent since equation (2.4) can be obtained from equation (2.3). We have

$$\mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du} / \frac{ds}{du} = \frac{\dot{\mathbf{r}}}{\dot{\mathbf{s}}}$$

$$\mathbf{r}' = \frac{\dot{s}.\ddot{r} - \dot{r}\,\ddot{s}}{\dot{s}^2}.\,\dot{s} \qquad \left[\because \quad \frac{ds}{du} = \dot{\mathbf{s}}\right]$$

Substituting in equation (2.3), we have

$$\left[\mathbf{R} - \mathbf{r}\,\frac{\dot{\mathbf{r}}}{\dot{\mathbf{s}}}, \frac{\dot{\mathbf{s}}\ddot{\mathbf{r}} - \dot{\mathbf{r}}\ddot{\mathbf{s}}}{\dot{\mathbf{s}}^3}\right] = 0$$

or $\qquad [\mathbf{R} - \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0$

which is the equation (2.4) of the osculating plane.

Also if we use the same parameter s in finding equation (2.4), then equations (2.3) and (2.4) coincide. Hence definitions 1 and 2 of osculating plane are equivalent.

Remark 2: The equation (2.3') of the osculating plane are may be written as

$$[\mathbf{R} - \mathbf{r}, \mathbf{t}, \mathbf{t}'] = 0$$

(B) Osculating plane at the point of inflexion

Theorem: Show that when the curve is analytic, there exist a definite osculating plane at a point of inflexion P provided the curve is not a straight line.

Proof: We know that $\mathbf{r}'(=\mathbf{t})$ is a unit tangent vector, therefore

$$\mathbf{r}'^2 = 1$$

Differentiating w.r.t. ‘t’, we get

$$\mathbf{r}' \cdot \mathbf{r}'' = 0$$

Again differentiating, we get

$$\mathbf{r}''\cdot\mathbf{r}''+\mathbf{r}'\cdot\mathbf{r}'''=0$$

or $$\mathbf{r}'\cdot\mathbf{r}'''=0 \qquad (2.5)$$

[$\because \mathbf{r}''=0$ at the point of inflexion]

If $\mathbf{r}'''\neq 0$, then $\mathbf{r}'$ is linearly independent of $\mathbf{r}'''$. Differentiating successively (2.5) and applying above argument, we shall get

$$\mathbf{r}''\cdot\mathbf{r}^m=0,\quad m\geq 2$$

where $\mathbf{r}^m$ is the first non zero derivative of $\mathbf{r}$ at point P. Therefore if $\mathbf{r}^m\neq 0$, we have from equation (2.2)

$$\mathbf{r}(s+\delta s)-\mathbf{r}(s)=\frac{(\delta s)^m}{m!}.\mathbf{r}^m(s)+O\{(\delta s)^{m+1}\} \qquad \text{as } \delta s\to 0$$

Hence the equation (2.1) of the osculating plane at P becomes

$$[\mathbf{R}-\mathbf{r}(s),\mathbf{r}'(s)\ ,\mathbf{r}^m(s)]=0 \qquad (2.6)$$

Again if for all $m\geq 2$ the derivative $\mathbf{r}^m=0$, we conclude $\mathbf{r}'(=\mathbf{t})$ is constant (since the curve under consideration is analytic) i.e. the tangent vector is same at each point of the curve and hence the curve is a straight line.

Hence equation (2.6) is the equation of osculating plane at a point of inflexion P when the curve is not straight line.

2.2 TO FIND THE TANGENT PLANE AT ANY POINT OF THE SURFACE f(x,y,z)=0

If s is the arc length measured from a fixed point upto the point $P(x,y,z)$, we have

$$\frac{\partial f}{\partial x}.\frac{dx}{ds}+\frac{\partial f}{\partial y}.\frac{dy}{ds}+\frac{\partial f}{\partial z}.\frac{dz}{ds}=0$$

or $$x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0 \Rightarrow \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right).(x'\,y',z')=0 \qquad (2.7)$$

We know that the vector $(x'\,y',z')$ i.e. $(\mathbf{r}'=x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})$ is the unit tangent vector to the curve at P, hence (2.7) shows that $\mathbf{r}'$ is perpendicular to the vector $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)(=\nabla f\ \ say)$. Thus all the tangent line to the surface at P are perpendicular to this vector ∇f, and therefore lie in the plane through P perpendicular to this vector. This plane is called the **tangent plane** to the surface at P. Hence if **R** is the position vector of a current point and **r** the position vector of P, then the equation of the **tangent plane** is given by

$$(\mathbf{R}-\mathbf{r})\cdot\nabla f = 0$$

(A) Normal Plane: The plane through P and perpendicular to the tangent line at P is called the **normal plane** at P of the curve. Clearly its equation will be

$$(\mathbf{R}-\mathbf{r}).\mathbf{r}' = 0 \qquad \text{or} \qquad (\mathbf{R}-\mathbf{r}).\mathbf{t} = 0 \tag{2.8}$$

Cor.1. : The equation of the **normal plane in Cartesian** is given by

$$(X-x)\,x' + (Y-y)\,y' + (Z-z)\,z' = 0 \tag{2.9}$$

Cor.2. : The normal plane is perpendicular to the osculating plane

We clearly see from (2.8), that the normal plane is perpendicular to the vector $\dot{\mathbf{r}}$, but the osculating plane is perpendicular to the vector $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$.

Now $\dot{\mathbf{r}}\cdot(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}) = \dot{\mathbf{r}}\times\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = 0$ $\qquad [\dot{\mathbf{r}}\times\dot{\mathbf{r}} = 0]$

which shows that the vectors $\dot{\mathbf{r}}$ and $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$ are at right angle. Hence the result follows.

2.3 TO FIND THE OSCULATING PLANE AT A POINT OF A SPACE CURVE GIVEN BY THE INTERSECTION OF SURFACES $\mathbf{F(r)} = 0;\ \boldsymbol{\psi}(\mathbf{r}) = 0$

The tangent plane to the given surfaces at $P(\mathbf{r})$ are given by

$$(\mathbf{R}-\mathbf{r})\cdot\nabla f = 0, \qquad (\mathbf{R}-\mathbf{r})\cdot\nabla\psi = 0$$

Let $\quad F \equiv (\mathbf{R}-\mathbf{r})\cdot\nabla f - \mu(\mathbf{R}-\mathbf{r})\cdot\nabla\psi = 0 \qquad (2.10)$

be the plane through the line of intersection of the two tangent planes i.e. through the tangent line to the curve of intersection of the two surfaces.

If (2.10) be the equation of the osculating plane at P, it must have three point contact with the curve at P. Therefore the required conditions are

$$F = 0, \dot{F} = 0, \ddot{F} = 0$$

dots denotes the differentiation w.r.t. 'μ'.

$\dot{F} = 0$ gives $\mathbf{R}\cdot\nabla f + (\mathbf{R}-\mathbf{r})\cdot\nabla f^{\bullet} - \mu\dot{\mathbf{R}}\cdot\nabla\psi - \mu(\mathbf{R}-\mathbf{r})\cdot(\nabla\psi)^{\bullet} = 0 \qquad (2.11)$

At P, $\mathbf{R} = \mathbf{r}$, condition (2.11) reduces to

$$\mathbf{R}\cdot\nabla f - \mu\,\mathbf{r}\cdot\nabla\psi = 0 \tag{2.12}$$

which is an identity, tangent and normal being orthogonal,

$$\therefore \qquad \overset{\bullet}{\mathbf{r}} \cdot \nabla f = 0, \quad \overset{\bullet}{\mathbf{r}} \cdot \nabla \psi = 0 \tag{2.13}$$

Similarly $\ddot{F} = 0$ yields

$$\ddot{\mathbf{r}} \cdot \nabla f - \mu \ddot{\mathbf{r}} \cdot \nabla \psi = 0 \tag{2.14}$$

or
$$\mu = \frac{\ddot{\mathbf{r}} \nabla f}{\ddot{\mathbf{r}} \nabla \psi} \tag{2.15}$$

Differentiating (2.13) w.r.t. u, we get

$$\ddot{\mathbf{r}} \cdot \nabla f + \dot{\mathbf{r}} \cdot (\nabla f)^{\bullet} = 0\,; \quad \ddot{\mathbf{r}} \cdot \nabla \psi + \dot{\mathbf{r}} \cdot (\nabla \psi)^{\bullet} = 0$$

which gives $\dfrac{\dot{\mathbf{r}} \cdot (\nabla f)^{\bullet}}{\dot{\mathbf{r}} \cdot (\nabla \psi)^{\bullet}} = \dfrac{\ddot{\mathbf{r}} \cdot (\nabla f)}{\ddot{\mathbf{r}} \cdot (\nabla \psi)} = \mu$ from (2.15)

Substituting of u in (2.10) yields

$$\frac{(\mathbf{R} - \mathbf{r}) \cdot (\nabla f)}{(\mathbf{R} - \mathbf{r}) \cdot (\nabla \psi)} = \frac{\dot{\mathbf{r}} \cdot (\nabla f)^{\bullet}}{\dot{\mathbf{r}} \cdot (\nabla \psi)^{\bullet}}$$

or
$$\frac{(\mathbf{R} - \mathbf{r}) \cdot (\nabla f)}{\dot{\mathbf{r}} \cdot (\nabla f)^{\bullet}} = \frac{(\mathbf{R} - \mathbf{r}) \cdot (\nabla \psi)}{\dot{\mathbf{r}} \cdot (\nabla \psi)^{\bullet}} \tag{2.16}$$

which is the required equation of the osculating plane at $P(\mathbf{r})$.

Cor. In Cartesians: Let $f(\mathbf{r}) = f(x, y, z)$, $\psi(\mathbf{r}) = \psi(x, y, z)$

$$\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}, \qquad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k} \qquad \left[\text{where, } f_x = \frac{\partial f}{\partial x}\right]$$

$$(\nabla f)^{\bullet} = \sum (f_{xx}\,\dot{x} + f_{yy}\,\dot{y} + f_{zz}\,\dot{z})\,\mathbf{i}$$

Substituting in (2.16) of the osculating plane, we get

$$\frac{(X - x) f_x + (Y - y) f_y + (Z - z) f_z}{(x^2 f_{xx} + \ldots\ldots + 2\dot{y}\dot{z} f_{yz})} = \frac{(X - x)\psi_x + (Y - y)\psi_y + (Z - z)\psi_z}{(x^2 \psi_{xx} + \ldots\ldots + 2\dot{y}\dot{z}\psi_{yz})} \tag{2.16'}$$

Example 2.1: For the curve $x = 3t, y = 3t^2, z = 2t^3$, show that any plane meets it in three points and deduce the equation to the osculating plane at $t = t_1$.

Solution. Let the equation of the plane be

$$Ax + By + Cz + D = 0$$

$\therefore \quad F(t) \equiv 3At + 3Bt^2 + 2Ct^3 + D = 0$

which is cubic in t. Hence the plane meets the given curve in three points.

Also $\quad \dot{x} = 3,\ \dot{y} = 6t, \dot{z} = 6t^2 \qquad [\dot{x} = dx/dt]$

$\ddot{x} = 0,\ \ddot{y} = 6, \ddot{z} = 12t$

Hence the equation of osculating plane at the point t_1 is

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 3t_1^3 \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix} = 0$$

or $2t_1^2 x - 2t_1 y + z = 2t_1^3$ is the required equation of the osculating plane at $t = t_1$.

2nd Method: The position vector **r** of any point on the curve is given by

$r = (3t, 3t^2, 2t^3) \qquad \therefore \ \dot{\mathbf{r}} = (3, 6t, 6t^2) = 3(1, 2t, 2t^2)$

$\ddot{\mathbf{r}} = 6(0, 1, 2t) \qquad$ [here, $\dot{\mathbf{r}} = dr/dt \ \ etc.$]

$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 18(2t^2, -2t, 1)$

The equation of the osculating plane at $t = t_1$ is given by

$\{(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - (3t_1\,\mathbf{i} + 3t_1^2\,\mathbf{j} + 2t_1^3\mathbf{k})\}.18(2t_1^2\mathbf{i} - 2t_1\mathbf{j} + \mathbf{k}) = 0$

{using $[\mathbf{R} - \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0$ and $\mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ }

or $\quad (x - 3t_1)(2t_1^2) + (y - 3t_1^2)(-2t) + (z - 2t_1^2)(1) = 0$

or $\quad 2t_1^2 x - 2t_1 y + z = 2t_1^3$

Example 2.2: Find the equation of the osculating plane at a general point on the curve given by $\mathbf{r} = (u, u^2, u^3)$. Show that the osculating planes at any three points of this curve meet at a point lying in the plane determined by these three points.

Solution. $\mathbf{r} = (u, u^2, u^3)$

$\dot{\mathbf{r}} = (1, 2u, 3u^2); \quad \ddot{\mathbf{r}} = (0, 2, 6u)$

$\therefore \qquad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (1, 2u, 3u^2) \times (0, 2, 6u)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 2(3u^2\,\mathbf{i} - 3u\,\mathbf{j} + \mathbf{k})$$

The equation of the osculating plane at a general point (u, u^2, u^3) is given by

$$[\mathbf{R}-\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0 \qquad \text{i.e. } (\mathbf{R}-\mathbf{r}).(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = 0$$

or $[(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - (u\,\mathbf{i} + u^2\,\mathbf{j} + u^3\mathbf{k})].2(3u^2\,\mathbf{i} - 3u\,\mathbf{j} + \mathbf{k}) = 0$

or $3(x-u)u^2 - 3(y-u^2)u + (z-u^3) = 0$

or $3u^2x - 3uy + z - u^3 = 0$ (1)

Let u, u^2, u^3 be the three distinct values of the parameter. The osculating planes at these three points are linearly independent and these planes meet at a point, say (x_0, y_0, z_0). The point (x_0, y_0, z_0) lies on (1) i.e. the parameters u, u^2, u^3 will satisfy the condition

$$u^3 - 3u^2x_0 + 3uy_0 - z_0 = 0 \qquad (2)$$

Suppose that the equation of the plane passing through these three points is given by

$$ax + by + cz + d = 0 \qquad (3)$$

The parameters must therefore satisfy the condition

$$au + bu^2 + cu^3 + d = 0$$

$$\text{or} \qquad cu^3 + bu^2 + au + d = 0 \qquad (4)$$

But the equation (4) has three distinct roots, therefore $c \neq 0$.

Comparing coefficients of like powers of u in (2) and (4), we have

$$\frac{1}{c} = \frac{-3x_0}{b} = \frac{3y_0}{a} = \frac{-z_0}{d} \quad \Rightarrow \quad a = 3cy_0, b = -3cx_0, d = -cz_0$$

Putting values in (3), the equation of the plane is

$$3cy_0\,x - 3cx_0y - cz - cz_0 = 0$$

$$\text{or} \qquad 3y_0\,x - 3x_0y - z - z_0 = 0$$

which clearly passes through (x_0, y_0, z_0).

Example 2.3: Show that the osculating plane at a point P has, in general three point contact (contact of second order) with the curve at P.

Solution. Let the equation of the curve be

$$\mathbf{r} = \mathbf{r}(s)$$

Let the arc length be measured from the point P, so that $s = 0$ at P. The equation of the osculating plane at P is given by

$$[\mathbf{R} - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0) = 0$$

Let $\quad F(s) = [\mathbf{r}(s) - \mathbf{r}(0), \mathbf{r}'(0), \mathbf{r}''(0)] = 0$

$$= \left[s\mathbf{r}'(0) + \frac{s^2}{2!}\mathbf{r}''(0) + \frac{s^3}{3!}\mathbf{r}'''(0) + 0(s^4), \mathbf{r}'(0), \mathbf{r}''(0) \right] = 0$$

[expending $\mathbf{r}(s)$ in power of s by Taylor's series]

$$= \frac{s^3}{3!}\left[\mathbf{r}'''(0), \mathbf{r}'(0), \mathbf{r}''(0)\right] + 0.(s^4) = 0$$

[since $\mathbf{r}'(0) \times \mathbf{r}'(0) = 0, \mathbf{r}''(0) \times \mathbf{r}''(0) = 0$]

Obviously $\quad F'(s) = 0$ and $F''(s) = 0$ at $s = 0$ i.e. at P.

Also $\quad F'''(s) \neq 0$ at $s = 0$ i.e. at P provided

$$[\mathbf{r}'''(0), \mathbf{r}''(0), \mathbf{r}'(0)] \neq 0$$

This shows that the osculating plane at P has in general three point contact with curve at P. In case $[\mathbf{r}'''(0), \mathbf{r}''(0), \mathbf{r}'(0)] = 0$ then $F'''(s) = 0$ and thus the osculating plane at P has at least four point contact with curve at P.

Example 2.4: Prove that there are three points on the cubic

$$x = at^3 + b, y = 3ct^2 + 3dt, z = 3et + f$$

such that the osculating planes pass through the origin, and that the points lie in the plane $3cex + afy = 0$.

Solution. $\mathbf{r} = (at^3 + b, 3ct^2 + 3dt, 3et + f)$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = 3(at^2, 2ct + d, e)$$

$$\ddot{\mathbf{r}} = 6(at, c, 0)$$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 18(-ec, aet, -(act^2 + adt))$$

Let $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$

Hence the equation of the osculating plane

$$[\mathbf{R}-\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0 \qquad \text{i.e. } (\mathbf{R}-\mathbf{r}).(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}) = 0 \text{ reduce to}$$

$$[(X - at^3 - b)\,\mathbf{i} + (Y - 3ct^2 - 3dt)\,\mathbf{j} + (Z - 3et - f)\mathbf{k}].$$
$$[-ec\,\mathbf{i} + aet\,\mathbf{j} - (act^2 + adt)\mathbf{k}] = 0$$

If it passes through the origin, then putting $X = Y = Z = 0$, we get

$$(at^3 + b)ec + (-3ct^2 - 3dt)(aet) + (3et + f)(act^2 + adt) = 0$$

or $$aect^3 + acft^2 + adft + bec = 0$$

which is cubic in t, giving three values of t, hence there are three such points. Let the parameter t for the points be t_1, t_2, t_3.The plane passing through these points is

$$\begin{vmatrix} X & Y & Z & 1 \\ at_1^3 + b & 3ct_1^2 + 3t_1 d & 3et_1 + f & 1 \\ at_2^3 + b & 3ct_2^2 + 3t_2 d & 3et_2 + f & 1 \\ at_3^3 + b & 3ct_3^2 + 3t_3 d & 3et_3 + f & 1 \end{vmatrix} = 0$$

or $$3ec\left(X + Y\frac{af}{3ec}\right)(t_1 - t_2)(t_{21} - t_3)(t_3 - t_1) = 0$$

or $$3ecX + afY = 0 \qquad [\therefore\ t_1 \neq t_2 \neq t_3]$$

Example 2.5: The normals are drawn from the point (α, β, γ) to the ellipsoid

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$$

Find the equation to the osculating plane at (α, β, γ) of the cubic curve through the feet of the normals.

Solution. The equation of the normal to the ellipsoid at (x, y, z) is

$$\frac{X-x}{x/a^2} = \frac{Y-y}{y/b^2} = \frac{Z-z}{z/c^2} = u\,(say)$$

where u is a parameter. If this line passes through (α, β, γ), then

$$\frac{\alpha-x}{x/a^2} = \frac{\beta-y}{y/b^2} = \frac{\gamma-z}{z/c^2} = u\,(say)$$

$$\therefore\ \alpha = x(a^2+u)/a^2,\ \beta = y(b^2+u)/b^2,\ \gamma = z(c^2+u)/c^2$$

or $x = a^2\alpha/(a^2+u), y = b^2\beta/(b^2+u), z = c^2\gamma(c^2+u)$ (1)

Now we are to find the osculating plane at (α,β,γ) of the curve given by (1). Putting $x=\alpha, y=\beta, z=\gamma$, we get $u=0$ in each case.

Now the position vector **r** of any point on the curve is given by

$$\mathbf{r} = \left(\frac{a^2\alpha}{a^2+u}, \frac{b^2\beta}{b^2+u}, \frac{c^2\gamma}{c^2+u}\right)$$

$$\therefore \quad \dot{\mathbf{r}} = -\left(\frac{a^2\alpha}{(a^2+u)^2}, \frac{b^2\beta}{(b^2+u)^2}, \frac{c^2\gamma}{(c^2+u)^2}\right)$$

$$\ddot{\mathbf{r}} = 2\left(\frac{a^2\alpha}{(a^2+u)^3}, \frac{b^2\beta}{(b^2+u)^3}, \frac{c^2\gamma}{(c^2+u)^3}\right)$$

At $u=0$, $\dot{\mathbf{r}} = \left(\frac{\alpha}{a^2}, \frac{\beta}{b^2}, \frac{\gamma}{c^2}\right)$; $\quad \ddot{\mathbf{r}} = 2\left(\frac{\alpha}{a^4}, \frac{\beta}{b^4}, \frac{\gamma}{c^4}\right)$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \left(\frac{(c^2-b^2)\beta\gamma}{b^4c^4}, \frac{(a^2-ch2)\gamma\alpha}{a^4c^4}, \frac{(b^2-a^2)\alpha\beta}{b^4c^4}\right) = 0$$

The equation of the osculating plane at the point **r** is

$$[\mathbf{R}-\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0$$

At the point (α,β,γ) it reduces to

$$\{(X-\alpha)\mathbf{i} + (Y-\beta)\mathbf{j} + (Z-\gamma)\mathbf{k}\}.\left(\frac{(c^2-b^2)\beta\gamma}{b^4c^4}\mathbf{i} + \frac{(a^2-ch2)\gamma\alpha}{a^4c^4}\mathbf{j} + \frac{(b^2-a^2)\alpha\beta}{b^4c^4}\mathbf{k}\right) = 0$$

or $\quad \sum \frac{X\beta\gamma}{c^4b^4}(c^4-b^4) - \alpha\beta\gamma\sum\frac{(c^2-b^2)}{c^4b^4} = 0$

or $\quad \sum \frac{Xa^4}{\alpha}(b^2-c^2) - \sum a^4(b^2-c^2) = 0$

or $\quad \sum \frac{Xa^4}{\alpha(a^2-b^4)(c^2-a^2)} - \sum\frac{a^4}{(a^2-b^2)(c^2-a^2)} = 0$

or $\quad \sum \frac{Xa^4}{\alpha(a^2-b^4)(c^2-a^2)} + 1 = 0$

Example 2.6: Prove that the osculating plane at (x_1, y_1, z_1) on the curve of intersection of the cylinders $x^2 + z^2 = a^2, \quad y^2 + z^2 = b^2$ is given by $(xx_1^3 - zz_1^3 - a^4)/a^2 = (yy_1^3 - zz_1^3 - b^4)/b^2$.

Solution. Let $f(x, y, z) = x^2 + z^2 - a^2 = 0$ (1)

$$\psi(x, y, z) = y^2 + z^2 - b^2 = 0 \qquad (2)$$

$\therefore f_x = 2x,\ f_y = 0,\ f_z = 2z, \quad \psi_x = 0, \psi_y = 2y, \psi_z = 2z$

At the point (x_1, y_1, z_1), the above values are

$$f_x = 2x_1,\ f_y = 0,\ f_z = 2z_1, \quad \psi_x = 0, \psi_y = 2y_1, \psi_z = 2z_1 \qquad (3)$$

Similarly $\therefore f_{xx} = 2,\ f_{xy} = 0, etc. \quad \psi_{xx} = 0, \psi_{xy} = 0\ etc.$

Let the equation of intersection of the surfaces (1) and (2) be $\mathbf{r} = \mathbf{r}(u)$. Differentiating (1) and (2) with respect to 'u', we have

$$x\dot{x} + z\dot{z} = 0, \quad y\dot{y} + z\dot{z} = 0$$

$\therefore$ At (x_1, y_1, z_1), $\quad x_1\dot{x} = y_1\dot{y} = -z_1\dot{z} = 0$

$$\text{or} \qquad \frac{\dot{x}}{1/x_1} = \frac{\dot{y}}{1/y_1} = -\frac{\dot{z}}{1/z_1} \qquad (3)$$

Now the equation of the osculating plane at (x_1, y_1, z_1) given by

$$\frac{(x - x_1) f_x + (y - y_1) f_y + (z - z_1) f_z}{\dot{x}^2 f_{xx} + \dot{y}^2 f_{yy} + \dot{z}^2 f_{zz} + 2\dot{y}\dot{z} f_{yz} + 2\dot{z}\dot{x} f_{zx} + 2\dot{x}\dot{y} f_{xy}} \qquad \text{[see (2.16')]}$$

$$= \frac{(x - x_1)\psi_x + (y - y_1)\psi_y + (z - z_1)\psi_z}{\dot{x}^2\psi_{xx} + \dot{y}^2\psi_{yy} + \dot{z}^2\psi_{zz} + 2\dot{y}\dot{z}\psi_{yz} + 2\dot{z}\dot{x}\psi_{zx} + 2\dot{x}\dot{y}\psi_{xy}}$$

Becomes [using the relation (2) and (3)]

$$\frac{(x - x_1)x_1 + (z - z_1)z_1}{\dot{x}^2 + \dot{z}^2} = \frac{(y - y_1)y_1 + (z - z_1)z_1}{\dot{y}^2 + \dot{z}^2}$$

$$\text{or} \qquad \frac{xx_1 + zz_1 - (x_1^2 + z_1^2)}{1/x_1^2 + 1/z_1^2} = \frac{yy_1 + zz_1 - (y_1^2 + z_1^2)}{1/y_1^2 + 1/z_1^2} \qquad \text{using (3)}$$

Since the point (x_1, y_1, z_1) lies on (1) and (2) both, hence

$$x_1^2 + z_1^2 = a^2\ , \quad y_1^2 + z_1^2 = b^2 \qquad (4)$$

Using (4) the equation of the osculating plane becomes

$$\frac{(xx_1 + zz_1 - a^2)x_1^2}{a^2} = \frac{(yy_1 + zz_1 - b^2)y_1^2}{b^2}$$

or $$\frac{xx_1^3 + (zz_1 - a^2)(a^2 - z_1^2)}{a^2} = \frac{yy_1^3 + (zz_1 - b^2)(b^{2-} z_1^2)}{b^2}$$

or $$[xx_1^3 - zz_1^3 - a^4 + a^2(z_1^2 + zz_1)]/a^2 = [yy_1^3 - zz_1^3 - b^4 + b^2(z_1^2 + zz_1)]/b^2$$

or $$(xx_1^3 - zz_1^3 - a^4)/a^2 + (z_1^2 + zz_1) = (yy_1^3 + zz_1^3 - b^4)/b^2 + (z_1^2 + zz_1)$$

or $$(xx_1^3 - zz_1^3 - a^4)/a^2 = (yy_1^3 + zz_1^3 - b^4)/b^2$$

This is the required equation of the osculating plane.

Example 2.7: Prove that the points of the curve of intersection of sphere and coincide $r\,x^2 + ry^2 + r\,z^2 = 1,\ a\,x^2 + by^2 + c\,z^2 = 1$, at which the osculating planes through the origin lie on the cone $\frac{a-r}{b-c}x^4 + \frac{b-r}{c-a}y^4 + \frac{c-r}{a-b}z^4 = 0$

Solution. Let $f(x, y, z) = r\,x^2 + ry^2 + r\,z^2 - 1 = 0$ (1)

$$\psi(x, y, z) = a\,x^2 + by^2 + c\,z^2 - 1 = 0 \qquad (2)$$

$$f_x = 2rx,\ f_y = 2ry,\ f_z = 2rz \quad \psi_x = 2ax, \psi_y = 2by, \psi_z = 2cz$$

$$f_{xx} = 2r,\ f_{yy} = 2r,\ f_{zz} = 2r \quad f_{xy} = 0\ etc.$$

Let the equation of the curve of intersection of (1) and (2) be $\mathbf{r} = \mathbf{r}(u)$. Differentiating (1) and (2) w.r.t. 'u', we have

$$rx\dot{x} + ry\dot{y} + rz\dot{z} = 0, \quad ax\dot{x} + by\dot{y} + cz\dot{z} = 0$$

or $$x\dot{x} + y\dot{y} + z\dot{z} = 0$$

Solving $\frac{x\dot{x}}{b-c} + \frac{y\dot{y}}{c-a} + \frac{z\dot{z}}{a-b}$

or $$\frac{\dot{x}}{(b-c)/x} + \frac{\dot{y}}{(c-a)/y} + \frac{\dot{z}}{(a-b)/z} \qquad (3)$$

Now the equation to the osculating plane [see (2.16')] is

$$\frac{(X-x)2rx+(Y-y)2by+(Z-z)2cy}{2r(\dot{x}^2+\dot{y}^2+\dot{z}^2)}$$

$$=\frac{(X-x)2rx+(Y-y)2by+(Z-z)2cy}{2(a\dot{x}^2+b\,\dot{y}^2+c\dot{z}^2)}$$

Since the osculating plane passes through the origin, hence

$$\sum[x^2(ax^2+by^2+cz^2-a\dot{x}^2+b\,\dot{y}^2+c\dot{z}^2)]=0$$

or $$\sum[x^2\{\dot{z}^2(c-a)+\dot{y}^2(a-b)\}]=0$$

or $$\sum\left[x^2\left\{\frac{(a-b)^2(c-a)}{z^2}-\frac{(c-a)^2(a-b)}{y^2}\right\}\right]=0 \qquad \text{using (3)}$$

or $$\sum\left[\left\{x^2\,\frac{(a-b)(c-a)}{y^2z^2}\{(a-b)y^2-(c-a)z^2\}\right\}\right]=0$$

or $$\sum\left[\left\{\frac{x^2}{(b-c)y^2z^2}\{a(y^2+z^2)-(by^2+cz^2)\}\right\}\right]=0$$

or $$\sum\left[\frac{x^2.x^2}{x^2y^2z^2(b-c)}\left\{a\left(\frac{1}{r}-x^2\right)-(1-ax^2)\right\}\right]=0$$

or $$\sum\left[\frac{x^4}{(b-c)}(a-r)\right]=0$$

2.4 THE PRINCIPAL NORMAL AND BI-NORMAL

A curve is the locus of a point whose position vector **r** relative to a fixed origin may be expressed as a function of a single variable parameter known as tangent. Then its Cartesian coordinates x, y, z are also functions of the same parameter. When the curve is not a plane curve it is said to be skew, tortuous or twisted. We shall confine our attention to those portions of the curve which are free from singularities of all kinds. All the normal to the curve at any point lie in the plane through the point perpendicular to the tangent to the curve i.e. in the normal plane. Two of these normals are of great importance and are given special names.

(i) Principal normal: The normal which lies in osculating plane at any point of a curve is called Principal Normal. Obviously, this normal is along the line of intersection of the osculating plane and the normal plane at the point (as shown in the figure 2.3).

(ii) Bi-normal: The normal which is perpendicular to the osculating plane at a point R(t) (as shown in the figure 2.3) is called the Bi-normal. Obviously, binormal is also perpendicular to the principal normal as well as tangent to the curve.

The unit vector along the principal normal and the bi-normal are denoted by **n** and **b** respectively.

(A) To find the direction of principal normal and binormal: Since bi-normal is perpendicular to the osculating plane (osculating plane is perpendicular to the vector $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$ and therefore bi-normal is parallel to the vector $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$.

The principal normal being perpendicular to the tangent (tangent is perpendicular to the vector $\dot{\mathbf{r}}$) and the bi-normal, will be parallel to the vector

$$(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\times\dot{\mathbf{r}} \quad i.e.\ (\dot{\mathbf{r}}^2)\dot{\mathbf{r}}-(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}}.$$

If the parameter be arc length s: We know **t** is the unit tangent vector at a point of space curve.

Hence $\mathbf{t}.\mathbf{t}=1$ *or* $\mathbf{r}'.\mathbf{r}'=1$ which on differentiation w.r.t. 's' gives $\mathbf{r}'.\mathbf{r}''=1$; showing that $\mathbf{r}''$ is perpendicular to the vector $\mathbf{r}'$ and hence principal normal will be parallel to the vector $\mathbf{r}''$, and bi-normal will be parallel to the vector $\mathbf{r}'\times\mathbf{r}''$. Since bi-normal is perpendicular to the tangent and the principal normal.

Corollary: In Cartesians

(i) When the parameter u

Here, $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\dot{\mathbf{r}}=\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k}=\sum\dot{x}\mathbf{i}$

$\ddot{\mathbf{r}}=\sum\ddot{x}\mathbf{i}$ $\qquad\therefore\qquad$ $(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\mathbf{i}=\sum(\dot{y}\ddot{z}-\dot{z}\ddot{y})\mathbf{i}$

Hence the principal normal being parallel to the vector

$(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\times\dot{\mathbf{r}}^*$ $\qquad$ *i.e.* $\qquad [\sum(\dot{y}\ddot{z}-\dot{z}\ddot{y})\mathbf{i}]\times\sum\dot{x}\mathbf{i}$

i.e. $\qquad \{\sum\dot{z}(\dot{z}\ddot{x}-\dot{x}\ddot{z})-\dot{y}(\dot{x}\ddot{y}-\dot{y}\ddot{x})\}\mathbf{i}$

the direction ratios of it are

$$\dot{z}(\dot{z}\ddot{x}-\dot{x}\ddot{z})-\dot{y}(\dot{x}\ddot{y}-\dot{y}\ddot{x}) \;;\; \dot{x}(\dot{x}\ddot{y}-\dot{y}\ddot{x})-\dot{z}(\dot{y}\ddot{z}-\dot{z}\ddot{y}) \;;\dot{y}(\dot{y}\ddot{z}-\dot{z}\ddot{y})-\dot{x}(\dot{z}\ddot{x}-\dot{x}\ddot{z})$$

Since the bi-normal is parallel to the vector

$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$ i.e. to $\sum(\dot{y}\ddot{z}-\dot{z}\ddot{y})\mathbf{i}$, its direction ratios are

$$\dot{y}\ddot{z}-\dot{z}\ddot{y} \;;\; \dot{z}\ddot{x}-\dot{x}\ddot{z} \;,\; \dot{x}\ddot{y}-\dot{y}\ddot{x}$$

(ii) When the parameter is arc-length s

Since principal normal is parallel to the vector $\mathbf{r}''$, its direction ratios are x'', y'', z''. The bi-normal being parallel to the vector $\mathbf{r}' \times \mathbf{r}''$ i.e. to $\sum (y'z'' - z'y'')\mathbf{i}$, the direction ratios of bi-normal are given by

$$(y'z'' - z'y'')\ ,\ (z'x'' - x'z'')\ ,\ (x'y'' - y'x'')$$

(B) The unit vector t, n, b

In the study of differential geometry the three unit vectors viz. unit tangent vector **t**, the unit principal normal vector **n**, and the unit bi-normal vector **b** play an important part. The sense of the unit vector **b** is chosen so that tried **t, n, b** form a right handed orthogonal system of axes i.e. as shown in figure 2.2.

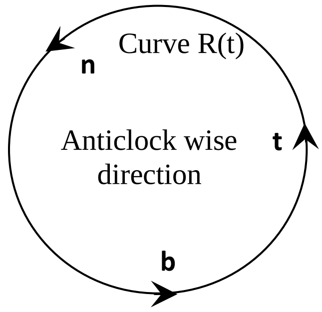

Figure 2.2

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \mathbf{b} = \mathbf{t} \times \mathbf{n}, \mathbf{n} = \mathbf{b} \times \mathbf{t} \text{ and } \mathbf{t}.\mathbf{n} = 0, \mathbf{n}.\mathbf{b} = 0, \mathbf{b}.\mathbf{t} = 0$$

This adjoining figure 2.3, shows that at each point of the curve there are three mutually perpendicular planes.

(C) Fundamental planes

(i) **The osculating plane** containing **t** and **n** and clearly its equation is $(\mathbf{R} - \mathbf{r}).\mathbf{b} = 0$.

(ii) **The normal plane** containing **n** and **b** and clearly its equation is $(\mathbf{R} - \mathbf{r}).\mathbf{t} = 0$.

(iii) **The rectifying plane** containing **b** and **t** and clearly its equation is $(\mathbf{R} - \mathbf{r}).\mathbf{n} = 0$

Where the point P(r) is on the space curve and **R** as indicated in the figure, lies in these planes, these planes are also called **fundamental planes** that explain in figure 2.3

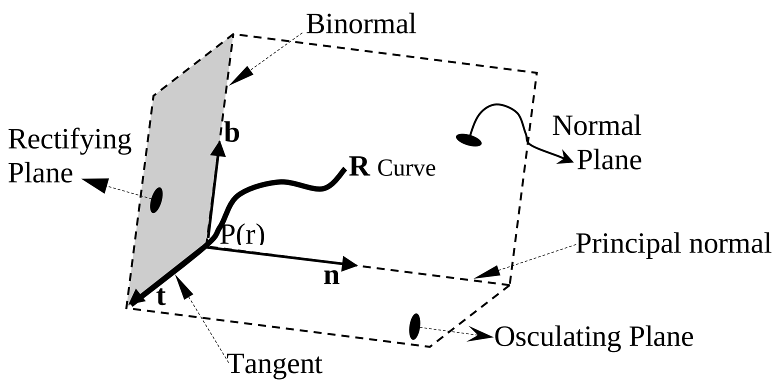

Figure 2.3

Corollary. In Cartesians: Let (ξ, η, ζ) be a current point and (x, y, z) the point where the planes are determined.

$$\mathbf{R} = \xi\,\mathbf{i} + \eta\,\mathbf{j} + \zeta\,\mathbf{k}\ , \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Again let (l_r, m_r, n_r) where $(r = 1,2,3)$ be the direction ratios of the tangent, principal normal and bi-normal, so that

$$\mathbf{t} = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k},\ \mathbf{n} = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k},\ \mathbf{b} = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}$$

Substituting values in the equations to three fundamental planes, we get

Normal plane as $\quad l_1(\xi - x) + m_1(\eta - y) + n_1(\zeta - z) = 0$

Rectifying plane as $\quad l_2(\xi - x) + m_2(\eta - y) + n_2(\zeta - z) = 0$

Osculating plane as $\quad l_3(\xi - x) + m_3(\eta - y) + n_3(\zeta - z) = 0$

(D) To find the equation of the principal normal and binormal.

Let **r** be the positive vector of a point P on the given curve C at which we are to find equations of principal normal and binormal (as shown in figure 2.4). Let **R** be the positive vector of current point R on the principal normal $\vec{\mathbf{PR}}$, then we have

$$\vec{\mathbf{OP}} = \mathbf{r},\ \vec{\mathbf{OR}} = \mathbf{R},\ \vec{\mathbf{PR}} = v\mathbf{n}$$

Since **n** is the unit vector along the principal normal $\vec{\mathbf{PR}}$ and v is some scalar. Therefore triangle law of vectors, namely

$$\vec{\mathbf{OR}} = \vec{\mathbf{OP}} + \vec{\mathbf{PR}} \text{ gives } \mathbf{R} = \mathbf{r} + v\,\mathbf{n}.$$

This is the require equation of the principal normal at a point P(r) on the curve C. Similarly if **R** denotes the position vector of a current point Q on the binomial, the equation of the binomial at a point P(r) on the curve C is given in figure 2.4

$$\mathbf{R}=\mathbf{r}+\lambda\,\mathbf{b}, \quad \text{where } \lambda \text{ is some scalar.}$$

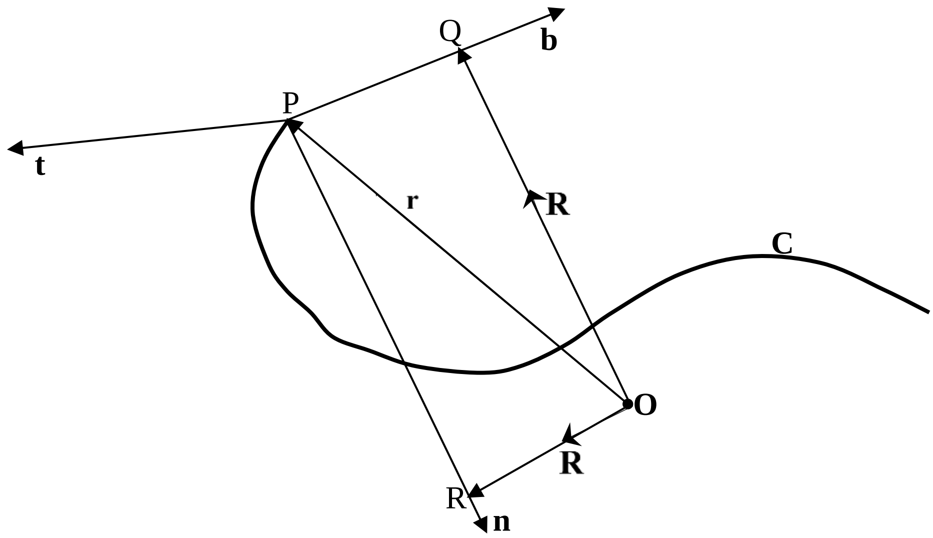

Figure 2.4

Example 2.8: Find the basic unit vector **t**, **n**, and **b** of the curve $\mathbf{r}=(u,u^2,u^3)$ at the point $u=1$. Also find the equation of tangent, the principal normal and binormal of this point.

Solution. The equation of the given curve is

$$\mathbf{r}=(u,u^2,u^3)$$

$$\therefore \quad \frac{d\mathbf{r}}{du}=\dot{\mathbf{r}}=(1,2u,3u^2)$$

or
$$\frac{d\mathbf{r}}{ds}.\frac{ds}{du}=(1,2u,3u^2)$$

or
$$\mathbf{t}.\frac{ds}{du}=(1,2u,3u^2) \qquad (1)$$

on squaring both side of equation (1), we have

$$\left(\frac{ds}{du}\right)^2=1+4u^2+9u^4 \Rightarrow \left(\frac{ds}{du}\right)=\sqrt{(1+4u^2+9u^4)} \qquad (2)$$

Using equation (2) in (1), we get

$$\mathbf{t} = \frac{1}{\sqrt{(1+4u^2+9u^4)}}(1, 2u, 3u^2) \tag{3}$$

Differentiating equation (1) w.r.t. 'u', we get

or $$\left(\frac{d\mathbf{t}}{ds}.\frac{ds}{du}\right)\frac{ds}{du} + \frac{ds}{du}\mathbf{t}\frac{d^2 s}{du^2} = (0, 2, 6u) \tag{4}$$

Taking vector product of equation (2.30) and (2.33), we get

$$\kappa \mathbf{b}\left(\frac{ds}{du}\right)^3 = (6u^2, -6u, 2) \qquad [\because\ \mathbf{t}\times\mathbf{n} = \mathbf{b},\ \mathbf{t}\times\mathbf{t} = 0]$$

On squaring $\kappa^2\left(\frac{ds}{du}\right)^6 = 4(1+9u^2+9u^4)$

$$\Rightarrow \quad \kappa\left(\frac{ds}{du}\right)^3 = 2\sqrt{(1+9u^2+9u^4)} \tag{5}$$

Hence, using equation (3), (4) becomes

$$\mathbf{b} = \frac{1}{\sqrt{(1+9u^2+9u^4)}}(3u^2, -3u, 1) \tag{6}$$

Now putting $u=1$, in (2) and (6), we have

$$\mathbf{t} = \frac{1}{\sqrt{14}}(1, 2, 3),\ \mathbf{b} = \frac{1}{\sqrt{19}}(3, -3, 1)$$

Also $$\mathbf{n} = \mathbf{b}\times\mathbf{t} = \frac{1}{\sqrt{266}}(-11, -8, 9)$$

The equation of the tangent at $u=1$ i.e. at $\mathbf{r} = (1,1,1)$ are

$$\frac{X-1}{1} = \frac{Y-1}{1} = \frac{Z-1}{3} \tag{7}$$

The equation of principal normal at $u=1$, are

$$\frac{X-1}{-11} = \frac{Y-1}{-8} = \frac{Z-1}{9} \tag{8}$$

The equation of binomial at $u=1$, are

$$\frac{X-1}{3} = \frac{Y-1}{-3} = \frac{Z-1}{1} \tag{9}$$

Example 2.9: Let C be a curve given by the equation $\mathbf{r}=(u,u^2,u^3)$, find the curvature and torsion of C at the point (0,0,0). Also find the equation of its binormal line and normal plane at the point (1,1,1).

Solution. At the point (0,0,0) of the curve $\mathbf{r}=(u,u^2,u^3)$, we have $u=0$. Putting $u=0$ in relation (1) and (2) of above example, we get

$$\kappa=2 \text{ and } \tau=3$$

Again at the point (1,1,1) of the curve C, $u=1$. See example (2.8) the binormal is given by equation (9).

The equation of normal plane at (1,1,1) using (**R-r**).**t**=0 is given by

$$\{(X-1)\mathbf{i}+(Y-1)\mathbf{j}+(Z-1)\mathbf{k}\}\frac{1}{\sqrt{(14)}}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0$$

or $$(X-1)+2(Y-1)+3(Z-1)=0$$

or $$X+2Y+3Z=6$$

2.5 (A) **CURVATURE**

Definition: (A)Tangent: It is usually convenient to choose as the scalar parameter the length s of the arc of the curve measured from a fixed point A on it. Then for points on one side A the value of s will be positive, for points on the other side, negative. The positive direction along the curve at any point is taken as that corresponding to algebraically increase of s. Thus the position vector **r** of a point on the curve is a function of s, regular within the range considered.

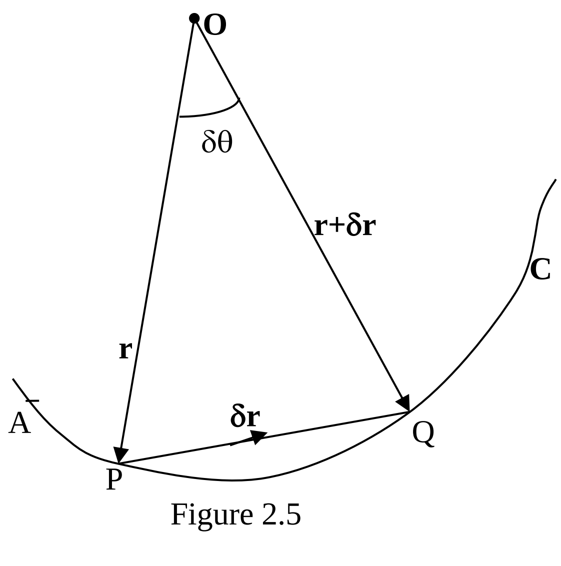

Figure 2.5

Its successive derivatives with respect to s will be denoted by **r**', **r**'', **r**''', and so on. Let P, Q be the points on the curve whose position vectors are **r, r+δr** corresponding to the values s, $s+\delta s$ of the parameter, then **δr** is the vector PQ. The quotient $\boldsymbol{\delta}\mathbf{r}/\delta s$ is a vector in the same direction as **δr** ; and in the limit, as δs tends to zero, this direction becomes that of the tangent at P. Moreover the ratio of the lengths of the chord PQ and the arc PQ tends to unity as Q moves up to coincidence with P. Therefore the limiting values of $\boldsymbol{\delta}\mathbf{r}/\delta s$ is a unit vector parallel to the tangent to the curve at P, and in the positive direction. We shall denote this by **t** and call it the unit tangent at P. Thus

$$\mathbf{t} = \underset{\delta s \to 0}{lt} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{r}' \qquad (2.5.1)$$

The vector equation of the tangent at P may be written down at once. For the position vector **R** of a current point on the tangent is given by

$$\mathbf{R} = \mathbf{r} + u\mathbf{t} \qquad (2.5.2)$$

where u is a variable number, negative or positive. Thus relation (2.5.2) is known as the equation of the tangent. In Cartesian coordinates equation of tangent can be written as

$$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0 \qquad (2.5.3)$$

Every line through P in the plane (2.5.3) is a normal to the curve.

(B) CURVATURE

The curvature of the curve at any point is the arc-rate of rotation of the tangent. Thus if $\delta\theta$ is the angle between the tangents at P and Q (figure 2.5), $\delta\theta/\delta s$ is the average curvature of the are PQ; and its limiting value as δs tends to zero is the curvature at the point P. This is sometimes called the first curvature or the circular curvature. We shall denote it by ***k***. Thus

$$k = \underset{\delta s \to 0}{lt} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} = \theta' \qquad (2.5.4)$$

The unit tangent is not a constant vector, for its direction changes from point to point of the

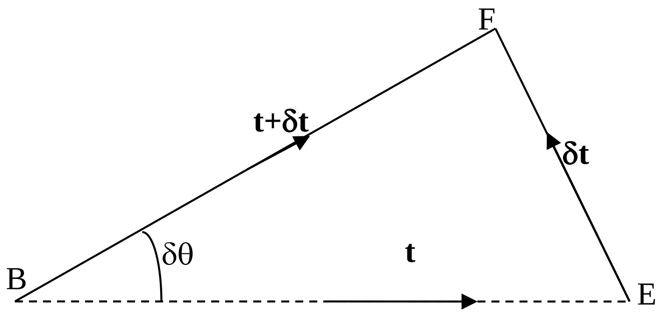

Figure 2.6

curve. Let ***t*** be its value at P and ***t+δt*** at Q. If the vector BE and BF are respectively equal to these, then ***δt*** is the vector EF and $\delta\theta$ the angle EBF. The quotient $\frac{\delta \mathbf{t}}{\delta\theta}$ is a vector parallel to ***δt***, and therefore in the limit as δs tends to zero its direction is perpendicular to the tangent at P. Moreover since BE and BF are of unit length, the modulus of the limiting value of $\frac{\delta \mathbf{t}}{\delta\theta}$ is the limiting value of $\frac{\delta\theta}{\delta s}$, which is ***k***. Hence the relation

$$\frac{d\mathbf{t}}{ds} = \underset{\delta s \to 0}{lt} \frac{\delta t}{\delta s} = k\mathbf{n} \qquad (2.5.5)$$

where **n** is a unit vector perpendicular to **t** and in the plane of the tangents at P and a consecutive point. This plane, containing two consecutive tangents and therefore three consecutive points at P, is called the plane of curvature or the osculating plane at P.

The arc rate at which the tangent changes direction *i.e.* $\left(\frac{d\mathbf{t}}{ds}\right)$ as the point P moves along the curve is called the curvature vector of the curve and its magnitude is denoted by k.

By definition we have $|\mathbf{t}'| = |k|$ where k is the curvature vector. In order to determine the sign of k, we have from (figure 2.3) $\mathbf{t}'(= r'')$ laying in the osculating plane and normal to **t** and therefore $\mathbf{t}'$ is proportional to **n**, i.e. we may write $\mathbf{t}' = \pm k\,\mathbf{n}$. But we choose the direction of **n** such that the curvature k is always positive i.e. we take $\mathbf{t}' = k\,\mathbf{n}$.

(C) TORSION

Among the normals at P to the curve that which is perpendicular to the osculating plane is called the binormal. Being perpendicular to both **t** and **n** it is parallel to $\mathbf{t}\times\mathbf{n}$. Denoting this unit vector by **b** we have to trio **t, n, b** forming a right-handed system (as shown in figure 2.2) of mutually perpendicular unit vectors, and therefore connected by the relations

$$\mathbf{t}\bullet\mathbf{n}=\mathbf{n}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{t}=0 \text{ and}$$

$$\mathbf{t}\times\mathbf{n}=\mathbf{b},\mathbf{n}\times\mathbf{b}=\mathbf{t},\mathbf{b}\times\mathbf{t}=\mathbf{n}$$

the cyclic order being preserved in the cross products. We may call **b** the unit binormal. The positive direction along the binormal is taken as that if **b**, just as the positive direction along the principal normal is that of **n.** The equation of the binormal is

$$\mathbf{R}=\mathbf{r}+u\mathbf{b} \text{ and } \mathbf{b}=\mathbf{t}\times\mathbf{n}=\boldsymbol{\rho}\,\mathbf{r}'\times\mathbf{r}''$$

Definition: The arc rate at which the changes direction bi- normal $i.e.\left(\frac{d\mathbf{b}}{ds}\right)$ as PI moves along the curve is called the torsion vector of the curve and its magnitude is denoted by τ. The torsion τ may have positive as well as negative direction. Therefore τ is determined both in magnitude and direction.

An alternative definition of Torsion: The angle between the osculating planes at any two points P, Q of a curve is called the whole torsion of the arc PQ. The limiting value of the ratio of the whole torsion to the arc PQ is called the torsion of the curve at the point P as $Q\to P$. The radius of the circle whose curvature is equal to the torsion of the curve at any point , is called the radius of torsion at that point and is denoted by σ.

Solved Examples

Example 2.10: Calculate the curvature and torsion of the cubic curve given by $\mathbf{r}=(u,u^2,u^3)$.

Solution. Here $\mathbf{r}=(u,u^2,u^3)$

$$\therefore\ \dot{\mathbf{r}}=(1,2u,3u^2)$$

$$\therefore\quad \dot{\mathbf{r}}\times\ddot{\mathbf{r}}=(\mathbf{i}+2u\mathbf{j}+3u^2\mathbf{k})\times(2\mathbf{j}+6u\mathbf{k})$$

$$=2\mathbf{k}-6u\mathbf{j}+12u^2\mathbf{i}-6u^2\mathbf{i}$$

$$=6u^2\mathbf{i}-6u\mathbf{j}+2\mathbf{k}=(6u^2,-6u,\ 2)$$

$$=2(3u^2,-3u,1)$$

$$\therefore \quad |\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = 2(9u^4+9u^2+1)^{1/2}$$

Also, $[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}] = \dot{\mathbf{r}}\times\ddot{\mathbf{r}}\bullet\dddot{\mathbf{r}} = 2(3u^2,-3u,1)\bullet(0,0,6)$

2(0+0+6)=12

$$\therefore \quad k = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{2(9u^4+9u^2+1)^{1/2}}{(1+4u^2+9u^4)^{3/2}} \tag{1}$$

And $\tau = \dfrac{|\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}|}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = \dfrac{12}{4(9u^4+9u^2+1)}$

$$\text{Or} \qquad \tau = \frac{3}{(9u^4+9u^2+1)} \tag{2}$$

Example 2.10: For the curve $x = a(3u-u^3)$, $y = 3au^2$ show that curvature and torsion are equal.

Solution. The positive vector $\mathbf{r}$ of any point of the curve is given by

$$\mathbf{r} = (a(3u-u^3), 3au^2, a(3u+u^3))$$

Differentiating w.r.t. 'u', we have

$$\dot{\mathbf{r}} = a(3-3u^2, 6u, 3u+3u^2)$$

$$\ddot{\mathbf{r}} = a(-6u, 6, 6u)$$

$$\dddot{\mathbf{r}} = a(-6, 0, 6) = 6a(-1,0,1)$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = 3a(1-u^2, 2u, 1+u^2)\times 6a(-u,1,u)$$

$$= 18a^2[(1-u^2)\mathbf{k} - u(1-u^2)\mathbf{j} - 2u^2\mathbf{k} + 2u^2\mathbf{i} - u(1+u^2)\mathbf{j} - (1+u^2)\mathbf{i}]$$

$$= 18a^2[-(1-u^2)\mathbf{i} - (u+u-u^3+u^3)\mathbf{j} + (1-u^2+2u^2)\mathbf{k}]$$

$$= 18a^2[-(1-u^2)\mathbf{i} - 2u\mathbf{j} + (1+u^2)\mathbf{k}]$$

$$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = 18a^2[(1-u^2)^2 + 4u^2 + (1+u^2)^2]^{1/2}$$

$$= 18a^2[(1+u^2)^2 + (1+u^2)^2]^{1/2} = 18\sqrt{2}a^2(1+u^2)$$

$$[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}] = \dot{\mathbf{r}}\times\ddot{\mathbf{r}}\bullet\dddot{\mathbf{r}} = 18a^2[-(1-u^2)^2, 2u(1+u^2).6a(-1,0,1)]$$

$$= 108a^3[+(1-u^2)^2 + 0 + (1+u^2)] = 216a^3$$

$$|\dot{\mathbf{r}}| = 3a\{(1-u^2)^2 + 4u^2 + (1+u^2)^2\}^{1/2}$$

$$= 3a\{(1+u^2)^2 + (1+u^2)^2\}^{1/2} = 3\sqrt{2}a(1+u^2)$$

$$\therefore \quad k = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{18\sqrt{2}a^2(1+u^2)}{3 \times 18\sqrt{2}a^3(1+u^2)^3} = \frac{1}{3a(1+u^2)}$$

And $$\tau = \frac{|\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}|}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{216a^3}{18 \times 18 \times 2a^4(1+u^2)^2} = \frac{1}{3a(1+u^2)^2}$$

Hence $k = \tau$.

Example 2.11: Find the radii of curvature and torsion of the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$.

Solution. Here $\mathbf{r} = (a\cos u, a\sin u, au\tan\alpha)$

Differentiating w.r.t. 'u' we get

$$\dot{\mathbf{r}} = a(-\sin u, \cos u, \tan\alpha)$$

$$\ddot{\mathbf{r}} = a(-\cos u, -\sin u, 0)$$

$$\dddot{\mathbf{r}} = a(\sin u, -\cos u, 0)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = a^2(\sin u\tan\alpha, \cos u\tan\alpha, 1)$$

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = a^2[(\sin^2 u\tan^2\alpha + \cos^2 u\tan^2\alpha + 1)]^{1/2} = a^2\sec\alpha$$

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \bullet \dddot{\mathbf{r}} = a^3(\sin^2 u\tan\alpha + \cos^2 u\tan\alpha + 0 = a^3\tan\alpha$$

Also, $$|\dot{\mathbf{r}}| = a[\sin^2 u + \cos^2 u + \tan^2\alpha]^{1/2} = a\sec\alpha$$

$$\therefore \quad k = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{a^2\sec\theta}{a^3se^3c\theta} = \frac{1}{a\sec^2\alpha} = \frac{\cos^2\alpha}{a}$$

$$\therefore \quad \rho = \frac{1}{k} = a\sec^2\alpha$$

and $$\tau = \frac{a^2\tan\alpha}{a^4\sec^2\alpha} = \frac{\sin\alpha}{a\sec\alpha}$$

$$\therefore \quad \sigma = \frac{1}{\rho} = a\,\mathrm{cosec}\alpha\sec\alpha$$

Example 2.12: For the curve $x = 4a\cos^3 u$, $y = 4a\sin^3 u$, $z = 3c\cos 2u$

Prove that $\kappa = \dfrac{a}{6(a^2 + c^2)\sin 2u}$

Solution. Here, $\mathbf{r} = (4a\cos^3 u, 4a\sin^3 u, 3c\cos 2u)$

Differentiating w.r.t. arc length 's'

$$\mathbf{r}' = \mathbf{t} = 6(-2a\cos^2 u\sin u, 2a\sin^2 u\cos u, -c\sin 2u)\frac{du}{ds}$$

$$\mathbf{t} = 12\sin u\cos u(-a\cos u, a\sin u, -c)\frac{du}{ds} \qquad (1)$$

Squaring equation (1) and using $\mathbf{t}^2 = 1$

$$1 = 144\sin^2 u\cos^2 u(a^2\cos^2 u + a^2\sin^2 u + c^2)\left(\frac{du}{ds}\right)^2$$

or $$\frac{ds}{du} = 12\sin u\cos u\sqrt{(a^2 + c^2)} \qquad (2)$$

Substituting equation (2) in (1), we get

$$\mathbf{t} = 12\sin u\cos u(-a\cos u, a\sin u, -c)\frac{1}{12\sin u\cos u\sqrt{(a^2 + c^2)}}$$

$$\therefore \quad \mathbf{t} = \frac{1}{\sqrt{(a^2 + c^2)}}(-a\cos u, a\sin u, -c) \qquad (3)$$

Differentiating equation (3) w.r.t. 's' and using $\mathbf{t}' = \kappa\,\mathbf{n}$, we have

$$\mathbf{t}' = \kappa\,\mathbf{n} = \frac{1}{\sqrt{(a^2 + c^2)}}(a\sin u, a\cos u, 0)\frac{du}{ds} \qquad (4)$$

$$|\kappa||\mathbf{n}| = \frac{1}{\sqrt{a^2 + c^2}}(a^2\sin^2 u + a^2\cos^2 u)^{1/2}\left(\frac{du}{ds}\right)$$

or $$\kappa = \frac{1}{\sqrt{a^2 + c^2}} \cdot \frac{a}{12\sin u\cos u\sqrt{a^2 + c^2}} \qquad [\because |\mathbf{n}| = 1 \text{ and using (2)}]$$

or $$\kappa = .\frac{a}{6(a^2 + c^2)\sin 2u}.$$

Example 2.13: Find the radii of curvature and torsion at any point of the curve

$$x^2 + y^2 = a^2, \quad x^2 - y^2 = az.$$

Solution. Let the parametric equation of the curve be

$$x = a\cos u,\ y = a\sin u,\ z = a\cos 2u$$

$$\therefore\ \mathbf{r} = a(\cos u, \sin u, \cos 2u)$$

$$\therefore\ \dot{\mathbf{r}} = a(-\sin u, \cos u, -2\sin 2u)$$

$$\therefore\ \ddot{\mathbf{r}} = a^2(-\cos u, -\sin u, -4\cos 2u)$$

$$\therefore\ \dddot{\mathbf{r}} = a^3(\sin u, -\cos u, -8\sin 2u) \qquad (1)$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = a^2(-4\cos u\cos 2u - 2\sin u\sin 2u,$$
$$2\sin 2u\cos u - 4\sin u\cos 2u, \cos^2 u + \sin^2 u)$$

$$= a^2[-2\cos u\cos 2u - 2\cos(2u-u),\ 2\sin(2u-u) - 2\sin u\cos 2u,\ 1]$$

$$= a^2[-2\cos u\cos 2u - 2\cos u,\ 2\sin u - 2\sin u\cos 2u,\ 1]$$

$$= a^2(-4\cos^3 u, 4\sin^3 u, 1) \qquad (2)$$

Taking scalar product of equation (2) with (1), we have

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = a^3\{-4\sin u\cos^3 u - 4\sin^3 u\cos u + 8\sin 2u\}$$

$$= a^3\{-4\sin u\cos u(\cos^2 + \sin^2 u) + 8\sin 2u\}$$

$$= a^3\{-4\sin u\cos u + 8\sin 2u\}$$

$$= a^3\{-4\sin 2u + 8\sin 2u\} = 6a^3\sin 2u$$

Also, $\quad |\dot{\mathbf{r}}|^2 = a^2(\cos^2 u + \sin^2 u + 4\sin^2 2u)$

$$= a^2\{1 + 4\sin^2 2u\} = a^2\{5 - 4\cos^2 2u\}$$

And $\quad |\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2 = a^4(16\cos^6 u + 16\sin^6 u + 1)$

$$= a^4[16\{\cos^2 u + \sin^2 u)(\cos^4 u + \sin^4 u - \cos^2 u\sin^2 u)\} + 1]$$

$$= a^4[16\{(\cos^2 u + \sin^2 u)^2 - 3\cos^2 u\sin^2 u\} + 1]$$

$$= a^4[16\{1 - 3\cos^2 u\sin^2 u\} + 1]$$

$$= a^4[17 - 48\cos^2 u\sin^2 u]$$

$$= a^4[17 - 12\sin^2 2u]$$

$$= a^4[5 + 12\cos^2 2u]$$

Now, $$\rho^2 = \frac{1}{k^2} = \frac{|\dot{\mathbf{r}}|^6}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{a^6(5 - 4\cos^2 2u)^3}{a^4(5 + 12\cos^2 2u)}$$

$$= \frac{a^2\left(5 - \dfrac{4z^2}{a^2}\right)^3}{\left(5 + \dfrac{12z^2}{a^2}\right)} \qquad \text{[since } z = a\cos 2u\text{]}$$

$$= \frac{a^2(5a^2 - 4z^2)^3}{a^2(5a^2 + 12z^2)}$$

And $$\sigma = \frac{1}{\tau} = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]} = \frac{a^4(5 + 12\cos^2 2u)}{6a^3 \sin 2u}$$

$$= \frac{a\left(5 + \dfrac{12z^2}{a^2}\right)}{6\sqrt{\left(1 - \dfrac{z^2}{a^2}\right)}} = \frac{5a^2 + 12z^2}{6\sqrt{(a^2 - z^2)}}$$

Example 2.14: A right helix of radius a and slope α has four point contact with a given curve at the point where its curvature and torsion are $1/\rho$ and $1/\sigma$. Prove that

$$a = \sigma^2 \rho / (\rho^2 + \sigma^2) \quad \text{and } \tan\alpha = \rho/\sigma.$$

Solution. For a three point contact between two curves, consecutive tangents to the curves are same. Hence ρ is the same.

If in addition there is contact at the fourth point also, consecutive osculating planes and hence consecutive binormals are the same. Hence σ is the same.

Thus ρ and σ for the curve are the same as ρ and σ for the helix.

For the helix, we have

$$\rho = a\sec^2\alpha \quad \text{or} \quad \frac{1}{\rho} = \frac{1 + \cos 2\alpha}{2a} \qquad (1)$$

and $\qquad \sigma = \pm \dfrac{a}{\sin\alpha\cos\alpha} \qquad$ or $\qquad \dfrac{1}{\sigma} = \pm \dfrac{\sin 2\alpha}{2a} \qquad (2)$

From (1) and (2), we have

$$\left(\frac{1}{\rho} - \frac{1}{2a}\right)^2 + \frac{1}{\sigma^2} = \frac{1}{4a^2}$$

Or $\qquad \dfrac{1}{\rho^2} + \dfrac{1}{\sigma^2} = \dfrac{1}{\rho a} \qquad$ or $\qquad a = \dfrac{\rho\sigma^2}{\rho^2 + \sigma^2}$

Also $\quad \dfrac{\rho}{\sigma} = \dfrac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan\alpha$.

2.6 CHECK YOUR PROGRESS

SP-1. Define the osculating plane at a point of a space curve and determine its equation.

SP-2. Show that if a curve is given in terms of a general parameter u, then equation of the osculating plane is $[\mathbf{R} - \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}] = 0$

SP-3. Define the osculating plane of a curve at a point and from this definition find its equation.

SP-4. Find the equation of the osculating plane at a point of a space curve given by the intersection of surfaces $f(\mathbf{r}) = 0$ and $g(\mathbf{r}) = 0$.

SP-5. Find the osculating plane at a point of intersection of the surface $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$.

SP-6. Define the normal plane to a space curve at a point and find its equation.

SP-7. Find the osculating plane at the point u on the helix $x = a\cos u, y = a\sin u, z = cu$.

SP-8. Show that oscillating planes at any three points on the curve $\quad x = te_1 + \dfrac{1}{2}t^2 e_2 + \dfrac{1}{3}t^3 e_3$

meet at a point laying in the plane determined by these three points.

SP-9. Show that a curve is a plane curve if all oscillating planes have a common point of intersection.

SP-10. Explain the concept of unit tangent vector, unit principal normal vector and unit binomial vector for a space curve.

SP-11. If the tangent and the binormal at a point of a curve makes angles θ, ϕ respectively with a fixed direction, show that $\dfrac{\sin\theta}{\sin\phi}\dfrac{d\theta}{d\phi} = -\dfrac{k}{\tau}$.

SP-12. Show that the principal normals at consecutive points do not intersect unless $\tau = 0$.

SP-13. Find the curvature along the curve $x = (t - \sin t)e_1 + (1 - \cos t)e_2 + te_3$.

SP-14. Show that radius of spherical curvature of a circular helix is equal to radius of circular curvature.

SP-15. Find the curvature and torsion of the curve $x = a(u - \sin u)$, $y = a(1 - \cos u)$, $s = bu$.

SP-16. Find the curvature, centre of curvature, and the torsion of the curve $x = a\cos u$, $y = a\sin u$, $z = a\cos 2u$.

SP-17. Establish briefly the Serret-Frenet formulae at a point of space curve.

SP-18. Define the curvature k and torsion τ of a twisted curve and establish Serret-Frenet formulae.

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CHAPTER-3

CURVES IN SPACE R^3(II)

Objectives: The students will learn about Screw curvature, Serret Frenet formulae, and how to use Serret Frenet formulae to find the curvatures and torsion of a skewed plane in this chapter, which is a continuation of the second chapter.

3.1 INTRODUCTION

In previous chapter we have studied the characterstics of continuous curve of class ≥ 2 and plane like, osculating plane, tangent plane, normal plane, rectifying plane, intersection of space curve and plane, principal normal and binormal, curvature, and torsion of the curve. In continuation of second chapter, to know about the more characterstics of the curves and surfaces the current chapter is required.

3.2 SCREW CURVATURE

Definition: The arc rate at which the principal normal changes direction $i.e.\left(\dfrac{d\mathbf{n}}{ds}\right)$ as PI moves along the curve is called the screw curvature and its magnitude is denoted by $\sqrt{k^2+\tau^2}$.

Note: We often say $k = 1/\rho$ where ρ is called the radius of curvature and $\tau = 1/\sigma$, where σ is called the radius of torsion.

(A) Serret Frenet formulae: The following three relations are known as Serret Frenet formulae

$$\mathbf{t}' = k\,\mathbf{n} \tag{1}$$

$$\mathbf{n}' = \tau\mathbf{b} - k\mathbf{t} \tag{2}$$

$$\mathbf{b}' = -\tau\,\mathbf{n} \tag{3}$$

where the symbols have their usual meaning.

Proof: Proof of formula (1). We know

$$\mathbf{t}^2 = 1$$

Differentiating w. r. t. the arc length 's'

$\mathbf{t}\cdot\mathbf{t}' = 0$ This implies $\mathbf{t}'$ is perpendicular to $\mathbf{t}$.

The equation of the osculating plane at a point PI of the curve is

$$[\mathbf{R}-\mathbf{r},\mathbf{t},\mathbf{t}'] = 0$$

The last equation shows that $\mathbf{t}'$ lie in the osculating plane and hence $\mathbf{t}'$ is perpendicular to the binomial **b** (since osculating plane is perpendicular to **b**).

Thus $\mathbf{t}'$ is parallel to $\mathbf{b}\times\mathbf{t}$ $\quad\Rightarrow\quad$ $\mathbf{t}'$ is parallel to **n**

We may write

$$\mathbf{t}' = \pm k\,\mathbf{n}$$

By conclusion, we have

$$\mathbf{t}' = k\,\mathbf{n}\,.$$

Note: One proof of first formula has also been given in (1) above.

Proof of formula (3). We know that

$$\mathbf{b}^2 = 1$$

Differentiating w. r. t. the arc length 's'

$\mathbf{b}\cdot\mathbf{b}' = 0$ showing that $\mathbf{b}'$ is perpendicular to $\mathbf{b}$ and thus $\mathbf{b}'$ lies in the osculating plane.

Also, $\quad \mathbf{b}\cdot\mathbf{t} = 0$

Differentiating w. r. t. 's'

$$\mathbf{b}'\cdot\mathbf{t} + \mathbf{b}\cdot\mathbf{t}' = 0 \quad i.e. \quad \mathbf{b}'\cdot\mathbf{t} + \mathbf{b}\cdot k\mathbf{n} = 0$$

i.e. $\quad \mathbf{b}'\cdot\mathbf{t} = 0$

Showing $\mathbf{b}'$ is perpendicular to **t**, but $\mathbf{b}'$ is also perpendicular to **b** and $\mathbf{b}'$ lies in the osculating plane and therefore $\mathbf{b}'$ must be parallel to **n**. [Alternatively $\mathbf{b}'$ is perpendicular to **b** and **t** both implies $\mathbf{b}'$ is parallel to $\mathbf{b}\times\mathbf{t}$ *i.e.* $\mathbf{b}'$ is parallel to **n**].

We may write $\quad \mathbf{b}' = \pm\tau\,\mathbf{n}$

By conclusion, we have $\quad \mathbf{b}' = -\tau\,\mathbf{n}\,.$

Where τ is the torsion of the curve and measures the arc rate of turning of **b**.

Proof of formula (2):

$$\mathbf{n}' = \tau\mathbf{b} - k\mathbf{t}$$

We know $\quad \mathbf{n} = \mathbf{b}\times\mathbf{t}$

Differentiating w. r. t. 's'

$$\mathbf{n}' = \mathbf{b}'\times\mathbf{t} + \mathbf{b}\times\mathbf{t}' = -\tau\mathbf{n}\times\mathbf{t} + \mathbf{b}\times k\mathbf{n} \qquad [\because\ \mathbf{b}' = -\tau\,\mathbf{n} \text{ and } \mathbf{t}' = k\,\mathbf{n}\,]$$

$= \tau \mathbf{b} - k\,\mathbf{t}$

Remark: Serret Frenet formulae may be represented in the form of matrix equation as follows

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

Corollary: Serret frenet formulae in Cartesians.

Let $l_1, m_1, n_1;\ l_2, m_2, n_2;\ and\ l_3, m_3, n_3$ be the direction cosines of the tangent, the principal normal and the binormal respectively at a point on the curve, so that

$\mathbf{t} = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k}$; $\mathbf{n} = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k}$; $\mathbf{b} = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}$;

Then substitution in Frenet's formulae provide

$$l_1' = \frac{l_2}{\rho},\ m_1' = \frac{m_2}{\rho},\ n_1' = \frac{n_2}{\rho}$$

$$l_2' = -\frac{l_1}{\rho} + \frac{l_3}{\rho},\ m_1' = -\frac{m_1}{\rho} + \frac{m_3}{\rho},\ n_1' = -\frac{n_1}{\rho} + \frac{n_3}{\rho}$$

$$l_3' = -\frac{l_2}{\rho},\ m_3' = -\frac{m_2}{\rho},\ n_3' = -\frac{n_2}{\rho}$$

Theorem 3.1: To show that a necessary and sufficient condition that a curve be a straight line is that $k = 0$ at all points.

Proof: Necessary Condition: The equation of a straight line $\mathbf{r} = \mathbf{a}s + \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

Hence $\mathbf{r}' = \mathbf{a}$ *i.e.* $\mathbf{t} = \mathbf{a}$; differentiating again $\mathbf{t}' = 0$ but $\mathbf{t}' = k\mathbf{n}$ (Frenet's formula)

$\therefore 0 = k\mathbf{n}$, squaring we get $k^2 = 0$ i.e. $k = 0$ which is, therefore, the necessary condition for a curve to be a straight line.

Sufficient Condition: Conversely if $k = 0$, Frenet's Formula $\mathbf{t}' = k\mathbf{n}$ gives $\mathbf{t}' = 0$ i.e. $\mathbf{r}'' = 0$ .Integrating we get $\mathbf{r}' = \mathbf{a}$ constant vector, integrating once more we get an equation of the type $\mathbf{r} = \mathbf{a}s + \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and represent a straight line. Hence $k = 0$ is also sufficient.

Theorem 3.2: If C is a curve for which **b** varies differentially with arc length. Then to show that a necessary and sufficient condition that C is a plane curve is that $\tau = 0$ at all points.

Proof: Necessary Condition: Let the curve lie in a plane. Since **b** is normal to the osculating plane, therefore the plane curve lies in the osculating plane, i.e. the plane considered is osculating plane and it must be fixed. Now we know **t** and **n** lie in this plane (osculating plane) and hence ($\mathbf{b} = \mathbf{t} \times \mathbf{n}$) is a constant vector.

$\therefore$ $\mathbf{b} = a$ constant vector and so $\mathbf{b}' = 0$

Hence Frenet's formula $\mathbf{b}' = -\tau\,\mathbf{n}$ gives $0 = -\tau\,\mathbf{n}$, squaring we get $\tau = 0$, as the necessary condition for the curve to be a plane curve.

Sufficient Condition: Conversely given $\tau = 0$ to prove that the curve is a plane. We have from Frenet's Formula ($\mathbf{b}' = -\tau\,\mathbf{n}$), $\mathbf{b}' = 0$; integrating it, **b=a** constant vector. If the equation of the curve is **r=r**(s), we have

$$(\mathbf{r}.\mathbf{b})' = \mathbf{r}'.\mathbf{b} + \mathbf{r}.\mathbf{b}' = \mathbf{t}.\mathbf{b} + \mathbf{r}.\mathbf{b}' = 0 + 0 \qquad [\because \mathbf{t}.\mathbf{b} = 0, \mathbf{b}' = 0]$$

i.e. $\mathbf{r}.\mathbf{b} = \mathbf{a}$

constant $\Rightarrow$ that any vector from the origin to the curve is a right angle s (taking this constant to be zero) to the fixed **b**. Hence the curve must be a plane curve. Hence $\tau = 0$ is also the sufficient condition for the curve to be the plane curve.

Theorem 3. 3 (i): To show that necessary and sufficient condition for the curve to be the plane curve is $[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = 0$

Proof: We have $\mathbf{r}' = \mathbf{t}$ (1)

Differentiating w.r.t. s,

$$\mathbf{r}'' = \mathbf{t}' \qquad \text{or} \qquad \mathbf{r}'' = k\mathbf{n} \tag{2}$$

Taking vector product of (1) and (2),

$$\mathbf{r}' \times \mathbf{r}'' = k\mathbf{b} \tag{3}$$

Differentiating (3) w.r.t. 's',

$$\mathbf{r}' \times \mathbf{r}''' + \mathbf{r}'' \times \mathbf{r}'' = k'\mathbf{b} + k\mathbf{b}'$$

$$\mathbf{r}' \times \mathbf{r}''' = k'\mathbf{b} - k\tau\,\mathbf{n} \qquad [\because \mathbf{r}'' \times \mathbf{r}'' = 0] \tag{4}$$

Taking scalar product of (1) and (4)

$$\mathbf{r}''.(\mathbf{r}'\times\mathbf{r}''') = kk'\mathbf{nb} - k^2\tau\,\mathbf{n.n}$$

or $$[-\mathbf{r}',\mathbf{r}'',\mathbf{r}'''] = -k^2\tau \qquad [\because\ \mathbf{n.b}=0,\ \mathbf{n.n}=1]$$

or $$[\mathbf{r}',\mathbf{r}'',\mathbf{r}'''] = k^2\tau \tag{5}$$

If the left hand member of the equation (5) is zero, then either $k=0$ or $\tau=0$.

Now let $\tau\neq 0$ at some points of the curve, then in the neighborhood of this point $\tau\neq 0$. Hence $k=0$ in this neighbourhood and hence the curve is a straight line (see Theorem 1) and therefore $\tau=0$ on this line and this is in contradiction to our hypothesis. Hence $\tau=0$ at all points and the curve is a plane.

Conversely: If $\tau=0$ i.e. the curve is a plane and therefore from the equation (5) $[\mathbf{r}',\mathbf{r}'',\mathbf{r}''']=0$. Therefore the condition is necessary as well as sufficient.

Theorem 3.3(ii): Theorem 3(a) may also be put as – To show that the necessary and sufficient condition for the curve to be plane is

$$[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]=0$$

Proof: We have

$$[\mathbf{r}',\mathbf{r}'',\mathbf{r}'''] = [\dot{\mathbf{r}}u' + \ddot{\mathbf{r}}u'^2 + \dddot{\mathbf{r}}u'^3 + \dot{\mathbf{r}}u''' + 3\ddot{\mathbf{r}}u'u'']$$

$$= u'^6[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]$$

or $$[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}] = \dot{\mathbf{s}}^6[\mathbf{r}',\mathbf{r}'',\mathbf{r}'''] \qquad \left[\because\ u'=\frac{du}{ds}=\frac{1}{ds/du}=\frac{1}{\dot{\mathbf{s}}}\right]$$

$$= \dot{\mathbf{s}}^6 k^2\tau$$

Now see theorem 3(i).

Example 3.1: Using Frenet's formula, for the curve $4a\cos^3 u$, $y=4a\sin^3 u$, $z=3c\cos 2u$

Prove that $\kappa = \dfrac{a}{6(a^2+c^2)\sin 2u}$

Solution. Here, $\mathbf{r}=(4a\cos^3 u, 4a\sin^3 u, 3c\cos 2u)$

Differentiating w.r.t. arc length 's'

$$\mathbf{r}'=\mathbf{t}=6(-2a\cos^2 u\sin u, 2a\sin^2 u\cos u, -c\sin 2u)\frac{du}{ds}$$

$$\mathbf{t} = 12\sin u \cos u(-a\cos u, a\sin u, -c)\frac{du}{ds} \tag{1}$$

Squaring equation (1) and using $\mathbf{t}^2 = 1$

$$1 = 144\sin^2 u\cos^2 u(a^2\cos^2 u + a^2\sin^2 u + c^2)\left(\frac{du}{ds}\right)^2$$

or
$$\frac{ds}{du} = 12\sin u\cos u\sqrt{(a^2 + c^2)} \tag{2}$$

Substituting equation (2) in (1), we get

$$\mathbf{t} = 12\sin u\cos u(-a\cos u, a\sin u, -c)\frac{1}{12\sin u\cos u\sqrt{(a^2 + c^2)}}$$

$$\therefore \quad \mathbf{t} = \frac{1}{\sqrt{(a^2 + c^2)}}(-a\cos u, a\sin u, -c) \tag{3}$$

Differentiating equation (3) w.r.t. ‘s’ and using $\mathbf{t}' = \kappa\,\mathbf{n}$, we have

$$\mathbf{t}' = \kappa\,\mathbf{n} = \frac{1}{\sqrt{(a^2 + c^2)}}(a\sin u, a\cos u, 0)\frac{du}{ds} \tag{4}$$

$$|\kappa||\mathbf{n}| = \frac{1}{\sqrt{a^2 + c^2}}(a^2\sin^2 u + a^2\cos^2 u)^{1/2}\left(\frac{du}{ds}\right)$$

or
$$\kappa = \frac{1}{\sqrt{a^2 + c^2}}\cdot\frac{a}{12\sin u\cos u\sqrt{a^2 + c^2}} \qquad [\because |\mathbf{n}| = 1 \text{ and using (2)}]$$

or
$$\kappa = .\frac{a}{6(a^2 + c^2)\sin 2u}.$$

Example 3.2: Given the curve $\mathbf{r} = (e^{-u}\sin u,\, e^{-u}\cos u, e^{-u})$. Find at any point ‘$u$’ of this curve

(i) Unit tangent vector **t**

(ii) The equation of tangent

(iii) The equation of normal plane

(iv) The curvature

(v) The unit principal normal vector **b**, and

(vi) The equation of the binormal.

Solution. The equation of given curve is

$$\mathbf{r} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})$$

$$\therefore\quad \mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du}\frac{du}{ds} = (e^{-u}\cos u - e^{-u}\sin u, -e^{-u}\sin u - e^{-u}\cos u, -e^{-u})\frac{du}{ds}$$

$$\text{or}\quad \mathbf{t} = e^{-u}(\cos u - \sin u, -\sin u - \cos u, -1)\frac{du}{ds} \qquad (1)$$

Squaring both sides of equation (1), we have

$$1 = e^{-2u}[(\cos u - \sin u)^2 + (-\sin u - \cos u)^2 + (-1)^2]\left(\frac{du}{ds}\right)^2 \qquad [\because \mathbf{t}^2 = 1]$$

$$\text{or } 1 = e^{-2u}.3\left(\frac{du}{ds}\right)^2 \qquad \Rightarrow \frac{du}{ds} = \frac{e^u}{\sqrt{3}} \qquad (2)$$

putting the value of du/ds in (1), we have

$$\mathbf{t} = e^{-u}.\frac{e^u}{\sqrt{3}}(\cos u - \sin u, -\sin u - \cos u, -1)$$

$$\text{or }\ \mathbf{t} = e(1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1) \qquad (3)$$

The relation (3) gives the unit tangent vector **t**.

The **equation of the tangent line** to the curve at the point 'u' is

$$\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$$

$$\text{or}\qquad \mathbf{R} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u}) + \lambda(1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1)$$

If **R**=(X,Y,Z) the equation of the tangent line in Cartesian coordinates are

$$\frac{X - e^{-u}\sin u}{\cos u - \sin u} = \frac{Y - e^{-u}\cos u}{-\sin u - \cos u} = \frac{Z - e^{-u}}{-1} \qquad (4)$$

The equation of the normal plane to the curve at point 'u' is

$$(\mathbf{R} - \mathbf{r})\bullet\mathbf{t} = 0$$

$$\text{or}\quad [\mathbf{R} - (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})]\bullet(1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1) = 0$$

$$\text{or}\quad [\mathbf{R} - (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})](-\cos u + \sin u, \sin u + \cos u, 1) = 0$$

If **R**=(X,Y,Z), the equation of normal plane in Cartesian form is

$$(X - e^{-u}\sin u)(\sin u - \cos u) + (Y - e^{-u}\cos u)(\sin u + \cos u) + (Z - e^{-u}).1 = 0$$

or $$X(\sin u - \cos u) + Y(\sin u + \cos u) + Z = 2e^{-u} \tag{5}$$

Now differentiating equation (3) w.r.t. 's' and using Frenet's formula, we have

$$\kappa\mathbf{n} = \frac{d\mathbf{t}}{du}.\frac{du}{ds} = \frac{1}{\sqrt{3}}(-\sin u - \cos u, -\cos u + \sin u, 0)\frac{du}{ds} \tag{6}$$

Squaring both sides of (6), we have

$$\kappa^2 = \frac{1}{3}[(-\sin u - \cos u)^2 + (-\cos u + \sin u)^2 + 0^2)]\left(\frac{du}{ds}\right)^2$$

or $$\kappa^2 = \frac{2}{3}\left(\frac{du}{ds}\right)^2 \quad \Rightarrow \kappa = \sqrt{\frac{2}{3}}\left(\frac{du}{ds}\right) \tag{7}$$

Using (2) in (7), we have

$$\kappa = \sqrt{\frac{2}{3}}\frac{e^u}{\sqrt{3}} = \frac{\sqrt{2}}{3}e^u \tag{8}$$

Relation (8) gives curvature of the curve.

Substituting the value of κ from (7) to (6), we have

$$\sqrt{\frac{2}{3}}\left(\frac{du}{ds}\right)\mathbf{n} = \frac{1}{\sqrt{3}}(-\sin u - \cos u, -\cos u + \sin u, 0)\frac{du}{ds}$$

$$\mathbf{n} = \frac{1}{\sqrt{2}}(-\sin u - \cos u, -\cos u + \sin u, 0)$$

The relation (9) gives us the unit principal normal vector **n**. The equation of the principal normal at the point 'u' is

$$\mathbf{R} = \mathbf{r} + v\mathbf{n}$$

or $$\mathbf{R} = (e^{-u}\sin u,\, e^{-u}\cos u, e^{-u})] - \frac{v}{\sqrt{2}}(\sin u + \cos u, -\cos u - \sin u, 0)$$

In cartesian form, the equation of principal normal are

$$\frac{X - e^{-u}\sin u}{\sin u + \cos u} = \frac{Y - e^{-u}\cos u}{\cos u - \sin u} = \frac{Z - e^{-u}}{0} \tag{10}$$

Now, we have

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{1}{\sqrt{3}}(\cos u - \sin u, -\sin u - \cos u, -1) \times \left(-\frac{1}{\sqrt{2}}\right)(\sin u + \cos u, -\cos u - \sin u, 0)$$

or $$\mathbf{b} = -\frac{1}{6}(\cos u - \sin u, -\sin u - \cos u, 2)$$

or $$\mathbf{b} = \frac{1}{\sqrt{6}}(\sin u - \cos u, \sin u + \cos u, -2) \qquad (11)$$

The relation (11) gives us the unit binormal vector **b**. The equation of the binormal at the point 'u' is

$$\mathbf{R} = \mathbf{r} + \lambda \mathbf{b}$$

or $$\mathbf{R} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})] + \frac{\lambda}{\sqrt{6}}(\sin u - \cos u, \sin u + \cos u, -2)$$

In Cartesian form, the equation of the binormal is

$$\frac{X - e^{-u}\sin u}{\sin u - \cos u} = \frac{Y - e^{-u}\cos u}{\sin u + \cos u} = \frac{Z - e^{-u}}{-2}$$

Example 3.3: Calculate the curvature and torsion of the cubic curve given by $\mathbf{r} = (u, u^2, u^3)$.

Solution. Here $\mathbf{r} = (u, u^2, u^3)$

$$\therefore\ \dot{\mathbf{r}} = (1, 2u, 3u^2)$$

$$\therefore\quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (\mathbf{i} + 2u\mathbf{j} + 3u^2\mathbf{k}) \times (2\mathbf{j} + 6u\mathbf{k})$$

$$= 2\mathbf{k} - 6u\mathbf{j} + 12u^2\mathbf{i} - 6u^2\mathbf{i}$$

$$= 6u^2\mathbf{i} - 6u\mathbf{j} + 2\mathbf{k} = (6u^2, -6u,\ 2)$$

$$= 2(3u^2, -3u, 1)$$

$$\therefore\qquad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = 2(9u^4 + 9u^2 + 1)^{1/2}$$

Also, $[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \bullet \dddot{\mathbf{r}} = 2(3u^2, -3u, 1) \bullet (0,0,6)$

2(0+0+6)=12

$$\therefore\qquad k = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{2(9u^4 + 9u^2 + 1)^{1/2}}{(1 + 4u^2 + 9u^4)^{3/2}} \qquad (1)$$

And $$\tau = \frac{|\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}|}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{12}{4(9u^4 + 9u^2 + 1)}$$

Or $$\tau = \frac{3}{(9u^4 + 9u^2 + 1)} \qquad (2)$$

Example 3.4: For the curve $x = a(3u - u^3)$, $y = 3au^2$ show that curvature and torsion are equal.

Solution. The positive vector **r** of any point of the curve is given by

$$\mathbf{r} = (a(3u - u^3), 3au^2, a(3u + u^3))$$

Differentiating w.r.t. 'u', we have

$$\dot{\mathbf{r}} = a(3 - 3u^2, 6u, 3u + 3u^2)$$

$$\ddot{\mathbf{r}} = a(-6u, 6, 6u)$$

$$\dddot{\mathbf{r}} = a(-6, 0, 6) = 6a(-1,0,1)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 3a(1-u^2, 2u, 1+u^2) \times 6a(-u, 1, u)$$

$$= 18a^2[(1-u^2)\mathbf{k} - u(1-u^2)\mathbf{j} - 2u^2\mathbf{k} + 2u^2\mathbf{i} - u(1+u^2)\mathbf{j} - (1+u^2)\mathbf{i}]$$

$$= 18a^2[-(1-u^2)\mathbf{i} - (u + u - u^3 + u^3)\mathbf{j} + (1 - u^2 + 2u^2)\mathbf{k}]$$

$$= 18a^2[-(1-u^2)\mathbf{i} - 2u\mathbf{j} + (1+u^2)\mathbf{k}]$$

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = 18a^2[(1-u^2)^2 + 4u^2 + (1+u^2)^2]^{1/2}$$

$$= 18a^2[(1+u^2)^2 + (1+u^2)^2]^{1/2} = 18\sqrt{2}a^2(1+u^2)$$

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \bullet \dddot{\mathbf{r}} = 18a^2[-(1-u^2)^2, 2u(1+u^2).6a(-1,0,1)]$$

$$= 108a^3[+(1-u^2)^2 + 0 + (1+u^2)] = 216a^3$$

$$|\dot{\mathbf{r}}| = 3a\{(1-u^2)^2 + 4u^2 + (1+u^2)^2\}^{1/2}$$

$$= 3a\{(1+u^2)^2 + (1+u^2)^2\}^{1/2} = 3\sqrt{2}a(1+u^2)$$

$$\therefore \qquad k = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{18\sqrt{2}a^2(1+u^2)}{3 \times 18\sqrt{2}a^3(1+u^2)^3} = \frac{1}{3a(1+u^2)}$$

And $$\tau = \frac{|\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}|}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{216a^3}{18 \times 18 \times 2a^4(1+u^2)^2} = \frac{1}{3a(1+u^2)^2}$$

Hence $k = \tau$.

Example 3.5: Find the radii of curvature and torsion of the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$.

Solution. Here $\mathbf{r} = (a\cos u, a\sin u, au\tan\alpha)$

Differentiating w.r.t. 'u' we get

$$\dot{\mathbf{r}} = a(-\sin u, \cos u, \tan\alpha)$$

$$\ddot{\mathbf{r}} = a(-\cos u, -\sin u, 0)$$

$$\dddot{\mathbf{r}} = a(\sin u, -\cos u, 0)$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = a^2(\sin u\tan\alpha, \cos u\tan\alpha, 1)$$

$$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = a^2[(\sin^2 u\tan^2\alpha + \cos^2 u\tan^2\alpha + 1)]^{1/2} = a^2\sec\alpha$$

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = \dot{\mathbf{r}}\times\ddot{\mathbf{r}}\bullet\dddot{\mathbf{r}} = a^3(\sin^2 u\tan\alpha + \cos^2 u\tan\alpha + 0 = a^3\tan\alpha$$

Also, $$|\dot{\mathbf{r}}| = a[\sin^2 u + \cos^2 u + \tan^2\alpha]^{1/2} = a\sec\alpha$$

$$\therefore \quad k = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{a^2\sec\theta}{a^3 se^3 c\theta} = \frac{1}{a\sec^2\alpha} = \frac{\cos^2\alpha}{a}$$

$$\therefore \quad \rho = \frac{1}{k} = a\sec^2\alpha$$

and $$\tau = \frac{a^2\tan\alpha}{a^4\sec^2\alpha} = \frac{\sin\alpha}{a\sec\alpha}$$

$$\therefore \quad \sigma = \frac{1}{\rho} = a\,\mathrm{cosec}\,\alpha\sec\alpha$$

Example 3.6: For the curve $x = 4a\cos^3 u$, $y = 4a\sin^3 u$, $z = 3c\cos 2u$

Prove that $\kappa = \dfrac{a}{6(a^2 + c^2)\sin 2u}$

Solution. Here, $\mathbf{r} = (4a\cos^3 u, 4a\sin^3 u, 3c\cos 2u)$

Differentiating w.r.t. arc length 's'

$$\mathbf{r}' = \mathbf{t} = 6(-2a\cos^2 u\sin u, 2a\sin^2 u\cos u, -c\sin 2u)\frac{du}{ds}$$

$$\mathbf{t} = 12\sin u\cos u(-a\cos u, a\sin u, -c)\frac{du}{ds} \qquad (1)$$

Squaring equation (1) and using $\mathbf{t}^2 = 1$

$$1 = 144\sin^2 u\cos^2 u(a^2\cos^2 u + a^2\sin^2 u + c^2)\left(\frac{du}{ds}\right)^2$$

or $$\frac{ds}{du} = 12\sin u\cos u\sqrt{(a^2 + c^2)} \qquad (2)$$

Substituting equation (2) in (1), we get

$$\mathbf{t} = 12\sin u\cos u(-a\cos u, a\sin u, -c)\frac{1}{12\sin u\cos u\sqrt{(a^2 + c^2)}}$$

$$\therefore \quad \mathbf{t} = \frac{1}{\sqrt{(a^2 + c^2)}}(-a\cos u, a\sin u, -c) \qquad (3)$$

Differentiating equation (3) w.r.t. 's' and using $\mathbf{t}' = \kappa\,\mathbf{n}$, we have

$$\mathbf{t}' = \kappa\,\mathbf{n} = \frac{1}{\sqrt{(a^2 + c^2)}}(a\sin u, a\cos u, 0)\frac{du}{ds} \qquad (4)$$

$$|\kappa||\mathbf{n}| = \frac{1}{\sqrt{a^2 + c^2}}(a^2\sin^2 u + a^2\cos^2 u)^{1/2}\left(\frac{du}{ds}\right)$$

or $$\kappa = \frac{1}{\sqrt{a^2 + c^2}}.\frac{a}{12\sin u\cos u\sqrt{a^2 + c^2}} \qquad [\because |\mathbf{n}| = 1 \text{ and using (2)}]$$

or $$\kappa = .\frac{a}{6(a^2 + c^2)\sin 2u}.$$

Example 3.7: Given the curve $\mathbf{r} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})$. Find at any point '$u$' of this curve

(vii) Unit tangent vector **t**

(viii) The equation of tangent

(ix) The equation of normal plane

(x) The curvature

(xi) The unit principal normal vector **b**, and

(xii) The equation of the binormal.

Solution. The equation of given curve is

$$\mathbf{r} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})$$

$$\therefore \quad \mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du}\frac{du}{ds} = (e^{-u}\cos u - e^{-u}\sin u, -e^{-u}\sin u - e^{-u}\cos u, -e^{-u})\frac{du}{ds}$$

$$\text{or} \quad \mathbf{t} = e^{-u}(\cos u - \sin u, -\sin u - \cos u, -1)\frac{du}{ds} \tag{1}$$

Squaring both sides of equation (1), we have

$$1 = e^{-2u}[(\cos u - \sin u)^2 + (-\sin u - \cos u)^2 + (-1)^2]\left(\frac{du}{ds}\right)^2$$

$$[\because \mathbf{t}^2 = 1]$$

$$\text{or} \quad 1 = e^{-2u}.3\left(\frac{du}{ds}\right)^2 \quad \Rightarrow \frac{du}{ds} = \frac{e^u}{\sqrt{3}} \tag{2}$$

putting the value of du/ds in (1), we have

$$\mathbf{t} = e^{-u}.\frac{e^u}{\sqrt{3}}(\cos u - \sin u, -\sin u - \cos u, -1)$$

$$\text{or} \quad \mathbf{t} = e(1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1) \tag{3}$$

The relation (3) gives the unit tangent vector **t**.

The **equation of the tangent line** to the curve at the point 'u' is

$$\mathbf{R} = \mathbf{r} + \lambda\,\mathbf{t}$$

$$\text{or} \quad \mathbf{R} = (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u}) + \lambda(1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1)$$

If **R**=(X,Y,Z) the equation of the tangent line in Cartesian coordinates are

$$\frac{X - e^{-u}\sin u}{\cos u - \sin u} = \frac{Y - e^{-u}\cos u}{-\sin u - \cos u} = \frac{Z - e^{-u}}{-1} \tag{4}$$

The equation of the normal plane to the curve at point 'u' is

$$(\mathbf{R} - \mathbf{r}) \bullet \mathbf{t} = 0$$

$$\text{or} \quad [\mathbf{R} - (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})] \bullet (1/\sqrt{3})(\cos u - \sin u, -\sin u - \cos u, -1) = 0$$

$$\text{or} \quad [\mathbf{R} - (e^{-u}\sin u,\ e^{-u}\cos u, e^{-u})](-\cos u + \sin u, \sin u + \cos u, 1) = 0$$

If **R**=(X,Y,Z), the equation of normal plane in Cartesian form is

$$(X - e^{-u}\sin u)(\sin u - \cos u) + (Y - e^{-u}\cos u)(\sin u + \cos u) + (Z - e^{-u}).1 = 0$$

or $\quad X(\sin u - \cos u) + Y(\sin u + \cos u) + Z = 2e^{-u}$ (5)

Now differentiating eequation (3) w.r.t. 's' and using Frenet's formula, we have

$$\kappa \mathbf{n} = \frac{d\mathbf{t}}{du}.\frac{du}{ds} = \frac{1}{\sqrt{3}}(-\sin u - \cos u, -\cos u + \sin u, 0)\frac{du}{ds} \tag{6}$$

Squaring both sides of (6), we have

$$\kappa^2 = \frac{1}{3}[(-\sin u - \cos u)^2 + (-\cos u + \sin u)^2 + 0^2)]\left(\frac{du}{ds}\right)^2$$

or $\quad \kappa^2 = \dfrac{2}{3}\left(\dfrac{du}{ds}\right)^2 \quad \Rightarrow \kappa = \sqrt{\dfrac{2}{3}}\left(\dfrac{du}{ds}\right)$ (7)

Using (2) in (7), we have

$$\kappa = \sqrt{\frac{2}{3}}\frac{e^u}{\sqrt{3}} = \frac{\sqrt{2}}{3}e^u \tag{8}$$

Relation (8) gives curvature of the curve.

Substituting the value of κ from (7) to (6), we have

$$\sqrt{\frac{2}{3}}\left(\frac{du}{ds}\right)\mathbf{n} = \frac{1}{\sqrt{3}}(-\sin u - \cos u, -\cos u + \sin u, 0)\frac{du}{ds}$$

$$\mathbf{n} = \frac{1}{\sqrt{2}}(-\sin u - \cos u, -\cos u + \sin u, 0)$$

The relation (9) gives us the unit principal normal vector **n**.

The equation of the principal normal at the point 'u' is

$$\mathbf{R} = \mathbf{r} + v\mathbf{n}$$

or $\quad \mathbf{R} = (e^{-u}\sin u,\, e^{-u}\cos u, e^{-u})] - \dfrac{v}{\sqrt{2}}(\sin u + \cos u, -\cos u - \sin u, 0)$

In cartesian form, the equation of principal normal are

$$\frac{X - e^{-u}\sin u}{\sin u + \cos u} = \frac{Y - e^{-u}\cos u}{\cos u - \sin u} = \frac{Z - e^{-u}}{0} \tag{10}$$

Now, we have

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{1}{\sqrt{3}}(\cos u - \sin u, -\sin u - \cos u, -1)$$

$$\times\left(-\frac{1}{\sqrt{2}}\right)(\sin u + \cos u, -\cos u - \sin u, 0)$$

or $$\mathbf{b} = -\frac{1}{6}(\cos u - \sin u, -\sin u - \cos u, 2)$$

or $$\mathbf{b} = \frac{1}{\sqrt{6}}(\sin u - \cos u, \sin u + \cos u, -2) \tag{11}$$

The relation (11) gives us the unit binormal vector **b**.

The equation of the binormal at the point 'u' is

$$\mathbf{R} = \mathbf{r} + \lambda \mathbf{b}$$

or $$\mathbf{R} = (e^{-u}\sin u,\, e^{-u}\cos u, e^{-u})] + \frac{\lambda}{\sqrt{6}}(\sin u - \cos u, \sin u + \cos u, -2)$$

In Cartesian form, the equation of the binormal is

$$\frac{X - e^{-u}\sin u}{\sin u - \cos u} = \frac{Y - e^{-u}\cos u}{\sin u + \cos u} = \frac{Z - e^{-u}}{-2}$$

Example 3.8: Find the basic unit vector **t**, **n**, and **b** of the curve $\mathbf{r} = (u, u^2, u^3)$ at the point $u = 1$. Also find the equation of tangent, the principal normal and binormal of this point.

Solution. The equation of the given curve is

$$\mathbf{r} = (u, u^2, u^3)$$

$$\therefore \quad \frac{d\mathbf{r}}{du} = \dot{\mathbf{r}} = (1, 2u, 3u^2)$$

or $$\frac{d\mathbf{r}}{ds}.\frac{ds}{du} = (1, 2u, 3u^2)$$

or $$\mathbf{t}.\frac{ds}{du} = (1, 2u, 3u^2) \tag{1}$$

on squaring equation (1), we have

$$\left(\frac{ds}{du}\right)^2 = 1 + 4u^2 + 9u^4 \Rightarrow \left(\frac{ds}{du}\right) = \sqrt{(1 + 4u^2 + 9u^4)} \tag{2}$$

Using equation (2) in (1), we get

$$\mathbf{t} = \frac{1}{\sqrt{(1+4u^2+9u^4)}}(1, 2u, 3u^2) \tag{3}$$

Differentiating equation (1) w.r.t. 'u', we get

or $$\left(\frac{d\mathbf{t}}{ds}.\frac{ds}{du}\right)\frac{ds}{du} + \frac{ds}{du}\mathbf{t}\frac{d^2s}{du^2} = (0, 2, 6u) \tag{4}$$

Taking vector product of equation (1) and (4), we get

$$\kappa\mathbf{b}\left(\frac{ds}{du}\right)^3 = (6u^2, -6u, 2) \qquad [\because\ \mathbf{t}\times\mathbf{n} = \mathbf{b},\ \mathbf{t}\times\mathbf{t} = 0]$$

On squaring $\kappa^2\left(\frac{ds}{du}\right)^6 = 4(1+9u^2+9u^4)$

$$\Rightarrow \quad \kappa\left(\frac{ds}{du}\right)^3 = 2\sqrt{(1+9u^2+9u^4)} \tag{6}$$

Hence, using equation (5), (6) becomes

$$\mathbf{b} = \frac{1}{\sqrt{(1+9u^2+9u^4)}}(3u^2, -3u, 1) \tag{7}$$

Now putting $u = 1$, in (3) and (7), we have

$$\mathbf{t} = \frac{1}{\sqrt{14}}(1, 2, 3),\ \mathbf{b} = \frac{1}{\sqrt{19}}(3, -3, 1)$$

Also $$\mathbf{n} = \mathbf{b}\times\mathbf{t} = \frac{1}{\sqrt{266}}(-11, -8, 9)$$

The equation of the tangent at $u = 1$ i.e. at $\mathbf{r} = (1,1,1)$ are

$$\frac{X-1}{1} = \frac{Y-1}{1} = \frac{Z-1}{3} \tag{8}$$

The equation of principal normal at $u = 1$, are

$$\frac{X-1}{-11} = \frac{Y-1}{-8} = \frac{Z-1}{9} \tag{9}$$

The equation of binomial at $u = 1$, are

$$\frac{X-1}{3} = \frac{Y-1}{-3} = \frac{Z-1}{1} \tag{10}$$

Example 3.9: Find the radii of curvature and torsion at any point of the curve

$$x^2+y^2=a^2, \quad x^2-y^2=az.$$

Solution. Let the parametric equation of the curve be

$$x=a\cos u,\ y=a\sin u,\ z=a\cos 2u$$

$$\therefore\ \mathbf{r}=a(\cos u, \sin u, \cos 2u)$$

$$\therefore\ \dot{\mathbf{r}}=a(-\sin u, \cos u, -2\sin 2u)$$

$$\therefore\ \ddot{\mathbf{r}}=a^2(-\cos u, -\sin u, -4\cos 2u)$$

$$\therefore\ \dddot{\mathbf{r}}=a^3(\sin u, -\cos u, -8\sin 2u) \qquad (1)$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=a^2(-4\cos u\cos 2u-2\sin u\sin 2u,$$
$$2\sin 2u\cos u-4\sin u\cos 2u, \cos^2 u+\sin^2 u)$$

$$=a^2[-2\cos u\cos 2u-2\cos(2u-u),\ 2\sin(2u-u)-2\sin u\cos 2u,\ 1]$$

$$=a^2[-2\cos u\cos 2u-2\cos u,\ 2\sin u-2\sin u\cos 2u,\ 1]$$

$$=a^2(-4\cos^3 u, 4\sin^3 u, 1) \qquad (2)$$

Taking scalor product of equation (2) with (1), we have

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]=a^3\{-4\sin u\cos^3 u-4\sin^3 u\cos u+8\sin 2u\}$$

$$=a^3\{-4\sin u\cos u(\cos^2+\sin^2 u)+8\sin 2u\}$$

$$=a^3\{-4\sin u\cos u+8\sin 2u\}$$

$$=a^3\{-4\sin 2u+8\sin 2u\}=6a^3\sin 2u$$

Also, $$|\dot{\mathbf{r}}|^2=a^2(\cos^2 u+\sin^2 u+4\sin^2 2u)$$

$$=a^2\{1+4\sin^2 2u\}=a^2\{5-4\cos^2 2u\}$$

And $$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2=a^4(16\cos^6 u+16\sin^6 u+1)$$

$$=a^4[16\{\cos^2 u+\sin^2 u)(\cos^4 u+\sin^4 u-\cos^2 u\sin^2 u)\}+1]$$

$$=a^4[16\{(\cos^2 u+\sin^2 u)^2-3\cos^2 u\sin^2 u\}+1]$$

$$=a^4[16\{1-3\cos^2 u\sin^2 u\}+1]$$

$$= a^4[17 - 48\cos^2 u \sin^2 u]$$

$$= a^4[17 - 12\sin^2 2u]$$

$$= a^4[5 + 12\cos^2 2u]$$

Now, $\rho^2 = \dfrac{1}{k^2} = \dfrac{|\dot{\mathbf{r}}|^6}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \dfrac{a^6(5 - 4\cos^2 2u)^3}{a^4(5 + 12\cos^2 2u)}$

$$= \frac{a^2\left(5 - \dfrac{4z^2}{a^2}\right)^3}{\left(5 + \dfrac{12z^2}{a^2}\right)} \qquad \text{[since } z = a\cos 2u\text{]}$$

$$= \frac{a^2\left(5a^2 - 4z^2\right)^3}{a^2\left(5a^2 + 12z^2\right)}$$

And $\sigma = \dfrac{1}{\tau} = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]} = \dfrac{a^4(5 + 12\cos^2 2u)}{6a^3 \sin 2u}$

$$= \frac{a\left(5 + \dfrac{12z^2}{a^2}\right)}{6\sqrt{\left(1 - \dfrac{z^2}{a^2}\right)}} = \frac{5a^2 + 12z^2}{6\sqrt{(a^2 - z^2)}}$$

Example 3.10: Find the osculating plane, curvature and torsion at any point of the curve $x = a\cos 2u$, $y = a\sin 2u$, $z = 2a\sin u$.

Solution. The position vector **r** of any point on the curve is given by

$$\mathbf{r} = a(\cos 2u,\ \sin 2u,\ 2\sin u)$$

$$\dot{\mathbf{r}} = 2a(-\sin 2u,\ \cos 2u,\ \cos u)$$

$$\ddot{\mathbf{r}} = -2a(2\cos 2u, 2\sin 2u,\ \sin u)$$

$$\dddot{\mathbf{r}} = -2a(-4\sin 2u, 4\cos 2u,\ \cos u)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 4a^2(\{\sin u + \sin 2u\cos u\}, -\{\cos u + \cos u\cos 2u\}, 2)$$

$$[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]=\dot{\mathbf{r}}\times\ddot{\mathbf{r}}\bullet\dddot{\mathbf{r}}=8a^3[4\sin 2u(\sin u+\sin 2u\cos u)]$$
$$+4\cos 2u(\cos u+\cos u\cos 2u)-2\cos u]$$

$$=8a^3[4(\cos 2u\cos u+\sin 2u\sin u)]$$
$$+4\cos 2u(\cos^2 2u+\sin^2 2u)-2\cos u]$$

$$=8a^3[4\cos u+4\cos u-2\cos u]=48a^3\cos u$$

$$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|=4a^2[(\sin u+\sin 2u\cos u)^2+(\cos u+\cos u\cos 2u)^2+4]^{1/2}$$

$$=4a^2[5+3\cos^2 u]^{1/2}$$

Thus osculating plane is

$$[\mathbf{R}-\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}]=0 \qquad i.e. \qquad (\mathbf{R}-\mathbf{r})\bullet\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=0$$

i.e.

$$(x-a\cos 2u)(\sin u+\sin 2u\cos u)-(y-a\sin 2u)\cos u(1+\cos 2u)+2(z-2a\sin u)=0$$

or $\quad (\sin u+\sin 2u\cos u)x-2\cos^3 u y+2z=3u\sin u$

Also $\quad \kappa=\dfrac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^2}=\dfrac{4a^2(5+3\cos^2 u)^{1/2}}{8a^3(1+\cos^2 u)^{3/2}}$

$$=\frac{(5+3\cos^2 u)^{1/2}}{(1+\cos^2 u)^{3/2}}$$

$$\tau=\frac{[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]}{[\dot{\mathbf{r}}\times\ddot{\mathbf{r}}]^2}=\frac{8a^3\cos u}{16a^4(5+3\cos^2 u)}=\frac{3}{a(5\sec u+3\cos u)}.$$

Example 3.11: A right helix of radius a and slope α has four point contact with a given curve at the point where its curvature and torsion are $1/\rho$ and $1/\sigma$. Prove that

$$a=\sigma^2\rho/(\rho^2+\sigma^2) \quad \text{and} \quad \tan\alpha=\rho/\sigma.$$

Solution. For a three point contact between two curves, consecutive tangents to the curves are same. Hence ρ is the same.

If in addition there is contact at the fourth point also, consecutive osculating planes and hence consecutive binormals are the same. Hence σ is the same.

Thus ρ and σ for the curve are the same as ρ and σ for the helix.

For the helix, we have

$$\rho = a\sec^2\alpha \quad \text{or} \quad \frac{1}{\rho} = \frac{1+\cos 2\alpha}{2a} \qquad (1)$$

and $$\sigma = \pm\frac{a}{\sin\alpha\cos\alpha} \quad \text{or} \quad \frac{1}{\sigma} = \pm\frac{\sin 2\alpha}{2a} \qquad (2)$$

From (1) and (2), we have

$$\left(\frac{1}{\rho} - \frac{1}{2a}\right)^2 + \frac{1}{\sigma^2} = \frac{1}{4a^2}$$

Or $$\frac{1}{\rho^2} + \frac{1}{\sigma^2} = \frac{1}{\rho a} \quad \text{or} \quad a = \frac{\rho\sigma^2}{\rho^2+\sigma^2}$$

Also $\dfrac{\rho}{\sigma} = \dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$.

Example 3.12: For the curve

$x = a\tan u$, $y = a\cot u$, $z = a\sqrt{2}\log\tan u$. Prove that $\rho = -\sigma = \dfrac{2\sqrt{2}a}{\sin^2 2u}$.

Solution. Here $\mathbf{r} = (a\tan u, a\cot u, a\sqrt{2}\log\tan u)$

Differentiating w.r.t.'s'

$$\mathbf{r}' = \mathbf{t} = a\left(\sec^2 u, -\operatorname{cosec}^2 u, \frac{\sqrt{2}}{\sin u\cos u}\right)\left(\frac{du}{ds}\right)^2 \qquad (1)$$

Squaring and using $\mathbf{t}^2 = 1$, we get

$$1 = a^2\left(\sec^4 u + \operatorname{cosec}^4 u + \frac{2}{\sin^2 u\cos^2 u}\right)\left(\frac{du}{ds}\right)^2$$

or $$\left(\frac{ds}{du}\right)^2 = a^2\left(\frac{\sin^4 u + \cos^4 u + 2\sin^2 u\cos^2 u}{\sin^4 u\cos^4 u}\right)$$

$$= a^2\frac{(\sin^2 u + \cos^2 u)^2}{\sin^4 u\cos^4 u}$$

$$\therefore \quad \frac{ds}{du} = \frac{a}{\sin^2 u\cos^2 u} \qquad (2)$$

$\therefore$ from (1) and (2)

$$\mathbf{r} = a\left(\sec^2 u, -\mathrm{cosec}^2 u, \frac{\sqrt{2}}{\sin u \cos u}\right)\frac{\sin^2 u \cos^2 u}{a}$$

$$= (\sin^2 u, -\cos^2 u, \sqrt{2}\sin u \cos u)$$

Differentiating again

$$\mathbf{r}' = \kappa \mathbf{n} = (2\sin u \cos u, 2\cos u \sin u, \sqrt{2}\cos 2u)\frac{\sin^2 u \cos^2 u}{a} \qquad (3)$$

$$\left[\because \frac{du}{ds} = \frac{\sin^2 u \cos^2 u}{a}\right] \qquad \text{[from (2)]}$$

or $$\kappa^2 = \frac{\sin^4 u \cos^4 u}{a^2}[4\sin^2 u \cos^2 u + 4\cos^2 u \sin^2 u + 2\cos^2 2u]$$

or $$\kappa = \frac{\sqrt{2}\sin^2 u \cos^2 u}{a}[4\sin^2 u \cos^2 u + (\cos^2 u - \sin^2 u)]^{1/2}$$

or $$\kappa = \frac{\sqrt{2}\sin^2 u \cos^2 u}{a}[\cos^2 u + \sin^2 u)]^2 = \frac{\sqrt{2}\sin^2 u \cos^2 u}{a}$$

or $$\frac{1}{\kappa} = \rho = \frac{4a}{\sqrt{2}\sin^2 2u} = \frac{2\sqrt{2}a}{\sin^2 2u}$$

Substituting for κ in (3), we get

$$\frac{\sqrt{2}\sin^2 u \cos^2 u}{a}\mathbf{n} = \frac{\sin^2 u \cos^2 u}{a}(\sin 2u, \sin 2u, \sqrt{2}\cos 2u)$$

or $$\mathbf{n} = \frac{1}{\sqrt{2}}(\sin 2u, \sin 2u, \sqrt{2}\cos 2u)$$

Differentiating w.r.t. ‘s’

$$\mathbf{n}' = \tau \mathbf{b} - \kappa \mathbf{t} = \frac{1}{\sqrt{2}}(2\cos 2u, 2\cos 2u, -2\sqrt{2}\sin 2u)\left(\frac{du}{ds}\right)$$

or $$\tau \mathbf{b} - \kappa \mathbf{t} = \frac{2\sin^2 u \cos^2 u}{a\sqrt{2}}(\cos 2u, \cos 2u, -\sqrt{2}\sin 2u)$$

Squaring $\tau^2 - \kappa^2 = \dfrac{2\sin^4 u \cos^4 u}{a^2}(\cos^2 2u, \cos^2 2u, -2\sin^2 2u)$

or $\quad \tau^2 + \dfrac{2\sin^4 u \cos^4 u}{a^2} = \dfrac{4\sin^4 u \cos^4 u}{a^2}$ [putting for κ]

or $\quad \tau = -\dfrac{\sqrt{2}\sin^2 u \cos^2 u}{a}$ [taking –ve sign]

or $\rho = \dfrac{1}{\tau} = \dfrac{2\sqrt{2}a}{\sin^2 2u}$.

Example 3.13: Prove that at the point of intersection of the surfaces

$$x^2 + y^2 = z^2,\ z = a\tan^{-1}\frac{y}{x}.$$

where $y = x\tan\theta$, the radius of curvature of the intersection is

$$a(2+\theta^2)^{2/3}/(8+5\theta^2+\theta^4)^{1/2}.$$

Solution. For the point of intersection

$$y = x\tan\theta,\ z = a\tan^{-1}\frac{y}{x} = a\theta$$

and $\quad x^2 + x^2\tan^2\theta = a^2\theta^2$

$\therefore \quad x^2\sec^2\theta = a^2\theta^2$

$\therefore \quad x = a\theta\cos\theta$

Also $\quad y = a\theta\sin\theta$

Hence the parametric equations of the curve is

$$\mathbf{r} = a\theta\cos\theta, a\theta\sin\theta, a\theta$$

$$\therefore \frac{d\mathbf{r}}{d\theta} = \dot{\mathbf{r}} = a(\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta, 0) \qquad (1)$$

$$\ddot{\mathbf{r}} = a(-2\sin\theta - \theta\cos\theta, 2\cos\theta - \theta\sin\theta, 0) \qquad (2)$$

$$\dddot{\mathbf{r}} = a(-3\cos\theta + \theta\sin\theta, -3\sin\theta - \theta\cos\theta, 0) \qquad (3)$$

Taking cross product of (1) and (2), we get

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = a^2(\theta\sin\theta - 2\cos\theta, -\theta\cos\theta, -\theta\cos\theta - 2\sin\theta, \theta^2 + 2) \qquad (4)$$

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = (\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\bullet\dddot{\mathbf{r}}$$

$$= a^3[\theta\sin\theta - 2\cos\theta](-3\cos\theta + \theta\sin\theta) + (\theta\cos\theta + 2\sin\theta)(3\sin\theta + \theta\cos\theta)$$

$$= a^3[-5\theta\sin\theta\cos\theta + 6\cos^2\theta + \theta^2\sin^2\theta + 5\theta\sin\theta\cos\theta + 6\sin^2\theta + \theta^2\cos^2\theta]$$

$$= a^3(\theta^2 + 6) \qquad (5)$$

Squaring (1), $|\dot{\mathbf{r}}|^2 = a^2[(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2 + 1$

$$= a^3[2 + \theta^2]$$

$\therefore \quad |\dot{\mathbf{r}}| = a[2 + \theta^2]^{1/2}$

From (4), $|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2 = a^4[(\theta\sin\theta - 2\cos\theta)^2 + (\theta\cos\theta 2\sin\theta)^2 + (\theta^2 + 2)^2]$

$$= a^4[\theta^2 + 4 + \theta^4 + 4\theta^2 + 4]$$

$$= a^4[8 + 5\theta^2 + \theta^4]$$

$$\therefore \quad \rho = \frac{1}{\kappa} = \frac{|\dot{\mathbf{r}}|^3}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|} = \frac{a^3[2+\theta^2]^{3/2}}{a^2[8+5\theta^2+\theta^4]^{1/2}} = \frac{a[2+\theta^2]^{3/2}}{[8+5\theta^2+\theta^4]^{1/2}}$$

and $\sigma = \dfrac{1}{\tau} = \dfrac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]} = \dfrac{a^4[8+5\theta^2+\theta^4]}{a^2[\theta^2+6]} = \dfrac{a[8+5\theta^2+\theta^4]}{[\theta^2+6]}$.

Example 3.13: A point moves on a sphere of radius, 'a' so that its latitude is equal to its longitude. Prove that at (x, y, z),

$$\rho = \frac{(2a^2 - z^2)^{3/2}}{a(8a^2 - 3z^2)^{1/2}}, \quad \sigma = \frac{8a^2 - 3z^2}{6(a^2 - z^2)^{1/2}}$$

Solution. For any point of sphere

$$x = a\sin\theta\cos\phi, \; y = a\sin\theta\sin\phi, \; z = a\cos\theta$$

If latitude=longitude, $\theta = \phi$

$\therefore \quad x = a\sin\theta\cos\theta, \; y = a\sin\theta\sin\theta, \; z = a\cos\theta$

$$\therefore \quad \mathbf{r} = \left(\frac{1}{2}a\sin 2\theta, a\sin^2\theta, a\cos\theta\right)$$

Differentiating w.r.t. 'θ' and let dot denote the differentiation w.r.t. 'θ'.

$$\dot{\mathbf{r}} = a(\cos 2\theta, \sin 2\theta, -\sin\theta)$$

$$\ddot{\mathbf{r}} = a(-2\sin 2\theta, 2\cos 2\theta, -\cos\theta)$$

$$\dddot{\mathbf{r}} = a(-4\cos 2\theta, -4\sin 2\theta, \sin\theta)$$

$$[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}] = 6a^3\sin\theta$$

$$\therefore \quad |\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2 = a^4[\sin^6\theta + \cos^2\theta(1+4\sin^4\theta+4\sin^2\theta)+4]$$

$$= a[4\sin^4\theta + \cos^2\theta + 4\sin^2\theta\cos^2\theta + 4]$$

$$= a^4[4\sin^2\theta + \cos^2\theta + 4] = a^4[8-3\cos^2\theta]$$

$$\therefore \quad |\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = a^2(8-3\cos^2\theta)^{1/2}$$

$$|\dot{\mathbf{r}}| = a[\cos^2 2\theta + \sin^2 2\theta + \sin^2\theta]^{1/2} = a(1+\sin^2\theta)^{1/2}$$

$$= a(2-\cos^2\theta)^{1/2}$$

$$\therefore \quad \kappa = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{a^2(8-3\cos^2\theta)^{1/2}}{a^3(2-\cos^2\theta)^{3/2}}$$

$$= \frac{\left(8-3\dfrac{z^2}{a^2}\right)^{1/2}}{a\left(2-\dfrac{z^2}{a^2}\right)^{3/2}} = \frac{a(8a^2-z^2)^{1/2}}{(2a^2-z^2)^{3/2}}$$

$$\therefore \quad \rho = \frac{1}{\kappa} = (2a^2-z^2)^{3/2}/\{a(8a^2-3z^2)^{1/2}\}$$

$$\sigma = \frac{1}{\tau} = \frac{[\dot{\mathbf{r}}\times\ddot{\mathbf{r}}]}{[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}]} = \frac{a^4(8-3\cos^2\theta)}{6a^3\sin\theta} = \frac{8a^2-3z^2}{6(a^2-z^2)^{1/2}}$$

Example 3.14: Prove that the principal normal to the helix is the normal to the cylinder.

Solution. Let the equation to the helix be

$$\mathbf{r} = (a\cos u, a\sin u, au\tan\alpha)$$

$$\dot{\mathbf{r}} = a(-\sin u, \cos u, \tan\alpha)$$

$$\ddot{\mathbf{r}} = a(-\cos u, -\sin u, 0)$$

Now d.c.'s of principal normal are

$$-(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}} + (\dot{\mathbf{r}})^2\ddot{\mathbf{r}}$$

i.e. $\quad 0 + a^2(\sin^2 u + \cos^2 u + \tan^2\alpha).a(-\cos u, -\sin u, 0)$

i.e. $\quad -(1+\tan^2\alpha).a^3(\cos u, \sin u, 0)$

Hence d.c.'s of principal normal are proportional to $(\cos u, \sin u, 0)$.

Also the equation of cylinder is

$$x^2 + y^2 = a^2$$

Tangent plane to the cylinder at $(a\cos u, a\sin u, 0)$ is

$$Xa\cos u + Ya\sin u = a^2$$

i.e. $\quad \cos u X + \sin u Y = a$

$\therefore$ d.c.'s of normal are proportional to $(\cos u, \sin u, 0)$.

Example 3.15: Show that if the space curve C, $\mathbf{r} = \mathbf{r}(s)$ has constant torsion τ than the curve C_1, $\mathbf{r}_1 = -\dfrac{\mathbf{n}}{\tau} + \int \mathbf{b}\, ds$ has constant curvature $\pm\tau$.

Solution. Let the quantities belonging to C_1 be distinguished by the use of suffix unit. Thus $\mathbf{r}_1$ denotes the position vector of a current point on C_1, etc.

Now we are given that $\mathbf{r}_1 = -\dfrac{\mathbf{n}}{\tau} + \int \mathbf{b}\, ds \qquad (1)$

Differentiating (1) w.r.t. 's', we get

$$\frac{d\mathbf{r}_1}{ds_1}\frac{ds_1}{ds} = -\frac{\mathbf{n}'}{\tau} + \mathbf{b} \qquad [\tau \text{ is constant }]$$

or $$\mathbf{t}_1\frac{ds_1}{ds} = -\frac{1}{\tau}(\tau\mathbf{b} - \kappa\mathbf{t}) + \mathbf{b}$$

or $$\mathbf{t}_1 \frac{ds_1}{ds} = -\frac{\kappa}{\tau}\mathbf{t} \tag{2}$$

This relation show that $\mathbf{t}_1$ is parallel to $\mathbf{t}$, we may take

$$\mathbf{t}_1 = \pm\mathbf{t} \tag{3}$$

Thus using (3) in (2), we get

$$\frac{d\mathbf{t}_1}{ds_1}\frac{ds_1}{ds} = \pm\mathbf{t}' = \pm\kappa\mathbf{n} \tag{4}$$

or $$\kappa_1\mathbf{n}_1\left(\pm\frac{\kappa}{\tau}\right) = \tau\mathbf{n}, \qquad \text{[using (4)]}$$

or $$\kappa_1\mathbf{n}_1 = \tau\mathbf{n} \tag{5}$$

The relation (5) shows that $\mathbf{n}_1$ is parallel to $\mathbf{n}$, we may take

$$\mathbf{n}_1 = \pm\mathbf{n} \tag{6}$$

Thus using (6) in (5), we get

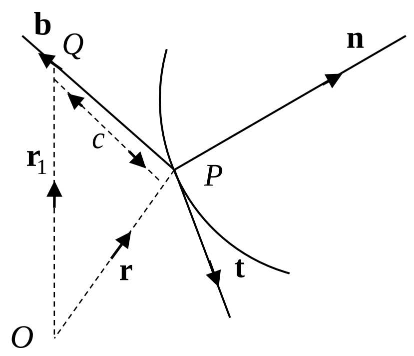
$$\kappa_1 = \pm\tau$$

$\therefore$ Curvature of C_1 is $\pm\tau$ which is constant.

Example 3.16: On the binormal to a given curve a point Q is taken at a constant distance c from the given curve. Prove that the curvature κ_1 of the locus of Q is given by

$$\kappa_1^2(1+c^2\tau^2)^3 = c^2\tau^4(1+c^2\tau^2)+(\kappa - c\tau' + c^2\kappa\tau^2)^2 .$$

Solution. Let the quantities belonging to the locus of Q be distinguished by the use of suffix unity. Let the position vector of P on the curve be $\mathbf{r}$ and that of Q at a distance c from P on the binormal PQ be $\mathbf{r}_1$.



Hence $\mathbf{r}_1 = \mathbf{r} + c\mathbf{b}$

Differentiating w.r.t. 's' and let dash denote differentiation w.r.t. 's'.

$$\therefore \qquad \mathbf{r}_1' = \frac{d\mathbf{r}_1}{ds_1}\frac{ds_1}{ds} = \mathbf{t}_1\frac{ds_1}{ds} = bt - c\tau\mathbf{n} \qquad [\because \mathbf{b}' = -\tau\mathbf{n}]$$

Squaring $\left(\frac{ds_1}{ds}\right)^2 = 1 + \tau^2 c^2$

or $\left(\frac{ds_1}{ds}\right) = \sqrt{(1+\tau^2 c^2)}$

Also differentiating equation (1), we get

$$\mathbf{r}_1'' = \mathbf{t}' - c(\tau\mathbf{n})'$$

or $$\mathbf{r}_1'' = \kappa\mathbf{n} - c\tau'\mathbf{n} - c\tau(\tau\mathbf{b} - \kappa\mathbf{t}) = c\kappa\tau\,\mathbf{t} + (\kappa - c\tau')\mathbf{n} - c\tau^2\mathbf{b}$$

Now $$\kappa_1 = \frac{\left|\frac{d\mathbf{r}_1}{ds} \times \frac{d^2\mathbf{r}_1}{ds^2}\right|}{\left(\frac{ds_1}{ds}\right)^3} = \frac{|\mathbf{r}_1' \times \mathbf{r}_1''|}{(s_1')h3}$$

$$\kappa_1 = \frac{|(\mathbf{t} - c\tau\mathbf{n}) \times [c\tau\kappa\mathbf{t} + (\kappa - c\tau')\mathbf{n} - c\tau^2\mathbf{b}]|}{(1+c^2\tau^2)^{3/2}}$$

or $$(1+c^2\tau^2)^3\kappa_1^2 = \left|c^2\tau^3\mathbf{t} + c\tau^2\,\mathbf{n} + (\kappa - c\tau' + c^2\kappa\tau^2)\mathbf{b}\right|^2$$

or $$= c^4\tau^6 + c^2\tau^4 + (\kappa - c\tau' + c^2\kappa\tau^2)^2$$

or $$(1+c^2\tau^2)^3\kappa_1^2 = c^2\tau^4(1+c^2\tau^2) + (\kappa - c\tau' + c^2\kappa\tau^2)^2$$

Example 3.17: A point Q is taken on the binormal at a variable point P of a curve of constant torsion τ so that PQ is of constant length c. Show that the binormal of the curve traced by Q makes an angle $\tan^{-1} c\rho/\{\sigma\sqrt{(c^2+\sigma^2)}\}$ with PQ.

Solution. Proceeding as in previous example as above, we have

$$\mathbf{r}_1' = (\mathbf{t} - c\tau\mathbf{n}),\ \mathbf{r}_1'' = c\tau\kappa\mathbf{t} + (\kappa - c\tau')\mathbf{n} - c\tau^2\mathbf{b}$$

and $$\mathbf{r}_1' \times \mathbf{r}_1'' = c^2\tau^3\mathbf{t} + c\tau^2\mathbf{n} + (\kappa + c^2\kappa\tau^2)\mathbf{b} \qquad [\because \tau = \text{const. } \tau' = 0].$$

We know that the binormal to a curve is parallel to $\mathbf{r}_1' \times \mathbf{r}_1''$.

Hence if θ is the angle between PQ (i.e. **b**) and binormal to the locus of Q (i.e. $\mathbf{r}_1' \times \mathbf{r}_1''$), then

$$\cos\theta = \mathbf{b} \bullet \frac{\mathbf{r}_1' \times \mathbf{r}_1''}{|\mathbf{r}_1' \times \mathbf{r}_1''|} - \frac{\mathbf{b} \bullet \{c^2\tau^3\mathbf{t} + c\tau^2\mathbf{n} + (\kappa + c^2\kappa\tau^2)\mathbf{b}\}}{\{c^2\tau^4(1+c^2\tau^2) + (\kappa + c^2\kappa\tau^2)^2\}^{1/2}}$$

$$= \frac{\kappa + c^2\kappa\tau^2}{\{c^2\tau^4(1+c^2\tau^2) + (\kappa + c^2\kappa\tau^2)^2\}^{1/2}}$$

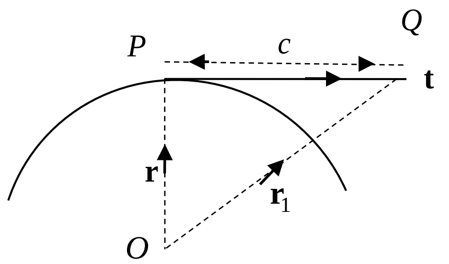
$$\tan\theta = \frac{\{c^2\tau^4(1+c^2\tau^2)\}^{1/2}}{(\kappa + c^2\kappa\tau^2)} = \frac{c\tau}{\kappa\sqrt{(1+c^2\tau^2)}} = \frac{c\rho}{\sigma\sqrt{(c^2+\sigma^2)}}$$

$$\therefore \qquad \theta = \tan^{-1}[c\rho/\{\sigma\sqrt{(c^2+\sigma^2)}\}]$$

Example 3.18: On the tangent to a given curve a point Q is taken at a constant distance c from the point contact. Prove that curvature κ_1 of the locus of Q is given by

$$\kappa_1^2(1+c^2\kappa^2)^3 = c^2\kappa^2\tau^2(1+c^2\kappa^2) + (\kappa + \kappa\tau' + c^2\kappa^3)^2.$$

Solution. Let the quantities belonging to the locus of Q be distinguished by the suffix unity. Let the position vector of the point P on the curve be **r** and that of Q on the tangent at P on the curve be .



Since PQ=c, $\mathbf{r}_1 = \mathbf{r} + c\mathbf{t}$

Differentiating w.r.t. 's' and let dashes denote differentiation w.r.t. 's'.

$$\mathbf{r}_1' = \frac{d\mathbf{r}_1}{ds_1}\frac{ds_1}{ds} = \mathbf{t}_1 \frac{ds_1}{ds} = \mathbf{t} + c\kappa\,\mathbf{n} \qquad (1)$$

Squaring $\left(\dfrac{ds_1}{ds}\right)^2 = (1 + c^2\kappa^2)$

or $s_1' = (1 + c^2\kappa^2)^{1/2}$

Differentiating equation (1) again

$$\mathbf{r}_1'' = \mathbf{t}' - c\kappa'\mathbf{n} + c\kappa\mathbf{n}' = \kappa\mathbf{n} + c\kappa'\mathbf{n} - c\kappa(\tau\mathbf{n} - \kappa\mathbf{t})$$

or $\mathbf{r}_1'' = -c\kappa^2\,\mathbf{t} + (\kappa + c\kappa')\mathbf{n} - c\kappa\tau\,\mathbf{b}$

$$\mathbf{r}' \times \mathbf{r}_1'' = (\mathbf{t} + c\kappa\mathbf{n}) \times [-c\kappa^2\,\mathbf{t} + (\kappa + c\kappa')\mathbf{n} - c\kappa\tau\,\mathbf{b}]$$

$$= (\kappa + c\kappa')\mathbf{b} - c\kappa\tau\,\mathbf{n} + c^2\kappa^3\,\mathbf{b} + c^2\kappa^2\tau\,\mathbf{t}$$

$$= c^2\kappa^2\tau\,\mathbf{t} - c\kappa\tau\,\mathbf{n} + (\kappa + c\kappa' + c^2\kappa^3)\mathbf{b}$$

$\therefore \quad |\mathbf{r}' \times \mathbf{r}_1''| = \tau^2 c^2\kappa^2(1 + c^2\kappa^2) + (\kappa + c\kappa' + c^2\kappa^3)^2$

Hence $\kappa_1' = \dfrac{|\mathbf{r}_1' \times \mathbf{r}_2''|}{(s_1')^3}$ $\left[\mathit{Formula}\ \kappa = \dfrac{|\mathbf{r}_1' \times \mathbf{r}_2''|}{\dot{s}_1^{\,3}} \right]$

or $(s_1')^6\kappa_1^2 = |\mathbf{r}_1' \times \mathbf{r}_2''|^2$

or $$(1 + c^2\kappa^2)^3\kappa_1^2 = c^2\kappa^2\tau^2(1 + c^2\kappa^2) + (\kappa + c\kappa' + c^2\kappa^3)^2 \qquad (2)$$

Example 3.19: A point Q is taken on the tangent **t** at the point P on a curve C so that PQ=c, a constant. Prove that the unit tangent vector $\mathbf{t}_1$ at a point on the locus C_1 of Q is parallel to the osculating plane of C at P. Also show that necessary and sufficient conditions for C_1 to be a straight line is

$$c^2\kappa^2\tau^2(1+c^2\kappa^2)+(\kappa+c\kappa'+c^2\kappa^3)^2=0.$$

Solution. Proceeding as in example 3.18 above, we get from (1)

$$\mathbf{t}_1\frac{ds_1}{ds}=\mathbf{t}+c\kappa\,\mathbf{n}$$

The equation shows that $\mathbf{t}_1$ is a linear combination of vectors **t** and **n**. But **t** and **n** lie in the osculating plane of at P. Hence it follows that $\mathbf{t}_1$ is parallel to the osculating plane of C at P.

Again we know that a necessary and sufficient condition for a curve C_1 to be a straight line is that its curvature $\kappa_1=0$. Thos putting $\kappa_1=0$, in equation (2) of above example, the required condition is given by

$$c^2\kappa^2\tau^2(1+c^2\kappa^2)+(\kappa+c\kappa'+c^2\kappa^3)^2=0.$$

Example 3.20: There is a one –to-one correspondence between the points of two curves, and the tangent at corresponding points are parallel, show that the principal normals are parallel and, also the binormals. Prove also that

$$\kappa_1/\kappa=ds/ds_1=\tau_1/\tau\,.$$

Solution. Suppose C and C_1 are two curves. Let the suffix unity be used to distinguish quantities belonging to C_1.

Since the tangents at corresponding points of C and C_1 are parallel, we may write

$$\mathbf{t}_1=\pm\mathbf{t} \qquad (1)$$

Differentiating (1) w.r.t. ‘s’, we get

$$\frac{d\mathbf{t}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\,\mathbf{t}}{ds}, \quad \text{or} \qquad \kappa_1\mathbf{n}_1\frac{ds_1}{ds} = \pm\kappa\,\mathbf{n} \tag{2}$$

Equation (2) shows that $\mathbf{n}_1$ is parallel to $\mathbf{n}$ i.e. principal normals at corresponding points on C and C_1 are parallel, we may write

$$\mathbf{n}_1 = \pm\mathbf{n} \tag{3}$$

Using (3) in (2), we get

$$\kappa_1\frac{ds_1}{ds} = \pm\kappa \qquad \text{or} \qquad \frac{ds_1}{ds} = \frac{\kappa_1}{\kappa} \tag{4}$$

Taking cross product of (1) and (3), we get

$$\mathbf{t}_1\times\mathbf{n}_1 = \pm\mathbf{t}\times\mathbf{n} \qquad \text{or} \qquad \mathbf{b}_1 = \mathbf{b} \tag{5}$$

This shows that the binormals at the corresponding points of C and C_1 are parallel.

Differentiating (3), w.r.t. 's', we get $\dfrac{d\mathbf{n}_1}{ds_1}\dfrac{ds_1}{ds} = \pm\dfrac{d\,\mathbf{n}}{ds}$

$$\text{or} \qquad (\tau_1\mathbf{b}_1 - \kappa_1\mathbf{t}_1)\frac{ds_1}{ds} = \pm(\tau\,\mathbf{b} - \kappa\,\mathbf{t}) \tag{6}$$

Taking dot product of (5) and (6), we get

$$\frac{ds_1}{ds}\tau_1 = \pm\tau \quad \text{or} \quad \frac{ds}{ds_1} = \frac{\tau_1}{\tau} \qquad \text{[taking +ve sign]} \tag{7}$$

$\therefore$ From (4) and (7), we get

$$\frac{\kappa_1}{\kappa} = \frac{ds}{ds_1} = \frac{\tau_1}{\tau}$$

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CHAPTER-4

CURVATURE AND ITS CHARACTERISTICS

Objectives: The students will learn about locus of centre of curvature, principal normal at consecutive points do not intersect unless $\tau = 0$, Spherical curvature, Locus of centre of spherical curvature, Helices, Spherical Indicatrix which create extra capacity to understand about curvature on the curve and plane.

4.1 INTRODUCTION

In previous chapter we have studied the characterstics of continuous curve of class ≥ 2 and plane like, osculating plane, tangent plane, normal plane, rectifying plane, intersection of space curve and plane, principal normal and binormal, curvature, torsion of the curve, screw curvature, Serret Frenet formulae and uses of these formula to find the curvature and torsion of a skewed plane, Helices and spherical indicatrix. In continuation of this study, to know about the locus of centre of curvature.

4.2 Locus of centre of curvature. Just as the arc-rate of turning of the tangent is called the curvature, and the arc-rate of turning of the binormal the torsion, so that the arc rate of turning of the principal normal is called the screw curvature. Its magnitude is the modulus of $\mathbf{n}'$. But we have seen that

$$\mathbf{n}' = \tau\,\mathbf{b} - \kappa\,\mathbf{t}. \tag{4.2.1}$$

Hence the magnitude of the screw curvature is $\sqrt{\kappa^2 + \tau^2}$. This quantity, however, does not play such an important part in the theory of curves as the curvature and torsion.

The centre of curvature at *P* is the point of intersection of the principal normal at *P* with that normal at the consecutive point P' which lies in the osculating plane at *P*. Consecutive principal normals do not in general intersect. It is worth nothing that the tangent to the locus of centre of curvature lies in the normal plane of the original curve. For the centre of curvature is the point whose position vector **c** is given by

$$\mathbf{c} = \mathbf{r} + \rho\,\mathbf{n} \tag{4.2.2}$$

The tangent to its locus, being parallel to $\frac{d\,\mathbf{c}}{ds}$, is therefore parallel to

$$\mathbf{t} + \rho'\,\mathbf{n} + \rho(\tau\,\mathbf{b} - \kappa\,\mathbf{t}),$$

that is , $\rho'\mathbf{n} + \rho\tau\mathbf{b}$.

It therefore lies in the normal plane of the original curve, and is inclined to the principal normal **n** at an angle β such that

$$\tan\beta = \frac{\rho\tau}{\rho'} = \frac{\rho}{\rho'\sigma} . \qquad (4.2.3)$$

If the original curve is one of constant curvature, $\rho' = 0$, and the tangent to the locus of *C* is then parallel to **b**. It will be proved that the locus of *C* has then the same constant curvature as the original curve, and that its torsion varies inversely as the torsion of the given curve.

4.2.1 Locus of centre of curvature in term of parameter. If *s* is the are-length measured from a fixed point ***A*** on the curve to the current point *P*, the position vector **r** of *P* is a function of *s*; and therefore, by Taylor's Theorem,

$$\mathbf{r} = \mathbf{r}_0 + s\,\mathbf{r}_0' + \frac{s^2}{2!}\mathbf{r}_0'' + \frac{s^3}{3!}\mathbf{r}_0''' + \ldots.. , \qquad (4.2.4)$$

where the suffix zero indicates the values of the quantity is to be taken for the point ***A***. If $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the unit tangent, principal normal, and binormal at ***A***, and κ, τ the curvature and the torsion at that point, we have

$$\mathbf{r}_0' = \mathbf{t},\ \mathbf{r}_0'' = \kappa\,\mathbf{n} , \qquad (4.2.5)$$

while the values of $\mathbf{r}_0'''$ and $\mathbf{r}_0''''$ are as given in Ex.1. Hence the above formula gives

$$\mathbf{r} = \mathbf{r}_0 + s\,\mathbf{t} + \frac{s^2}{2!}\kappa\,\mathbf{n} + \frac{s^3}{3!}(\kappa'\mathbf{n} - \kappa^2\mathbf{t} + \kappa\,\tau\,\mathbf{b})$$

$$+ \frac{s^4}{4!}\{(\kappa''\mathbf{n} - \kappa^3 - \kappa\,\tau^2)\mathbf{n} - 3\kappa\,\kappa'\,\mathbf{t} + (2\kappa'\,\tau + \kappa\,\tau')\mathbf{b}\} + \ldots..$$

If then ***A*** is taken as origin, and the tangent, principal normal and binormal it ***A*** as coordinate axes, the coordinates of *P* are the coefficients of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in the above expansion. Thus since $\mathbf{r}_0$ is now zero, we have

$$x = s - \frac{\kappa^2}{6}s^3 - \frac{\kappa\,\kappa'}{8}s^4 + \ldots.. ,$$

$$y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{1}{24}(\kappa'' - \kappa^3 - \kappa\tau^2)s^4 + ...$$

$$z = \frac{1}{6}\kappa\tau s^3 + \frac{1}{24}(2\kappa'\tau + \kappa\tau')s^4 + ... \qquad (4.2.6)$$

From the last equation it follows that, for sufficiently small values of s, changes sign with s (unless κ or τ is zero). Hence at an ordinary point of the curve, the curve crosses the osculating plane. On the other hand, for sufficiently small values of s, y does not change sign with s $(\kappa \neq 0)$. Thus in the neighbourhood of an ordinary point, the curve lies on one side of the plane determined by the tangent and binormal. Thus plane is called the rectifying plane.

Example 4.1: Show that Serret-Frenet formulae can be written in the form

$\mathbf{t}' = \mathbf{w} \times \mathbf{t}, \quad \mathbf{n}' = \mathbf{w} \times \mathbf{n}, \quad \mathbf{b}' = \mathbf{w} \times \mathbf{b}$ and determine $\mathbf{w}$.

Solution: $\mathbf{w}$ is called Darbouxe vector of the curve.

We have from Frenet's formulae

$$\mathbf{t}' = \kappa\mathbf{n} = \tau\mathbf{t} \times \mathbf{t} + \kappa\mathbf{b} \times \mathbf{t}$$

$$= (\tau\mathbf{t} + \kappa\mathbf{b}) \times \mathbf{t} \qquad [\because \mathbf{t} \times \mathbf{t} = 0, \quad \mathbf{b} \times \mathbf{t} = \mathbf{n}]$$

$$= \mathbf{w} \times \mathbf{t}, \qquad \text{where } \mathbf{w} = \tau\mathbf{t} + \kappa\mathbf{b} \qquad (1)$$

which proves that first result

$$\mathbf{n}' = \tau\mathbf{b} - \kappa\mathbf{t} = \tau(\mathbf{t} \times \mathbf{n}) + \kappa(\mathbf{b} \times \mathbf{n})$$

$$= (\tau\mathbf{t} + \kappa\mathbf{b}) \times \mathbf{n} = \mathbf{w} \times \mathbf{n}$$

$$\mathbf{b}' = -\tau\mathbf{n} = \tau(\mathbf{t} \times \mathbf{b}) + \kappa(\mathbf{b} \times \mathbf{b}) \qquad [\because \mathbf{b} \times \mathbf{b} = 0, -\mathbf{n} = \mathbf{t} \times \mathbf{b}]$$

$$= (\tau\mathbf{t} + \kappa\mathbf{b}) \times \mathbf{b} = \mathbf{w} \times \mathbf{b} \qquad \text{where } \mathbf{w} = \tau\mathbf{t} + \kappa\mathbf{b} \text{ from (1).}$$

Example 4.2: If the position vector $\mathbf{r}$ of a current point on a curve is a function of any parameter u and dots denote differentiation with respect to u, then prove that

$$\dot{\mathbf{r}} = \dot{s}\mathbf{t}, \quad \ddot{\mathbf{r}} = \ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}$$

and $\dddot{\mathbf{r}} = (\dddot{s} - \kappa^2\dot{s}^3)\mathbf{t} + \dot{s}(3\kappa\ddot{s} + \kappa\dot{s})\mathbf{n} + \kappa\tau\dot{s}^3\mathbf{b}$

Hence deduce that

$$\mathbf{b}=\frac{\dot{\mathbf{r}}\times\ddot{\mathbf{r}}}{\kappa\dot{s}^3},\quad \mathbf{n}=\frac{\dot{s}\ddot{\mathbf{r}}-\ddot{s}\dot{\mathbf{r}}}{\kappa\dot{s}^3},\ \kappa^2=\frac{\ddot{\mathbf{r}}^2-\ddot{s}^2}{\dot{s}^4},\ \tau=\frac{[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]}{\kappa^2\dot{s}^6}$$

Solution: We have

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{du}=\frac{d\mathbf{r}}{ds}\frac{ds}{du}=\mathbf{r}'\dot{s}=\dot{s}\mathbf{t}$$

or $\dot{\mathbf{r}}=\dot{s}\mathbf{t}$ (1)

Differentiating again w.r.t. 'u ', we get

$\ddot{\mathbf{r}}=\ddot{s}\mathbf{t}+\dot{s}^2\mathbf{t}'$ or $\ddot{\mathbf{r}}=\ddot{s}\mathbf{t}+\kappa\dot{s}^2\mathbf{n}$ $[\because \mathbf{t}'=\kappa\mathbf{n}]$ (2)

where dash represents differentiation w.r.t. 's' . Differentiating (2) again w.r.t. 'u ', we get

$$\dddot{\mathbf{r}}=\dddot{s}\mathbf{t}+\ddot{s}\dot{s}\mathbf{t}'+(\dot{\kappa}\dot{s}^2+2\kappa\dot{s}\ddot{s})\mathbf{n}+\kappa\dot{s}^3\mathbf{n}'$$

$$=\dddot{s}\mathbf{t}+\ddot{s}\dot{s}\kappa\mathbf{n}+(\dot{\kappa}\dot{s}^2+2\kappa\dot{s}\ddot{s})\mathbf{n}+\kappa\dot{s}^3(\tau\mathbf{b}-\kappa\mathbf{t})$$

or $\dddot{\mathbf{r}}=(\dddot{s}-\kappa^2\dot{s}^3)\mathbf{t}+\dot{s}(3\kappa\ddot{s}+\dot{\kappa}\dot{s})\mathbf{n}+\kappa\tau\dot{s}^3\mathbf{b}$ (3)

Now cross product of (1) and (2) gives

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=\dot{s}\ddot{s}\mathbf{t}\times\mathbf{t}+\kappa\dot{s}^3\mathbf{t}\times\mathbf{n}$$

or $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=\kappa\dot{s}^3\mathbf{b}$ $[\because \mathbf{t}\times\mathbf{n}=\mathbf{b},\ \mathbf{t}\times\mathbf{t}=0\ ]$ (4)

or $\mathbf{b}=\dfrac{\dot{\mathbf{r}}\times\ddot{\mathbf{r}}}{\kappa\dot{s}^3}$ (5)

Now multiplying (2) by $\dot{s}$ and (1) by $\ddot{s}$ and and subtracting, we have

$$\dot{s}\ddot{\mathbf{r}}-\ddot{s}\dot{\mathbf{r}}=\kappa\dot{s}^3\mathbf{n}$$

or $\mathbf{n}=\dfrac{\dot{s}\ddot{\mathbf{r}}-\ddot{s}\dot{\mathbf{r}}}{\kappa\dot{s}^3}$

squaring (2), we get

$$\ddot{\mathbf{r}}=\ddot{s}^2\mathbf{t}^2+\kappa^2\dot{s}^4\mathbf{n}^2+2\kappa\dot{s}^2\ddot{s}\mathbf{t}.\mathbf{n}=\ddot{s}^2+\kappa^2\dot{s}^4$$

$$\kappa^2=\frac{\ddot{\mathbf{r}}-\ddot{s}^2}{\dot{s}^4} \qquad (6)$$

Now finally taking the dot product of (4) and (3), we get

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}.\dddot{\mathbf{r}}=\kappa^2\tau\,\dot{s}^6 \qquad [\because \mathbf{b}.\mathbf{t}=0,\ \mathbf{b}.\mathbf{n}=0,\ \mathbf{b}.\mathbf{b}=1]$$

$$[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]=\kappa^2\tau\,\dot{s}^6 \qquad \text{or} \qquad \tau=\frac{[\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}]}{\kappa^2\,\dot{s}^6}$$

Example 4.3: Show that principal normal at consecutive points do not intersect unless $\tau=0$.

Solution: Let us consider two consecutive points on the curve with position vector $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ with respect to some fixed origin O. Again let $\mathbf{n}$ and $\mathbf{n}+d\mathbf{n}$ be the unit principal normals at these two points respectively. Now the principal normals will intersect if the three vectors $d\mathbf{r}$, $\mathbf{n}$, $\mathbf{n}+d\mathbf{n}$ are coplanar *i.e.* if their scalar triple product is zero.

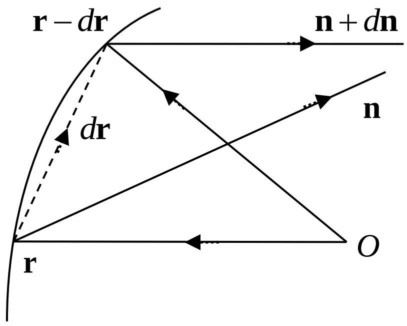

Figure 4.1

i.e. if $[d\mathbf{r},\mathbf{n},\mathbf{n}+d\mathbf{n}]=0$

or $\quad [d\mathbf{r},\mathbf{n},d\mathbf{n}]=0 \qquad [\because \mathbf{n}\times\mathbf{n}=0]$

or $\quad [\frac{d\mathbf{r}}{ds},\mathbf{n},\frac{d\mathbf{n}}{ds}]=0$

or $\quad [\mathbf{t},\mathbf{n},\mathbf{n}']=0 \qquad$ or $\qquad [\mathbf{t},\mathbf{n},\tau\mathbf{b}-\kappa\mathbf{t}]=0$

or $\quad [\mathbf{t},\mathbf{n},\tau\mathbf{b}]=0 \qquad [\because \mathbf{t}\times\mathbf{t}=0]$

or $\quad \tau[\mathbf{t},\mathbf{n},\mathbf{b}]=0 \qquad (1)$

Since $[\mathbf{t},\mathbf{n},\mathbf{b}]=1\neq 0$. Hence eqn. (1) hold only when $\tau=0$.

Example 4.4: Prove that for any curve $\mathbf{t}'.\mathbf{b}'=-\kappa\tau$.

Solution. It immediately follows from Frenet's formulae.

We have $\mathbf{t}' = \kappa\,\mathbf{n}$ and $\mathbf{b}' = -\tau\,\mathbf{n}$

$$\therefore \qquad \mathbf{t}'.\mathbf{b}' = -\kappa\tau \qquad [\because \mathbf{n}.\mathbf{n} = 1]$$

Example 4.5: If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction, show that $\dfrac{\sin\theta}{\sin\phi}\dfrac{d\theta}{d\phi} = -\dfrac{\kappa}{\tau}$ where κ and τ have their usual meaning.

Solution. Let the tangent and binormal at a point of a curve make angles θ and ϕ with fixed direction say, **d**, then

$$\mathbf{t} \bullet \mathbf{d} = d\cos\theta \qquad \text{where } |\mathbf{d}| = d$$

$$\mathbf{b} \bullet \mathbf{d} = d\cos\phi$$

Differentiating w.r.t. 's' we get

$$\mathbf{t}' \bullet \mathbf{d} = -d\sin\theta(d\theta/ds) \qquad [\because \ \mathbf{b} \text{ is constant vector}]$$

$$i.e. \qquad \kappa\mathbf{n} \bullet \mathbf{d} = -d\sin\theta(d\theta/ds) \qquad (1)$$

$$\text{and} \qquad \mathbf{b}' \bullet \mathbf{d} = -d\sin\phi(d\phi/ds)$$

$$i.e. \qquad -\tau\,\mathbf{n} \bullet \mathbf{d} = -d\sin\phi(d\phi/ds) \qquad (2)$$

Dividing (1) by (2) we get the required result as

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}.$$

Example 4.6: If a particle moving in a space have velocity **V** and acceleration **f**, show that radius of curvature ρ is given by

$$v^3/(|\mathbf{V} \times \mathbf{f}|).$$

Solution. Since we know that the velocity at a point of curve is along the tangent to the curve at the point. Hence the velocity **V** is given by

$$\mathbf{V} = v\mathbf{t} \qquad \text{where } \mathbf{t} \text{ is the unit tangent vector.}$$

Differentiating w.r.t. parameter **t** (time) and let dots denotes differentiation w.r.t. 't'.

$$\mathbf{V} = \dot{v}\mathbf{t} + v\frac{d\mathbf{t}}{ds}.\frac{ds}{dt}$$

i.e. $\mathbf{f} = \dot{v}\,\mathbf{t} + v\kappa\,\mathbf{n}v$

$\therefore$ $\mathbf{V}\times\mathbf{f} = v^3\kappa\mathbf{b}$ *i.e.* $|\mathbf{V}\times\mathbf{f}| = v^3\kappa$ $[\because |\mathbf{b}| = 1]$

or $\rho = \dfrac{v^3}{|\mathbf{V}\times\mathbf{f}|}$ [as $\kappa = 1/\rho$]

Example 4.7: Show that the principal normals at the consecutive points do not intersect unless $\tau = 0$.

Solution. Let the consecutive points be $\mathbf{r}$, $\mathbf{r}+d\mathbf{r}$ and the unit principal normal $\mathbf{n}$, $\mathbf{n}+d\,\mathbf{n}$. For intersection of the principal normals the necessary condition is that the three vectors $d\,\mathbf{r}, \mathbf{n}, \mathbf{n}+d\mathbf{n}$ be coplanar that is, that $\mathbf{r}', \mathbf{n}, \mathbf{n}'$ be coplanar. This requires

$$[\mathbf{t}, \mathbf{n}, \tau\,\mathbf{b} - \kappa\,\mathbf{t}] = 0,$$

that is $\tau[\mathbf{t}, \mathbf{n}, \mathbf{b}] = 0$,

which holds only when τ vanishes.

Example 4.8: Parameters other than *s*.

Solution. If the position vector $\mathbf{r}$ of the current point is a function of any parameter u, and dashes denote differentiations with respect to u, we have

$$\mathbf{r}' = \frac{d\,\mathbf{r}}{ds}\cdot\frac{ds}{du} = s'\mathbf{t},$$

$$\mathbf{r}'' = s''\mathbf{t} + \kappa s'^2\mathbf{n}$$

$$\mathbf{r}''' = (s''' - \kappa^2 s'^3)\mathbf{t} + s'(3\kappa s'' + \kappa' s')\mathbf{n} + \kappa\tau s'^3\mathbf{b}.$$

Hence prove that

$$\mathbf{b} = \mathbf{r}'\times\mathbf{r}''/\kappa s'^3,$$

$$\mathbf{n} = (s'\mathbf{r}'' - s''\mathbf{r}')/\kappa s'^3,$$

$$\kappa^2 = (\mathbf{r}''^2 - s''^2)/s'^4,$$

$$\tau = [\mathbf{r}', \mathbf{r}'', \mathbf{r}''']/\kappa^2 s'^6.$$

Example 4.9: In the case of a curve of constant curvature find the curvature and torsion of the locus of its centre of curvature *C*.

Solution. The position vector of *C* is equal to

$$\mathbf{c} = \mathbf{r} + \rho\,\mathbf{n}$$

Hence, since ρ is constant,

$$d\,\mathbf{c} = \{\mathbf{t} + \rho(\tau\,\mathbf{b} - \kappa\,\mathbf{t})\}ds = \frac{\tau}{\kappa}\mathbf{b}\,d\,s\,.$$

Let the suffix unity distinguish quantities belonging to the locus of *C*. Then $d\,\mathbf{c} = \mathbf{t}_1 d\,s_1$. We may take the positive direction along this locus so that $\mathbf{t}_1 = \mathbf{b}$, then it follows that

$$\frac{ds_1}{ds} = \frac{\tau}{\kappa}\,.$$

Next differentiating the relation $\mathbf{t}_1 = \mathbf{b}$, we obtain

$$\kappa_1\mathbf{n}_1 = -\tau\,\mathbf{n}\frac{ds}{ds_1} = -\kappa\,\mathbf{n}\,.$$

Therefore the two principal normal are parallel. We may choose

$$\mathbf{n}_1 = -\mathbf{n}, \text{ and therefore } \kappa_1 = \kappa\,.$$

Thus the locus of *C* has the same constant curvature as the given curve. The unit binormal $\mathbf{b}_1$ is now fixed : for

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \mathbf{b} \times (-\mathbf{n}) = \mathbf{t}\,.$$

Differentiating this result we may obtain

$$-\tau_1\mathbf{n}_1 = \kappa\,\mathbf{n}\frac{ds}{ds_1} = \frac{\kappa^2}{\tau}\mathbf{n}\ \text{, and therefore}\quad \tau_1 = \kappa^2/\tau\,.$$

Example 4.10: Prove that $\mathbf{r}''' = \kappa'\mathbf{n} - \kappa^2\mathbf{t} + \kappa\tau\,\mathbf{b}$, and hence that

$$\mathbf{r}'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} - 3\kappa\kappa'\,\mathbf{t} + (2\kappa'\tau + \tau'\kappa)\mathbf{b}$$

(using Serret Frenet formulae)

Example 4.11: Prove the relations

$$\mathbf{r}' \bullet \mathbf{r}'' = 0, \qquad \mathbf{r}' \bullet \mathbf{r}''' = -\kappa^2, \qquad \mathbf{r}' \bullet \mathbf{r}'''' = -3\kappa\kappa'$$

$$\mathbf{r}'' \bullet \mathbf{r}''' = \kappa\kappa', \qquad \mathbf{r}'' \bullet \mathbf{r}'''' = \kappa(\kappa'' - \kappa^3 - \kappa\tau^2)$$

$$\mathbf{r}''' \bullet \mathbf{r}'''' = \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa^2\tau\tau' + \kappa\kappa'\tau^2 .$$

Example 4.12: If $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are the moment about the origin of unit vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ localized in the tangent, normal and binormal, and dashes denote differentiations with respect to *s*, show that $\mathbf{m}_1' = \kappa\mathbf{m}_2$, $\mathbf{m}_2' = \mathbf{b} - \kappa\mathbf{m}_1 + \tau\mathbf{m}_3$, $\mathbf{m}_3' = -\mathbf{n} - \tau\mathbf{m}_2$.

Solution: If $\mathbf{r}$ is the current point, we have

$$\mathbf{m}_1 = \mathbf{r}\times\mathbf{t}, \ \mathbf{m}_2 = \mathbf{r}\times\mathbf{n}, \ \mathbf{m}_3 = \mathbf{r}\times\mathbf{b}$$

Therefore $\mathbf{m}_1' = \mathbf{t}\times\mathbf{t} + \mathbf{r}\times(\kappa\mathbf{n}) = \kappa\mathbf{m}_2$

and similarly for the others.

4.3 Spherical curvature. The sphere of closest contact with θ curve at *P* is that which passes through four points on the ultimately coincident with *P*. This is called the osculating sphere or sphere of curvature at *P*.

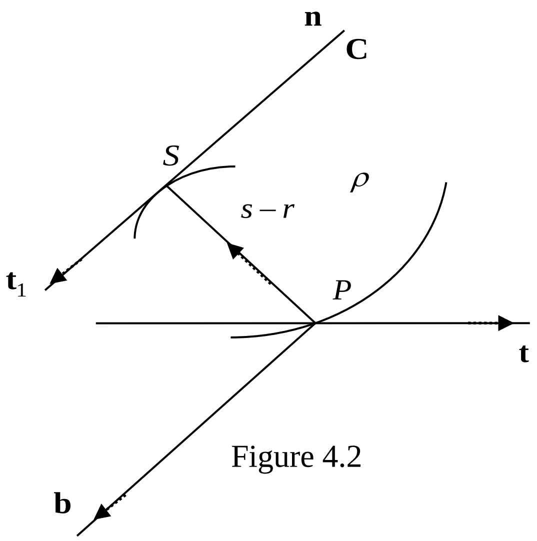

Figure 4.2

Its centre *S* and radius ***R*** are called the centre and radius of spherical curvature. The centre of sphere through *P* and an adjacent point *Q* on the curve lies on the plane which is the perpendicular bisector of *PQ*; the limiting position of this plane is the normal plane at *P*. The centre of spherical curvature is the limiting position of the intersection of three normal planes at adjunct points. Now the normal plane at the point $\mathbf{r}$ is

$$(\mathbf{s} - \mathbf{r}) \bullet \mathbf{t} = 0 \tag{4.3.1}$$

being the current point on the plane. The limiting position the line of intersection of this plane and an adjacent normal plane is determined by (i) and the equation obtained by differentiating it with respect to ar-length *s*,

$$\kappa(\mathbf{s}-\mathbf{r})\bullet\mathbf{n}-1=0$$

or its equivalent

$$\kappa(\mathbf{s}-\mathbf{r})\bullet\mathbf{n}=\rho \qquad (4.3.2)$$

The limiting position of the point of intersection of three adjacent normal planes is then found from (4.2.1), (4.3.2) and the equation obtain by differentiating (4.3.2), viz.

$$(\mathbf{s}-\mathbf{r})\bullet(\tau\,\mathbf{b}-\kappa\,\mathbf{t})=\rho' \qquad (4.3.3)$$

which in virtue of (4.3.1), is equivalent to

$$\kappa(\mathbf{s}-\mathbf{r})\bullet\mathbf{b}=\sigma\,\rho' \qquad (4.3.4)$$

The vector $(\mathbf{s}-\mathbf{r})$, satisfying (4.3.1), (4.3.2), and (4.3.4) is clearly

$$\mathbf{s}-\mathbf{r}=\rho\,\mathbf{n}+\sigma\,\rho'\,\mathbf{b} \qquad (4.3.5)$$

and this equation determines the position vector ***s*** of the centre of spherical curvature. Now $\rho\,\mathbf{n}$ is the vector *PC,* and therefore σ is the vector *CS*. Thus the centre of spherical curvature is on the axis of the circle of curvature, at a distance $\sigma\,\rho'$ from the centre of curvature. On squaring both sides of the last equation, we have for determining the radius of spherical curvature

$$\mathbf{R}^2=\rho^2+\sigma^2\rho'^2 \qquad (4.3.6)$$

Another formula for $\mathbf{R}^2$, maybe determines as follows. On squaring the expansion for $\mathbf{r}'''$, we find

$$\mathbf{r}'''^2=\kappa'^2+\kappa^4+\kappa^2\tau^2$$

$$=\kappa^4(1+\tau^2\mathbf{R}^2)\text{, by (4.3.6).}$$

Hence the formula

$$\mathbf{R}^2=\rho^4\sigma^2\mathbf{r}'''-\sigma^2, \qquad (4.3.7)$$

which is, however, not so important as (4.3.6).

For a curve of constant curvature, $\rho'=0$ and the centre of spherical curvature coincides with the centre of circular curvature.

4.3.1 Locus of centre of spherical curvature. The position vector **s** of the centre of spherical curvature has been shown to be

$$\mathbf{s} = \mathbf{r} + \rho\,\mathbf{n} + \sigma\,\rho'\,\mathbf{b}\,. \tag{4.3.8}$$

Hence, for a small displacement *ds* of the current point *P* along the original curve, the displacement of *S* is

$$d\,\mathbf{s} = \{\mathbf{t} + \rho'\,\mathbf{n} + \rho(\tau\,\mathbf{b} - \kappa\,\mathbf{t}) + \sigma'\rho'\,\mathbf{b} + \sigma\,\rho''\,\mathbf{b} - \rho'\,\mathbf{n}\}d\,s$$

$$= d\,s\left(\frac{\rho}{\sigma} + \sigma'\,\rho' + \sigma\rho''\right)\mathbf{b}\,. \tag{4.3.9}$$

Thus the tangent to the locus of *S* is parallel to **b**. We may measure the arc-length s_1 of the locus of *S* in that direction which makes its unit tangent $\mathbf{t}_1$ have the same direction as **b**.

Thus $\mathbf{t}_1 = \mathbf{b}$, and, since $d\,\mathbf{s} = \mathbf{t}_1\,d\,\mathbf{s}_1$, it follows that

$$\frac{d\,s_1}{d\,s} = \frac{\rho}{\sigma} + \rho'\,\sigma' + \sigma\,\rho''$$

$$= \frac{\rho}{\sigma} + \frac{d}{d\,s}(\sigma\,\rho') \tag{4.3.10}$$

To find the curvature κ_1 of the locus of *S* differentiating the equation

$$\kappa_1\,\mathbf{n}_1 = \frac{d\,\mathbf{b}}{d\,s}\frac{d\,s}{d\,s_1} = -\tau\,\mathbf{n}\frac{d\,s}{d\,s_1}$$

Thus the ***principal normal to the locus of S is parallel to the principal normal of the original curve***. We may choose the direction of $\mathbf{n}_1$ as opposite to $\mathbf{n}$. Thus

$$\mathbf{n}_1 = -\mathbf{n}\,.$$

The unit binormal $\mathbf{b}_1$ of the locus of *S* is then

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \mathbf{b} \times (-\mathbf{n}) = \mathbf{t}$$

and is thus equal to the unit tangent of the original curve. The curvature κ_1 as found above is thus equal to

$$\kappa_1 = \tau\,\frac{d\,s}{d\,s_1}\,.$$

The torsion τ_1 is obtained by differentiating $\mathbf{b}_1 = \mathbf{t}$. Thus

$$-\tau_1 \mathbf{n}_1 = \frac{d\mathbf{t}}{ds}\frac{ds}{ds_1} = \kappa \mathbf{n} \frac{ds}{ds_1}$$

so that $$\tau_1 = \kappa \frac{ds}{ds_1}. \qquad (4.3.11)$$

From the last two results it follows that

$$\kappa \kappa_1 = \tau \tau_1$$

so that the product of the curvatures of the two curves is equal to the product of their torsions. The binormal of each curve is parallel to the tangent to the other, and their principal normals are parallel but in opposite directions.

If the original curve is one of constant curvature, $\rho' = 0$, and S coincides with the centre of circular curvature. Then

$$\frac{ds_1}{ds} = \frac{\rho}{\sigma} = \frac{\tau}{\kappa},$$

and $$\kappa_1 = \kappa. \qquad (4.3.12)$$

Thus the locus of two centers of curvature has the same (constant) curvature as the original curve. Also

$$\tau_1 = \kappa^2 / \tau \qquad (4.3.13)$$

So that the product of the torsions of two curves is equal to the square of their common curvature.

Example 4.12: Prove that in order that the principal normals of a curve be binormals of another, the relation $a(\kappa^2 + \tau^2) = b\kappa$ must hold, where a and b are constant.

Solution. Let the quantities belonging to the other curve C_1(say) be denoted by the use of suffix unity.

We have $\mathbf{b}_1 = \mathbf{n}$ (given) (1)

P($\mathbf{r}$) and Q($\mathbf{r}_1$) be two points on the curves C and C_1 respectively.

$\mathbf{r}_1 = \mathbf{r} + c\mathbf{n}$ where c is some scalar (2)

Differentiating (2) w.r.t. 's'

$$\frac{d\mathbf{r}_1}{ds_1}\frac{ds_1}{ds} = \mathbf{r}' + c\mathbf{n}'$$

or $$\mathbf{t}_1 \frac{ds_1}{ds} = \mathbf{t} + c(\tau\mathbf{b} - \kappa\mathbf{t})$$ (using Frenet's formulae)

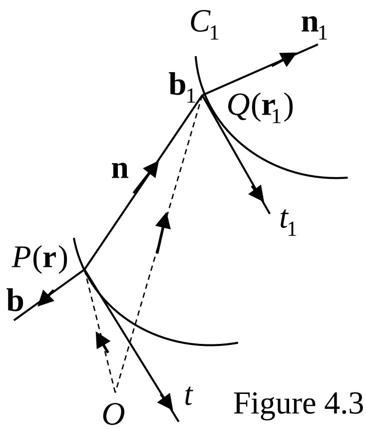

Figure 4.3

Differentiating again w.r.t. 's' and making use of Frenet's formulae

$$\mathbf{t}_1 \frac{d^2 s_1}{ds^2} + \kappa_1 \mathbf{n}_1 \left(\frac{ds_1}{ds}\right)^2 = \kappa\mathbf{n} + c\{\tau'\mathbf{b} + \tau(-\tau\mathbf{n}) - \kappa'\mathbf{t} - \kappa^2\mathbf{n}\}$$

or $$\mathbf{t}_1 \frac{d^2 s_1}{ds^2} + \kappa_1 \mathbf{n}_1 \left(\frac{ds_1}{ds}\right)^2 = \kappa = (c - c\tau^2 - c\kappa^2)\mathbf{n} + c\tau'\mathbf{b} - c\kappa'\mathbf{t} \qquad (3)$$

Now multiplying scalarly the L.H.S. of (3) by $\mathbf{b}_1$ and R.H.S. by $\mathbf{n}$, [since $\mathbf{b}_1 = \mathbf{n}$ by (1)], we have

$$0 = \kappa - c(\tau^2 + \kappa^2)$$

or $$c(\tau^2 + \kappa^2) = \kappa$$

we can write $$a(\tau^2 + \kappa^2) = b\kappa$$ [where $c = a/b$]

Example 4.12: On the binormal of a curve of constant torsion τ, appoint Q is taken at a constant distance c from the curve. Show that the bi-normal to the locus of Q is inclined to the bi-normal of the given curve at an angle $\tan^{-1}[c\tau^2/\kappa\sqrt{(c^2\tau^2+1)}]$.

Solution. Let the quantities belonging to the locus of Q be denoted by the use of suffix unity.

$$\mathbf{r}_1 = \mathbf{r} + c\mathbf{b}$$ (see figure 4.4)

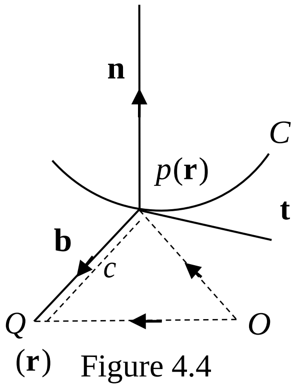

Figure 4.4

Differentiating w.r.t. 's'

$$\mathbf{t}_1\left(\frac{ds_1}{ds}\right)=\mathbf{t}-c\tau\,\mathbf{n} \qquad \text{(Frenet's formulae)}$$

Squaring both sides, we get

$$\left(\frac{ds_1}{ds}\right)^2=1+c^2\tau^2 \qquad \text{or} \qquad \left(\frac{ds_1}{ds}\right)=(1+c^2\tau^2)^{1/2} \qquad (2)$$

Hence substituting $\frac{ds_1}{ds}$ in (1), we get

$$\mathbf{t}_1=(\mathbf{t}-c\tau\mathbf{n})/(1+c^2\tau^2)^{1/2}$$

Differentiating again w.r.t. 's'

$$\kappa_1\mathbf{n}_1\frac{ds_1}{ds}=[\kappa\mathbf{n}-c\tau(\tau\,\mathbf{b}-\kappa\,\mathbf{t})/(1+c^2\tau^2)^{1/2}] \qquad (3)$$

[using Frenet's formulae and τ =constant given]

Squaring both sides, we get

$$\kappa_1^2(1+c^2\tau^2)=[\kappa^2+c^2\tau^4+c^2\tau^2\kappa^2]^2/(1+c^2\tau^2) \qquad \text{[using (2)]}$$

or $\quad \kappa_1^2=\dfrac{A^2}{(1+c^2\tau^2)^2} \qquad$ *i.e.* $\qquad \kappa_1=\dfrac{A}{(1+c^2\tau^2)}$

[where $A^2=\kappa^2(1+c^2\tau^2)+c^2\tau^4$]

Hence from (3); $\mathbf{n}_1=(c\tau\kappa\,\mathbf{t}+\kappa\mathbf{n}-c\tau^2\,\mathbf{b})/A$.

Let α be the angle of inclination of **b** with $\mathbf{b}_1$

$$\therefore \qquad \cos\alpha = \mathbf{b} \bullet \mathbf{b}_1 = \mathbf{b} \bullet [\mathbf{t}_1 \times \mathbf{n}_1] = [\mathbf{b}, \mathbf{t}_1, \mathbf{n}_1]$$

but $$\mathbf{t}_1 \times \mathbf{n}_1 = \frac{1}{\sqrt{(1+c^2\tau^2)}}\left(c^2\tau^3, c\tau^2, \kappa\{1+c^2\tau^2\}\right)$$

$$\therefore \quad [\mathbf{b}, \mathbf{t}_1, \mathbf{n}_1] = \mathbf{b} \bullet \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\kappa(1+c^2\tau^2)}{A\sqrt{(1+c^2\tau^2)}} = \frac{\kappa\sqrt{(1+c^2\tau^2)}}{A}$$

$$\therefore \qquad \cos\alpha = (\kappa / A)\sqrt{(1+c^2\tau^2)}$$

$$\therefore \quad \tan\alpha = \sqrt{(\sec^2\alpha - 1)} = \sqrt{\left[\frac{A^2}{\kappa^2(1+c^2\tau^2)} - 1\right]}$$

$$= \sqrt{\left[\frac{\kappa^2(1+c^2\tau^2) + c^2\tau^4 - \kappa^3(1+c^2\tau^2)}{\kappa^2(1+c^2\tau^2)}\right]} = \frac{c\tau^2}{\kappa\sqrt{(1+c^2\tau^2)}}$$

$$\therefore \quad \alpha = \tan^{-1}\left\{\frac{c\tau^2}{\kappa\sqrt{(c^2\tau^2+1)}}\right\}$$

Example 4.13: If $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are the moments about the origin of unit vector **t**, **n**, **b** localized in the tangent, principal normal and binormal, show that

$$\mathbf{m}_1' = \kappa\mathbf{m}_2,\ \mathbf{m}_2' = \mathbf{b} - \kappa\mathbf{m}_1 + \tau\mathbf{m}_3',\ \mathbf{m}_3' = -\mathbf{n} - \tau\mathbf{m}_2 .$$

Solution. Let **r** be the position vector of any current point.

We have $\mathbf{m}_1 = \mathbf{r} \times \mathbf{t}$, $\mathbf{m}_2 = \mathbf{r} \times \mathbf{n}$, $\mathbf{m}_3 = \mathbf{r} \times \mathbf{b}$

$$\therefore \qquad \mathbf{m}_1' = \mathbf{t} \times \mathbf{t} + \mathbf{r} \times \kappa\mathbf{n} \qquad [\because \mathbf{r}' = \mathbf{t},\ \mathbf{t}' = \kappa\mathbf{n}]$$

$$= \kappa\mathbf{r} \times \mathbf{n} \qquad [\because \mathbf{t} \times \mathbf{t} = 0]$$

$$= \kappa\mathbf{m}_2 \qquad \text{[using (1)]}$$

Also $$\mathbf{m}_2' = \mathbf{t} \times \mathbf{n} + \mathbf{r} \times (\mathbf{n}')$$

$$= \mathbf{t} \times \mathbf{n} + \mathbf{r} \times (\tau\mathbf{b} - \kappa\mathbf{t}) = \mathbf{b} + \tau\mathbf{m}_3 - \kappa\mathbf{m}_1$$

Similarly find $\mathbf{m}_3'$.

Example 4.14: Prove that for any curve

$$[\mathbf{t}',\mathbf{t}'',\mathbf{t}'''] = [\mathbf{r}'',\mathbf{r}''',\mathbf{r}''''] = \kappa^3(\kappa\tau' - \kappa'\tau) = \kappa^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right) \text{ and}$$

$$[\mathbf{b}',\mathbf{b}'',\mathbf{b}'''] = \tau^3(\kappa\tau' - \kappa'\tau) = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

Solution. (i) We have by Frenet's formulae,

$$\mathbf{r}' = \mathbf{t},\ \mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n} \qquad (1)$$

$$\mathbf{r}''' = \mathbf{t}'' = \kappa\mathbf{n}' + \kappa'\mathbf{n} = \kappa(\tau\mathbf{b} - \kappa\mathbf{t}) + \kappa'\mathbf{n} = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b} \qquad (2)$$

$$\mathbf{r}'''' = \mathbf{t}''' = \kappa^2\mathbf{t}' - 2\kappa\kappa'\mathbf{t} + \kappa''\mathbf{n} + \kappa'\mathbf{n}' + (\kappa'\tau + \kappa\tau')\mathbf{b} + \kappa\tau\mathbf{b}'$$

$$= -\kappa^3\mathbf{n} - 2\kappa\kappa'\mathbf{t} + \kappa''\mathbf{n} + \kappa'(\tau\mathbf{b} - \kappa\mathbf{t}) + (\kappa'\tau + \kappa\tau')\mathbf{b} - \kappa\tau^2\mathbf{n}$$

$$= -3\kappa\kappa'\mathbf{t} + (\kappa'' = \kappa\tau^2 - \kappa^3)\mathbf{n} + (2\kappa'\tau + \kappa\tau')\mathbf{b} \qquad (3)$$

$$\mathbf{r}''\times\mathbf{r}''' = \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b} = \mathbf{t}'\times\mathbf{t}'' \qquad \text{[using (1) and (2)]} \qquad (4)$$

∴ From (1), (2), (3) and (4), we have

$$[\mathbf{t}',\mathbf{t}'',\mathbf{t}'''] = [\mathbf{r}'',\mathbf{r}''',\mathbf{r}''''] = \mathbf{r}''\times\mathbf{r}'''\bullet\mathbf{r}''''$$

$$= -3\kappa^3\kappa'\tau + \kappa^3(2\kappa'\tau + \kappa\tau') \qquad \text{[Taking scalar product of (3) and (4)]}$$

$$= \kappa^3(\kappa\tau' - \kappa'\tau)$$

$$\kappa^5\frac{(\kappa\tau' - \kappa'\tau)}{\kappa^2} = \kappa^5\frac{d}{ds}\left(\frac{\kappa}{\tau}\right)$$

(ii) We have by Frenet's formulae

$$\mathbf{b}' = -\tau\mathbf{n} \qquad (5)$$

$$\mathbf{b}'' = -\tau(\tau\mathbf{b} - \kappa\mathbf{t}) - \tau'\mathbf{n} = \kappa\tau\mathbf{b} - \tau'\mathbf{n} - \tau^2\mathbf{b} \qquad (6)$$

$$\mathbf{b}''' = (\kappa'\tau + \kappa\tau')\mathbf{t} + \kappa\tau\mathbf{t}' - \tau''\mathbf{n} - \tau'\mathbf{n}' - 2\tau\tau'\mathbf{b} - \tau^2\mathbf{b}'$$

$$= (\kappa'\tau + \kappa\tau')\mathbf{t} + \kappa^2\tau\mathbf{n} - \tau''\mathbf{n} - \tau'(\tau\mathbf{b} - \kappa\mathbf{t}) - 2\tau\tau'\mathbf{b} + \tau^3\mathbf{n}$$

$$= (2\kappa\tau' + \kappa'\tau)\mathbf{t} + (\kappa^2\tau - \tau'' + \tau^3)\mathbf{n} - 3\tau\tau'\mathbf{b} \qquad (7)$$

$$\mathbf{b}' \times \mathbf{b}'' = \tau^3\mathbf{t} + \kappa\tau^2\mathbf{b} \quad \text{[Taking cross product of (5) and (6)]} \qquad (8)$$

Taking scalar product of (7) and (8), we get

$$[\mathbf{b}', \mathbf{b}'', \mathbf{b}'''] = \mathbf{b}' \times \mathbf{b}'' \bullet \mathbf{b}'''$$

$$= \tau^3(2\kappa\tau' + \kappa'\tau) - 3\kappa\tau^3\tau' = \tau^3(\kappa'\tau - \kappa\tau')$$

$$\tau^5 \frac{(\kappa'\tau - \kappa\tau')}{\tau^2} = \tau^5 \frac{d}{ds}\left(\frac{\kappa}{\tau}\right). \qquad (9)$$

Example 4.15: Prove that the position vector of the current point on a curve satisfies the differential equation

$$\frac{d}{ds}\left\{\sigma \frac{d}{ds}\left(\rho \frac{d^2\mathbf{r}}{ds^2}\right)\right\} + \frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{d\mathbf{r}}{ds}\right) + \frac{\sigma}{\rho}\frac{d^2\mathbf{r}}{ds^2} = 0.$$

Solution. We know that

$$\tau = \frac{1}{\sigma}, \ \kappa = \frac{1}{\rho}$$

$$\frac{d\mathbf{r}}{ds} = \mathbf{r}' = \mathbf{t}, \ \frac{d^2\mathbf{r}}{ds^2} = \mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}$$

The left hand side of the given differential equation may be rewritten as

$$\frac{d}{ds}\left\{\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa} \bullet \kappa\mathbf{n}\right)\right\} + \frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right) + \frac{\tau}{\kappa}\kappa\mathbf{n}$$

$$= \frac{d}{ds}\left\{\frac{1}{\tau}\frac{d}{ds}(\mathbf{n})\right\} + \tau\mathbf{n} + \frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right)$$

$$= \frac{d}{ds}\left\{\frac{1}{\tau}\mathbf{n}'\right\} + \tau\mathbf{n} + \frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right)$$

$$= \frac{d}{ds}\left\{\frac{\tau\mathbf{b} - \kappa\mathbf{t}}{\tau}\right\} + \frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right) + \tau\mathbf{n}$$

$$=\frac{d}{ds}\left\{\mathbf{b}-\frac{\kappa}{\tau}\mathbf{t}\right\}+\frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right)+\tau\,\mathbf{n}$$

$$=\frac{d\mathbf{b}}{ds}-\frac{d}{ds}\left\{\frac{\kappa}{\tau}\mathbf{t}\right\}+\frac{d}{ds}\left(\frac{\kappa}{\tau}\mathbf{t}\right)+\tau\,\mathbf{n}$$

$$=\mathbf{b}'+\tau\,\mathbf{n}=-\tau\,\mathbf{n}+\tau\,\mathbf{n}=0\,.$$

Example 4.16: Show that the position vector **r**(s) of any space curve of the class satisfies the differential equation

$$\mathbf{r}''''-\left(\frac{2\kappa'}{\kappa}+\frac{\tau'}{\tau}\right)\mathbf{r}'''+\left(\kappa^2+\tau^2+\frac{\kappa'\tau'}{\kappa\tau}+\frac{2\kappa'^2-\kappa\kappa''}{\kappa h2}\right)\mathbf{r}''+\kappa^2\left(\frac{\kappa'}{\kappa}-\frac{\tau'}{\tau}\right)\mathbf{r}'=0\,.$$

Solution. We have from example 10 above.

$$\mathbf{r}'=\mathbf{t}\,,\ \mathbf{r}''=\mathbf{t}'=\kappa\,\mathbf{n}$$

$$\mathbf{r}'''=-\kappa^2\mathbf{t}+\kappa'\,\mathbf{n}+\kappa\,\tau\,\mathbf{b}$$

$$\mathbf{r}''''=-3\kappa\,\kappa'\,\mathbf{t}+(\kappa''-\kappa\,\tau^2-\kappa^3)\mathbf{n}+(2\kappa'\tau+\kappa\,\tau')\mathbf{b}$$

Substituting these values in L.H.S. of given differential equation, we have left hand side

$$=-3\kappa\,\kappa'\,\mathbf{t}+(\kappa''-\kappa\,\tau^2-\kappa^3)\mathbf{n}+(2\kappa'\tau+\kappa\,\tau')\mathbf{b}-\left(\frac{2\kappa'}{\kappa}+\frac{\tau'}{\tau}\right)(-\kappa^2\mathbf{t}+\kappa'\,\mathbf{n}+\kappa\,\tau\,\mathbf{b})$$
$$+\left(\kappa^2+\tau^2+\frac{\kappa'\tau'}{\kappa\tau}+\frac{2\kappa'^2-\kappa\kappa''}{\kappa h2}\right)\kappa\,\mathbf{n}+\kappa^2\left(\frac{\kappa'}{\kappa}-\frac{\tau'}{\tau}\right)\mathbf{t}$$

$$=\left\{-3\kappa\,\kappa'+2\kappa'\kappa+\frac{\kappa^2\tau'}{\tau}+\kappa'\kappa-\frac{\kappa^2\tau'}{\tau}\right\}\mathbf{t}+\left\{\begin{matrix}\kappa''-\kappa\,\tau^2-\kappa^3-\dfrac{2\kappa'^2}{\kappa}-\dfrac{\kappa'\tau'}{\tau}+\kappa^3+\tau^2\kappa+\dfrac{\kappa'\tau'}{\tau}\\ +\dfrac{2\kappa'}{\kappa}-\kappa''\end{matrix}\right\}\mathbf{n}$$
$$+(2\kappa'\tau+\kappa\,\tau'-2\kappa^2\mathbf{t}-\kappa'\tau)\mathbf{b}$$

$$=0\mathbf{t}+0\mathbf{n}+0\mathbf{b}=0$$

Hence the positive vector **r**(s) of the current point satisfies the given differential equation.

Example 4.17: If the n^th^ derivative of **r** with respect to s is given by

$$\mathbf{r}^{(n)}=a_n\mathbf{t}+b_n\mathbf{n}+c_n\mathbf{b}$$

Prove the following reduction formulae

$$a_{n+1} = a_n' - \kappa b_n \ , \ b_{n+1} = b_n' - \kappa a_n - \tau c_n \ , \ c_{n+1} = c_n' + \tau b_n$$

Solution. $\mathbf{r}^{(n)} = a_n \mathbf{t} + b_n \mathbf{n} + c_n \mathbf{b}$ (given) (1)

Replacing n by (n+1) in (1), we get

$$\mathbf{r}^{(n+1)} = a_{n+1}\mathbf{t} + b_{n+1}\mathbf{n} + c_{n+1}\mathbf{b} \tag{2}$$

Now differentiating equation (1), w.r.t. s , we get

$$\mathbf{r}^{(n+1)} = a^n \mathbf{t}' + a_n'\mathbf{t}' + b_n \mathbf{n}' + b_n'\mathbf{n} + c_n \mathbf{b}' + c_n'\mathbf{b}$$

or $$\mathbf{r}^{(n+1)} = a^n (\kappa \mathbf{n}) + a_n'\mathbf{t}' + b_n(\tau \mathbf{b} - \kappa \mathbf{t}) + b_n'\mathbf{n} + c_n(-\tau \mathbf{n}) + c_n'\mathbf{b}$$

or $$\mathbf{r}^{(n+1)} = (a_n' - \kappa \mathbf{b}_n)\mathbf{t} + (b_n' + \kappa a_n - \tau c_n)\mathbf{n} + (\tau b_n + c_n')\mathbf{b} \tag{3}$$

Since L.H.S. of (2) and (3) are same , therefore, R.H.S. will also be same. Hence equating coefficient of **t**, **n** and **b** in R.H.S. of (2) and (3), we have

$$a_{n+1} = a_n' - \kappa b_n$$

$$b_{n+1} = b_n' - \kappa a_n - \tau c_n$$

$$c_{n+1} = c_n' + \tau b_n$$

Example 4.18: A curve is uniquely determined, except as to position in space, when its curvature and torsion are given functions of its arc-length *s*.

Consider two curves having equal curvatures κ and equal torsion τ for the same values of *s*. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ refer to one curve and $\mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1$ to the other. Then at point on the curve determined by the same value of *s*, we have

$$\frac{d}{ds}(\mathbf{t} \bullet \mathbf{t}_1) = \mathbf{t} \bullet (\kappa \mathbf{n}_1) + \kappa \mathbf{n} \bullet \mathbf{t}_1,$$

$$\frac{d}{ds}(\mathbf{n} \bullet \mathbf{n}_1) = \mathbf{n} \bullet (\tau \mathbf{b}_1 - \kappa \mathbf{t}_1) + (\tau \mathbf{b} - \kappa \mathbf{t}) \bullet \mathbf{n}_1$$

$$\frac{d}{ds}(\mathbf{b}\bullet\mathbf{b}_1)=\mathbf{b}\bullet(-\tau\,\mathbf{n}_1)+(-\tau\,\mathbf{n})\bullet\mathbf{b}_1$$

Now the sum of second members of these equations is zero.

Hence $$\frac{d}{ds}(\mathbf{t}\bullet\mathbf{t}_1+\mathbf{n}\bullet\mathbf{n}_1+\mathbf{b}\bullet\mathbf{b}_1)=0$$

and therefore $(\mathbf{t}\bullet\mathbf{t}_1+\mathbf{n}\bullet\mathbf{n}_1+\mathbf{b}\bullet\mathbf{b}_1)$=constant.

Suppose now that the two curves are placed so that their initial points, from which s is measured, coincide, and are then turned (without deformation) till their principal planes at the initial point also coincide. Then, at that point $\mathbf{t}=\mathbf{t}_1$, $\mathbf{n}=\mathbf{n}_1$, $\mathbf{b}=\mathbf{b}_1$, and the value of the constant in the last equation is 3. Thus

$$\mathbf{t}\bullet\mathbf{t}_1+\mathbf{n}\bullet\mathbf{n}_1+\mathbf{b}\bullet\mathbf{b}_1=3$$

But the sum of the cosines of three angles can be equal to 3 only when each of the angles vanishes, or is an integral multiple of 2π. This requires that, at all pairs of corresponding points,

$$\mathbf{t}=\mathbf{t}_1,\ \mathbf{n}=\mathbf{n}_1,\ \mathbf{b}=\mathbf{b}_1$$

So that the principal planes of the two curves are parallel. Moreover, the relation $\mathbf{t}=\mathbf{t}_1$ may be written

$$\frac{d}{d\,s}(\mathbf{r}-\mathbf{r}_1)=0,$$

So that $\mathbf{r}-\mathbf{r}_1$=constant vector.

But this difference vanishes at the initial point; and therefore it vanishes throughout. Thus $\mathbf{r}-\mathbf{r}_1$ at all corresponding points and the two curves coincide.

In making the initial points and the principal planes there coincident, we altered only the position and orientation of the curves in space; and the theorem has thus been proved. When a curve is specified by equations giving the curvature and torsion as function of s

$$\kappa=f(s),\quad \tau=F(s),$$

These are called the intrinsic equations of the curve.

4.4. Helices: A curve traced on the surface of a cylinder, and cutting the generators at a constant angle to a fixed direction. If then $\mathbf{t}$ is the unit tangent to the helix, and '**a**' a constant vector parallel to the generators of the cylinder, we have

$$\mathbf{t}.\mathbf{a}=\text{const} \qquad (4.4.1)$$

and therefore, on differentiation with respect to *s*,

$$\kappa.\mathbf{n} \bullet \mathbf{a}=0. \qquad (4.4.2)$$

Thus, since the curvature of the helix does not vanish, the principal normal is everywhere perpendicular to the generators. Hence the fixed direction of the generators is parallel to the plane of **t** and **b**; and since it makes a constant angle with **t**, it also makes a constant angle with **b**.

An important property of all helix is that the curvature and torsion are in a constant ratio. To prove this, we differentiate the relation $\mathbf{n} \bullet \mathbf{a}$=0, obtaining

$$(\tau\,\mathbf{b} - \kappa\,\mathbf{t}) \bullet \mathbf{a}=0. \qquad (4.4.3)$$

Thus **a** is perpendicular to the vector $\tau\,\mathbf{b} - \kappa\,\mathbf{t}$. But **a** is parallel to the plane of **t** and **b**, and must therefore be parallel to the vector $\tau\,\mathbf{t} + \kappa\,\mathbf{b}$, which is inclined to **t** at an angle $\tan^{-1}\kappa/\tau$. But this angle is constant. Therefore the curvature and the torsion are in a constant ratio.

Conversely, we may prove that a curve whose curvature and torsion are in a a constant ratio is a helix. Let $\tau = c\,\kappa$, where *c* is constant. Then since

$$\mathbf{t}' = \kappa\,\mathbf{n},$$

and $$\mathbf{b}' = -\tau\,\mathbf{n} = -c\kappa\,\mathbf{n},$$

it follows that $$\frac{d}{ds}(\mathbf{b} + c\,\mathbf{t}) = 0,$$

and therefore $$\mathbf{b} + c\,\mathbf{t} = \mathbf{a},$$

where **a** is a constant vector. Forming the scalar product of each side with **t** we have

$$\mathbf{t} \bullet \mathbf{a}=c. \qquad (4.4.4)$$

Thus **t** is inclined at a constant angle to the fixed direction of **a**, and the curve is therefore a helix.

Finally we may show that the curvature and the torsion of a helix are in a constant ratio to the curvature κ_0 of the plane section of the cylinder perpendicular to the generators. Take the z-axes

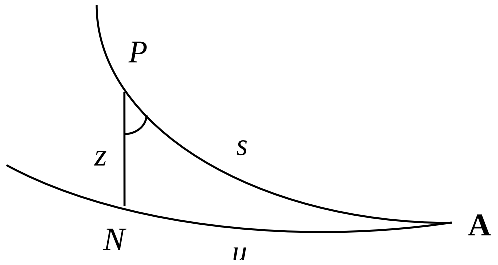

Figure 4.5

Parallel to the generators, and let s be measured from the intersection $\boldsymbol{A}$ of the curve with xy plane. Let u be the arc length of the normal section of the cylinder by the xy plane, measured from the same point $\boldsymbol{A}$ up to the generator through the current point (x, y, z). Then, if β is the constant angle at which the curve cuts the generators, we have

$$u = s\sin\beta$$

and therefore $$u' = \sin\beta .$$

The coordinate (x, y) are function of u, while $z = s\cos\beta$. Hence for the current point on the helix, we have

$$\mathbf{r} = (x, y, s\cos\beta),$$

so that $$\mathbf{r}' = \left(\frac{dx}{du}\sin\beta, \frac{dy}{du}\sin\beta, \cos\beta\right),$$

and $$\mathbf{r}'' = \left(\frac{d^2x}{du^2}\sin^2\beta, \frac{d^2y}{du^2}\sin^2\beta, 0\right)$$

Hence the curvature of helix is given by

$$\kappa^2 = \mathbf{r}''^2 = \left\{\left(\frac{d^2x}{du^2}\right)^2 + \left(\frac{d^2y}{du^2}\right)^2\right\}\sin^4\beta = \kappa_0^2\sin^4\beta,$$

so that $$\kappa = \kappa_0\sin^2\beta . \qquad (4.4.5)$$

For the torsion, we have already proved that

$$\beta = \tan^{-1}\kappa/\tau,$$

so that $$\tau = \kappa\cot\beta = \kappa_0\sin\beta\cos\beta .$$

From these results it is clear that the only curve whose curvature and torsion are both constant is the circular helix. For such a curve must be a helix, since the ratio of its curvature to its torsion is also constant. And since κ_0 is constant, so that the cylinder on which the helix is drawn is a circular cylinder.

4.5. Spherical Indicatrix. The locus of a point, whose position vector is equal to the unit tangent **t** of a given curve, is called the spherical indicatrix of the tangent to the curve. Such a locus lies on the surface of a unit sphere, hence the name. Let the suffix unity be used to distinguish quantities belonging to this locus.

Then $$\mathbf{r}_1 = \mathbf{t} \tag{4.5.1}$$

and therefore $\mathbf{t}_1 = \dfrac{d\mathbf{r}_1}{ds_1} = \dfrac{d\,\mathbf{t}}{ds}\cdot\dfrac{ds}{ds_1} = \kappa\,\mathbf{n}\dfrac{ds}{ds_1}$,

showing that the tangent to the spherical indicatrix is parallel to the principal normal of the given curve. We may measure s_1 so that

$$\mathbf{t}_1 = \mathbf{n} \tag{4.5.2}$$

and therefore $$\frac{ds_1}{ds} = \kappa\,.$$

For the curvature κ_1 of the indicatrix, on differentiating the relation $\mathbf{t}_1 = \mathbf{n}$, we find the formula

$$\kappa_1\,\mathbf{n}_1 = \frac{d\,\mathbf{n}}{ds}\cdot\frac{ds}{ds_1} = \frac{1}{\kappa}(\tau\,\mathbf{b} - \kappa\,\mathbf{t})\,.$$

Squaring both sides we obtained the result

$$\kappa_1^2 = (\kappa^2 + \tau^2)/\kappa^2 \tag{4.5.3}$$

So that the curvature of the indicatrix is the ratio of the screw curvature to the circular curvature of the curve. The unit binormal of the indicatrix is

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\tau\,\mathbf{t} + \kappa\,\mathbf{b}}{\kappa\,\kappa_1}\,. \tag{4.5.4}$$

The torsion could be obtained by differentiating this equation; but the result follows more easily from the equation.

$$\kappa_1^2\,\tau_1 = \left(\frac{d\,s_1}{ds}\right)^6 = [\mathbf{r}_1', \mathbf{r}_2'', \mathbf{r}_3'''] = [\mathbf{t}', \mathbf{t}'', \mathbf{t}''']$$

$$= \kappa^3(\kappa\,\tau' - \kappa'\tau)$$

which reduces to $$\tau_1 = \frac{(\kappa\,\tau' - \kappa'\tau)}{\kappa(\kappa^2 + \tau^2)}. \qquad (4.5.5)$$

Similarly the spherical indicatrix of the binormal of the given curve is the locus of a point whose position vector is **b**. Using the suffix unity to distinguish quantities belonging to this locus, we have

$$\mathbf{r}_1 = \mathbf{b} \qquad (4.5.6)$$

and therefore $$\mathbf{t}_1 = \frac{d\,\mathbf{b}}{ds}\cdot\frac{ds}{ds_1} = -\tau\,\mathbf{n}\frac{ds}{ds_1}.$$

We may measure s_1 so that $\mathbf{t}_1 = -\mathbf{n}$

and therefore, $\dfrac{ds_1}{ds} = \tau$.

To find the curvature, differentiating the equation $\mathbf{t}_1 = -\mathbf{n}$. Then

$$\kappa_1\mathbf{n}_1 = \frac{d}{ds}(-\mathbf{n})\frac{ds}{ds_1} = \frac{1}{\tau}(\kappa\,\mathbf{t} - \tau\,\mathbf{b})$$

giving the direction of the principal normal. On squaring this result, we have

$$\kappa_1^2 = (\kappa^2 + \tau^2)/\tau^2. \qquad (4.5.7)$$

Thus the curvature of the indicatrix is the ratio of the screw curvature to the torsion of the given curve.

The unit binormal is $\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \dfrac{\tau\,\mathbf{t} + \kappa\,\mathbf{b}}{\tau\,\kappa_1}$,

And the torsion, found as in the previous case, is equal to

$$\tau_1 = \frac{(\tau\,\kappa' - \kappa\,\tau')}{\tau(\kappa^2 + \tau^2)}. \qquad (4.5.8)$$

4.6. Surfaces.

We have seen that a curve is the locus of a point whose coordinates x, y, z are functions of a single parameter. We now define a surface as the locus of a point whose coordinates are functions of two independent parameters u, v . Thus

$$x = f_1(u,v), \qquad y = f_2(u,v), \qquad z = f_3(u,v) \qquad (4.6.1)$$

Are parametric equations of a surface. In particular cases one, or even two, of the functions may involve only a single parameter. If now u, v are eliminated from the equation (1), we obtain a relation between the coordinates which may be written

$$F(x, y, z) = 0 \tag{4.6.2}$$

This is the oldest form of the equation of surface. The two parametric representation of a surface as given in (4.6.1) is due to Gauss. In subsequent chapters it will form the basis of our investigation. But for the discussion in the present chapter the form (4.6.2) of the equation of a surface will prove more convenient.

4.7. Normal to the Tangent plane. Consider any curve drawn on the surface

$$F(x, y, z) = 0 \tag{4.7.1}$$

Let *s* be the arc-length measured from a fixed point up to the current point (x, y, z). Then, since the function ***F*** has the same value at all points of the surface, it remains constant along the curve as *s* varies. Thus

$$\frac{\partial \mathbf{F}}{\partial x}\frac{dx}{ds} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{ds} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{ds} = 0,$$ which may written more briefly

$$\mathbf{F}_x x' + \mathbf{F}_y y' + \mathbf{F}_z z' = 0. \tag{4.7.2}$$

Now the vector (x', y', z') is the unit tangent to the curve at the point (x, y, z); and the last equation shows that it is perpendicular to the vector $(\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z)$. The tangent to any curve drawn on surface is called a tangent line to the surface. Thus all tangent lines to the surface at the point (x, y, z) are perpendicular to the vector $(\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z)$, and therefore lie in the plane through (x, y, z) perpendicular to this vector. This plane is called the tangent plane to the surface at that point, and the normal to the plane at the point of contact is called the normal to the surface at that point. Since the line joining any point (X,Y,Z) on the tangent plane to the point of contact is perpendicular to the normal, it follows that

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0 \tag{4.7.3}$$

This is the equation of the tangent plane. Similarly if (X,Y,Z) is a current point on the normal, get

$$\frac{(X-x)}{\frac{\partial F}{\partial x}}+\frac{(Y-y)}{\frac{\partial F}{\partial y}}+\frac{(Z-z)}{\frac{\partial F}{\partial z}} \qquad (4.7.4)$$

These are the equations of the normal at the point (x, y, z).

4.8 CHECK YOUR PROGRESS

SA 1: Prove that the tangent plane to the surface $xyz = a^3$, and the coordinate planes, bound a tetrahedron of constant volume.

SA 2: At points common to the surface $a(yz + zx + xy) = xyz$ and a sphere whose centre is the origin, the tangent plane to the surface makes intercepts on the axes whose sum is constant.

SA 3: Any tangent plane to the surface $a(x^2 + y^2) + xyz = 0$ meets it again in a conio whose projection on the plane of xy is a rectangular hyperbola.

SA 4: If κ is zero at all points, the curve is a straight line. If τ is zero at all points, the curve is plane. The necessary and sufficient condition that the curve be plane is

$$[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] = 0.$$

SA 5: If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction, show that $\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}$.

SA 6: Prove that the shortest distance between the principal normals at consecutive points s apart, is $s\rho\sqrt{\rho^2+\sigma^2}$, and that it divides the radius of curvature in the ratio $\rho^2 : \sigma^2$.

SA 7: Find the curvature, the centre of curvature, and the torsion of the curve

$$x = a\cos u,\ y = a\sin u,\ z = a\cos 2u.$$

SA 8: Prove that the position vector of the current point on a curve satisfies the differential equation

$$\frac{d}{ds}\left\{\sigma\frac{d}{ds}\left(\rho\frac{d^2\mathbf{r}}{ds^2}\right)\right\}+\frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{d\mathbf{r}}{ds}\right)+\frac{\sigma}{\rho}\frac{d^2\mathbf{r}}{ds^2}=0.$$

SA 9: If s_1 is the arc length of the locus of centre of curvature, show that

$$\frac{ds_1}{ds}=\frac{1}{\kappa^2}\sqrt{\kappa^2\tau^2}+\kappa'^2=\sqrt{\left(\frac{\rho}{\sigma}\right)^2}+\rho'^2.$$

SA 10: Prove that, for curve drawn on the surface of a sphere, $\frac{\rho}{\sigma}+\frac{d}{d\,s}(\sigma\,\rho')=0$; that is

$$\rho+\frac{d^{2\rho}}{d\,\psi^2}=0\text{, where } d\psi=\tau\,ds.$$

SA 11: Show that the radius of spherical curvature of a circular helix is equal to the radius of circular curvature.

SA 12: The normal plane to the locus of centre of circular curvature of a curve *C* bisects the radius of spherical curvature at the corresponding point of *C*.

SA 13: A curve is drawn on a right circular cone everywhere inclined at the same angle α to the axes. Prove that $\kappa=\tau\tan\alpha$.

SA14: On the tangent to a given curve a point Q is taken at a constant distance *c* from the point of contact. Prove that the curvature κ_1 of the locus of Q is given by

$$\kappa_1^2(1+c^2\kappa^2)^3=c^2\kappa^2\tau^2(1+c^2\kappa^2)+(\kappa+c\kappa'+c^2\kappa^3)^2.$$

SA 15: On the tangent to a given curve a point Q is taken at a constant distance *c* from the point of contact. Prove that the curvature κ_1 of the locus of Q is given by

$$\kappa_1^2(1+c^2\kappa^2)^3=c^2\tau^4(1+c^2\tau^2)+(\kappa-c\tau'+c^2\kappa\tau^2)^2.$$

4.9 SELF ASSESSMENT TEST

i) The normal at a point *P* on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}1$ meets the coordinate planes in G_1, G_2, G_3. Prove that the ratios $PG_1 : PG_2\,;\, PG_3$ are constant.

ii) If ψ is such that $d\psi=\tau\,ds$, show that

$$\rho_1=\frac{1}{\kappa_1}=\rho+\frac{d^2\rho}{d\psi^2},\quad \sigma_1=\frac{1}{\tau_1}=\frac{\rho}{\sigma}\left(\rho+\frac{d^2\rho}{d\psi^2}\right),$$

$$\mathbf{R}^2=\rho^2+\left(\frac{d\rho}{d\psi}\right)^2,\quad \mathbf{R}\frac{d\,\mathbf{R}}{d\rho}=\rho+\frac{d^2\rho}{d\psi^2}=\rho_1.$$

iii) Show that for any curve $[\mathbf{r}'', \mathbf{r}''', \mathbf{r}'''']=\kappa^5 \dfrac{d}{ds}\left(\dfrac{\tau}{\kappa}\right)$.

iv) Prove that for any curve $[\mathbf{t}', \mathbf{t}'', \mathbf{t}''']=[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']=\kappa^3(\kappa\tau'-\kappa'\tau)=\kappa^5 \dfrac{d}{ds}\left(\dfrac{\tau}{\kappa}\right)$

and also that $[\mathbf{b}', \mathbf{b}'', \mathbf{b}''']=\tau^3(\kappa'\tau-\kappa\tau')=\tau^5 \dfrac{d}{ds}\left(\dfrac{\kappa}{\tau}\right)$.

v) For the curve $x=4a\cos^3 u$, $y=4a\sin^3 u$, $s=3c\cos 2u$ prove that $\mathbf{n}=(\sin u, \cos u, 0)$ and $\kappa=\dfrac{a}{6(a^2+c^2)\sin 2u}$.

vi) The shortest distance between consecutive radii of spherical curvature divides the radius in ratio $\sigma^2 : \rho^2\left(\dfrac{dR}{d\rho}\right)^2$.

vii) On the binormal of a curve of constant torsion τ a point Q is taken at a constant distance c from the curve. Show that the binormal to the locus of Q is inclined to the binormal of the given curve at an angle $\tan^{-1}\dfrac{c\tau^2}{\sqrt{(c^2\tau^2+1)}}$.

ix) Prove that the curvature κ_1 of the locus of centre of (circular) curvature of a given curve is given by $\kappa_1^2=\left\{\dfrac{\rho^2\sigma}{R^3}\dfrac{d}{ds}\left(\dfrac{\sigma\rho'}{\rho}\right)-\dfrac{1}{R}\right\}^2+\dfrac{\rho'^2\sigma^4}{\rho^2R^4}$, where the symbols have their usual meanings.

x) If there is one-one correspondence between the points of two curves, and the tangents at corresponding points are parallel, show that the principal normals are parallel, and therefore also the binormals. Prove also that $\dfrac{\kappa_1}{\kappa}=\dfrac{ds}{ds_1}=\dfrac{\tau_1}{\tau}$. Two curves so related are said to be deducible from each other by a combescure transformation.

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CHAPTER-5

ENVELOPE, DEVELOPABLE SURFACES(I)

Objectives: In this chapter the students will learn about family of surfaces, and its characteristics, Envelope, Equation of envelope, Edge of regression, Equation of edge of regression of the envelope, Envelope of the system of surfaces with two parameters, Ruled Surface (Developable and skew), Equation of the tangent plane to a ruled surface , Developable surface, Osculating plane of the edge of regression at any point is the tangent plane to the developable at that point, Locus of the centre of osculating spheres, Characterization of a developable surface.

5.1 INTRODUCTION

Family of Surfaces: An equation of the form

$$f(x, y, z, a) = 0 \tag{1}$$

Where 'a' is the constant, represent a surface. If 'a' can takes all real values i.e. if 'a' is a parameter, then (1) represent the equation of one parameter family of surfaces with 'a' as parameter. Giving different values to 'a' we shall get different surfaces (members) of this family of surfaces. Similarly the equation of the form

$$f(x, y, z, a, b) = 0 \tag{2}$$

where 'a' and 'b' are parameters represents the family of surfaces with two parameters 'a' and 'b'.

5.1.1 Characteristic: Let $f(x, y, z, a) = 0$ be the equation of one parameter family of surfaces, a being the parameter and which is constant for any given surface. Let the two members of the family be

$$f(x, y, z, \alpha) = 0\,,\ \ f(x, y, z, \alpha + \delta\alpha) = 0$$

The curve of intersection of these two surfaces may be given by

$$f(x, y, z, \alpha) = 0\,,\ \ \frac{f(x, y, z, \alpha + \delta\alpha) - f(x, y, z, \alpha)}{\delta\alpha} = 0$$

Now the limiting position of the curve as $\delta\alpha \to 0$, becomes

$$f(x, y, z, \alpha) = 0\,,\ \frac{\partial f}{\partial \alpha} = 0$$

this limiting position is called the characteristic corresponding to the value α .

5.1.2 Envelope : The locus of characteristic for all values of the parameter is called the envelope of the system of surfaces.

The equation of the envelope: Eliminating α between $f = 0$, $\frac{\partial f}{\partial \alpha} = 0$, we get the equation of the envelope.

Theorem 5.1: The envelope of a family of surfaces touches each member of the family, at all points of its characteristic.

Proof: Let the equation of given family of surfaces be $f(x, y, z, \alpha) = 0$ where α is a parameter.

The equation of the envelope is obtained by eliminating α from the equations.

$$f(x, y, z, \alpha) = 0\,,\ \frac{\partial f(x, y, z, \alpha)}{\partial \alpha} = 0 \qquad (1)$$

Thus the equation of the envelope may be regarded as

$$f(x, y, z, \alpha) = 0 \qquad (2)$$

in which α is not constant but a function of x, y, z given by

$$\frac{\partial f(x, y, z, \alpha)}{\partial \alpha} = 0 \qquad (3)$$

The normal to the envelope is parallel to the vector

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial z} \right)$$

But using equation (3), this vector reduces to

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

which is a vector to which the normal to the surface $f(x, y, z, \alpha) = 0$ is parallel. This means that all common points, the surface and envelope have the same normal, and therefore, the same tangent plane; so that they touch each other at all points of characteristic.

5.1.3 The edge of regression : We have seen that the curve in which any surface is met by the consecutive surface is called the characteristic of the envelope. Every characteristic will meet the next in one or more points, and the locus of these points is called the edge of regression cumspidal edge of envelope. Since each characteristic lies on the envelope, therefore the edge of regression is a curve which lies on the envelope.

Alternate definition of edge of regression: It is the locus of the ultimate points of intersection of consecutive characteristics of one parameter family of surfaces.

5.1.4 To find the equation of edge of regression of the envelope: The equation of the characteristic corresponding to the surface $f(x, y, z, \alpha) = 0$ are

$$f(x, y, z, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha} = 0 \qquad (1)$$

The equations of the next consecutive characteristic are therefore,

$$f(x, y, z, \alpha + \delta\alpha) = 0 \text{ and } \frac{\partial}{\partial \alpha} f(x, y, z, \alpha + \delta\alpha) = 0 \qquad (2)$$

or $$f + (\delta\alpha)\frac{\partial f}{\partial \alpha} + ... = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha} + \frac{\partial^2 f}{\partial \alpha^2}\partial\alpha = 0 \qquad (3)$$

Hence at any point of the regression, we must have

$$f = 0, \ \frac{\partial f}{\partial \alpha} = 0, \ \frac{\partial^2 f}{\partial \alpha^2} = 0$$

[$\because$ for points of intersection of (2) with (1), all four equations must be satisfied].

Eliminating α from the equations, we get the required equations to the edge of regression (curve).

5.1.5 To prove that each characteristic touches the edges of regression:

The edge of regression is given by

$$f(x,y,z,\alpha)=0\,,\ \frac{\partial f}{\partial \alpha}(x,y,z,\alpha)=0\,,\ \frac{\partial^2 f}{\partial \alpha^2}(x,y,z,\alpha)=0$$

so it may be considered to be the curve given by

$$f(x,y,z,\alpha)=0\,,\ \frac{\partial f}{\partial \alpha}(x,y,z,\alpha)=0$$

where α is a function of x,y,z, given by $\dfrac{\partial^2 f}{\partial \alpha^2}=0$.

The tangent at (x,y,z) to the edge of regression is the line of intersection of the tangent planes of the surfaces. It is therefore perpendicular to each of the vectors

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x},\ \frac{\partial f}{\partial y}+\frac{\partial f}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y},\ \frac{\partial f}{\partial z}+\frac{\partial f}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial z}\right)$$

and $$\left(\frac{\partial^2 f}{\partial x\partial \alpha}+\frac{\partial^2 f}{\partial \alpha^2}\cdot\frac{\partial \alpha}{\partial x},\ \frac{\partial^2 f}{\partial y\partial \alpha}+\frac{\partial^2 f}{\partial \alpha^2}\cdot\frac{\partial \alpha}{\partial y},\ \frac{\partial^2 f}{\partial z\partial \alpha}+\frac{\partial^2 f}{\partial \alpha^2}\cdot\frac{\partial \alpha}{\partial z}\right)$$

For the edge of regression $\dfrac{\partial f}{\partial \alpha}=0$, $\dfrac{\partial^2 f}{\partial \alpha^2}=0$.

Therefore, the vectors to which the tangent is perpendicular reduce to

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) \quad \text{and} \quad \left(\frac{\partial^2 f}{\partial x\partial \alpha},\frac{\partial^2 f}{\partial y\partial \alpha},\frac{\partial^2 f}{\partial z\partial \alpha}\right)$$

which are the vectors perpendicular to the tangent planes at (x,y,z) to

$f(x,y,z,\alpha)=0$, $\dfrac{\partial f}{\partial \alpha}(x,y,z,\alpha)=0$ *i.e.* to the characteristic. This means, the tangent to the edge of regression is parallel to the tangent to the characteristic and hence the two curves touch at their common points.

Example 5.1: Find the envelope of the plane $3xt^2 - 3yt + z = t^3$ and show that its edge of regression is the curve of intersection of the surfaces $y^2 = zx$, $xy = z$.

Solution: $f(x, y, z, t) = 3xt^2 - 3yt + z - t^3 = 0$ (1)

$$\frac{\partial f}{\partial t} = 6xt - 3y - 3t^2 = 0 \quad (2)$$

$$\frac{\partial^2 f}{\partial t^2} = 6x - 6t = 0 \quad (3)$$

Eliminating 't' from (1) and (2), the equation of the envelope is obtained.

Multiply (1) by 3 and (2) by t and subtract, we get

$$3xt^2 - 6yt + 3z = 0 \quad \text{or} \quad xt^2 - 2yt + z = 0 \quad (4)$$

Solving (2) and (4), we get

$$\frac{t^2}{2y^2 - 2zx} = \frac{t}{xy - z} = \frac{1}{2x^2 - 2y}$$

Hence eliminating 't' the envelope is

$$(xy - z)^2 = 4(y^2 - zx)(x^2 - y)$$

For edge of regression, we need to eliminating 't' between (1), (2) and (3),

$x = 1$ from (3),

From (2), $6x^2 - 3y = 3x^2$ $\quad\therefore\quad t^2 = x^2 = y$

From (1), $3xy - 3yx + z - y.x = 0$ or $xy = z$

Also $x^2 y = y^2$ $\quad [\because \quad x^2 = y \Rightarrow xh2y = y.y]$

or $x(xy) = y^2$ $\quad$ or $xz = y^2$

$\therefore$ The edge of regression is the curve of intersection of $xy = z$, $xz = y^2$.

Example 5.2: The envelope of surfaces $f(x, y, z, a, b, c) = 0$ where a, b, c are parameters connected by the equation $\phi(a,b,c) = 0$ and f and ϕ are homogeneous with respect to a, b, c is found by eliminating a, b, c between the equations

$$f = 0,\ \phi = 0,\ \frac{\partial f/\partial a}{\partial\phi/\partial a} = \frac{\partial f/\partial b}{\partial\phi/\partial b} = \frac{\partial f/\partial c}{\partial\phi/\partial c}.$$

Solution: We have from the given equations

$$\frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db + \frac{\partial f}{\partial c}dc = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial a}da + \frac{\partial \phi}{\partial b}db + \frac{\partial \phi}{\partial c}dc = 0 \tag{2}$$

Multiplying (2) by λ and adding to (1).

$$\left(\frac{\partial f}{\partial a} + \lambda\frac{\partial \phi}{\partial a}\right)da + \left(\frac{\partial f}{\partial b} + \lambda\frac{\partial \phi}{\partial b}\right)db + \left(\frac{\partial f}{\partial c} + \lambda\frac{\partial \phi}{\partial c}\right)dc = 0$$

Now λ is at our choice, so we choose λ, such that

$$\left(\frac{\partial f}{\partial a} + \lambda\frac{\partial \phi}{\partial a}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial a}/\frac{\partial \phi}{\partial a} = -\lambda$$

Again 'b' and 'c' can be treated as independent variables, so coefficients of db; dc are separately zero.

$$\therefore \quad \frac{\partial f/\partial b}{\partial\phi/\partial b} = \frac{\partial f/\partial c}{\partial\phi/\partial c} = -\lambda$$

$$\text{where} \quad \frac{\partial f/\partial a}{\partial\phi/\partial a} = \frac{\partial f/\partial b}{\partial\phi/\partial b} = \frac{\partial f/\partial c}{\partial\phi/\partial c} \tag{3}$$

So envelope will be obtained by eliminating a, b, c between the equations (3) and $f = 0$, $\phi = 0$.

Example 5.3: Find the envelope of the plane $lx+my+nz=0$, where $al^2+bm^2+cn^2=0$.

Solution. Refer last question, the envelope is obtained by eliminating l,m,n between

$$f=lx+my+nz=0,\ \phi=al^2+bm^2+cn^2=0$$

and $\dfrac{\partial f/\partial l}{\partial\phi/\partial l}=\dfrac{\partial f/\partial m}{\partial\phi/\partial m}=\dfrac{\partial f/\partial n}{\partial\phi/\partial n}$

or $\dfrac{x}{al}=\dfrac{y}{bm}=\dfrac{z}{cn}=\dfrac{x/\sqrt{a}}{\sqrt{a}l}=\dfrac{y/\sqrt{b}}{\sqrt{b}m}=\dfrac{z\sqrt{c}}{\sqrt{c}n}=\dfrac{\sqrt{(x^2/a+y^2/b+z^2/c)}}{\sqrt{(al^2+bm^2+cn^2)}}$

Since $al^2+bm^2+cn^2=0$, the required envelope is

$$x^2/a+y^2/b+z^2/c=0.$$

Example 5.4: Prove that the envelope of the normal planes drawn through the generators of the cone $ax^2+by^2+cz^2=0$ is given by

$$a^{1/3}(b-c)^{2/3}x^{2/3}+b^{1/3}(c-a)^{2/3}y^{2/3}+c^{1/3}(a-b)^{2/3}z^{2/3}=0.$$

Solution. Let (ξ,η,ζ) be any point on the cone, the equation of the tangent plane is

$$ax\xi+by\eta+cz\zeta=0$$

The equation of the generator through this point is

$$x/\xi=y/\eta=z/\zeta$$

Let $Lx+My+Nz=0$, be the normal plane.

Since the generator lies on it

$$L\xi+M\eta+N\zeta=0 \tag{1}$$

Since it is perpendicular to tangent plane

$$La\xi + Mb\eta + Nc\zeta = 0 \tag{2}$$

From (1) and (2),

$$\frac{L\xi}{(b-c)} = \frac{M\eta}{(c-a)} = \frac{N\zeta}{(a-b)}$$

$\therefore$ equation of the normal plane

$Lx + My + Nz = 0$ becomes

$$\frac{b-c}{\xi}x + \frac{c-a}{\eta}y + \frac{a-b}{\zeta}z = 0 \tag{3}$$

Also $a\xi^2 + b\eta^2 + c\zeta^2 = 0$ [since ξ, η, ζ is a point the given cone]

Here $\dfrac{\partial f/\partial\xi}{\partial\phi/\xi} = \dfrac{\partial f/\partial\eta}{\partial\phi/\partial\eta} = \dfrac{\partial f/\partial\zeta}{\partial\phi/\partial\zeta}$ gives

$$\frac{(b-c)x}{\xi^2 2a\xi} = \frac{(c-a)y}{\eta^2 2b\eta} = \frac{(a-b)z}{\zeta^2 2c\zeta}$$

or $$\frac{(b-c)^{1/3}x^{1/3}}{a^{1/3}\xi} = \frac{(c-a)^{1/3}y^{1/3}}{b^{1/3}\eta} = \frac{(a-b)^{1/3}z^{1/3}}{c^{1/3}\zeta}$$

or $$\frac{(b-c)^{2/3}x^{2/3}a^{1/3}}{a^{1/3}a^{2/3}\xi^2} = \frac{(c-a)^{2/3}y^{2/3}b^{1/3}}{b^{1/3}b^{2/3}\eta^2} = \frac{(a-b)^{2/3}z^{2/3}c^{1/3}}{c^{1/3}c^{2/3}\zeta^2}$$

$$= \sum \frac{(b-c)^{2/3}x^{2/3}a^{1/3}}{a^{1/3}a^{2/3}\xi^2}$$

$\therefore$ $= \sum (b-c)^{2/3}x^{2/3}a^{1/3} = 0$ since $a\xi^2 + b\eta^2 + c\zeta^2 = 0$

Hence the required envelope is

$$= \sum a^{1/3}(b-c)^{2/3}x^{2/3} = 0.$$

5.2 To find the envelope of the system of surfaces whose equations involve two parameters.

Let the equation of the surface be $f(x,y,z,a,b)=0$ where a,b are the parameters. A consecutive surface of the system is

$$f(x,y,z,a+\delta a,b+\delta b)=0$$

or $$f(x,y,z,a)+\delta a\frac{\delta f}{\delta a}+\delta b\frac{\delta f}{\delta b}+.....=0$$

Hence when $\delta a,\ \delta b$ are infinitely small we must have a point of ultimate intersection:

$$f=0\,,\ \delta a\frac{\partial f}{\partial a}+\delta b\frac{\partial f}{\partial b}=0$$

Since $\delta a,\ \delta b$ are independent,

$$f=0\,,\ \frac{\partial f}{\partial a}=0\,,\ \frac{\partial f}{\partial b}=0$$

Hence the curve of intersection of $f=0$ with any surface consecutive to it goes through the point which satisfy the equation

$$f=0\,,\ \frac{\partial f}{\partial a}=0\,,\ \frac{\partial f}{\partial b}=0$$

By eliminating $a,\ b$ from the above equations we get the equation of the envelope.

5.2.1 To prove that the envelope touches each surface of the system at the corresponding characteristic points.

Consider the surface $f(x,y,z,a,b)=0$. The normal at (x,y,z) to a surface of the family $f=0$ is parallel to the vector

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) \qquad (1)$$

The characteristics points are given by $f=0$, $\dfrac{\partial f}{\partial a}=0$, $\dfrac{\partial f}{\partial b}=0$ and the equation of the envelope can be regarded as $f(x,y,z,a,b)=0$ where a, b are functions of (x,y,z) given by

$$\frac{\partial f}{\partial a}=0\,,\ \frac{\partial f}{\partial b}=0 \tag{2}$$

The normal at (x,y,z) to envelope is parallel to the vector

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial x}+\frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial x},\frac{\partial f}{\partial y}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial y}+\frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial y},\frac{\partial f}{\partial z}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial z}+\frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial z}\right) \tag{3}$$

Since $\dfrac{\partial f}{\partial a}=0$, $\dfrac{\partial f}{\partial b}=0$ hence using (2), (3) reduces to

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

which means that the envelope has the normal line and therefore, the envelope and the surface of the family have the same tangent plane at a characteristic point.

Example 5.5: Find the envelope of the plane

$$(x/a)\cos\theta\sin\phi+(y/b)\sin\theta\sin\phi+(z/c)\cos\phi=1.$$

Solution. Differentiating partially w.r.t. θ and ϕ,

$$-(x/a)\sin\theta\sin\phi+(y/b)\cos\theta\sin\phi=0 \tag{1}$$

$$(x/a)\cos\theta\cos\phi+(y/b)\sin\theta\cos\phi-(z/c)\sin\phi=0 \tag{2}$$

From (1), $\tan\theta=ay/bx$.

From (2), $(z/c)\tan\phi=(x/a)\cos\theta+(y/b)\sin\theta$ (3)

Also $(x/a)\cos\theta\sin\phi+(y/b)\sin\theta\sin\phi=1-(z/c)\cos\phi$ (4)

$\therefore$ $(z/c)\tan\phi\sin\phi=1-(z/c)\cos\phi$ from (3) and (4)

$$\therefore \qquad \frac{z}{c}\left(\frac{\sin^2\phi+\cos^2\phi}{\cos\phi}\right)=1 \qquad \therefore \qquad \cos\phi = z/c \tag{5}$$

$\therefore$ Eliminating ϕ from (3) and (5), we have

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\frac{z}{c}\cdot\frac{\sqrt{(c^2-z^2)}}{z} \tag{6}$$

Also from (1), $(x/a)\sin\theta-(y/b)\cos\theta=0$ (7)

Squaring (6) and (7) and adding

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Example 5.6: Prove that the envelope of the surfaces $f(x,y,z,a,b,c,d)=0$ where a,b,c,d are parameters connected by the equation $\phi(a,b,c,d)=0$ and f and ϕ are homogeneous with respect to a,b,c,d is found by eliminating a,b,c,d between the equations $f=0$, $\phi=0$ and

$$\frac{\partial f/\partial a}{\partial\phi/\partial a}=\frac{\partial f/\partial b}{\partial\phi/\partial b}=\frac{\partial f/\partial c}{\partial\phi/\partial c}=\frac{\partial f/\partial d}{\partial\phi/\partial d}.$$

Solution. We have from the given equations

$$\frac{\partial f}{\partial a}da+\frac{\partial f}{\partial b}db+\frac{\partial f}{\partial c}dc+\frac{\partial f}{\partial d}dd=0 \tag{1}$$

$$\frac{\partial\phi}{\partial a}da+\frac{\partial\phi}{\partial b}db+\frac{\partial\phi}{\partial c}dc+\frac{\partial\phi}{\partial d}dd=0 \tag{2}$$

Multiplying (2) by λ and adding to (1).

$$\left(\frac{\partial f}{\partial a}+\lambda\frac{\partial\phi}{\partial a}\right)da+\left(\frac{\partial f}{\partial b}+\lambda\frac{\partial\phi}{\partial b}\right)db+\left(\frac{\partial f}{\partial c}+\lambda\frac{\partial\phi}{\partial c}\right)dc+\left(\frac{\partial f}{\partial d}+\lambda\frac{\partial\phi}{\partial d}\right)dd=0$$

here λ is at our choice, we choose λ, such that

$$\left(\frac{\partial f}{\partial a}+\lambda\frac{\partial \phi}{\partial a}\right)=0 \quad i.e. \qquad \frac{\partial f}{\partial a}/\frac{\partial \phi}{\partial a}=-\lambda$$

Now out of the four variables a,b,c,d one variable can be eliminated between the equations $f=0$, $\phi=0$ and therefore three are independent, coefficients of db, dc, dd must be separately zero.

$$\therefore \quad \frac{\partial f}{\partial b}=-\lambda\frac{\partial \phi}{\partial b} \text{ etc.}$$

$$\therefore \quad \frac{\partial f/\partial a}{\partial \phi/\partial a}=\frac{\partial f/\partial b}{\partial \phi/\partial b}=\frac{\partial f/\partial c}{\partial \phi/\partial c}=\frac{\partial f/\partial d}{\partial \phi/\partial d}$$

Eliminating a,b,c,d from these ratios and $f=0$, $\phi=0$, the envelope can be found.

Example 5.7: Find the envelope of $lx+my+nz=p$ when

$$a^2l^2+b^2m^2+c^2n^2=p^2.$$

Solution. Refer last question,

here $f=lx+my+nz-p=0$ (1)

$$\phi=a^2l^2+b^2m^2+c^2n^2-p^2=0 \qquad (2)$$

$$\therefore \quad \frac{\partial f/\partial l}{\partial \phi/\partial l}=\frac{\partial f/\partial m}{\partial \phi/\partial m}=\frac{\partial f/\partial n}{\partial \phi/\partial n}=\frac{\partial f/\partial p}{\partial \phi/\partial p}$$

gives $\dfrac{x^2}{a^2l}+\dfrac{y^2}{b^2m}+\dfrac{z^2}{c^2n}=\dfrac{1}{p}$ $\quad\therefore\quad l=\dfrac{px}{a^2}$ etc. (3)

$\therefore$ The envelope is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ [Eliminating l,m,n,p from (2) and (3)].

Example 5.8: Find the envelope of a plane that forms with the rectangular coordinates planes a tetrahedron of constant volume $c^3/6$.

Solution. Let the plane be $x/\alpha+y/\beta+z/\gamma=1$ (1)

Also equations of three coordinate planes are

$$x = 0,\ y = 0,\ z = 0$$

Thus to obtained vertices of tetrahedron, we solve above four equations taken three at a time.

Equation (1) with $y = 0,\ z = 0$ gives $x = \alpha$.

$\therefore$ Point of intersection is $A(\alpha,0,0)$. Similarly we get $B(0,\beta,0)$, $C(0,0,\gamma)$ and $x = 0,\ y = 0,\ z = 0$ gives $O(0,0,0)$.

$\therefore$ Volume of tetrahedron OABC is given by

$$\frac{c^2}{6} = -\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 1 \\ 0 & \beta & 0 & 1 \\ 0 & 0 & \gamma & 1 \end{vmatrix}$$

or $\quad \frac{1}{6}c^3 = \frac{1}{6}\alpha\beta\gamma \quad$ or $\quad \alpha\beta\gamma = c^3 \qquad (2)$

For envelope $\dfrac{x/\alpha^2}{\beta\gamma} = \dfrac{y/\beta^2}{\gamma\alpha} = \dfrac{z/\gamma^2}{\alpha\beta}$ [Example 5.6]

or $\quad \dfrac{x}{\alpha} = \dfrac{y}{\beta} = \dfrac{z}{\gamma} = \dfrac{x/\alpha + y/\beta + z/\gamma}{1+1+1} = \dfrac{1}{3}$ [Using (1)]

Substituting in (2), the equation of the required envelope is

$$27xyz = c^3.$$

Example 5.9: A plane makes intercept a,b,c on the axes, so that $a^{-2} + b^{-2} + c^{-2} = k^{-2}$. So that its envelopes is a coincoid which has axes as equal conjugate diameters.

Solution. The equation of the plane is

$$x/\alpha + y/\beta + z/\gamma = 1 \qquad (1)$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{k^2} \qquad \text{(given)} \qquad (2)$$

$\therefore$ For the envelope $\dfrac{x/a^2}{2/a^3}=\dfrac{y/b^2}{2/b^3}=\dfrac{z/c^2}{2/c^3}$

$$\begin{vmatrix} 0 & \dfrac{yb^2(c^2-a^2)}{\beta} & \dfrac{zc^2(a^2-b^2)}{\gamma} \\ r^2 & \beta^2 & \gamma^2 \\ 1 & \dfrac{\beta^2}{b^2} & \dfrac{\gamma^2}{c^2} \end{vmatrix}=0$$

or $\dfrac{yb^2(c^2-a^2)}{\beta}\left(\gamma^2-\dfrac{r^2\gamma^2}{c^2}\right)+\dfrac{zc^2(a^2-b^2)}{\gamma}\left(\dfrac{r^2\beta^2}{b^2}-\beta^2\right)=0$

or $z\beta^3c^4(a^2-b^2)(r^2-b^2)=y\gamma^3b^4(c^2-a^2)(r^2-c^2)$

or $\dfrac{\beta^3(r^2-b^2)}{yb^4(c^2-a^2)}=\dfrac{\gamma^3(r^2-c^2)}{zc^4(a^2-b^2)}=\dfrac{\alpha^3(r^2-\alpha^2)}{xa^4(b^2-c^2)}$ (4)

Eliminating α,β,γ between (1) and (4), the equation of envelope is

$$\sum\frac{[xa(b^2-b^2)]^{2/3}}{(r^2-a^2)^{-1/3}}=0.$$

Example 5.10: Show that the edge of regression of the envelope of the plane

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1 \quad \text{is the cubic curve given by}$$

$$x=\frac{(a+\lambda)^3}{(c-a)(b-a)},\ y=\frac{(b+\lambda)^3}{(c-b)(a-b)},z=\frac{(c+\lambda)^3}{(a-c)(b-c)}.$$

Solution. For the edge of regression of

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1 \qquad (1)$$

Differentiating w.r.t. λ, we get

$$-\frac{x}{(a+\lambda)^2}-\frac{y}{(b+\lambda)^2}-\frac{z}{(c+\lambda)^2}=0 \qquad (2)$$

Again differentiating w.r.t. λ, we get

$$-\frac{2x}{(a+\lambda)^3}-\frac{2y}{(b+\lambda)^3}-\frac{2z}{(c+\lambda)^3}=0 \qquad (3)$$

Multiplying (2) by $2/(c+\lambda)$ and subtracting from (3),

$$\frac{2x}{(a+\lambda)^2}\left[\frac{1}{c+\lambda}-\frac{1}{a+\lambda}\right]+\frac{2y}{(b+\lambda)^2}\left[\frac{1}{c+\lambda}-\frac{1}{a+\lambda}\right]=0,$$

$\therefore$ $$\frac{x(b+\lambda)(c+\lambda)}{(a+\lambda)^2(b-c)}=\frac{y(c+\lambda)(a+\lambda)}{(b+\lambda)^2(c-a)}=\frac{z(a+\lambda)(b+\lambda)}{(c+\lambda)^2(a-b)} \qquad \text{[by symmetry]}$$

or $$\frac{x}{(a+\lambda)^3(b-c)}=\frac{y}{(b+\lambda)^3(c-a)}=\frac{z}{(c+\lambda)^3(a-b)}$$

[Dividing by $(a+\lambda)(b+\lambda)(c+\lambda)$]

or $$\frac{x/(a+\lambda)}{(a+\lambda)^2(b-c)}=\frac{y/(b+\lambda)}{(b+\lambda)^2(c-a)}=\frac{z/(a+\lambda)}{(c+\lambda)^2(a-b)}$$

$$=\frac{\sum x/(a+\lambda)}{(a+\lambda)^2(b-c)+(b+\lambda)^2(c-a)+(c+\lambda)^2(a-b)}$$

$$=\frac{1}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

$$=\frac{-1}{(b-c)(c-a)(a-b)}$$

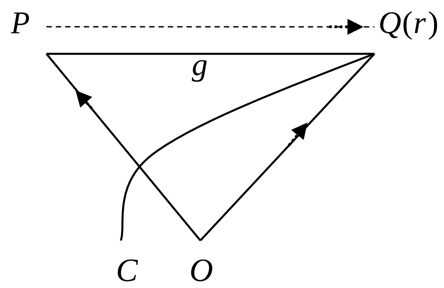
$$\therefore \qquad x=\frac{(a+\lambda)^3}{(c-a)(b-a)},\ y=\frac{(b+\lambda)^3}{(c-a)(a-b)},\ z=\frac{(c+\lambda)^3}{(a-c)(b-c)}.$$

5.3 Ruled Surface (Developable and skew)

Definition: A ruled surface is a surface which is generated by the motion of one parameter family of straight line and straight line itself is called its generating line, generating or ruling. Cones, Cylinders and coincoids are special form of ruled surfaces.

There are two distinct classes of ruled surfaces, namely those on which consecutive generators intersect and those on which generators do not intersect; these are called **developable and skew surfaces respectively**. Skew surfaces are also named scrolls.

To find equation of the ruled surfaces: If C is any curve on the ruled surface such that it meets each generator precisely once, then C is called a **base curve or directrix** (these are many in number) and the ruled surface is determined by any curve C and the direction of the generator at their point of meeting with the curve. Let $\mathbf{g}(u)$ be any unit vector along the generator at a curved point Q on C and $\mathbf{r}(u)$ the position vector of Q,



then **R** the position vector of general point P on ruled surface is given by

$$\mathbf{R}=\mathbf{r}+vg$$

where v is the parameter and determines directed distance along the generator from C.

5.3.1 To find weather the surface generated is developable or skew: Let g_1 and g_2

be two consecutive generators through $Q_1(\mathbf{r}_0)$ and $Q_1(\mathbf{r}_0 + \mathbf{t}\,ds)$ on the directrix C and let their direction be along the unit vector g and $g + g'\,ds$.

Let MN be the line of shortest distance between the generators g_1 and g_2. The shortest distance MN being perpendicular to both g_1 and g_2, will be parallel to the vector

$$(g + g'\,ds) \times g \;\; i.e. \;\; g' \times g$$

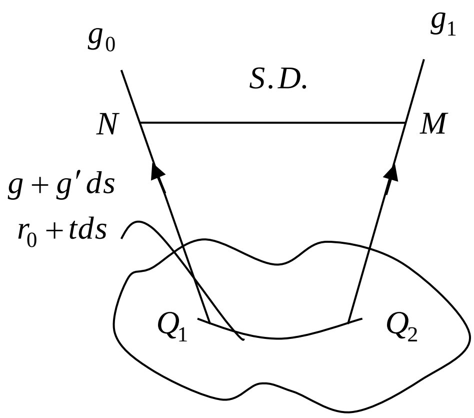
Also unit vector along MN

$$= \frac{g' \times g}{|g' \times g|}$$

Shortest distance MN=Projection of $Q_1 Q_2 = (\mathbf{t}\,ds)$ on MN

$$= \mathbf{t}\,ds.\frac{g' \times g}{|g' \times g|} = [\mathbf{t}, \mathbf{g}', \mathbf{g}]\frac{ds}{a} \qquad \text{where} \qquad |g' \times g| = a$$

We know that if consecutive generators intersect the surface is a developable and then the



S.D. is zero. Hence the necessary and sufficient condition that the ruled surface is developable is

$$[\mathbf{t}, \mathbf{g}', \mathbf{g}] = 0$$

If however the ruled surface is skew, the S.D. MN must not be zero *i.e.*

$$[\mathbf{t}, \mathbf{g}', \mathbf{g}] \neq 0$$

Example 5.11: To find the condition that the surface generated by $x = az + \alpha$ x, $y = bz + \beta$ is developable or skew, where a, b, α, β are function of **t**.

Solution. Let the two consecutive generators be

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z}{1}, \quad \frac{x-\alpha-\delta\alpha}{a+\delta a} = \frac{y-\beta-\delta\beta}{b+\delta b} = z$$

They will intersect if the shortest distance between above lines is zero *i.e.*

if $$\begin{vmatrix} \delta\alpha & \delta\beta & 0 \\ a+\delta a & b+\delta b & 1 \\ a & b & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} \delta\alpha & \delta\beta & 0 \\ \delta a & \delta b & 1 \\ a & b & 1 \end{vmatrix} = 0$$

or $\delta\alpha\,\delta b - \delta\beta\,\delta a = 0$

$\therefore$ $(\alpha' b' - \beta' a')dt^2 = 0$ where $a' = da/dt$ *etc.*

$\therefore$ They intersect *i.e.* the surface is developable if $\alpha' b' - \beta' a' = 0$, and they do not intersect, *i.e.* the surface is skew if $\alpha' b' - \beta' a' \neq 0$.

Alternate Method: Here $\mathbf{r} = (\alpha, \beta, 0)$

$\therefore$ $\mathbf{t} = \mathbf{r}' = (\alpha', \beta', 0)$

$\mathbf{g} = (a, b, 1)$, and $\mathbf{g}' = (a', b', 0)$

$\mathbf{g} \times \mathbf{g}' = (b' - a', a'b - ab')$

$[\mathbf{t}, \mathbf{g}', \mathbf{g}] = \mathbf{t}.(\mathbf{g}' \times \mathbf{g}) = \alpha' b' - \beta' a'$

Therefore the surface is developable if $\alpha' b' - \beta' a' = 0$ and the surface is skew if $\alpha' b' - \beta' a' \neq 0$.

Example 5.12: Show that the line given by $y = tx - t^3$, $z = t^3 y - t^6$ generate a developable surface.

Solution. Write the equation as

$$x = \frac{1}{t} y + t^2, \quad y = \frac{1}{t^3} z + t^3$$

$$\therefore \qquad x = \frac{z}{t^4} + 2t^2 \quad \text{and} \qquad y = \frac{1}{t^3} z + t^3$$

Refer (Example 5.11) above, we have

$$\alpha = 2t^2,\ a = \frac{1}{t^4},\ \beta = t^3,\ b = \frac{1}{t^3}$$

$$\therefore \qquad \alpha' b' - \beta' a' = 4t \times \frac{-3}{t^4} - 3t^2 \times \frac{-4}{t^5} = 0$$

Hence the surface is developable.

Example 5.13: Show that the line

$x = 3t^2 z + 2t(1 - 3t^4),\ y = -2tz + t^2(3 + 4t^2)$ generate a skew surface.

Solution. Refer (Example 5.11) above, we have

$$a = 3t^2,\ \alpha = 2t - 6t^5,\ b = -2t,\ \beta = 3t^2 + 4t^4$$

$$\therefore \qquad \alpha' b' - \beta' a' = (2 - 30t^4)(-2) - (6t + 16t^3) \times 6t = 0$$

$$= -4 + 60t^4 - 3t^2 - 96t^4.$$

5.3.2 To find the equation of the tangent plane to a ruled surface

When the equation to the surface is $\mathbf{r} = \mathbf{r}(u, v)$, the normal to the surface is $\mathbf{r}_1 \times \mathbf{r}_2$, the tangent plane at the point $\mathbf{r}$ is

$$(\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = 0 \text{ or } [\mathbf{R} - \mathbf{r}, \mathbf{r}_1, \mathbf{r}_2] = 0$$

Now let the ruled surface be generated by $\mathbf{r} = \mathbf{r}_0 + \mu \mathbf{g}$ where $\mathbf{r}_0$ and $\mathbf{g}$ are function of u and $\mathbf{r}$ is a function of u and μ. Let the suffix (1) and (2) denote the the differentiation w.r.t. u and μ respectively.

$$\mathbf{r}_1 = \mathbf{r}_0' + \mu \mathbf{g}', \mathbf{r}_2 = \mathbf{g}$$

and
$$\begin{bmatrix} \mathbf{r}_1 = \dfrac{\partial \mathbf{r}}{\partial u}, & \mathbf{r}_{21} = \dfrac{\partial \mathbf{r}}{\partial \mu} \\ \mathbf{r}_0' = \partial \mathbf{r}_0 / \partial \mu, & etc. \end{bmatrix}$$

Hence the equation to the tangent plane reduces to

$$[\mathbf{R} - \mathbf{r}, \mathbf{r}_0' + \mu \mathbf{g}', \mathbf{g}] = 0 \tag{1}$$

which is the required equation to the tangent plane.

Cor. 5.1: Cartesian form: Let us suppose

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \ \mathbf{r}_0 = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k},$$

$$\mathbf{g} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}, \ \mathbf{g}' = l'\mathbf{i} + m'\mathbf{j} + n'\mathbf{k}$$

Substituting these values in (1), we get

$$\begin{vmatrix} x-(\alpha+\mu l) & y-(\beta+\mu m) & z-(\gamma+\mu n) \\ l & m & n \\ \alpha'+\mu l' & \beta'+\mu m' & \gamma'+\mu n' \end{vmatrix} = 0$$

$$[\because \ \mathbf{r} = \mathbf{r}_0 + \mu \mathbf{g} = (\alpha+\mu l)\mathbf{i} + (\beta+\mu m)\mathbf{j} + (\gamma+\mu n)\mathbf{k}\ ] \tag{2}$$

Cor.5.2: If (ξ,η,ζ) be a point on the surface which is generated by straight lines $x = az + \alpha$, $y = bz + \beta$, where a, α, b, β are function of *t*.

$\because$ ξ,η,ζ can be regarded as the function of *t* and z given by

$$\xi = az + \alpha, \ \eta = bz + \beta, \ \zeta = z$$

The equation of the tangent plane at (t, z) is

$$\begin{vmatrix} x-\xi & y-\eta & z-\zeta \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial x}{\partial z} & \dfrac{\partial y}{\partial z} & \dfrac{\partial z}{\partial z} \end{vmatrix} = 0$$

or $$\begin{vmatrix} X-(az+\alpha) & Y-(bz+\beta) & Z-z \\ a'z+\alpha' & b'z+\beta' & 0 \\ a & b & 1 \end{vmatrix} = 0$$

Subtracting 'a' times column 3 from column one and 'b' times column 3 from two, we have

$$\begin{vmatrix} X-az-\alpha & Y-bz-\beta & Z-z \\ a'z+\alpha' & b'z+\beta' & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

or (i) $(X-az-\alpha)(b'z+\beta') - (Y-bz-\beta)(a'z+\alpha') = 0$

This is a plane passing through the line $X = az+\alpha$, $Y = bz+\beta$ which is generator through (t, z). Thus the tangent plane at any point of a ruled surface must contain the generator through the point.

Again if the surface is developable, *i.e.* $\alpha' b' - \beta' a' = 0$

or $$\frac{a'}{b'} = \frac{\alpha'}{\beta'} = \frac{a'z}{b'z} = \frac{a'z+\alpha'}{b'z+\beta'} = k \text{ (say)}$$

'k' being some function of t.

The equation of the tangent plane becomes

$$(X-az-\alpha) - k(Y-bz-\beta) = 0$$

which is independent of z and involve 't' only. Since 't' is given for a given generator, the tangent pane will be same at all points of generator.

If $\alpha' b' - \beta' a' \neq 0$, the tangent plane given by (1) will change if 't' is fixed and 'z' varies. Hence if the surface be a developable surface, the tangent plane is the same at all points of generator and contain only one parameter and if the surface be a skew, the tangent plane are different at different point of generator.

Note: From the above we conclude two important points.

(i) Tangent plane to a developable surface contains only one parameter.

(ii) If the ruled surface is developable, the tangent plane at all points of any generator must be same.

5.3.3 Developable Surface

An important example of one parameter family of surfaces is furnished by one parameter family of planes. "A surface generated by one parameter family of planes is called a developable surfaces or simply a developable."

If **N** denotes unit normal vector to the plane, the equation to such a family of planes is given by

$$\mathbf{r}.\mathbf{N}(u) = p(u)$$

where u is parameter and p is the length of the perpendicular from the origin to the plane.

(A) To prove that the envelope of a developable plane whose equation involves one parameter is a developable surface:

Let $\mathbf{V} = \mathbf{r}.\mathbf{N} - p = 0$ where **N** and p are functions of a single parameter u. Now by varying parameter u, a characteristics of this form of planes is given by

$$\text{V}=0,\ \dot{\mathbf{V}} = 0 \qquad \text{where} \qquad \dot{\mathbf{V}} = \mathbf{r}.\dot{\mathbf{N}} - \dot{p} \qquad (1)$$

dots denoting differentiation w.r.t. u.

Now since equation (1) represent planes, the characteristics are straight lines and therefore the envelope is a ruled surface.

Again two consecutive characteristics are given by

$$V=0,\ \dot{\mathbf{V}}=0;\ \ \mathbf{V}+\dot{\mathbf{V}}\delta u=0,\ \ \dot{\mathbf{V}}+\ddot{\mathbf{V}}\delta u=0$$

and these clearly lie in the plane $\dot{\mathbf{V}}+\ddot{\mathbf{V}}\delta u=0$ and therefore intersect. Hence the envelope is a developable surface.

(B) To prove that the osculating plane of the edge of regression at any point P is the tangent plane to the developable at P.

The edge of regression of the envelope is given by

$$V=0,\ \dot{\mathbf{V}}=0;\ \ddot{\mathbf{V}}=0 \ i.e.$$

$$\mathbf{r}.\mathbf{N}-p=0 \quad (1) \qquad \mathbf{r}.\dot{\mathbf{N}}-\dot{p}=0 \quad (2)$$

$$\mathbf{r}.\ddot{\mathbf{N}}-\ddot{p}=0 \quad (3)$$

Where $\mathbf{r}$ is regarded as function of u.

Differentiating (1), we obtained

$$\dot{\mathbf{r}}.\mathbf{N}+\mathbf{r}.\dot{\mathbf{N}}=\dot{p}$$

or $\dot{\mathbf{r}}.\mathbf{N}=0$ [using (2] (4)

Differentiating (2),

$$\dot{\mathbf{r}}.\dot{\mathbf{N}}+\mathbf{r}.\ddot{\mathbf{N}}=\ddot{p}$$

or $\dot{\mathbf{r}}.\dot{\mathbf{N}}=0$ [using (3)] (5)

Differentiating (4)

$$\ddot{\mathbf{r}}.\mathbf{N}+\dot{\mathbf{r}}.\dot{\mathbf{N}}=0$$

or $\ddot{\mathbf{r}}.\mathbf{N}=0$ [using (5)] (6)

From equation (4) and (6) we notice that both $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ are perpendicular to **N**, it follows that $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$ is perpendicular to the osculating plane and **N** being perpendicular to the plane $\mathbf{r}.\mathbf{N} - p = 0$ and hence the osculating plane of edge of regression at any point is the tangent plane to the developable at the same point.

(C) To find the condition that $z = f(x, y)$ may represent a developable surface.

If (x, y, z) is a point on it, the equation of the tangent plane at this point is

$$(X - x)\frac{\partial z}{\partial x} + (Y - y)\frac{\partial z}{\partial y} + (Z - z) = 0$$

or
$$X\frac{\partial z}{\partial x} + Y\frac{\partial z}{\partial y} - Z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 0 \qquad \text{(say)}$$

or
$$pX + qY - Z = px + qy - z = \phi \qquad \text{where } \left[p = Y\frac{\partial z}{\partial x},\ q = Y\frac{\partial z}{\partial y}\right]$$

If $z = f(x, y)$ is developable surface then tangent plane to it should involve only one parameter say u .

$$\therefore \qquad p = f_1(u),\ q = f_2(u),\ \phi = f_3(u)$$

By eliminating u , we can express p and ϕ as function of q. By differential calculus we know that if p is a function of q.

$$\frac{\partial(p, q)}{\partial(x, y)} = 0 \qquad \text{[since } p, q \text{ are functions of } x \text{ and } y]$$

or
$$\begin{vmatrix} \dfrac{\partial p}{\partial x} & \dfrac{\partial p}{\partial y} \\ \dfrac{\partial q}{\partial x} & \dfrac{\partial q}{\partial y} \end{vmatrix} = 0 \qquad rt - s^2 = 0$$

where $\left[r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, \, s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}, \, t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} \right]$

Also when ϕ as function of q, we have $\frac{\partial(\phi, q)}{\partial(x, y)} = 0$

or $$\begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} rx + sy & sx + ty \\ s & t \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} r & s \\ s & t \end{vmatrix} = 0 \quad \Rightarrow \quad rt - s^2 = 0, \text{ if } \phi \text{ as function of q.}$$

Hence $rt - s^2 = 0$ is the necessary and sufficient x condition for the surface $z = f(x, y)$ to be developable.

5.3.4 Developable Associated with Space Curves

Since the equation to their principal planes namely osculating plane, normal plane and rectifying plane contains only a single parameter which is usually taken to be the arc length s , hence their envalopes and developable surfaces and they are called osculating developable or tangential developable , polar developable and rectifying developable. Also the generators of the polar developable and rectifying developable are called polar lines and rectifying lines respectively.

(A) To prove that the curve itself is the edge of regression of the osculating developable.

At any point $\mathbf{r}$ on the curve $\mathbf{r} = \mathbf{r}(s)$ the equation of the osculating plane is

$$(\mathbf{R} - \mathbf{r}).\mathbf{b} = 0 \qquad \text{[where } \mathbf{r} \text{ and b are function of } s.] \qquad (1)$$

Differentiating (1) w.r.t. s, we get

$$-\mathbf{t}.\mathbf{b} - (\mathbf{R} - \mathbf{r}).\tau \mathbf{n} = 0$$

or $$(\mathbf{R} - \mathbf{r}).\,\mathbf{n} = 0 \qquad (\because \mathbf{t}.\,\mathbf{b} = 0, \tau \neq 0) \qquad (2)$$

The characteristic is given as the intersection of (1) and (2) *i.e.* at the intersection of osculating plane and the rectifying plane and is therefore the tangent to the curve at **r**.

For edge of regression differentiating (2) ahain w.r.t. '*s*'

$$-\mathbf{t}.\mathbf{n}+(\mathbf{R}-\mathbf{r})\bullet(\tau\,\mathbf{b}-\kappa\,\mathbf{t})=0$$

or $$(\mathbf{R}-\mathbf{r})\bullet(\tau\,\mathbf{b}-\kappa\,\mathbf{t})=0 \qquad (\because\ \mathbf{t}.\,\mathbf{n}=0) \qquad (3)$$

or $$(\mathbf{R}-\mathbf{r}).\mathbf{t}=0 \qquad \text{[using (1)]} \qquad (3')$$

The edge of regression being given by (1), (2), and $(3')$ is given by

$$(\mathbf{R}-\mathbf{r})=0 \quad \Rightarrow \mathbf{R}=\mathbf{r}$$

Thus the points of curve coincide with the point of the edge of regression and the space curve itself is the edge of regression.

(B) To prove that the edge of regression of the polar developable (*i.e.* envelope of the normal planes) is the locus of the centre of osculating spheres.

At any point **r** on the curve $\mathbf{r}=\mathbf{r}(s)$ the equation of the normal plane is

$$[\mathbf{R}-\mathbf{r}]\cdot\mathbf{t}=0 \qquad (1)$$

Differentiating w.r.t. *s*, we get

$$(\mathbf{R}-\mathbf{r})\cdot\kappa\mathbf{n}-\mathbf{t}\cdot\mathbf{t}=0$$

or $$(\mathbf{R}-\mathbf{r})\cdot\mathbf{n}-\rho=0 \qquad [\because\ \mathbf{t}\cdot\mathbf{t}=1] \qquad (2)$$

or $$(\mathbf{R}-\mathbf{r}-\rho\mathbf{n})\cdot\mathbf{n}=0 \qquad [\because\ \mathbf{n}\cdot\mathbf{n}=1] \qquad (2')$$

Characteristics is given as the intersection of (1) and $(2')$ and is clearly a straight line parallel to **b** and passing through the centre of circular curvature.

For edge of regression, differentiating (2), w.r.t. '*s*' we get

$$-\mathbf{t}\cdot\mathbf{n}+(\mathbf{R}-\mathbf{r})\cdot(\tau\,\mathbf{b}-\kappa\,\mathbf{t})=\rho'$$

or $$(\mathbf{R}-\mathbf{r})\cdot\tau\,\mathbf{b}=\rho' \qquad [\because\ \mathbf{t}\cdot\mathbf{n}=1 \text{ and using (1)}]$$

or $$(\mathbf{R}-\mathbf{r})\cdot\mathbf{b}=\sigma\rho'$$

edge of regression of is given as the intersection of (1), ($2'$) and (3), equation (1) shows that $(\mathbf{R}-\mathbf{r})$ is the normal plane. Equation (2) implies component of $(\mathbf{R}-\mathbf{r})$ along $\mathbf{n}$ is ρ .And equation (3) implies component of $(\mathbf{R}-\mathbf{r})$ along $\mathbf{b}$ is $\sigma\rho'$. Hence equation of edge of regression is given by

$$\mathbf{R}-\mathbf{r}=\rho\mathbf{n}+\sigma\rho'\mathbf{b} \qquad \Rightarrow \mathbf{R}=\mathbf{r}+\rho\mathbf{n}+\sigma\rho'\mathbf{b}$$

Thus $\mathbf{R}$ coincides with the centre of spherical curvature. Thus the edge of regression of the polar developable is the locus of the centre of spherical curvature (osculating spheres).

The tangent to the locus are the polar lines, which are the generators of the developable.

(C) To prove that the edge of regression of the rectifying developable has equation

$$\mathbf{R}=\mathbf{r}+\kappa\frac{(\tau\,\mathbf{t}+\kappa\mathbf{b})}{\kappa'\tau-\kappa\,\tau}.$$

The equation of the rectifying plane at any point $\mathbf{r}$ is given by

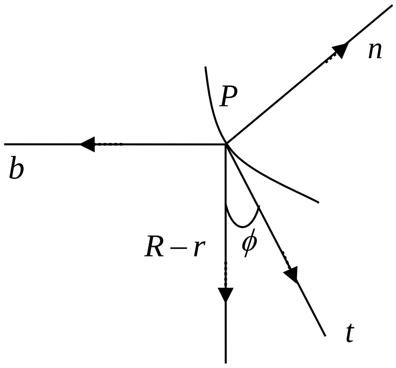
$$(\mathbf{R}-\mathbf{r})\cdot\mathbf{n}=0 \qquad (1)$$

where $\mathbf{r}$ and $\mathbf{n}$ are function of s. Differentiating (1) w.r.t. 's', we get

$$(\mathbf{R}-\mathbf{r})\cdot(\tau\,\mathbf{b}-\kappa\,\mathbf{t})=0 \qquad [\,\mathbf{r}'.\mathbf{n}=\mathbf{t}.\mathbf{n}=0\,] \qquad (2)$$

The characteristic (rectifying lines) is given as intersection of (1) and (2) and is clearly a straight line perpendicular to both $\mathbf{n}$ and $(\tau\,\mathbf{b}-\kappa\,\mathbf{t})$ and therefore it is parallel to the direction $(\tau\,\mathbf{t}+\kappa\,\mathbf{b})$. Hence it is included to the tangent at an angle ϕ, such that

$$\tan\phi=\frac{\kappa}{\tau} \qquad (3)$$



To find the edge of regression, differentiating (2) w.r.t. s , we have

$$(\mathbf{R}-\mathbf{r})\cdot(\tau'\mathbf{b}-\kappa'\mathbf{t})+\kappa=0 \qquad (4)$$

Again since the rectifying line is parallel to the direction $(\tau\,\mathbf{t}+\kappa\mathbf{b})$, the point $\mathbf{R}$ on the edge ofd regression is given by

$$\mathbf{R}-\mathbf{r}=\mu(\tau\,\mathbf{t}+\kappa\mathbf{b}) \qquad (5)$$

where μ is some scalar.

Substituting the value of $(\mathbf{R}-\mathbf{r})$ from (5), in (4), we get

$$\mu(\tau\,\mathbf{t}+\kappa\mathbf{b})(\tau'\mathbf{b}-\kappa'\mathbf{t})+\kappa=0$$

or $$\mu(\kappa\tau'-\tau\kappa')+\kappa=0$$

or $$\mu=\kappa/(\tau\kappa'-\kappa\tau') \qquad (6)$$

Hence from (6) and (5), we have

$$\mathbf{R}-\mathbf{r}=\kappa\frac{(\tau\,\mathbf{t}+\kappa\mathbf{b})}{\kappa'\tau-\kappa\tau} \quad \text{or} \quad \mathbf{R}=\mathbf{r}+\kappa\frac{(\tau\,\mathbf{t}+\kappa\mathbf{b})}{\kappa'\tau-\kappa\tau}.$$

Which is the required equation of edge of regression.

The term '**rectifying**' used for this developable is justified, when the surface is developed into a plane by unfolding about consecutive generators, the original curve becomes a straight line.

We also know that if the curve is a helix κ/τ is constant and from equation (3) the angle ϕ is equal to the angle β. Thus the rectifying lines are generators of the cylinder on which the helix is drawn, and rectifying developable is cylinder itself.

5.3.5 Characterization of a developable surface

Developable surface: The surfaces, for which the Gaussian curvature K is zero, are called developable surfaces. Hence the surface will be developable if and only if

$$LN - M^2 = 0 \qquad (1)$$

Theorem 5.1: The necessary and sufficient condition for a surface to be developable surface is that its Gaussian curvature should be zero.

Proof: Necessary condition

Let the surface be developable, to prove that the Gaussian curvature K is zero. For a developable surface to be a cylinder or cone, the Gaussian curvature K=0; and if these be excluded, the general equation of developable may be written

$$\mathbf{R} = \mathbf{r}(s) + v\,\mathbf{t} = \mathbf{r} + v\,\mathbf{t} \qquad (2)$$

If the suffix 1 and 2 denote partial differentiation w.r.t. s and v respectively, we have

$$\mathbf{R}_1 = \mathbf{t} + \kappa v\mathbf{n}, \quad \mathbf{R}_2 = \mathbf{t}, \quad \mathbf{R}_1 \times \mathbf{R}_2 = -\kappa v\mathbf{b}$$

$$\mathbf{R}_{11} = \kappa\mathbf{n} + \kappa v(\tau\mathbf{b} - \kappa\mathbf{t}) + \kappa' v\mathbf{n}, \quad \mathbf{R}_{12} = \mathbf{R}_{21} = \kappa\mathbf{n}, \ \mathbf{R}_{22} = 0$$

$$\mathbf{N} = \frac{\mathbf{R}_1 \times \mathbf{R}_2}{\mathbf{H}} = \frac{-\kappa v\mathbf{b}}{\mathbf{H}}$$

$$\therefore \ L = \mathbf{N}.\mathbf{R}_{11} = \frac{-v^2\kappa^2\tau}{\mathbf{H}},\ M == \mathbf{N}.\mathbf{R}_{12} = 0,\ N = \mathbf{N}.\mathbf{R}_{22} = 0$$

Hence Gaussian curvature $K = \dfrac{LN - M^2}{EG - F^2} = 0$ since $M=N=0$.

Sufficient condition

If the Gaussian curvature K=0 for the given surface $\mathbf{r} = \mathbf{r}(u, v)$ to prove that the given surface is developable. We have

$$K = 0 \qquad \text{or} \qquad (LN - M^2)/H^2 = 0$$

or $\quad LN - M^2 = 0$

or $\quad (\mathbf{r}_1 \cdot \mathbf{N}_1)(\mathbf{r}_2 \cdot \mathbf{N}_2) - (\mathbf{r}_1 \cdot \mathbf{N}_2)(\mathbf{r}_2 \cdot \mathbf{N}_1) = 0$

or $\quad (\mathbf{r}_1 \times \mathbf{r}_2)(\mathbf{N}_1 \times \mathbf{N}_2) = 0$ [by vector identity due to Lagrange]

or $\quad H\,\mathbf{N}.(\mathbf{N}_1 \times \mathbf{N}_2) = 0$ or $H[\mathbf{N}, \mathbf{N}_1, \mathbf{N}_2] = 0$

or $\quad [\mathbf{N}, \mathbf{N}_1, \mathbf{N}_2] = 0$ [since $H \neq 0$] (1)

In view of relation (1) above, there arise following four possibilities:

(iii) **N**, $\mathbf{N_1}$ and $\mathbf{N_2}$ are coplanar.

(iv) $\mathbf{N}_1 = 0$ or (iii) $\mathbf{N}_2 = 0$

(v) $\mathbf{N}_1 = \mu \mathbf{N}_2$

Now we shall discuss these possibilities:

(i) **N**, $\mathbf{N_1}$ and $\mathbf{N_2}$ are coplanar.

Since **N** is a vector of unit length (*i.e.* constant length) and so $\mathbf{N} \bullet \mathbf{N}_1 = 0$ and $\mathbf{N} \bullet \mathbf{N}_2 = 0$. These relations together imply that **N** is perpendicular to both $\mathbf{N_1}$ and $\mathbf{N_2}$. Hence **N**, $\mathbf{N_1}$ and $\mathbf{N_2}$ can not be coplanar.

(ii) $\mathbf{N}_1 = 0$. The equation of the tangent plane at any point $\mathbf{r}(u,v)$ on the surface $\mathbf{r} = \mathbf{r}(u,v)$ is

$$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{N} = 0$$

We have $\frac{\partial}{\partial u}\{(\mathbf{R}-\mathbf{r})\cdot\mathbf{N}\} = -\mathbf{r}_1\cdot\mathbf{N} + (\mathbf{R}-\mathbf{r})\cdot\mathbf{N}_1 = 0$ [since $\mathbf{r}_1\cdot\mathbf{N}=0,\ \mathbf{N}_1=0$]

Thus $(\mathbf{R}-\mathbf{r})\cdot\mathbf{N}$ is independent of parameter u and hence the equation of the tangent plane contains only one parameter v. Therefore, the surface is envelope of a single parameter family of planes and hence it is a developable surface.

(iii) $\mathbf{N}_2 = 0$. Proceeding similarly as in (ii) above, the equation of the tangent plane contains only one parameter u. Hence in this case also the surface is developable.

(iv) $\mathbf{N}_1 = \mu\mathbf{N}_2$. Let us change the parameters u, v to u', v' by the transformation

$u = u' + v', \quad v = u' - \mu v'$, we obtained

$$\mathbf{N}_1 = \frac{\partial \mathbf{N}}{\partial u'} = \frac{\partial \mathbf{N}}{\partial u}.\frac{\partial u}{\partial u'} + \frac{\partial \mathbf{N}}{\partial v}.\frac{\partial v}{\partial v'} = \mathbf{N}_1\cdot 1 + \mathbf{N}_2\cdot 1 = \mathbf{N}_1 + \mathbf{N}_2$$

$$\mathbf{N}_2' = \frac{\partial \mathbf{N}}{\partial v'} = \frac{\partial \mathbf{N}}{\partial u}.\frac{\partial u}{\partial v'} + \frac{\partial \mathbf{N}}{\partial v}.\frac{\partial v}{\partial v'} = \mathbf{N}_1\cdot 1 + \mathbf{N}_2\cdot(-\mu) = \mathbf{N}_1 - \mu\mathbf{N}_2 = 0 \quad [\because \mathbf{N}_1 = \mu\mathbf{N}_2]$$

These relations shows that **N**, the surface normal depends on only one parameter. Hence by case (iii) above, the surface is developable. Thus the theorem is completely proved.

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CHAPTER-6

ENVELOPE, DEVELOPABLE SURFACES (II)

Objectives: In continuation of the fifth chapter in the current chapter the students will learn about property of lines of curvatures on developable surfaces, Condition that a curve on a surface be a line of curvature is that the surface normal along the curve is developable, Application of developable surfaces with examples.

6.1 Property of lines of curvatures on developable

Monge's Theorem: A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normal along the curve is developable.

Proof: Suppose $\mathbf{r} = \mathbf{r}(s)$ is the equation of a curve on the surface $\mathbf{r} = \mathbf{r}(u,v)$. Consider the unit vector $\mathbf{N}$ along the normal to the surface $\mathbf{r} = \mathbf{r}(u,v)$ at point $P(\mathbf{r})$ on the curve $\mathbf{r} = \mathbf{r}(s)$. Then $\mathbf{r}$ is a function of s and also $\mathbf{N}$ may be regarded as a function of s. The position vector $\mathbf{R}$ of a point Q on the such a normal is represented by

$$\mathbf{R} = \mathbf{r}(s) + \mu\mathbf{N}(s) = \mathbf{r} + \mu\mathbf{N} \tag{1}$$

where s and μ are two parameters. Thus equation (1) is the equation of the surface formed by the normal to the surface $\mathbf{r} = \mathbf{r}(u,v)$ along the given curve.

Let suffix (1) and (2) be used to denote partial differentiation w.r.t. s and μ respectively.

From (1) we have

$$\mathbf{R}_1 = \mathbf{r}' + \mu\mathbf{N}' = \mathbf{t} + \mu\mathbf{N}' \qquad \left[\text{where, } \mathbf{r}' = \frac{d\mathbf{r}}{ds}, \mathbf{N}' = \frac{d\mathbf{N}}{ds}\right]$$

$$\mathbf{R}_2 = \mathbf{N} \tag{2}$$

From (2) $\mathbf{R}_{21} = \mathbf{N}' = \mathbf{R}_{12};\ \mathbf{R}_{22} = 0$

If M and N are fundamental coefficients of second order of surface (1), we have

$$HM = [\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_{12}]$$

and $$HN = [\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_{22}]$$

$$HM = [\mathbf{t} + \mu \mathbf{N}', \mathbf{N}, \mathbf{N}'] = [\mathbf{t}, \mathbf{N}, \mathbf{N}'] \qquad \text{since } [\mathbf{N}', \mathbf{N}, \mathbf{N}'] = 0 \qquad (3)$$

and $$HN = [\mathbf{t} + \mu \mathbf{N}', \mathbf{N}, 0] = 0$$

i.e. $HN = 0$ Implies N=0, as $\mathbf{H} \neq 0$

Hence the Gaussian curvature K of (1) is given by

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{M^2}{H^2} \qquad \text{(since } N\text{=0)} \qquad (4)$$

Now we know that a surface is developable if and only if its Gaussian curvature K is zero.

i.e. if and only if $-\dfrac{M^2}{H^2}$ is zero

i.e. if and only if M is zero.

i.e. if and only if $[\mathbf{t}, \mathbf{N}, \mathbf{N}']$ is zero [using (3) and $\mathbf{H} \neq 0$] .

Hence we are to prove only

$$[\mathbf{t}, \mathbf{N}, \mathbf{N}'] = 0$$

If and only if the curve $\mathbf{r} = \mathbf{r}(s)$ is a line of curvature on the surface $\mathbf{r} = \mathbf{r}(u, v)$.

Necessary condition: If the condition

$$[\mathbf{t}, \mathbf{N}, \mathbf{N}'] = 0$$

hold, then to prove that the curve $\mathbf{r} = \mathbf{r}(s)$ is a line of curvature on the surface

$$\mathbf{r} = \mathbf{r}(u, v)$$

Now $$[\mathbf{t}, \mathbf{N}, \mathbf{N}'] = 0$$

i.e. $[\mathbf{t}, \mathbf{N}', \mathbf{N}] = 0$ *i.e.* $(\mathbf{t} \times \mathbf{N}') \cdot \mathbf{N} = 0$ (5)

We know that $\mathbf{N}'$ is perpendicular to **N**, therefore $\mathbf{N}'$ lies in the tangent plane $\mathbf{r} = \mathbf{r}(u, v)$ and so $(\mathbf{t} \times \mathbf{N}')$ is parallel to the unit surface normal vector **N**. Thus in order equation (5) hold $(\mathbf{t} \times \mathbf{N}')$ must be zero since $\mathbf{N} \neq 0$.

$\therefore$ equation (5) implies $(\mathbf{t} \times \mathbf{N}') = 0$

$\Rightarrow$ $\mathbf{N}' = -\kappa \mathbf{t}$ for some function κ.

$$\Rightarrow \quad \frac{d\mathbf{N}}{ds} = -\kappa \frac{d\mathbf{r}}{ds} \qquad \Rightarrow \quad d\mathbf{N} + \kappa\, d\mathbf{r} = 0$$

Which is Rodrigue's formula characteristics only of lines of curvature and therefore the curve $\mathbf{r} = \mathbf{r}(s)$ is a line of curvature on the surface $\mathbf{r} = \mathbf{r}(u, v)$.

Sufficient condition: Conversely if the given curve $\mathbf{r} = \mathbf{r}(s)$ is a line of curvature on the surface $\mathbf{r} = \mathbf{r}(u, v)$, then we have

$$d\mathbf{N} + \kappa\, d\mathbf{r} = 0 \qquad \text{[Rodrigue's formula]}$$

$$\Rightarrow \quad \frac{d\mathbf{N}}{ds} + \kappa \frac{d\mathbf{r}}{ds} = 0 \qquad \Rightarrow \quad \mathbf{N}' + \kappa \mathbf{t} = 0$$

$$\Rightarrow \quad \mathbf{N}' = -\kappa \mathbf{t}$$

Hence $[\mathbf{t}, \mathbf{N}, \mathbf{N}'] = [\mathbf{t}, \mathbf{N}, -\kappa \mathbf{t}] = 0$

i.e. $[\mathbf{t}, \mathbf{N}, \mathbf{N}'] = 0$

So that the condition of developability is satisfied.

Exercise 6.1: Prove that surface $xy = (z - c)^2$ is developable.

Solution. $(z - c)^2 = xy$

$$(z-c)=\sqrt{xy} \qquad \text{or} \qquad (z=c+\sqrt{xy}$$

$$p=\frac{\partial z}{\partial x}=\frac{1}{2}\sqrt{\frac{y}{x}}, \qquad q=\frac{\partial z}{\partial y}=\frac{1}{2}\sqrt{\frac{x}{y}}$$

$$r=\frac{\partial^2 z}{\partial x^2}=-\frac{1}{4}y^{1/2}x^{-3/2}, \qquad t=\frac{\partial^2 z}{\partial y^2}=-\frac{1}{4}x^{1/2}y^{-3/2}$$

and $$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{4}x^{-1/2}y^{-1/2}$$

$$\therefore \qquad rt-s^2=\frac{1}{16}x^{-1}y^{-1}-\frac{1}{16}x^{-1}y^{-1}=0$$

Hence the surface is developable.

Exercise 6.2: By considering the volume of $rt-s^2$, determine, if the surface $xyz=a^3$ is developable.

Solution. Here $z=\dfrac{a^3}{xy}$

$$p=\frac{\partial z}{\partial x}=-\frac{a^3}{x^2y}, \qquad q=\frac{\partial z}{\partial y}=-\frac{a^3}{xy^2}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{2a^3}{x^3y}, \qquad t=\frac{2a^3}{xy^3}, \qquad s=\frac{\partial^2 z}{\partial x\partial y}=\frac{2a^3}{x^2y^2}$$

Now $$rt-s^2=\frac{4a^6}{x^4y^4}-\frac{a^6}{x^4y^4}=\frac{3a^6}{x^4y^4}\neq 0$$

Hence the surface is not developable.

Exercise 6.3: Prove that $z=y\sin x$ is a ruled surface.

Solution. $p=\dfrac{\partial z}{\partial x}=y\cos x$, $\qquad q=\dfrac{\partial z}{\partial y}=\sin x$

$$r = \frac{\partial^2 z}{\partial x^2} = -y\sin x \quad , \quad s = \frac{\partial^2 z}{\partial x \partial y} = \cos x \quad , \quad t = \frac{\partial^2 z}{\partial y^2} = 0$$

$\therefore \qquad rt - s^2 = -\cos^2 x \qquad (1)$

which is finite. Hence the given surface is ruled surface.

If $x = (2n+1)\pi/2$, $rt - s^2 = 0$ from (1), and surface is developable, where n is any integer.

Exercise 6.4: Prove that the equations of developable surface are tangent to a curve.

Solution. Let $x = az + \alpha$, $y = bz + \beta$

$$x = (a + a\delta u)z + \alpha + \alpha\delta u\ ,\ \ y = (b + b\delta u)z + \beta + \beta\delta u$$

be the equations of the generators of a developable surface. Their point of intersection is given by

$$x = \alpha - \frac{a\dot{\alpha}}{\dot{a}}\ , \quad y = \beta - \frac{b\dot{\beta}}{\dot{b}}\ ,\ z = -\frac{\dot{\alpha}}{\dot{a}} = -\frac{\dot{\beta}}{\dot{b}}$$

i.e. the coordinates contain only one parameter 'u' and so the locus of the points of intersection of consecutive generators of a developable is curve.

Note: This curve is called edge of regression.

Now by differentiating, we have

Similarly $y = b\ddot{z}$.

$\therefore$ The equation to the tangent at (ξ, η, ζ) a point on the surface is

$$\frac{x-\xi}{\dot{x}} = \frac{y-\eta}{\dot{y}} = \frac{z-\zeta}{\dot{z}} \qquad i.e. \qquad \frac{x-\xi}{a\dot{z}} = \frac{y-\eta}{b\dot{z}} = \frac{z-\zeta}{\dot{z}}$$

or $\quad x = az - a\zeta + \xi = az + \alpha$

$y = bz - b\zeta + \eta = bz + \beta \qquad (1)$

Since $\xi = a\zeta + \alpha$, and $\eta = b\xi + \beta$

Equations (1) represent the generator. Hence the theorem is proved.

Exercise 6.5: Find the tangent to the edge of regression of the developable

$$y = tx - t^2,\ z = t^3 y - t^6.$$

Solution. We know that the edge of the developable surface is tangent to the edge of regression refer exercise 6.4 above *i.e.*, the point of intersection of the two consecutive generators of a developable surfaces is a point on the edge of regression.

The two consecutive generators are

$$y = tx - t^2,\ z = t^3 y - t^6 \tag{1}$$

$$y = x(t+\delta t) - (t+\delta t)^3, \quad z = (t+\delta t)^3 y - (t+\delta t)^6 \tag{2}$$

Solving $x\delta t - 3t^2 \delta t = 0$, neglecting higher powers of δt.

$$x = 3t^2 \tag{3}$$

Now from second of (1) and (2),

$$3t^2 \delta t . y - 6t^5 . \delta t = 0 \quad \text{or} \quad y = 2t^3, \quad z = t^3 . 2t^2 - t^6 = t^6$$

$\therefore$ $x = 3t^2,\ y = 2t^3,\ z = t^6$ may be taken to represent the edge of regression.

Exercise 6.6: Find the equation to a developable surface which has $x = 6t,\ y = 3t^2,\ z = 2t^3$ for edge of regression.

Solution. The equation of edge of regression is

$$\mathbf{r} = (6t, 3t^2, 2t^3) \tag{1}$$

$$\therefore \qquad \dot{\mathbf{r}} = d\mathbf{r}/dt = (6, 6t, 6t^2)$$

We know that $\dot{\mathbf{r}}$ is the tangent vector to the curve $\mathbf{r} = \mathbf{r}(t)$ at the point $\mathbf{r}$.

The developable is a surface generated by the tangents to the edge of the regression (1). Let $\mathbf{R}$ be the position vector of a current point (x, y, z) on the developable, the equation of tangent at the point $\mathbf{r}$ is

$$\mathbf{R} = \mathbf{r} + \mu\dot{\mathbf{r}}$$

or $$(x, y, z) = (6t, 3t^2, 2t^3) + \mu(6, 6t, 6t^2)$$

or $$(x-6t, y-3t^2, z-2t^3) = \mu(6, 6t, 6t^2)$$

Hence, $x - 6t = 6\mu, y - 3t^2 = 6t\mu,\ z - 2t^3 = 6t^2\mu$

or $$\frac{x-6t}{6} = \frac{y-3t^2}{6t} = \frac{z-2t^3}{6t^2} = \mu$$

or $$\frac{x-6t}{1} = \frac{y-3t^2}{t} = \frac{z-2t^3}{t^2} = k \text{ (say)}$$

or $$x = k + 6t,\ y = 3t^2 + kt, z = 2t^3 + kt^2$$

and $xt - y = 3t^2$ and $ty - z = t^3$

$\therefore \quad xt - y = 3t^2 = 3(ty - z) \qquad \therefore \quad xt^2 - 4ty + 3z = 0$

Also $3t^2 - xt + y = 0$

whence, $$\frac{t^2}{-4y^2 + 3zx} = \frac{t}{xy - 9z} = \frac{i}{-x^2 + 12y}$$

$\therefore \quad (xy - 9t^2) = (3xz - 4y^2)(12y - x^2)$ is the required equation.

Exercise 6.7: Find equations to the developable surface which has the helix

$x = a\cos u,\ y = a\sin u,\ z = cu$ for its edge of regression.

Solution. The equation of regression is

$$\mathbf{r} = (a\cos u, a\sin u, cu) \qquad (1)$$

$$\therefore \qquad \mathbf{r} = d\mathbf{r}/du = (-a\sin u, a\cos u, c)$$

We know that the tangent vector to the curve $\mathbf{r} = \mathbf{r}(u)$ at the point. The required developable is a surface generated by the tangents to the edge of regression (1). Let **R** be the position vector of a current point (x, y, z) on the developable, then

$$\mathbf{R} = \mathbf{r} + \mu\dot{\mathbf{r}}$$

or $$(x, y, z) = (a\cos u, a\sin u, cu) + \mu(-a\sin u, a\cos u, c)$$

or $$(x, y, z) = (a\cos u - a\mu\sin u,\ a\sin u + a\mu\cos u, cu + c\mu)$$

$$\therefore \quad x = (a\cos u - a\mu\sin u) \quad , \ y = (a\sin u + a\mu\cos u) \qquad z = (cu + c\mu)$$

are the equations of developable surface, where u and μ are two parameters.

Exercise 6.8: Show that a developable surface can be found to pass through two given curves.

Solution. The equation to any plane contain 3 parameters, if it touches the first curve, one parameter is eliminated and if it touches second , one more parameter goes.

Hence only one parameter is left and its envelope is a developable surface.

Exercise 6.9: Find the equation of the developable surface which pass through the curves

$$z = 0,\ y^2 = 4ax; x = 0, y^2 = 4bz.$$

Solution. Any line tangent to $z = 0,\ y^2 = 4ax$ is

$$y = mx + a/m,\ z = 0$$

$\therefore$ any plane touching the first parabola is

$$y - mx - a/m + \lambda z = 0 \qquad (1)$$

Now the plane (1) will touch the curve

$$x = 0, y^2 = 4bz \qquad (2)$$

If $\left(-\lambda z + \frac{a}{m}\right)^2 - 4bz = 0$ has coincident roots.

i.e. $\lambda^2 z^2 - \left(4b + \frac{2a\lambda}{m}\right)z + \frac{a^2}{m^2} = 0$ has equal roots.

$\therefore \quad \left(4b + \frac{2a\lambda}{m}\right)z - 4\lambda^2 \frac{a^2}{m^2} = 0 \qquad [B^2 - 4AC = 0]$

i.e. $b = -\lambda a/m$ or $\lambda = -bm/a$

$\therefore$ The equation of the plane touching the two curves is

$$f(m) = y - mx - \frac{a}{m} - \frac{bm}{a}z = 0 \qquad (3)$$

[using the value of λ in (1)]

Now developable is given by

$$f(m) = 0, \; f'(m) = 0$$

$\therefore$ Differentiating (3) partially w.r.t. 'm'

$$-x + \frac{a}{m^2} - \frac{b}{a}z = 0 \qquad (4)$$

Now write (3) as, $y - m(x + \frac{a}{m^2} + \frac{bz}{a}) = 0$

or $\quad y - m\left(\frac{2a}{m^2}\right) = 0 \qquad$ with the help of (4).

or $\quad m = 2a/y$

Putting this value of m in (4), we get

$$y^2 = 4ax + 4bz$$

which is the required equation of developable.

Exercise 6.10: Find the equation of developable surface whose generating lines pass through the curves $y^2 = 4ax,\ z = 0$; $x^2 = 4ay,\ z = c$ and show that its edge of regression is given by

$$cx^2 - 3ayz = 0 = cy^2 - 3ax(c-z).$$

Solution. Any plane touching $z = 0, y^2 = 4ax$ is

$$\lambda z + y - mx - \frac{a}{m} = 0 \qquad \text{[see exercise 6.9 above]}$$

Its section by $z = c$ is

$$z = c.\lambda c + y - mx - \frac{a}{m} = 0$$

or
$$x = \frac{1}{m}\lambda + \left(\frac{\lambda c}{m} - \frac{a}{m^2}\right),\ z = c \qquad (1)$$

If it touches $z = c,\ x^2 = 4ay$, (1) should be of the form

$$x = M\ y + \frac{a}{M} \qquad (1')$$

$$\therefore \quad M = \frac{1}{m} \quad \text{and} \quad \frac{a}{m} = \frac{\lambda c}{m^2} - \frac{a}{m^2} \qquad \text{[comparing coefficients of (1) and (1}'\text{)]}$$

$$\therefore \quad \frac{\lambda c}{m} = am + \frac{a}{m^2} \qquad \text{or} \qquad \lambda = \frac{am^2}{c} + \frac{a}{mc}$$

$\therefore$ The plane touching both the curve is

$$\left(\frac{am^2}{c} + \frac{a}{mc}\right)z + y - mx - \frac{a}{m} = 0$$

or $$f(m) = (am^3 + a)\frac{z}{c} + m\,y - m^2 x - a = 0 \qquad (2)$$

Differentiating (2) partially w.r.t. m

$$f'(m) = (3am^2)\frac{z}{c} + y - 2mx = 0 \qquad (3)$$

Eliminating m between (2) and (3), the equation of the developable is obtained. The edge of regression is given by $f(m) = 0,\ f'(m) = 0$ and $f''(m) = 0$, therefore differentiating (3), we get

$$6am\frac{z}{c} - 2x = 0 \qquad \text{or } m = \frac{cx}{3az} \qquad (4)$$

Putting the value of m in (3) and (2), we have

$$\frac{3ac^2x^2}{9a^2z^2}.\frac{z}{c} + y - 2\frac{cx^2}{3az} = 0$$

or $$cx^2 + 3ayz - 2cx^2 = 0 \qquad \text{or } cx^2 - 3ayz = 0 \qquad (5)$$

and $$\left(\frac{c^3x^3}{27a^2z^3} + a\right)\frac{z}{c} + \frac{y.cx}{3az} - \frac{c^2x^3}{9a^2z^2} - a = 0$$

or $$\frac{(c^3x^3 + 27a^3z^3) - 3c^3x^3}{27a^2z^2c} + \frac{cxy}{3az} - a = 0$$

or $$-2c^3x^3 + 27a^3z^3 + 9azc^2xy - 27a^3z^2c = 0$$

Using (5), we have

$$-27a^3z^2(c - z) - 2c^2x \times 3ayz + 9azc^2xy = 0$$

or $$-3a\frac{c^2x^4}{y^2}(c - z) + c^2x.cx^2 = 0$$

or $$cy^2 - 3ax(c - z) = 0 \qquad (6)$$

Hence the edge of regression is given by (5) and (6).

Exercise 6.11: Show that the edge of regression of the developable that passes through the parabolas $x=0,\ z^2=4ay$; $x=a,\ y^2=4az$ is given by

$$\frac{3x}{y}=\frac{y}{z}=\frac{z}{3(a-x)}.$$

Solution. Any plane touching $x=0,\ z^2=4ay$ is

$$-\lambda x+z-my-\frac{a}{m}=0$$

or $$y=z/m-(a/m+\lambda x)/m$$

Its section by $x=a$ is

$$y=z/m-a\left(\frac{1}{m}+\lambda\right)/m \qquad (1)$$

If it touches $x=a, y^2=4az$, (1) should be of the form

$$y=Mz+\frac{a}{M}$$

$$\therefore\quad M=\frac{1}{m} \qquad \text{and} \qquad \frac{a}{M}=-\frac{a}{m}\left(\frac{1}{m}+\lambda\right) \qquad (2)$$

$$am=-\frac{a}{m^2}-\frac{a\lambda}{m} \qquad \text{or} \qquad \lambda=-\frac{1+m^3}{m}$$

$\therefore$ The plane touching both the parabolas is

$$y=\frac{1}{m}z-\frac{a}{m^2}+\frac{(1+m^3)}{m^2}x$$

or $$m^3x-m^2y+mz+(x-a)=0$$

The edge of regression is given by

$$F(m) = m^3x - m^2y + mz + (x-a) = 0, \quad F'(m) = 0,\ F''(m) = 0$$

$$m^3x - m^2y + mz + (x-a) = 0 \qquad (3)$$

Differentiating partially w.r.t. 'm'

$$3m^2x - 2my + z = 0 \qquad (4)$$

Again differentiating partially w.r.t. 'm'

$$2mx - 2y = 0 \qquad (5)$$

From (5), $y = 3mx$ or $m = y/(3x)$ (6)

Dividing (4) by 3x, we get

$$m^2 - 2m\frac{y}{3x} + \frac{z}{3x} = 0$$

or $$m^2 - 2m^2 + \frac{z}{3x} = 0 \qquad \text{[using (6)]}$$

or $$m^2 = z/3x \qquad (7)$$

Dividing (3) by x, we get

$$m^3 - m^2\frac{y}{x} + m\frac{z}{x} + 1 - \frac{a}{x} = 0$$

$$m^3 - 3m^3 + 3m^3 + 1 - \frac{a}{x} = 0 \qquad \text{[using (6) and (7)]}$$

or $$m^3 = (a-x)/x \qquad (8)$$

Now we have $m^3m = (m^2)^2$

i.e. $\frac{a-x}{x}.\frac{y}{3x}=\frac{z^2}{9x^2}$ [using (6), (7), (8)]

or $y/z = z/\{3(a-x)\}$ (9)

Again $m^3 = m^2.m$ gives

$$\frac{a-x}{x}=\frac{z}{3x}.\frac{y}{3x} \quad \text{or} \quad \frac{y}{3x}=\frac{3(a-x)}{z}$$

or $\frac{3x}{y}=z/\{3(a-x)\}$ (10)

$\because$ From (9) and (10), the edge of regression is given by

$$\frac{3x}{y}=\frac{y}{z}=\frac{z}{3(a-x)}.$$

Exercise 6.12: Show that two cones pass through the curves $x^2+y^2=4a^2,\ z=0$; $x=0, y^2=4a(z+a)$; and that their vertices are the points $(2a,0,-2a)$, $(-2a,0,2a)$.

Solution. A tangent plane to the first is

$$\lambda z+y-mx-2a\sqrt{(1+m^2)}=0$$

[$\because$ tangent to the first is $z=0,\ y=mx+2a\sqrt{(1+m^2)}$]

If it touches the second

$$\lambda z+y-2a\sqrt{(1+m^2)}=0$$

and $y-M(z+a)-\frac{a}{M}=0$ are equivalent,

or $\lambda=-M,\quad 2a\sqrt{(1+m^2)}=aM+\frac{a}{M}$

or $$M^2 - 2\sqrt{(1+m^2)}M + 1 = 0$$

or $$\{M - \sqrt{(1+m^2)}\}^2 - (1+m^2-1) = 0$$

or $$M = m + \sqrt{(1+m^2)}\ ,\ \ M = m - \sqrt{(1+m^2)}$$

$\therefore$ The plane is $[\pm m - \sqrt{(1+m^2)}]z + y - mx - 2a\sqrt{(1+m^2)} = 0$

or $$m(\pm z - x) - \sqrt{(1+m^2)}(z+2a) = 0 \tag{1}$$

or Differentiating (1) w.r.t. 'm'

$$(\pm z - x) - \frac{m}{\left(\sqrt{(1+m^2)}\right)^2}(z+2a) = 0 \tag{2}$$

or $$m(z+2a) - (1+m^2)(\pm z - x) = 0 \tag{3}$$

From (1) and (3)

$$\frac{m}{-y(\pm z - x)} = \frac{\sqrt{(1+m^2)}}{-y(z+2a)} = \frac{1}{[(\pm z - x)^2 - (z+2a)^2]}$$

Eliminating m in between these equations

$$1 + \frac{y^2(\pm z - x)^2}{[(\pm z - x)^2 - (z+2a)^2]^2} = \frac{y^2(z+2a)^2}{[(\pm z - x)^2 - (z-2a)^2]^2}$$

or $$i = \frac{y^2}{(z+2a)^2 - (\pm z - x)^2}$$

i.e. $y^2 = (z+2a)^2 - (x+z)^2$ and $y^2 = (z+2a)^2 - (x-z)^2$

both these equations represent cones with vertices at

$$(2a, 0 - 2a) \text{ and } (-2a, 0, 2a).$$

Exercise 6.13: Find the equation of the developable which passes through $z=0, 4a^3y^3=b^2c^2x^2$; $y=0, 4a^3z^2=bc^4x$. Also find the edge of the regression.

Solution. Let the equation of any plane be

$$lx+my+nz=1$$

If this plane touches the first curve

$$m^3b^2c^2=27a^3l^2 \qquad \text{[using } B^2-4AC=0\text{]}$$

Also it touches the second

$$27a^3l+bc^4n^3=0 \qquad \left[l=\frac{m^{3/2}bc}{3\sqrt{3}a^{3/2}}==-\frac{bc^4n^3}{27a^3}\right]$$

The equation of the plane is

$$lx+\frac{3al^{2/3}}{b^{2/3}c^{2/3}}y-\frac{z3al^{1/3}}{b^{2/3}c^{4/3}}=1$$

Replacing *l* by L^3

$$L^3x+\frac{3aL^2}{b^{2/3}c^{2/3}}y-\frac{3aLz}{b^{2/3}c^{4/3}}=1 \qquad (1)$$

Differentiating (1) w.r.t. *L*,

$$L^2x+\frac{2ayL}{b^{2/3}c^{2/3}}-\frac{az}{b^{2/3}c^{4/3}}=0 \qquad (2)$$

Multiplying (2) by *L* and subtracting from (1)

$$\frac{ayL^2}{b^{2/3}c^{2/3}}-\frac{2azL}{b^{1/3}c^{4/3}}-1=0 \qquad (3)$$

From (2) and (3)

$$\frac{L^2}{\dfrac{-2ay}{b^{2/3}c^{2/3}}-\dfrac{2a^2z^2}{b^{2/3}c^{8/3}}}=\frac{L}{-x+\dfrac{a^2yz}{bc^2}}=\frac{1}{-\dfrac{2axz}{b^{1/3}c^{4/3}}-\dfrac{2a^2y^2}{b^{4/3}c^{4/3}}}$$

Eliminating L, get

$$4a^2(bzx+ay^2)(c^2y+az^2)=(a^2yz-bc^2x)^2$$

which is the required equation of the developable.

For the edge of regression, differentiating (2)

$$Lx=-\frac{ay}{b^{2/3}c^{2/3}}$$

Substituting for L in (2)

$$\left(\frac{ay}{b^{2/3}c^{2/3}x}\right)^2x-\frac{ay}{b^{2/3}c^{2/3}}.\frac{2ay}{b^{2/3}c^{2/3}x}-\frac{az}{b^{1/3}c^{4/3}}=0$$

or $$\frac{a^2y^2}{b^{4/3}c^{4/3}}=-\frac{axz}{b^{1/3}c^{4/3}} \quad \text{or} \quad ay^2=-bxz \qquad (4)$$

Also substituting for L in (3),

$$\frac{ay}{x^2b^{2/3}c^{2/3}}.\frac{a^2y^2}{b^{4/3}c^{4/3}}+\frac{2az}{b^{1/3}c^{4/3}}.\frac{ay}{b^{2/3}c^{2/3}x}-1=0$$

or $$a^3y^3+2a^2yz.bx-b^2c^2x^2=0$$

or $$a^3y^3-2a^2yay^2-c^2.\frac{a^2y^4}{z^2}=0$$

or $$a^3y^3z+a^2c^2y^4=0 \quad \text{or} \quad az^2+c^2y=0$$

which is the required equation of the edge of regression.

Exercise 6.14: Prove that the edge of regression of the developable that passes through the circles $z=0,\ x^2+y^2=a^2$; $x=0,\ y^2+z^2=b^2$ lies on the cylinder $(x/a^2)^{2/3}-(z/b^2)^{2/3}=(1/a^2-1/b^2)^{4/3}$.

Solution. Let a plane be $lx+my+nz=1$ (1)

This touches $z=0,\ x^2+y^2=a^2$, if

$$a^2(l^2+m^2)=1 \tag{2}$$

and (1) touches $x=0,\ y^2+z^2=b^2$, if

$$b^2(m^2+n^2)=1 \tag{3}$$

with the help of (2) and (3), (1) becomes

$$\sqrt{(1-a^2m^2)}.\frac{x}{a}+my+\sqrt{(1-b^2m^2)}.\frac{z}{b}=1$$

$$\sqrt{(1-a^2m^2)}.\frac{x}{a}+my+\sqrt{(1-b^2m^2)}.\frac{z}{b}-1=0 \tag{4}$$

Then for edge of regression, we are to eliminating *m* between $f(m)=0,\ f'(m)=0,\ f''(m)=0$, where dashes denote differentiation w.r.t. *m*.

Differentiating (4) w.r.t. *m*,

$$-\frac{ax}{\sqrt{(1-a^2m^2)}}+\frac{y}{m}-\frac{bz}{\sqrt{(1-b^2m^2)}}=0 \tag{5}$$

Differentiating (5) w.r.t. ‘*m*’

$$\frac{axa^2m}{(1-a^2m^2)^{3/2}}+\frac{y}{m^2}-\frac{bz}{(1-b^2m^2)^{3/2}}=0 \tag{6}$$

Multiplying equation (6) by *m* and subtracting (5) from it

$$\frac{ax}{(1-a^2m^2)^{3/2}}+\frac{bz}{(1-b^2m^2)^{3/2}}=0 \qquad (7)$$

Also multiplying (5) by m^2 and subtracting it from (4)

$$\frac{x}{a\sqrt{(1-a^2m^2)}}+\frac{z}{b\sqrt{(1-b^2m^2)}}=0 \qquad (8)$$

Equation (7) can also be written as

$$\frac{(ax)^{1/3}}{(1-a^2m^2)^{1/2}}+\frac{(bz)^{1/3}}{(1-b^2m^2)^{1/2}}=0 \qquad (9)$$

From (8) and (9)

$$1-a^2m^2=\frac{\left\{(ax)^{1/3}.\frac{z}{b}+\frac{x}{a}(-bz)^{1/3}\right\}^2}{(-bz)^{2/3}} \qquad (10)$$

and $$1-a^2m^2=\frac{\left\{(ax)^{1/3}.\frac{z}{b}+\frac{x}{a}(-bz)^{1/3}\right\}^2}{(ax)^{2/3}} \qquad (11)$$

Multiplying (10) by b^2 and (11) by a^2 and subtracting and simplifying

$$\left(\frac{x}{a^2}\right)^{2/3}-\left(\frac{z}{b^2}\right)^{2/3}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^{1/3}.$$

Exercise 6.15: Prove that the section by the *xy*-plane of the developable generated by the tangents to the curve

$$x^2+y^2+z^2=r^2,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2} \text{ is given by}$$

$$z=0,\ \frac{a^2(a^2+c^2)}{x^2}=\frac{b^2(b^2+c^2)}{y^2}=\frac{(a^2-b^2)}{r^2}.$$

Solution. Let (x, y, z) be a point on the curve.

Then the tangent plane is

$$Xx + Yy + Zz = r^2,\ \frac{Xx}{a^2} + \frac{Yy}{b^2} = \frac{Zz}{c^2} \tag{1}$$

where,
$$x^2 + y^2 + z^2 = r^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \tag{2}$$

The developable required is the surface obtained by eliminating (x, y, z) between (1) and (2), its section by the *xy*-plane is obtained by putting $z = 0$ or what is the same thing, we may eliminate *x* and and *y* between

$$Xx + Yy = r^2,\ \frac{Xx}{a^2} + \frac{Yy}{b^2} = 0 \tag{3}$$

$$x^2 + y^2 + c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = r^2 \tag{4}$$

From (3),
$$\frac{Xx}{r^2/b^2} + \frac{Yy}{-r^2/a^2} = \frac{a^2b^2}{a^2 - b^2}$$

or
$$x = \frac{a^2r^2}{(a^2 - b^2)X},\ y = \frac{-b^2r^2}{(a^2 - b^2)Y} \tag{5}$$

Substituting from (5) in (4), we get required result.

Exercise 6.16: An ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is surrounded by a luminous ring $x = 0,\ y^2 + z^2 = a^2$. Show that the boundary of the shadow cast on the plane $z = 0$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 - c^2} = \frac{a^2}{a^2 - c^2}.$$

Solution. Here it is required to find the equation of developable which touches the ellipsoid and luminous ring.

Let $lx + my + nz = 1$ (1)

be a plane touching both the ellipsoid and the ring, then

$$a^2l^2 + b^2m^2 + c^2n^2 = 1 \qquad (2)$$

and $$a^2(m^2 + n^2) = 1 \qquad (3)$$

Eliminating n^2 between (2) and (3)

$$a^2.a^2l^2 + a^2(b^2 - c^2)m^2 = a^2 - c^2 \qquad (4)$$

Now (1) intersects $z = 0$ in

$$lx + my = 1 \qquad (5)$$

Therefore the required curve is the envelope of (5) subject to the condition (4).

$$\left.\begin{aligned} & x + y\frac{dm}{dl} = 0 && (\,from\,(5)) \\ & a^4l + ma^2(b^2 - c^2)\frac{dm}{dl} = 0 && (\,from(4)) \end{aligned}\right\} \qquad (6)$$

Eliminating $\frac{dm}{dl}$ between equation (6)

$$ya^2l = (b^2 - c^2)mx$$

or $$\frac{x}{a^2l} + \frac{y}{m(b^2 - c^2)} = \frac{lx + my}{\frac{a^2 - c^2}{a^2}} = \frac{a^2}{a^2 - c^2} \qquad \text{[using (4) and (5)]} \qquad (7)$$

Substituting from (7) in (5),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 - c^2} = \frac{a^2}{a^2 - c^2}.$$

Exercise 6.17: Prove that the developable surface that envelopes the sphere $x^2 + y^2 + z^2 = c^2$ and the hyperboloid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ meet the plane $y = 0$ in the conic $x^2/(b^2 - a^2) + z^2/(b^2 + c^2) = c^2/(b^2 - c^2)$.

Solution. Tangent plane to the sphere is

$$lx + my + nz = c \tag{1}$$

$$l^2 + m^2 + n^2 = 1 \tag{2}$$

If it is tangential to the hyperboloid also

$$a^2l^2 + b^2m^2 - c^2n^2 = c^2 \tag{3}$$

$$\therefore \qquad x\frac{dl}{dn} + y\frac{dm}{dn} + z = 0$$

$$a^2l\frac{dl}{dn} + mb^2\frac{dm}{dn} - c^2n = 0$$

$$l\frac{dl}{dn} + m\frac{dm}{dn} + n = 0$$

or

$$\begin{vmatrix} x & y & z \\ l & m & n \\ a^2l & b^2m & -c^2n \end{vmatrix} = 0 \tag{4}$$

Putting $y = 0$ and eliminating *m*, (1) to (4) can be written down as

$$lx + ny - c = 0 \qquad \text{[from (1)]} \tag{5}$$

$$a^2l^2 + b^2(1 - l^2 - n^2) - c^2n^2 = c^2 \qquad \text{[from (2) and (3)]}$$

or

$$l^2(b^2 - a^2) + n^2(b^2 + c^2) = b^2 - c^2 \tag{6}$$

$$lz(b^2 - a^2) - nx(b^2 + c^2) = 0 \tag{7}$$

From (5) and (7),

$$\frac{l}{cx(b^2+c^2)}=\frac{n}{-cz(b^2-a^2)}=\frac{1}{-x^2(b^2+c^2)-z^2(b^2-a^2)}$$

Substituting in (6)

$$\frac{c^2x^2(b^2+c^2)^2(b^2-a^2)+c^2z^2(b^2-a^2)^2(b^2+c^2)}{\{x^2(b^2+c^2)+z^2(b^2-a^2)\}^2}=b^2-c^2$$

or $$c^2(b^2+c^2)(b^2-a^2)=(b^2-c^2)\{x^2(b^2+c^2)+z^2(b^2-a^2)\}$$

or $$\frac{x^2}{b^2-a^2}+\frac{z^2}{b^2+c^2}=\frac{c^2}{b^2-a^2}.$$

Exercise 6.18: A developable surface is drawn through the curves

$$\lambda^2x^2+y^2=\lambda^2,\ z=c;\ x^2+y^2=1,\ z=-c$$

show that its section by the plane $z=0$ is given by

$$2x=\sin\phi+\sin\theta,\ 2y=\cos\theta+\lambda\cos\phi$$

where $\tan\theta=\tan\phi$.

Solution. Any point on the first curve is $(\sin\phi,\lambda\cos\phi,c)$ and a point on the second curve is $(\sin\theta,\cos\theta,-c)$.

A line through these two points is

$$\frac{x-\sin\theta}{\sin\phi-\sin\theta}=\frac{y-\cos\theta}{\lambda\cos\phi-\cos\theta}=\frac{z+c}{-2c} \tag{1}$$

By eliminating θ and ϕ between two equations (1), we get the ruled surface generated by the line .

The section of this surface by the plane $z=0$ is evidently

$$x = \frac{1}{2}(\sin\phi - \sin\theta) + \sin\theta = \frac{\sin\phi + \sin\theta}{2}$$

$$y = \frac{1}{2}(\lambda\cos\phi - \cos\theta) + \cos\theta = \frac{\lambda\cos\phi + \cos\theta}{2}$$

Since the ruled surface is developable, consecutive generators intersect if

$$\cos\theta\left(-\sin\phi\frac{d\phi}{d\theta} + \sin\theta\right) + \sin\theta\left(-\cos\phi\frac{d\phi}{d\theta} - \cos\theta\right) = 0 \qquad [\alpha' b' - \beta' a' = 0]$$

or $\quad \lambda\sin\phi\cos\theta = \sin\theta\cos\phi$

or $\quad \lambda\tan\phi = \tan\theta$.

Exercise 6.19: If the coincoids $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $x^2/a_1^2 + y^2/b_1^2 + z^2/c_1^2 = 1$ are confocal and a developable is circumscribed to the first along its curve of intersection with the second, the edge of regression lies on

$$\sum x^{2/3}(b^2 - c^2)^{2/3} a^{-2/3} = 1.$$

Solution. The plane $lx + my + nz = 1$ (1)

is a tangent plane to $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, if $a^2l^2 + b^2m^2 + c^2n^2 = 1$ (2)

Also the point of contact is to lie on. Since this point of contact is to lie on, we get

$$x^2/a_1^2 + y^2/b_1^2 + z^2/c_1^2 = 1$$

$$\frac{a^4l^2}{a_1^2} + \frac{b^4m^2}{b_1^2} + \frac{c^4n^2}{c_1^2} = 1 \qquad (3)$$

Differentiating (1), (2), and (3); and eliminating dl, dm, dn

$$\begin{vmatrix} \dfrac{x(b^2 - c^2)}{a^2b_1^2c_1^2l_1^2} & \dfrac{y(c^2 - a^2)}{b^2c_1^2a_1^2m_1^2} & \dfrac{z(a^2 - b^2)}{c^2a_1^2b_1^2n_1^2} \\ a^2l & b^2m & c^2n \\ \dfrac{a^4l^2}{a_1^2} & \dfrac{b^4m^2}{b_1^2} & \dfrac{c^4n^2}{c_1^2} \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & \dfrac{y(c^2-a^2)}{b^2c_1^2a_1^2m_1^2} & \dfrac{z(a^2-b^2)}{c^2a_1^2b_1^2n_1^2} \\ 1 & b^2m & c^2n \\ 1 & \dfrac{b^4m^2}{b_1^2} & \dfrac{c^4n^2}{c_1^2} \end{vmatrix} = 0$$

or $\dfrac{m^3b^4}{b_1^4 y(c^2-a^2)} = \dfrac{n^3c^4}{c_1^4 z(a^2-b^2)} = \dfrac{l^3a^4}{a_1^4 x(b^2-c^2)}$ (5)

Substituting values of l, m, n from (5) in (4), we get the required equation.

Exercise 6.20: A developable surface passes through the curves

$$y=0, x^2=(a-=b)(2z-b);\; x=0, y^2=(a-b)(2z-a).$$

Prove that the edge of regression lies on the cylinder $x^{2/3}-y^{2/3}+(a-b)^{2/3}=0$.

Solution. Consider the plane $lx+my+nz=1$ (1)

If this plane touches the first curve

$$(a-b)l^2=bn^2+2n \tag{2}$$

And, if it touches the second

$$(a-b)m^2=an^2+2n \tag{3}$$

Differentiating (1), (2), and (3) w.r.t. l, m, n); and eliminating dl, dm, dn

$$\begin{vmatrix} x & y & z \\ (a-b)l & 0 & -(1+bn) \\ x & y & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & -1 \\ (a-b)l & 0 & n \\ x & (a-y)m & n \end{vmatrix} = 0$$

or $\dfrac{x}{l}+\dfrac{y}{m}+\dfrac{(a-b)}{n}=0$ (4)

For edge of regression, differentiating (4) and eliminating dl, dm, dn and differential of (2) and (3),

$$\begin{vmatrix} \frac{x}{l^2} & \frac{y}{m^2} & \frac{(a-b)}{n^2} \\ (a-b)l & 0 & -(1+bn) \\ 0 & (a-b)m & -(1+na) \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & y/m & (a-b)/n \\ m & 0 & -n(1+nb) \\ n & (a-b)m & -n(1+an) \end{vmatrix} = 0$$

or $$\frac{(a-b)}{n^3} = \frac{y}{-m^3} = \frac{x}{l^3}$$

Hence, $$\frac{l}{x^{1/3}} = \frac{m}{-y^{1/3}} = \frac{n}{(a-b)^{1/3}} \tag{5}$$

Substituting from (5) in (4), the required result is obtained.

Exercise 6.21: Find the equation of developable surface which contains the two curves

$$y^2 = 4ax, z = 0 \text{ and } (y-b)^2 = 4cz, x = 0$$

and show that its edge of regression lies on the surface

$$(ax+by+cz)^2 = 3abx(y+b).$$

Solution. Any line tangent to $y^2 = 4ax, z = 0$ is

$$y = mx + \frac{a}{m}, z = 0$$

any plane touching this parabola is

$$y - mx - \frac{a}{m} + \lambda z = 0$$

If it touches $x = 0, (y-b)^2 = 4cz$, *i.e.* if

$$y + \lambda z - \frac{a}{m} = 0$$

touches $(y-b)^2 = 4cz$

$\left(-\lambda z + \frac{a}{m} - b\right)^2 - 4cz = 0$ should have coincident roots, so

$$\left[-2\lambda\left(\frac{a}{m}-b\right)-4c\right]^2-4\lambda^2\left(\frac{a}{m}-b\right)^2=0$$

or $$16c^2+16\lambda c\left(\frac{a}{m}-b\right)=0$$

or $$\lambda = mc/(bm-a)$$

The plane touching second is

$$y-mx-\frac{a}{m}+\frac{mcz}{bm-a}=0$$

or $$bym^2-amy-bm^3x+am^2x-abm+a^2+m^2cz=0$$

or $$bym^3x-m^2(ax+by+cz)+am(y+b)-a^2=0 \qquad (1)$$

Differentiating w.r.t. m,

$$3bym^2x-2m(ax+by+cz)+a(y+b)=0 \qquad (2)$$

Multiplying (1) by 3 and (2) by m and subtracting

$$m\{2(ax+by+cz)^2-6abx(b+y)\}-(ax+by+cz)a(b+y)+9a^2bx=0$$

or $$m=\frac{a(b+y)(ax+by+cz)-9a^2bx}{2(ax+by+cz)^2-6abx(b+y)}$$

putting this value of m in (2), the equation of the developable is obtained.

Now for edge of regression, differentiating (2) w.r.t. 'm'

$$6bmx-2(ax+by+cz)=0$$

or $$m=\frac{ax+by+cz}{3bx}$$

putting this value in (2),

$$\left(\frac{ax+by+cz}{3bx}\right)^2-2\left(\frac{ax+by+cz}{3bx}\right)=-a(b+y)$$

Hence edge lies on two surfaces.

Exercise 6.22: Show that the radius of curvature of the edge of the regression of rectifying developable is equal to $\operatorname{cosec}\phi\frac{d}{d\phi}\left(\sin^2\phi\frac{ds}{d\phi}\right)$, where $\tan\phi=\frac{\sigma}{\rho}$

and that the radius of torsion is equal to $-p\dfrac{d}{ds}\left(\sin^2\phi\dfrac{ds}{d\phi}\right)$.

Solution. At a point **r** on the curve, the equation of rectifying plane is

$$(\mathbf{R}-\mathbf{r}).\mathbf{n}=0 \qquad (1)$$

Differentiating w.r.t. s , we get

$$(\mathbf{R}-\mathbf{r})\bullet(\tau\mathbf{b}-\kappa\mathbf{t})-\mathbf{t}.\mathbf{n}=0$$

or $$(\mathbf{R}-\mathbf{r})\bullet(\tau\mathbf{b}-\kappa\mathbf{t})=0 \qquad [\because \mathbf{t}.\mathbf{n}=0]$$

or $$(\mathbf{R}-\mathbf{r})\bullet(\cot\phi\,\mathbf{b}-\mathbf{t})=0 \qquad [\because \tan\phi=\sigma/\rho \; i.e. \cot\phi=\tau/\kappa] \qquad (2)$$

Differentiating (2) w.r.t. 's' we get

$$(\mathbf{R}-\mathbf{r})\left(-\mathrm{cos}ec^2\phi\frac{d\phi}{ds}\mathbf{b}-\cot\phi\,\tau\,\mathbf{n}-\kappa\mathbf{n}\right)-\mathbf{t}(\cot\phi\,\mathbf{b}-\mathbf{t})=0$$

or $$(\mathbf{R}-\mathbf{r})\left(-\mathrm{cos}ec^2\phi\frac{d\phi}{ds}\mathbf{b}\right)+1=0 \quad [\text{using (1), and } \mathbf{t}\bullet\mathbf{b}=0, \mathbf{t}\bullet\mathbf{t}=1]$$

or $$(\mathbf{R}-\mathbf{r})\bullet\mathbf{b}=+\sin^2\phi\frac{ds}{d\phi}=\mu \; (say) \qquad (4)$$

when, $$1/\mu=+\mathrm{cos}ec^2\phi\frac{d\phi}{ds}=-\frac{d}{ds}(\cot\phi) \qquad (5)$$

The edge of regression is clearly given by (1), (2) and (3).

Now from (2), $(\mathbf{R}-\mathbf{r})\bullet\mathbf{t}-(\mathbf{R}-\mathbf{r})\mathbf{b}\cot\phi=\mu\cot\phi$ [by (4)]

Hence, $$(\mathbf{R}-\mathbf{r})=\mu(\cot\phi\,\mathbf{t}+\mathbf{b}) \qquad (6)$$

Equation (6) clearly; satisfies (1) and (4), hence point **R** on the edge of regression is given by

$$\mathbf{R}=\mathbf{r}+\mu(\cot\phi\,\mathbf{t}+\mathbf{b}) \qquad (7)$$

Now it is required to find curvature and torsion of the locus of **R**. Let the quantities belonging to the locus of **R** be distinguished by the use of suffix unity.

Differentiation of (7) w.r.t. 's' provides

$$\mathbf{t}_1\frac{ds_1}{ds}=\mathbf{t}+\mu'(\cot\phi\,\mathbf{t}+\mathbf{b})+\mu\left(\cot\phi\,\kappa\mathbf{n}-\frac{1}{\mu}\mathbf{t}-\tau\mathbf{n}\right) \qquad [\text{using equation (5)}]$$ or

$$\mathbf{t}_1\frac{ds_1}{ds}=\mathbf{t}+\mu'(\cot\phi\,\mathbf{t}+\mathbf{b}) \qquad [\text{using } \cot\phi=\tau/\kappa]$$

$$= \mu'(\cos\phi\,\mathbf{t} + \mathbf{b}\sin\phi)/\sin\phi$$

On squaring,

$$\left(\frac{ds_1}{ds}\right)^2 = \frac{\mu'^2}{\sin^2\phi} \quad \text{or} \quad \frac{ds_1}{ds} = \frac{\mu'}{\sin\phi} \qquad (7')$$

and therefore, $\mathbf{t}_1 = \cos\phi\,\mathbf{t} + \sin\phi\,\mathbf{b}$ (8)

Now again differentiating (8), w.r.t. '*s*', we get

$$\kappa_1\mathbf{n}_1\frac{ds_1}{ds} = (\cos\phi\kappa\mathbf{n} - \sin\phi\tau\mathbf{n}) + (-\sin\phi\,\mathbf{t} + \cos\phi\,\mathbf{b})\frac{d\phi}{ds}$$

$$= (-\sin\phi\,\mathbf{t} + \cos\phi\,\mathbf{b})\frac{d\phi}{ds}$$ [$\because$ first bracket is zero for $\cot\phi = \tau/\kappa$]

Squaring, $$\left(\kappa_1\frac{ds_1}{ds}\right)^2 = \left(\frac{d\phi}{ds}\right)^2 \quad i.e. \quad \kappa_1\frac{ds_1}{ds} = \frac{d\phi}{ds} \qquad (9)$$

Hence, $\mathbf{n}_1 = -\sin\phi\,\mathbf{t} + \cos\phi\,\mathbf{b}$ (10)

From (9); $$\rho_1 = \frac{ds_1}{ds}.\frac{ds}{d\phi} = \mu'\operatorname{cosec}\phi\frac{ds}{d\phi}$$ [from (7′)]

or $$\rho_1 = \frac{d}{ds}\left(\sin^2\phi\frac{ds}{d\phi}\right)\operatorname{cosec}\phi\frac{ds}{d\phi}$$ [from (4)]

$$= \operatorname{cosec}\phi\frac{d}{d\phi}\left(\sin^2\phi\frac{ds}{d\phi}\right)$$

Again $\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = -\mathbf{n}$ [takng cross product of (8) and (10)]

Differentiating w.r.t. '*s*'

$$-\tau_1\mathbf{n}_1\frac{ds_1}{ds} = -(\tau\,\mathbf{b} - \kappa\,\mathbf{t}) = \kappa\,\mathbf{t} - \kappa\cot\phi\,\mathbf{b}$$ [using $\cot\phi = \tau/\kappa$]

$$= -\frac{\kappa}{\sin\phi}[-\sin\phi\,\mathbf{t} + \cos\phi\,\mathbf{b}]$$

$$= \frac{\kappa}{\sin\phi}\mathbf{n}_1$$ [from (10)]

Hence, $$\tau_1\frac{ds_1}{ds} = \frac{\kappa}{\sin\phi} \quad \text{or} \quad \sigma_1 = \rho\sin\phi\frac{ds_1}{ds}$$

or $$\sigma_1 = \rho \sin\phi \frac{\mu'}{\sin\phi} \qquad \text{[from (7')]}$$

$$= \rho \frac{d}{ds}\left(\sin^2\phi \frac{ds}{d\phi}\right)$$

$\therefore$ $$\sigma_1 = \rho \frac{d}{ds}\left(\sin^2\phi \frac{ds}{d\phi}\right)$$ choosing such direction of $\mathbf{n}_1$.

6.2 CHECK YOUR PROGRESS

SA1: Exercise the terms (i) characteristic of the surface $f(x, y, z, \alpha) = 0$ for the parameter α, (ii) envelope of the family of surface $f(x, y, z, \alpha) = 0$; (iii) edge of regression and prove that

(a) The envelope touches each member of the family of surfaces at all points of its characteristics;

(b) The characteristic of a family of surface of one parameter are tangent to the edge of regression .

SA2: Prove that the edge of regression of the envelope of the normal planes of a curve is the locus of the centre of spherical curvature.

SA3: Prove that the generator of the rectifying developable of a skew curve makes with the tangent to the curve an angle θ where $\tan\theta = \kappa/\tau$.

SA4: Show that on a developable surface the generators form one system of curvature.

SA5: Show that the envelope of the plane $\frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi = 1$, where, ϕ are independent parameters, is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

SA6: Prove that the envelope of a plane which forms with the coordinate planes a tetrahedron of constant volume is a surface $xyz = 1$.

SA7: The envelope of a plane, the sum of the squares of whose intercepts on the axes is constant, is a surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = const.$

SA8: The envelope of the plane $(u-v)bcx+(1+uv)cay+(1-uv)abz=abc(u+v)$ where u, v are parameters, is the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

SA9: The envelope of the plane $lx+my+nz=p$, where $p^2=a^2l^2+b^2m^2+c^2n^2$ is an ellipsoid.

SA10: Find the envelope of the plane $\frac{x}{a+u}+\frac{y}{b+u}+\frac{z}{c+u}=1$, where u is the parameter, and determine the edge of regression.

SA11: A fixed point O on the x-axis is joined to a variable point P on the yz-plane. Find the envelope of the plane through P at right angle to OP.

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CHAPTER-7

FUNDAMENTAL FORMS-I

Objectives: In continuation of the previous chapters in the current chapter the **s**tudents will learn Curvilinear coordinates, Parametric form of the curve, Tangent plane and Normal, First fundamental form and their geometrical representation, Relationship between coefficients of first fundamental form, Directional coefficients and ratios, Normals with some solved example.

7.1. Curvilinear Coordinates. We have seen that a surface may be regarded as the locus of a point whose position vector **r** is a function of two independent parameters u, v. The Cartesian coordinates x, y, z of the point are then known functions of u, v, and the elimination of the two parameters leads to a single relation between x, y, z which is usually called the equation of the surface. We shall confine our attention to surfaces, or portions of surfaces, which present no singularities of any kind.

Any relation between the parameters say $f(u,v) = 0$ represents a curve on the surface. For **r** then becomes a function of only one independent parameter, so that the locus of the point is a curve. In particular, the curves on the surface, along, which one of the parameters remains constant, are called the parametric curves.

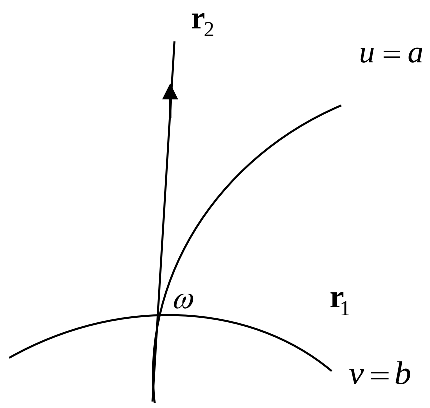

Figure 7.1

The surface can be mapped out by a double infinitude of values that can be assigned to each of the parameters. The parameters u, v thus constitute a system of curvilinear coordinates for points on the surface, the position of the point being determined by the value of u and v.

Suppose for example, that we are dealing with the surface of a sphere of radius *a*, and that three mutually perpendicular diameters are chosen as coordinate axes. The latitude λ of a point *P* on the surface may be defined as the inclination of the plane containing *P* and the *z*-axes to the *xz*-plane. Then the coordinates of *P* are given by

$$x = a\cos\lambda\cos\phi,\ y = a\cos\lambda\sin\phi,\ z = a\sin\lambda. \qquad (7.1)$$

Thus λ and ϕ may be taken as parameters for the surface. The parametric curves λ=constant are the small circles called parallels of latitude; the curve ϕ=constant are the great circle called the meridians of longitude. As these two systems of curves cut each other at right angles, we say the parametric curves are orthogonal.

As another example, consider the osculating developable of a twisted curve. The generators of this surface are the tangents to the curve. Hence the position vector of a point on the surface is given by

$$\mathbf{R} = \mathbf{r} + u\,\mathbf{t} \qquad (7.2)$$

Where u is the distance of the point from the curve measured along the tangent at the point **r**. But **r**, **t** are functions of the arc-length *s* of the given curve. Hence s, u may be taken as parameters for the osculating developable. The parametric curves *s*=constant, are the generators; and the curves u =constant, cut the tangents at a constant distance from the given curve.

If the equation of the surface is given in Monge's form

$$z = f(x, y)$$

the coordinates x, y may be taken as parameters. In this case, the parametric curves are the intersections of the surface with the planes x =constant and y =constant.

Or, let us consider a surface $\mathbf{r} = \mathbf{r}(u, v)$ defined on a domain *D* and if u and v are functions of single parameter *t*, then the position vector **r** becomes a function of single parameter *t*, and hence its locus is a curve laying on the surface $\mathbf{r} = \mathbf{r}(u, v)$. Let $u = u(t)$ and $v = v(t)$, then $\mathbf{r} = \mathbf{r}\{u(t), v(t)\}$ is a curve laying on the surface $\mathbf{r} = \mathbf{r}(u, v)$ in *D*.

It should be noted that the surface $\mathbf{r} = \mathbf{r}(u,v)$ and curves $u = u(t)$ $v = v(t)$ lie in the same domain and if m and n are class of the $\mathbf{r} = \mathbf{r}\{u(t), v(t)\}$ on the surface is smaller than m and n. The equations $u = u(t)$ $v = v(t)$ are called curvilinear equations of the curve on the surface.

7.2 Parametric curves. Let $\mathbf{r} = \mathbf{r}(u,v)$ be the equation of the surface defined on a domain D. Now by keeping u =constant or v =constant, we get curves of special importance which are called parametric curves. Thus:

If v =constant say (c), then as u varies, the point $\mathbf{r} = \mathbf{r}(u,c)$ describes a parametric curve called the u curve or the parametric curves curve $v = c$. Similarly, if u =constant say (c), then as v varies, the point $\mathbf{r} = \mathbf{r}(c,v)$ traces a parametric curve called the v curve or the parametric curves curve $u = c$. For u curve, u is the parameter and determines a sense along the curve. The tangent to the curve in the sense of u increasing is along the vector $\mathbf{r}_1$.

Similarly, the tangent to v the curve in the sense of v increasing is along the vector $\mathbf{r}_2$. Thus we have two systems of parametric curves viz., u curve and v curve, and since we know that $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$, therefore the parametric curves of the different systems can not touch each other.

If $\mathbf{r}_1 \bullet \mathbf{r}_2 = 0$ at a point P, the two parametric curves through the point P are orthogonal. If this condition is satisfied at every point *i.e.* for all values of u and v in the domain D, the two systems of parametric curves are orthogonal.

7.3. Tangent Plane and Normal

7.3.1 Tangent plane. Let the equation of the curve be $u = u(t)$, $v = v(t)$ then the tangent is parallel to the vector $\dot{\mathbf{r}}$.

where $$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}.\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}.\frac{dv}{dt}$$

or $$\frac{d\mathbf{r}}{dt} = \mathbf{r}_1 \frac{du}{dt} + \mathbf{r}_2 \frac{dv}{dt}$$ [or therefore, $d\mathbf{r} = \mathbf{r}_1 du + \mathbf{r}_2 dv$]

But $\mathbf{r}_1$ and $\mathbf{r}_2$ are non-zero and independent vectors, therefore the tangent to a curve (on the surface) through a point P lies in the plane which contains the two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. This plane is called the tangent plane at P. Hence $\mathbf{r}_1 \times \mathbf{r}_2$ gives the direction of the normal to the tangent plane is

$$(\mathbf{R}-\mathbf{r}).(\mathbf{r}_1 \times \mathbf{r}_2) = 0 \qquad (7.3)$$

where $\mathbf{r}$ is the position vector of P and $\mathbf{R}$ that of a current point on the plane.

7.3.2 Normal. The normal to the surface at the point P is a line passing through P and perpendicular to the tangent plane at P. Thus, if $\mathbf{r} = \mathbf{r}(u,v)$ is the equation of the surface, then clearly normal to the tangent plane at P is perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$ hence parallel to $\mathbf{r}_1 \times \mathbf{r}_2$ and therefore the equation of the normal line at P to the surface is

$$\mathbf{R} = \mathbf{r} + \lambda(\mathbf{r}_1 \times \mathbf{r}_2) \qquad (7.4)$$

where $\mathbf{R}$ is the position vector of a current point on the normal.

The normal to the surface at P is the same as normal to the tangent plane at P and therefore the unit surface normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} \qquad \text{where } H = \mathbf{r}_1 \times \mathbf{r}_2 \neq 0.$$

Note. The sense of the unit surface normal vector N is fixed by considering that $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{N}$ in this order form a right-handed system.

7.3.3 Cartesian Formulation. If the equation of the surface be $F(x,y,z)$ then the equation of the tangent plane at any point (x,y,z) is given by

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0 \qquad (1)$$

And the equations of the normal line at any point (x,y,z) are given by

$$\frac{(X-x)}{\frac{\partial F}{\partial x}} = \frac{(Y-y)}{\frac{\partial F}{\partial y}} = \frac{(Z-z)}{\frac{\partial F}{\partial z}} \qquad (2)$$

where, (X,Y,Z) is the current point on the tangent plane or the normal lines.

7.3.4 Theorem. To show that a proper parametric transformation either leaves every normal unchanged or reverses every normal.

Proof. Let $\mathbf{r} = \mathbf{r}(u,v)$ be the equation of the surface and the relationship $u' = \phi(u,v)$ $v' = \psi(u,v)$ gives a proper parametric transformation. By calculus we have

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{r}}{\partial u'}.\frac{\partial u'}{\partial u} + \frac{\partial \mathbf{r}}{\partial v'}.\frac{\partial v'}{\partial u} \qquad (3)$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u'}.\frac{\partial u'}{\partial v} + \frac{\partial \mathbf{r}}{\partial v'}.\frac{\partial v'}{\partial v} \qquad (4)$$

Cross product of the (3) and (4), yield the following

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial u'}{\partial u}\frac{\partial v'}{\partial v} - \frac{\partial u'}{\partial v}\frac{\partial v'}{\partial u} \right) \frac{\partial \mathbf{r}}{\partial u'} \times \frac{\partial \mathbf{r}}{\partial v'}$$

or $$H\mathbf{N} = \frac{\partial(u',v')}{\partial(u,v)} H'\mathbf{N}'$$

where the symbols $H, \mathbf{N}; H', \mathbf{N}'$ have usual meaning in the two systems of parameters u, v and u', v' respectively.

Since the parametric transformation is proper, hence we have

$$\frac{\partial(u',v')}{\partial(u,v)} \neq 0$$

i.e. $J \neq 0$ where $J = \dfrac{\partial(u',v')}{\partial(u,v)}$.

We know that H and H' are always positive. Therefore relation (5) implies that the unit vectors $\mathbf{N}$ and $\mathbf{N}'$ are in the same directions if $J > 0$ and are in the opposite direction if $J < 0$. But the jacobian J is continuous over the domain D of parameters u, v and does not vanish and hence J is of invariable sigh throughout D. Therefore throughout D the vectors $\mathbf{N}$ and $\mathbf{N}'$ are either the same vectors or are opposite vectors. Hence a proper parametric transformation either leaves every normal unchanged or reverses every normal.

7.4. First fundamental form or Metric

Let $\mathbf{r} = \mathbf{r}(u,v)$ be the equation of a surface. The quadratic differential form

$$E du^2 + 2F du dv + G dv^2$$

in du, dv where $E = \mathbf{r}_1^2$, $F = \mathbf{r}_1 \bullet \mathbf{r}_2$, $G = \mathbf{r}_2^2$, is called metric or first fundamental form. The quantities E, F, G are called first-order fundamental magnitudes, or first fundamental coefficients, and are of great importance. The values E, F, G will generally vary from point to point on the surface these quantities are a function of u, v

7.4.1 Geometrical Interpretation of Metric

Consider a curve $u = u(t)$, $v = v(t)$ on the surface $\mathbf{r} = \mathbf{r}(u,v)$. Let $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ corresponding to the parameter values u,v and $u+du, v+dv$ respectively be the position vectors of two neighboring points *P* and *Q* on the surface.

We have

$$\partial\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u}du + \frac{\partial\mathbf{r}}{\partial v}dv = \mathbf{r}_1 du + \mathbf{r}_2 dv$$

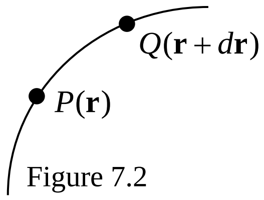

Figure 7.2

Let the arc *PQ* be *ds*. Since the points *P* and *Q* are neighboring points, therefore

$$ds = |d\mathbf{r}| \qquad \text{or} \qquad ds^2 = d\mathbf{r}^2$$

or $$ds^2 = (d\mathbf{r}_1 du + d\mathbf{r}_2 dv)^2$$

$$= \mathbf{r}_1^2 du^2 + 2\mathbf{r}_1.\mathbf{r}_2 dudv + \mathbf{r}_2^2 dv^2$$

or $$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \qquad (1)$$

ds is the 'infinitesimal distance' from the point u,v to the point ($u+du, v+dv$).

The name metric is assigned to the first fundamental form as mainly it is used to calculate the arc lengths of the curves on the surface. The arc length *s* of the curve has the following relation with parameter *t*

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

Special Case. On the parametric curve u=constant, *du*=0 and hence the metric (1) reduces to $ds^2 = Gdv^2$ and on the parametric curve v=constant, *dv*=0 and therefore the metric (1) reduces to $ds^2 = Edu^2$.

7.4.2 Relation between the coefficients *E*, *F*, *G*, and *H*.

We have $(\mathbf{r}_1 \times \mathbf{r}_2)^2 = \mathbf{r}_1^2\mathbf{r}_2^2 - (\mathbf{r}_1.\mathbf{r}_2)^2$

$$= EG - F^2 > 0 \quad \text{for} \quad [E>0,\ G>0\ ]$$

$\therefore$ $EG - F^2 = H^2$ (say), is always positive quantity and H is taken as the positive square root of $= EG - F^2$.

7.4.3 Important Properties

Following are two important properties of the first fundamental form or metric.

Property I: The metric is a positive quadratic form in *du, dv*.

Since $E > 0$ we may write $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$= \frac{1}{E}[E^2du^2 + 2EFdudv + EGdv^2]$$

$$= \frac{1}{E}[(Edu + Fdv)^2 + (EG - F)^2 dv^2]$$

$$= \frac{1}{E}[(Edu + Fdv)^2 + H^2dv^2]$$

≥ 0 as $H^2 > 0$ and for all real values of du and dv.

Again if $Edu^2 + 2Fdudv + Gdv^2 = 0$

i.e. $(Edu + Fdv)^2 + H^2dv^2 = 0$, then

$(Edu + Fdv)^2 = 0$ and $H^2dv^2 = 0$

i.e. $Edu + Fdv = 0$ and $dv = 0$ as $H^2 > 0$

i.e. $Edu = 0$ and $dv = 0$

i.e. $du = 0$ and $dv = 0$ as $E > 0$

But both *du* and *dv* can not vanish together. Hence Metric *i.e.*

$$Edu^2 + 2Fdudv + Gdv^2$$

is a positive definite quadratic form in *du*, *dv*.

Property II: The invariance Property.

The metric remains invariant if the parameters u, v are transformed to the parameters u', v' by the relation (say).

$$u' = \phi(u,v), \quad v' = \psi(u,v)$$

Then
$$\left.\begin{aligned}\mathbf{r}_1' &= \frac{\partial \mathbf{r}}{\partial u'} = \frac{\partial \mathbf{r}}{\partial u}.\frac{\partial u}{\partial u'} + \frac{\partial \mathbf{r}}{\partial v}.\frac{\partial v}{\partial u'} = \mathbf{r}_1 \frac{\partial u}{\partial u'} + \mathbf{r}_2 \frac{\partial v}{\partial u'} \\ \mathbf{r}_2' &= \frac{\partial \mathbf{r}}{\partial v'} = \mathbf{r}_1 \frac{\partial u}{\partial v'} + \mathbf{r}_2 \frac{\partial v}{\partial v'}\end{aligned}\right\} \qquad (1)$$

Again
$$\left.\begin{aligned}du &= \frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv' \\ dv &= \frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\end{aligned}\right\} \qquad (2)$$

Now we have

$$E'du'^2 + 2F'du'dv' + G'dv'^2$$
$$= \mathbf{r}_1'^2 du'^2 + 2\mathbf{r}_1'.\mathbf{r}_2' du'dv' + \mathbf{r}_2'^2 dv'^2$$
$$= (\mathbf{r}_1' du' + \mathbf{r}_2' dv')^2$$
$$= \left[\left(\mathbf{r}_1 \frac{\partial u}{\partial u'} + \mathbf{r}_2 \frac{\partial v}{\partial u'}\right)du' + \left(\mathbf{r}_1 \frac{\partial u}{\partial v'} + \mathbf{r}_2 \frac{\partial v}{\partial v'}\right)dv'\right]^2 \text{ [putting } \mathbf{r}_1 \text{ and } \mathbf{r}_2' \text{ from equation (1)]}$$
$$= \left[\mathbf{r}_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + \mathbf{r}_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right]^2$$
$$= (\mathbf{r}_1 du + \mathbf{r}_2 dv)^2 \qquad \text{[using equation (2)]}$$
$$= \mathbf{r}_1^2 du^2 + 2\mathbf{r}_1.\mathbf{r}_2 dudv + \mathbf{r}_2^2 dv^2$$
$$= Edu^2 + 2Fdudv + Gdv^2$$

Hence the metric is invariant.

7.4.4 Element of area. Consider the figure *ABCD* whose vertices *A*, *B*, *C*, *D* have parameter values (u,v), $(u+du,v)$, $(u+du,v+dv)$, $(u,v+dv)$ respectively. If *du* and *dv* are small and positive, then the figure *ABCD* is approximately a parallelogram. Let $\mathbf{r}$ denote the position vector of any point on the figure, then

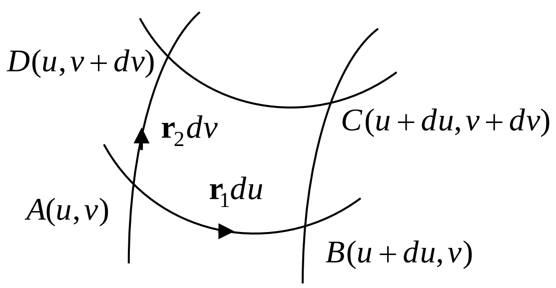

Figure 7.3

AD=Position vector of *D* –Position vector of *A*

$$= \mathbf{r}(u, v+dv) - \mathbf{r}(u,v)$$

$$= \mathbf{r}(u,v) + \frac{\partial \mathbf{r}}{\partial v} dv - \mathbf{r}(u,v)$$

$$\frac{\partial \mathbf{r}}{\partial v} dv = \mathbf{r}_2 dv.$$

Similarly $\mathbf{AB} = \mathbf{r}_1 du$

Hence the area *ds* of the parallelogram *ABCD* is given by

$$dS = \left| \mathbf{r}_1 du \times \mathbf{r}_2 dv \right| = \left| \mathbf{r}_1 \times \mathbf{r}_2 \right| du dv = H du dv.$$

Thus the element of the area on the surface at the point (u,v) is taken to be $Hdudv$.

7.5. First Order magnitudes. The suffix 1 will be used to indicate the partial differentiation with respect to u, and the suffix 2 partial differentiation with respect to v. Thus

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \ \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v}$$

$$\mathbf{r}_{11} = \frac{\partial^2 \mathbf{r}}{\partial u^2}, \ \mathbf{r}_{12} = \frac{\partial^2 \mathbf{r}}{\partial u \, \partial v}, \ \mathbf{r}_{22} = \frac{\partial^2 \mathbf{r}}{\partial v^2}$$

and so on. The vector $\mathbf{r}_1$ is tangential to the curve v=constant at the point **r**, for its direction is that of the displacement $d\mathbf{r}$ due to a variation in du the first parameter only. We take the positive direction along the parametric curve v=constant as that for which u increases. This is the direction of the vector $\mathbf{r}_1$. Similarly for $\mathbf{r}_2$ is tangential for the curve u=constant in the positive sense, which corresponds to increase of v.

Consider the neighboring points on the surface, with position vectors **r** and $\mathbf{r} + d\mathbf{r}$, corresponding to the parameter values u, v and $u+du$, $v+dv$ respectively. Then

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv$$

$$\mathbf{r}_1 du + \mathbf{r}_2 dv.$$

Since the two points are adjacent points on a curve passing through them, the length ds of the element of arc joining them is equal to their actual distance $|d\mathbf{r}|$ apart. Thus

$$ds^2 = d\mathbf{r}^2 = (\mathbf{r}_1 du + \mathbf{r}_2 dv)^2$$

$$\mathbf{r}_1^2 du^2 + 2\mathbf{r}_1 \bullet \mathbf{r}_2 du dv + \mathbf{r}_2^2 dv^2$$

If then we write $E = \mathbf{r}_1^2$, $F = \mathbf{r}_1 \bullet \mathbf{r}_2$, $G = \mathbf{r}_2^2$ (1)

We have the formula

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \qquad (2)$$

The quantities denoted by E, F, G are called the fundamental magnitudes of the first order. They are of the greatest importance and will occur throughout the remainder of this book. The quantities $EG\text{-}F^2$ are positive on a real surface when u and v real. For $\sqrt{E}$ and $\sqrt{G}$ are the modules of $\mathbf{r}_1$ and $\mathbf{r}_2$, and if ω denote the angle between these vectors, $F = \sqrt{EG}\cos\omega$, and therefore $EG\text{-}F^2$ is positive. We shall use the notation

$$H^2 = EG - F^2 \qquad (3)$$

and let H denote the positive square root of this quantity.

The length of an element of the parametric curve v =constant found from (2) by putting $dv = 0$.

Its value is, therefore $\sqrt{E}$. The unit vector tangential to the curve v =constant. Thus

$$\mathbf{a} = \frac{1}{\sqrt{E}}\frac{\partial \mathbf{r}}{\partial u} = E^{-1/2}\mathbf{r}_1.$$

Similarly, the length of an element of the curve u =constant, $\sqrt{G}dv$ and the unit tangent to this curve is

$$\mathbf{b} = \frac{1}{\sqrt{G}}\frac{\partial \mathbf{r}}{\partial v} = G^{-1/2}\mathbf{r}_2$$

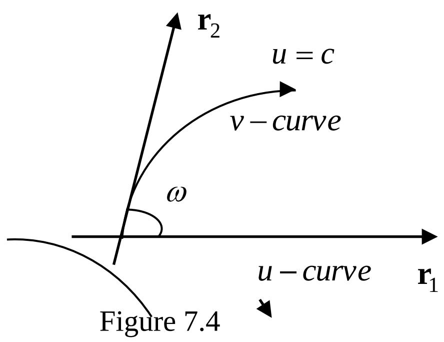

Figure 7.4

The two parametric curves through any point on the surface at an angle ω such that

$$\cos\omega = \mathbf{a} \bullet \mathbf{b} = \frac{\mathbf{r}_1 \bullet \mathbf{r}_2}{\sqrt{EG}} = \frac{F}{\sqrt{EG}}$$

Therefore $$\sin\omega = \sqrt{\frac{EG - F^2}{EG}} = \frac{H}{\sqrt{EG}} \qquad (4)$$

and $$\tan\omega = \frac{H}{F}.$$

Also since $$\sin\omega = |\mathbf{a} \times \mathbf{b}| = \frac{1}{\sqrt{EG}} |\mathbf{r}_1 \times \mathbf{r}_2|$$

it follows that $$|\mathbf{r}_1 \times \mathbf{r}_2| = H \qquad (5)$$

The parametric curves will cut at right angles at any point $F = 0$ at that point, and they will do so at all points if $F = 0$ on the surface. In this case, they are said to be orthogonal. The $F = 0$ is the necessary and sufficient condition that the parametric curves may form an orthogonal system.

7.5.1 Theorem. To prove that F=0, is the necessary and sufficient condition for the parametric curves on a surface to be orthogonal.

Solution. We know that $\omega = H / F$ (1)

where ω is the angle between the parametric curves.

Necessary condition: Suppose the system of parametric curves is orthogonal *i.e.* $\omega = \frac{1}{2}\pi$.

Equation (1) gives $\infty = H / F$ *i.e.* $F = 0$.

Sufficient condition: Suppose $F = 0$.

Equation (1) gives $\tan\omega = \infty$ which implies $\omega = \frac{1}{2}\pi$ and hence the system of parametric curves is orthogonal.

7.5.2 Theorem. To prove through every point of the surface there passes one and only one parametric curve of each system.

Solution. Consider a point $P(u_0, v_0)$; then u_0 and v_0 are uniquely determined by *P* and there are just the two parametric curves $u = u_0$ and $v = v_0$ through *P*. It follows that no two parametric curves of the same system intersect and that the curves $u = u_0$ and $v = v_0$ intersect once but not more than once if (u_0, v_0) belong to *D*, the domain of the surface over which the curves lie.

7.6. Direction coefficients

At a point $P(\mathbf{r}(u,v))$ on a surface, $\mathbf{r} = \mathbf{r}(u,v)$ there are independent vectors, **N** (the unit normal to the surface) and $\mathbf{r}_1, \mathbf{r}_2$ (the tangential vectors) so that an arbitrary vector **a** at *P* can therefore be exposed in the form

$$\mathbf{a} = a_n \mathbf{N} + (\lambda \mathbf{r}_1 + \mu \mathbf{r}_2) \qquad (1)$$

Where $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ is the tangential part of **a** and a_n, λ, μ are scalars defined uniquely by (1).

Let $\mathbf{T} = \lambda \mathbf{r}_1 + \mu \mathbf{r}_2 \qquad (2)$

Now λ, μ are the components of tangential vector $\lambda \mathbf{r}_1 + \mu \mathbf{r}_2$ written as vector (λ, μ). We have

$$|\mathbf{T}| = |\lambda \mathbf{r}_1 + \mu \mathbf{r}_2| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2} \qquad (3)$$

The scalar 'a_n' is called the normal component of **a**, and is given by

$$a_n = \mathbf{a}.\mathbf{N} \qquad \text{[Taking the scalar product of (1) with } \mathbf{N}]$$

To describe the direction of the tangent plane at *P*, it is convenient to use the component of a unit vector **e**, say, in the direction. These components are written as l, m and are called direction coefficients. Thus we have the identity

$$\mathbf{e}^2 = 1 = |l\mathbf{r}_1 + m\mathbf{r}_2|^2 = El^2 + 2Flm + Gm^2$$

$$i.e. \qquad El^2 + 2Flm + Gm^2 = 1 \qquad (4)$$

Now we may define the direction coefficients of a direction on a surface as follows.

Definition. Let **T** be a tangential vector to the surface at the point *P*. Suppose that the unit vector in the direction of **T** is **e**. If (l, m) are the component of the unit vector **e**. *i.e.* $\mathbf{e} = l\mathbf{r}_1 + m\mathbf{r}_2$, then (l, m) are termed as the direction coefficients of the direction represented by **T**. The direction coefficient of the direction, opposite to the direction whose coefficients are (l, m) are $(-l, -m)$.

7.6.1 To find the angle between two directions (tangential direction) on a surface at point *P*.

Solution. Now let the direction coefficients of the two directions at a point P be (l,m) and $(-l,-m)$ then the corresponding unit vectors **e** and $\mathbf{e}'$ are given by

$$\mathbf{e} = l\,\mathbf{r}_1 + m\mathbf{r}_2 \qquad \text{and} \qquad \mathbf{e}' = l'\,\mathbf{r}_1 + m'\,\mathbf{r}_2$$

If θ be the angle between these two directions, then

$$\cos\theta = \mathbf{e}.\mathbf{e}'$$

or $$\cos\theta = (l\,\mathbf{r}_1 + m\mathbf{r}_2).(l'\,\mathbf{r}_1 + m'\,\mathbf{r}_2)$$

$$= ll'\mathbf{r}_1.\mathbf{r}_1 + (lm' + l'm)\mathbf{r}_1.\mathbf{r}_2 + mm'\mathbf{r}_2.\mathbf{r}_2$$

$$= Ell' + F(lm' + l'm) + Gmm'$$

$$\sin\theta = |\mathbf{e}\times\mathbf{e}'|$$

or $$\sin\theta = |(l\,\mathbf{r}_1 + m\mathbf{r}_2)\times(l'\,\mathbf{r}_1 + m'\,\mathbf{r}_2)|$$

$$= |l(lm' - m\,l')(\mathbf{r}_1\times\mathbf{r}_2)|$$

$$= (lm' - m\,l')H \qquad [\text{where } \mathbf{N}H = \mathbf{r}_1\times\mathbf{r}_2]$$

Also $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(lm' - m\,l')H}{Ell' + F(lm' + l'm) + Gmm'}$$

Remark: If θ is such that $0 \le \theta \le \pi$ then

$$\left.\begin{aligned} &\cos\theta = Ell' + F(lm' + l'm) + Gmm' \\ \textit{and} \quad &\sin\theta = |lm' - ml'|H \end{aligned}\right\} \qquad (1)$$

7.7. Direction ratios.

If (l,m) are the direction coefficients of a direction on the surface, then the numbers λ,μ which are proportional to l,m respectively are called direction ratios.

It is convenient to use (λ,μ) as direction ratios and (l,m) as actual direction cosines.

To find the relation between *l, m* and (λ,μ)

By definition, direction ratios (λ,μ) are proportional to direction coefficients (l,m), therefore, let

$$\frac{l}{\lambda} = \frac{m}{\mu} = C\text{, giving} \qquad l = C\lambda,\ m = C\mu$$

But we know $$El^2 + 2Flm + Gm^2 = 1$$

$\therefore \qquad EC^2\lambda^2 + 2FC^2\lambda\mu + GC^2\mu^2 = 1$

or $\qquad C^2 = \dfrac{1}{E\lambda^2 + 2F\lambda\mu + G\mu^2}$

or $\qquad C = \dfrac{1}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}}$

$\therefore \qquad l = C\lambda = \dfrac{\lambda}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}}$

and $\qquad m = C\mu = \dfrac{\lambda}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}}$

Thus the relation between (λ, μ) and (l, m) is expressed as

$$(l, m) = (\lambda, \mu)/(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2} \qquad (5)$$

Example 7.1 For a surface of revolution

$$\mathbf{r} = (u\cos v, u\sin v, f(u)),$$

$$\mathbf{r}_1 = (\cos v, \sin v, f'(u)),$$

$$\mathbf{r}_2 = (-u\sin v, u\cos v, 0);$$

$$\therefore \qquad E = \mathbf{r}_1^2 = 1 + f'^2,$$

$$F = \mathbf{r}_1 \bullet \mathbf{r}_2 = 0$$

$$G = \mathbf{r}_2^2 = u^2$$

are the parametric curves are orthogonal, and

$$ds^2 = (1 + f'^2)du^2 + u^2 dv^2.$$

Example 7.2 Calculate the same quantities for the surface of the preceding example.

Example 7.3 Find the direction coefficients making an angle $\pi/2$ with the direction coefficients (l, m)

.

Solution. Let (l', m') be the required coefficients.

Since $\theta = \pi/2$,

Then $\qquad Ell' + F(lm' + l'm) + Gmm' = 0 \qquad (1)$

$\qquad (lm' - ml')H = 1 \qquad (2)$

Equation (1) may be written as

$$l'(El+Fm)+m'(Fl+Gm)=0$$

or $$\frac{l'}{(Fl+Gm)}=\frac{m'}{(El+Fm)}=a \text{ (say)}$$

$\therefore$ $$l'=a(Fl+Gm),\ m'=a(El+Fm) \qquad (3)$$

Substituting these values in equation (2), we get

$$Hal(El+Fm)+Ham(Fl+Gm)=1$$

or $$\frac{1}{a}=H(El^2+2Flm+Gm^2)=H$$

or $$a=\frac{1}{H}.$$

Substituting these values of a in equation (3), we have

$$l'=\frac{Fl+Gm}{H}, \qquad m'=\frac{El+Fm}{H}.$$

Example 7.4 Calculate the curve $u=u(t)$ and $v=v(t)$. Find the direction coefficients of the tangent to the curve.

Solution. The position vector of a current point is given by

$$\mathbf{r}=\mathbf{r}(u,v)$$

And $\frac{d\mathbf{r}}{dt}$ represent a tangent vector given by

$$\frac{d\mathbf{r}}{dt}=u\mathbf{r}_1+v\mathbf{r}_2.$$

Since $\frac{d\mathbf{r}}{dt}$ is not a unit vector, therefore the components $(\dot{u},\dot{v})$ of $\frac{d\mathbf{r}}{dt}$ are direction ratios of the tangents to the given (u',v').

7.8 Directions on a surface. Any direction on the surface on a given point (u,v) is determined by the increments du, dv of the parameters for a small displacement in that direction. Let ds be the length of the displacement $d\mathbf{r}$ corresponding to increments du, dv and let δs be the length of another displacement $\delta\mathbf{r}$ due to increments $\delta u, \delta v$. Then

$$d\mathbf{r} = \mathbf{r}_1 du + \mathbf{r}_2 dv,$$

$$\delta\mathbf{r} = \mathbf{r}_1 \delta u + \mathbf{r}_2 \delta v$$

have an inclination ψ of these directions is then given by

$$ds\,\delta s\cos\psi = d\,\mathbf{r} \bullet \delta\,\mathbf{r}$$

$$= E du\delta u + F(du\delta v + dv\delta u) + G dv\delta v,$$

$$ds\,\delta\, s\sin\psi = |d\,\mathbf{r} \times \delta\,\mathbf{r}|$$

$$= |du\,\delta v - dv\delta u|\,|\mathbf{r}_1 \times \mathbf{r}_2|$$

$$= H\,|du\,\delta v - dv\delta u|.$$

These two directions are perpendicular if $\cos\psi = 0$, that is if

$$E\frac{du}{dv}\frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G = 0 \qquad (6)$$

As an important particular case, the angle θ between the direction du, dv and that of the curve v =constant may be deduced from the above results by putting $\delta v = 0$ and $\delta s = \sqrt{E}\delta u$.

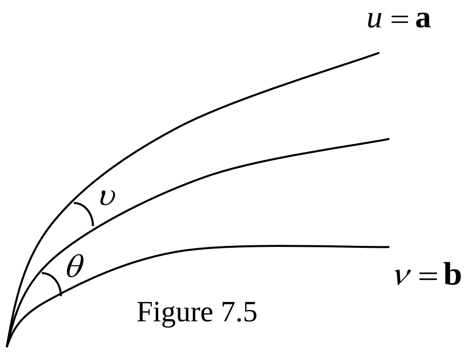

Figure 7.5

Thus
$$\left.\begin{aligned} \cos\theta &= \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right) \\ \sin\theta &= \frac{H}{\sqrt{E}}\left|\frac{dv}{ds}\right| \end{aligned}\right\} \qquad (7)$$

Similarly, its inclination υ to the parametric curve u =constant is obtained by putting δu =0 and $\delta s = \sqrt{G}\delta v$. Thus

$$\left.\begin{aligned}\cos\upsilon &= \frac{1}{\sqrt{G}}\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)\\ \sin\upsilon &= \frac{H}{\sqrt{G}}\left|\frac{du}{ds}\right|\end{aligned}\right\} \qquad (8)$$

The formula (6) leads immediately to the differential equation of the orthogonal trajectories of the family of curves given by

$$P\,\delta u + Q\delta v = 0$$

where *P, Q* are functions of u, v. For the given family of the curves, we have

$$\frac{\delta u}{\delta v} = -\frac{Q}{P}$$

And therefore from (6), if du/dv refer to the orthogonal trajectories, it follows that

$$(EQ - FP)du + (FQ - GP)dv = 0 \qquad (9)$$

This is the required differential equation. If instead of the differential equation of the original family of the curves, we are given their equation in the form

$$\phi(u,v) = c,$$

Where *c* is the arbitrary constant, it follows that

$$\phi_1\delta u + \phi_2\delta v = 0,$$

the suffixes as usual denoting partial derivatives with respect to u and v. The differential equation of the orthogonal trajectories is then obtained from the preceding result by putting $P = \phi_1$ and $Q = \phi_2$, which gives

$$(E\phi_2 - F\phi_1)du + (F\phi_2 - G\phi_1)dv = 0 \qquad (10)$$

An equation of the form

$$Pdu^2 + Qdudv + Rdv^2 = 0$$

Determines two directions on the surface, for it is quadratic in du/dv. Let the roots of the quadratic be denoted by du/dv and $\delta u/\delta v$. Then

$$\frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P},$$

and $$\frac{du}{dv}\frac{\delta u}{\delta v}=\frac{R}{P}$$

and substituting these values in (6), we see that the two directions will be at right angles if

$$ER-FQ+GP=0 \tag{11}$$

7.9 Normal. The normal to the surface at any point is perpendicular to every tangent line through that point and is

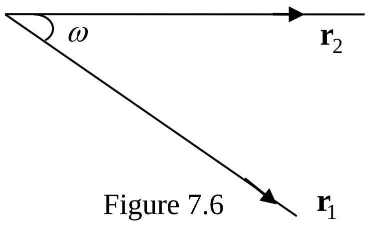

Figure 7.6

Therefore perpendicular to each of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Hence it is parallel to the vector $\mathbf{r}_1\times\mathbf{r}_2$, and we take the direction of this vector as the positive direction of the normal. The unit vector **n** parallel to the normal is therefore

$$\mathbf{n}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1\times\mathbf{r}_2|}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{H} \tag{12}$$

This may be called the unit normal to the surface. Since it is perpendicular to each of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, we have

$$\mathbf{n}\bullet\mathbf{r}_1=0,\ \mathbf{n}\bullet\mathbf{r}_2=0 \tag{13}$$

The scalar triple product of these three vectors has the value

$$[\mathbf{n},\mathbf{r}_1,\mathbf{r}_2]=\mathbf{n}\bullet\mathbf{r}_1\times\mathbf{r}_2=H\mathbf{n}^2=H \tag{14}$$

For the cross product of **n** with $\mathbf{r}_1$ and $\mathbf{r}_2$, we have

$$\mathbf{r}_1\times\mathbf{n}=\frac{1}{H}\mathbf{r}_1\times(\mathbf{r}_1\times\mathbf{r}_2)=\frac{1}{H}(F\,\mathbf{r}_1-E\,\mathbf{r}_2),$$

$$\mathbf{r}_2\times\mathbf{n}=\frac{1}{H}\mathbf{r}_2\times(\mathbf{r}_1\times\mathbf{r}_2)=\frac{1}{H}(G\,\mathbf{r}_1-F\mathbf{r}_2).$$

Example 7.5 Find an equation for the tangent plane to the surface $z=x^2+y^2$ at the point (1,-1, 2).

Solution. Let the parametric equation for the surface be $x = u$, $y = v$, $z = x^2 + y^2$, so that at the point (1, -1, 2), $u = 1,\ v = -1$.

Now position vector of any point on the surface is

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}$$

$$\therefore \qquad \mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + 2u\mathbf{k}\ ;\ \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + 2v\mathbf{k}$$

At the point $u = 1,\ v = -1$ *i.e.* at (1, -1, 2)

$$\mathbf{r}_1 = \mathbf{i} + 2\mathbf{k}\ ;\ \mathbf{r}_2 = \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{r}_1 \times \mathbf{r}_2 = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ at $u = 1,\ v = -1$

$\therefore$ The equation of the tangent plane at (1, -1, 2) is

$$(\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = 0$$

or $\quad \{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\} = 0$

or $\quad -2(x-1) + 2(y+1) + (z-2) = 0$

or $\quad -2x + 2y + z + 2 = 0$

Example 7.6 Find the equation of the tangent plane and normal to the surface $xyz = 4$ at the point (1, 2, 2).

Solution. The equation of the surface is

$$F(x, y, z) = xyz - 4 = 0 \qquad (1)$$

Differentiating (1) partially w.r.t. *x*, *y*, *z* respectively, we have

$$\frac{\partial F}{\partial x} = yz,\ \frac{\partial F}{\partial y} = xz,\ \frac{\partial F}{\partial z} = xy$$

$\therefore$ At the point (1, 2, 2), we have

$$\frac{\partial F}{\partial x} = 4,\ \frac{\partial F}{\partial y} = 2,\ \frac{\partial F}{\partial z} = 2$$

$\therefore$ The equation of the tangent plane at (1, 2, 2) is given by

$$(x-1)4 + (y-2)2 + (z-2)2 = 0$$

or $\quad 2x + y + z = 6$

The equation of the normal line at the point (1, 2, 2) are

$$\frac{(x-1)}{4}=\frac{(y-2)}{2}=\frac{(z-2)}{2}$$

or $$\frac{(x-1)}{2}=\frac{(y-2)}{1}=\frac{(z-2)}{1}$$

Example 7.7 Find the equation of the tangent plane and normal to the surface $z = xy$ at the point (2, 3, 6).

Solution. Here $F(x, y, z) = xy - z$

$$\frac{\partial F}{\partial x}=y\,,\ \frac{\partial F}{\partial y}=x\,,\ \frac{\partial F}{\partial z}=-1$$

$\therefore$ At the point (2, 3, 6), we have

$$\frac{\partial F}{\partial x}=3\,,\ \frac{\partial F}{\partial y}=2\,,\ \frac{\partial F}{\partial z}=-1$$

$\therefore$ Equation of the tangent plane at (2, 3, 6) is given by

$$(x-2)3+(y-3)2+(z-6)(-1)=0$$

or $$3x+2y-z=6$$

The equation of the normal line at the point (2, 3, 6) are

$$\frac{(x-2)}{3}=\frac{(y-2)}{2}=\frac{(z-6)}{-1}.$$

Example 7.8 Prove that the tangent to the surface $xyz = a^3$ and the coordinate planes bound a tetrahedron of constant volume.

Solution. The equation of the surface is

$$F(x, y, z) = xyz - a^3 = 0 \qquad (1)$$

Differentiating (1) partially w.r.t. *x*, *y*, *z* respectively, we have

$$\frac{\partial F}{\partial x}=yz\,,\ \frac{\partial F}{\partial y}=xz\,,\ \frac{\partial F}{\partial z}=xy$$

The equation of the tangent plane at any point x, y, z to the given surface (1) is

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0$$

or $(X-x)yz+(Y-y)xz+(Z-z)xy=0$

or $yzX+xzY+xyZ=3xyz$

or $$\frac{X}{x}=\frac{Y}{y}=\frac{Z}{z}=3 \qquad (2)$$

The intercepts made by (2) on the co-ordinate axes are 3*x*, 3*y*, 3*z* respectively.

$\therefore$ The volume of the tetrahedron formed by the tangent plane (2) and the coordinate plane is

$$=\frac{1}{6}3x.3y.3z=\frac{9}{2}xyz=\frac{9}{2}a^3, \text{ which is constant.}$$

Example 7.9 Show that the sum of the squares of the intercepts on the co-ordinate axes made by the tangent plane to the surface

$$x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3} \text{ is constant.}$$

Solution. The equation of the surface is

$$f(x,y,z)=x^{2/3}+y^{2/3}+z^{2/3}-a^{2/3}=0 \qquad (1)$$

$$\frac{\partial F}{\partial x}=\frac{2}{3}x^{-1/3},\ \frac{\partial F}{\partial y}=\frac{2}{3}y^{-1/3},\ \frac{\partial F}{\partial z}=\frac{2}{3}z^{-1/3}$$

The equation of the tangent plane at any point (*x*, *y*, *z*) to the given surface (1) is

$$(X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0$$

$$(X-x)\frac{2}{3}x^{-1/3}+(Y-y)\frac{2}{3}y^{-1/3}+(Z-z)\frac{2}{3}z^{-1/3}=0$$

or $$Xx^{-1/3}+Yy^{-1/3}+Zz^{-1/3}=x^{2/3}+y^{2/3}+z^{2/3}$$

or $$\frac{X}{x^{1/3}}=\frac{Y}{y^{1/3}}=\frac{Z}{z^{1/3}}=a^{2/3} \qquad \text{using (1)} \qquad (2)$$

Let x' be the intercept made by (2) on the *x*-axes then point (x', 0, 0) will lie on (2), thus giving

$$\frac{x'}{x^{1/3}}=a^{2/3} \quad \text{or} \quad x'=a^{2/3}x^{1/3}$$

Similarly, $y'=a^{2/3}y^{1/3},\ z'=a^{2/3}z^{1/3}$

where, y' and z' are intercepts made by (2) on *y* and *z* axes respectively.

Therefore the sum of the square of intercepts

$$= x' + y' + z' = (a^{2/3}x^{1/3})^2 + (a^{2/3}y^{1/3})^2 + (a^{2/3}z^{1/3})^2$$

$$a^{4/3}(x^{2/3} + y^{2/3} + z^{2/3} = a^{4/3}.a^{2/3} = a^2$$

Example 7.10 Prove that any tangent plane to the surface

$$a(x^2 + y^2) + xyz = 0$$

meets it again in a conic whose projection on the *xy*-plane is a rectangular hyperbola.

Solution. The equation of the surface is

$$f(x, y, z) = a(x^2 + y^2) + xyz = 0 \tag{1}$$

$$\frac{\partial F}{\partial x} = 2ax + yz\,,\ \frac{\partial F}{\partial y} = 2ay + xz\,,\ \frac{\partial F}{\partial z} = xy$$

Therefore at any point (x_1, y_1, z_1), we have

$$\frac{\partial F}{\partial x} = 2ax_1 + y_1z_1,\ \frac{\partial F}{\partial y} = 2ay_1 + x_1z_1,\ \frac{\partial F}{\partial z} = x_1y_1$$

The equation of the tangent plane at (x_1, y_1, z_1) to the given surface is

$$(X - x_1)\frac{\partial F}{\partial x} + (Y - y_1)\frac{\partial F}{\partial y} + (Z - z_1)\frac{\partial F}{\partial z} = 0$$

or $$(x - x_1)(2ax_1 + y_1z_1) + (y - y_1)(2ay_1 + x_1) + (z - z_1)(x_1y_1) = 0 \tag{2}$$

Now eliminating *z* between (1) and (2) we have

$$(x - x_1)(2ax_1 + y_1z_1) + (y - y_1)(2ay_1 + x_1) - \left(\frac{a(x^2 + y^2)}{xy} + z_1\right).x_1y_1 = 0$$

or $$xy[x(2ax_1 + y_1z_1) + y(2ay_1 + x_1z_1)] - (2ax_1^2 + 2ay_1^2 + 2x_1y_1z_1) = \{a(x^2 + y^2) + xyz_1\}x_1y_1$$

or $$xy\left[\frac{x}{x_1}(2ax_1^2 + x_1y_1z_1) + \frac{y}{y_1}(2ay_1^2 + x_1y_1z_1) - (2ax_1^2 + 2ay_1^2 + 2x_1y_1z_1)\right] = a(x^2 + y^2)x_1y_1 + xyx_1y_1z_1 \tag{3}$$

Now the point (x_1, y_1, z_1) also lies on (1)

$$= a(x^2 + y^2) + x_1y_1z_1 \tag{4}$$

Also $2ax_1^2 + x_1y_1z_1 = 2ax_1^2 - (ax_1^2 + ay_1^2)$ [using (4)]

or $2ax_1^2 + x_1y_1z_1 = ax_1^2 - ay_1^2$ (5)

Similarly $2ay_1^2 + x_1y_1z_1 = ay_1^2 - ax_1^2$ (6)

Using (4), (5), and (6) in (3) we have

$$xy\left[\frac{x}{x_1}(ax_1^2 - ay_1^2) + \frac{y}{y_1}(ay_1^2 - ax_1^2) - 0\right] = a(x^2 + y^2)x_1y_1 - a(x_1^2 + y_1^2)xy$$

or $(xy)a(x_1^2 - y_1^2)\left[\frac{x}{x_1} - \frac{y}{y_1}\right] = a[(x^2x_1y_1 - x_1^2x_1y_1) - (y_1^2x_1y_1 - y^2x_1y_1)]$

or $(xy)(x_1^2 - y_1^2)\left[\frac{x}{x_1} - \frac{y}{y_1}\right] = xx_1^2y_1\left(\frac{x}{x_1} - \frac{y}{y_1}\right) - yy_1^2x_1\left(\frac{x}{x_1} - \frac{y}{y_1}\right)$

or $(xy)(x_1^2 - y_1^2) = x(x_1^2y_1) - y(x_1y_1^2)$

or $z = 0,\ (xy)(x_1^2 - y_1^2) = x(x_1^2y_1) - y(x_1y_1^2)$

Clearly (7) represents a cylinder that passes through the curve of intersection of (1) and (2). The projection of conic (7) on *xy*-plane (*i.e.* *z*=0). This is the equation of the rectangular hyperbola.

Example 7.11 Find a unit normal vector to the surface

$2xz^2 - 3xy - 4x = 7$ at the point (1, -1, 2).

Solution. Here $F(x, y, z) = 2xz^2 - 3xy - 4x - 7 = 0$

$\therefore \quad \frac{\partial F}{\partial x} = 2z^2 - 3y - 4$ at (1, -1, 2)

$\frac{\partial F}{\partial y} = -3x = -3$ at (1, -1, 2)

$\frac{\partial F}{\partial z} = 4xz = 8$ at (1, -1, 2)

The vector normal to the surface is

$(\partial F/\partial x,\ \partial F/\partial y,\ \partial F/\partial z)$ *i.e.* (7, -3, 8)

Module of the normal vector

$$\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}=\sqrt{(49+9+64)}=\sqrt{122}$$

$\therefore$ Unit vector normal to the surface is

$$\left(\frac{7}{\sqrt{122}},-\frac{3}{\sqrt{122}},\frac{8}{\sqrt{122}}\right).$$

Example 7.12 Prove that at points common to the surface

$$a\,(yz+zx+xy)=xyz$$

and a sphere whose center is the origin, the tangent plane to the surface makes intercepts on axes whose sum is constant.

Solution. The equation of the surface is

$$F(x,y,z)=a(yz+zx+xy)-xyz=0 \qquad (1)$$

$$\frac{\partial F}{\partial x}=a(z+y)-yz\,;\ \frac{\partial F}{\partial y}=a(x+z)-xz\,;\ \frac{\partial F}{\partial z}=a(x+y)-xy$$

Let the equation of the sphere of radius **r** and center at the origin be

$$x^2+y^2+z^2=r^2 \qquad (2)$$

Let $P(x,y,z)$ be a common point to the surface (1) and (2). The equation to the tangent plane to (1) at $P(x,y,z)$ is

$$(X-x)[a(y+z)-yz]+(Y-y)[a(z+x)-zx]+(Z-z)[z(x+y)-xy]=0$$

or $$X[a(y+z)-yz]+Y[a(z+x)-zx]+Z[z(x+y)-xy]=2a(yz+zx+xy)-3xyz$$

or $$\frac{X}{x}[a(y+z)-yz]+\frac{Y}{y}[a(z+x)-zx]+\frac{Z}{z}[z(x+y)-xy]=2(xyz)-3xyz$$

or $$\frac{X}{x}[-ayz]+\frac{Y}{y}[-azx]+\frac{Z}{z}[-axy]=-xyz \qquad \text{using (1)}$$

Now dividing throughout by (xyz), we get

$$\frac{X}{\frac{x^2}{a}}+\frac{Y}{\frac{y^2}{a}}+\frac{Z}{\frac{z^2}{a}}=1 \qquad (3)$$

Clearly, the intercept made by the tangent plane (3) on the coordinate axes are

$$(x^2/a,\ y^2/a,\ z^2/a)$$

The sum of the intercepts

$$\frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{a}=\frac{x^2+y^2+z^2}{a}=\frac{r^2}{a} \qquad \text{using (3)}$$

which is constant.

Example 7.13 Deduce the formulae

$$H\mathbf{N}\times\mathbf{N}_1 = M\mathbf{r}_1 - L\mathbf{r}_2 \quad \text{and} \quad H\mathbf{N}\times\mathbf{N}_2 = N\mathbf{r}_1 - M\mathbf{r}_2 .$$

Solution. We know that

$$H\mathbf{N}=\mathbf{r}_1\times\mathbf{r}_2 \tag{1}$$

Differentiating equation (1) w.r.t. 'u' we have

$$H\mathbf{N}_1 + H_1\mathbf{N} = \mathbf{r}_{11}\times\mathbf{r}_2 + \mathbf{r}_1\times\mathbf{r}_{12} \tag{2}$$

Taking the cross product of both sides of equation (2) with **N**, we have

$$H\mathbf{N}\times\mathbf{N}_1 = \mathbf{N}\times(\mathbf{r}_{11}\times\mathbf{r}_2)+\mathbf{N}\times(\mathbf{r}_1\times\mathbf{r}_{12}) \tag{3}$$

Using the vector identity

$\mathbf{a}\times(\mathbf{b}\times c)=(\mathbf{a}.c)\mathbf{b}-(\mathbf{a}.\mathbf{b})c$ in the right-hand side of (3), we get

$$H\mathbf{N}\times\mathbf{N}_1 = [(\mathbf{N}.\mathbf{r}_2)\mathbf{r}_{11}-(\mathbf{N}.\mathbf{r}_{11})\mathbf{r}_2]+[(\mathbf{N}.\mathbf{r}_{12})\mathbf{r}_1-(\mathbf{N}.\mathbf{r}_1)\mathbf{r}_{12}]$$

$$= -L\mathbf{r}_2 + M\mathbf{r}_1 \quad \text{as } (\mathbf{N}.\mathbf{r}_{11})=L,\ (\mathbf{N}.\mathbf{r}_{12})=M$$

$$\mathbf{N}.\mathbf{r}_1 = 0 = \mathbf{N}.\mathbf{r}_2$$

Again differentiating (1) w.r.t. 'v' we have

$$H\mathbf{N}_2 + H_2\mathbf{N} = \mathbf{r}_{12}\times\mathbf{r}_2 + \mathbf{r}_1\times\mathbf{r}_{22} \tag{4}$$

Taking the cross product of both sides of equation (4) with **N**, we have

$$H\mathbf{N}\times\mathbf{N}_2 = \mathbf{N}\times(\mathbf{r}_{12}\times\mathbf{r}_2)+\mathbf{N}\times(\mathbf{r}_1\times\mathbf{r}_{22})$$

$$=[(\mathbf{N}.\mathbf{r}_2)\mathbf{r}_{12}-(\mathbf{N}.\mathbf{r}_{12})\mathbf{r}_2]+[(\mathbf{N}.\mathbf{r}_{22})\mathbf{r}_1-(\mathbf{N}.\mathbf{r}_1)\mathbf{r}_{22}]$$

$$= -M\mathbf{r}_2 + N\mathbf{r}_1 \quad \text{as } (\mathbf{N}.\mathbf{r}_{12})=M \quad (\mathbf{N}.\mathbf{r}_{22})=N .$$

Example 7.14 Shows that if L, M, N vanish everywhere on a surface, then the surface is a part of a plane.

Solution. We know that the surface normals at every point of a plane surface are parallel and hence the surface normal **N** is constant for a plane surface. Thus in these equations, we are to show that **N** is constant at every point of a surface.

Given $L = M = N = 0$

But $L = -\mathbf{N}_1.\mathbf{r}_1,\ M = -\mathbf{N}_1.\mathbf{r}_2 = -\mathbf{N}_2.\mathbf{r}_1,\ N = -\mathbf{N}_2.\mathbf{r}_2$

Therefore we have

$$\mathbf{N}_1.\mathbf{r}_1 = 0,\ \mathbf{N}_1.\mathbf{r}_2 = 0 \qquad (1)$$

$$\mathbf{N}_2.\mathbf{r}_1 = 0,\ \mathbf{N}_2.\mathbf{r}_2 = 0 \qquad (2)$$

Now $\mathbf{r}_1 \neq 0,\ \mathbf{r}_2 \neq 0$, the equation (1) implies that either,

$$\mathbf{N}_1 = 0$$

or $\mathbf{N}_1$ is perpendicular to the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ both

$\Rightarrow$ $\mathbf{N}_1$ is parallel to the vector $\mathbf{r}_1 \times \mathbf{r}_2$

$\Rightarrow$ $\mathbf{N}_1$ is parallel to the vector $\mathbf{N}H$

$\Rightarrow$ $\mathbf{N}_1$ is parallel to the vector $\mathbf{N}$.

But this is not true as $\mathbf{N}_1$ is perpendicular to $\mathbf{N}$, $\mathbf{N}$ being the vector of constant magnitude. Hence from (1), we must have

$\mathbf{N}_1 = 0$ *i.e.* $\mathbf{N}$ is independent of *u*.

Applying similar reasoning to equation (2), we get

$\mathbf{N}_2 = 0$ *i.e.* $\mathbf{N}$ is independent of *v*.

Hence **N** is a constant vector at every point of the surface. Hence proved.

Example 7.15 A real surface for which the equations

$E/L = F/M = G/N$ hold is either plane or spherical.

Solution. We know that

$$L = -\mathbf{N}_1.\mathbf{r}_1,\ M = -\mathbf{N}_1.\mathbf{r}_2 = -\mathbf{N}_2.\mathbf{r}_1,\ N = -\mathbf{N}_2.\mathbf{r}_2$$

Let $E/L = F/M = G/N = 1/\mu$ then

$$L = \mu E,\ M = \mu F,\ N = \mu G \qquad (1)$$

Consider the equation

$$L = -\mathbf{N}_1.\mathbf{r}_1$$

or $L + \mathbf{N}_1.\mathbf{r}_1 = 0$

or $\mu E + \mathbf{N}_1.\mathbf{r}_1 = 0$

or $\mu\mathbf{r}_1.\mathbf{r}_1 + \mathbf{N}_1.\mathbf{r}_1 = 0$ since $E = \mathbf{r}_1.\mathbf{r}_1$

or $(\mu\mathbf{r}_1 + \mathbf{N}_1)\mathbf{r}_1 = 0$

Similarly from relations $M = -\mathbf{N}_1.\mathbf{r}_2 = -\mathbf{N}_2.\mathbf{r}_1$ and $N = -\mathbf{N}_2.\mathbf{r}_2$ equation (1), get

$$(\mu\mathbf{r}_2 + \mathbf{N}_2)\mathbf{r}_1 = 0, \quad (\mu\mathbf{r}_1 + \mathbf{N}_1)\mathbf{r}_2 = 0, \quad (\mu\mathbf{r}_2 + \mathbf{N}_2)\mathbf{r}_2 = 0.$$

Since $\mathbf{N}$ is perpendicular to both $\mathbf{N}_1$ and $\mathbf{N}_2$, therefore each of $\mathbf{N}_1$ and $\mathbf{N}_2$ lies in the plane of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Hence the vector $\mu\mathbf{r}_1 + \mathbf{N}_1$ and $\mu\mathbf{r}_2 + \mathbf{N}_2$ lies in the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$.

Now since $\mathbf{r}_1 \neq 0$ and $\mathbf{r}_2 \neq 0$ and the vector $\mu\mathbf{r}_1 + \mathbf{N}_1$ is not perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$, therefore the equations

$$(\mu\mathbf{r}_1 + \mathbf{N}_1).\mathbf{r}_1 = 0 \quad \text{and} \quad (\mu\mathbf{r}_1 + \mathbf{N}_1).\mathbf{r}_2 = 0$$

will hold if and only if

$$(\mu\mathbf{r}_1 + \mathbf{N}_1) = 0 \quad i.e. \quad \mathbf{N}_1 = -\mu\mathbf{r}_1 \tag{2}$$

Similarly from other two equations namely

$$(\mu\mathbf{r}_2 + \mathbf{N}_2).\mathbf{r}_1 = 0 \quad \text{and} \quad (\mu\mathbf{r}_2 + \mathbf{N}_2).\mathbf{r}_2 = 0$$

we conclude that

$$(\mu\mathbf{r}_2 + \mathbf{N}_2) = 0 \quad i.e. \quad \mathbf{N}_2 = -\mu\mathbf{r}_2 \tag{3}$$

Differentiating equation (2) w.r.t. 'v' and (3) w.r.t. 'u' we get

$$\mathbf{N}_{12} = -\mu\mathbf{r}_{12} - \mu_2\mathbf{r}_1 \tag{4}$$

$$\mathbf{N}_{21} = -\mu\mathbf{r}_{12} - \mu_1\mathbf{r}_2 \tag{5}$$

Subtracting (5) from (4), we get

$$\mu_1\mathbf{r}_2 = \mu_2\mathbf{r}_1 \qquad [\text{as } \mathbf{N}_{12} = \mathbf{N}_{21}]$$

Now $\mathbf{r}_1$ and $\mathbf{r}_2$ are not parallel, therefore (6) will hold if

$$\mu_1 = 0, \quad \mu_2 = 0$$

i.e. μ is constant, being independent of μ and v.

Now if $\mu \neq 0$, equation (2) gives

$$\mathbf{r}_1 = \frac{1}{\mu}\mathbf{N}_1$$

Integrating it, $\mathbf{r} = \frac{1}{\mu}\mathbf{N} + \mathbf{a}$ [where **a** is constant]

or $(\mathbf{r} - \mathbf{a}) = -\frac{1}{\mu}\mathbf{N}$

squaring, $(\mathbf{r} - \mathbf{a})^2 = -\frac{1}{\mu^2}\mathbf{N}^2 = 1$

This shows that the locus of **r** is a sphere. Hence the surface is spherical.

Again if $\mu = 0$, then (2) and (3) gives

$$\mathbf{N}_1 = 0, \quad \text{and} \quad \mathbf{N}_2 = 0$$

This shows that **N** is constant being independent of μ and v. Thus **N** is a constant vector at any point on the surface. Hence the surface is a plane surface.

7.9 CHECK YOUR PROGRESS

SA1: On the surface of revolution $x = u\cos\phi$, $y = u\sin\phi$, $s = f(u)$, what are the parametric curves u =constant, and what are the curve ϕ =constant.

SA2: On the right helicoids given by $x = u\cos\phi$, $y = u\sin\phi$, $s = c\phi$, show that parametric curves are circular helices and straight lines.

SA3: On the hyperboloid of one sheet $\frac{x}{a} = \frac{\lambda + \mu}{1 + \lambda\mu}$, $\frac{y}{b} = \frac{1 - \lambda\mu}{1 + \lambda\mu}$, $\frac{z}{c} = \frac{\lambda - \mu}{1 + \lambda\mu}$. The parametric curves are the generators. What curves are represented by $\lambda = \mu$, and by $\lambda\mu$ =constant.

SA4: If ψ is the angle between the two directions given by

$$Pdu^2 + Qdudv + Rdv^2 = 0,$$

show that $$\tan\psi = \frac{H\sqrt{Q^2 - 4PR}}{ER - FQ + GP}.$$

SA5: If the parametric curves are orthogonal, show that the differential equation of lines on the surface cutting the curves u =constant at a constant angle β is

$$\frac{du}{dv} = \tan\beta\sqrt{\frac{G}{E}}.$$

SA6: Prove that the differential equations of the curves which bisect the angles between the parametric curves are $\sqrt{E}du - \sqrt{G}dv = 0$ and $\sqrt{E}du + \sqrt{G}dv = 0$.

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CHAPTER-8

FUNDAMENTAL FORMS-II

Objectives: In continuation of the previous chapters in the current chapter the **s**tudents will learn intrinsic and non-intrinsic properties of a surface, Second fundamental form and their geometrical meaning, hyperbolic, parabolic, elliptic, and planar, derivative of unit normal, the curvature of a normal section in terms of fundamental magnitudes, orthogonal trajectory and its condition, double family of curves .

INTRODUCTION:

We recall that a curve in E^3 is uniquely determined by two local invariant quantities curvature and torsion, as functions of arc length. Similarly, a surface of E^3 is uniquely determined by certain local invariant quantities called the first and second fundamental forms.

8.1. Intrinsic and non-intrinsic properties of a surface:

Any property or formula of a surface which can be deduced from the metric of the vector function $\mathbf{r}(u,v)$ [*i.e.* without knowing the equation of the surface] is called an intrinsic property. Those properties which are not intrinsic are called non-intrinsic properties of the surface.

8.2. Second fundamental form

We now suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on a surface of class ≥ 2. Then at each point on the patch there is a unit normal $\mathbf{N} = \dfrac{\mathbf{x_u} \times \mathbf{x_v}}{|\mathbf{x_u} \times \mathbf{x_v}|}$, which is a function of u and v of class C^1 with differential $d\mathbf{N} = N_u du + N_v dv$. Observe in figure 8.1 that $d\mathbf{N}$ is

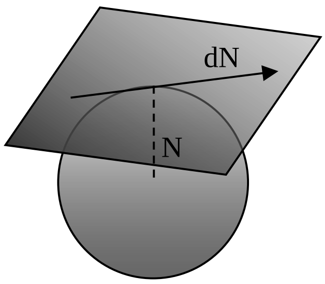

Figure 8.1

orthogonal to $\mathbf{N}$ since it is parallel to the tangent plane of the spherical image of $\mathbf{N}$. This also follows from $0 = d(1) = d(\mathbf{N.N}) = 2d\mathbf{N.N}$. Thus $d\mathbf{N}$ is a vector parallel to the tangent plane at $\mathbf{x}$ as shown in the figure 8.1.

Now consider the quantity $-d\mathbf{x}\cdot d\mathbf{N} = -(\mathbf{x_u}du + \mathbf{x_v}dv)\cdot(\mathbf{N_u}du + \mathbf{N_v}dv)$

$$= -\mathbf{x_u}\cdot\mathbf{N_u}du^2 - (\mathbf{x_u}\cdot\mathbf{N_v} + \mathbf{x_v}\cdot\mathbf{N_u})dudv - \mathbf{x_v}\cdot\mathbf{N_v}dv^2$$

$$= Ldu^2 + 2Mdudv + Ndv^2 \qquad (8.1)$$

where $L = -\mathbf{x_u}\cdot\mathbf{N_u}$, $M = -\frac{1}{2}(\mathbf{x_u}\cdot\mathbf{N}_v + \mathbf{x}_v\cdot\mathbf{N}_u)$, $N = -\mathbf{x}_v\cdot\mathbf{N}_v$

Let $\mathbf{r} = \mathbf{r}(u,v)$ be the equation of the surface. The quadratic differential form

$$Ldu^2 + 2Mdu\,dv + Ndv^2$$

in *du*, *dv* is called the second fundamental form. The quantities *L*, *M*, *N* are called second order fundamental magnitudes or fundamental coefficients and are explained as follow:

We know $\mathbf{r}_{11} = \dfrac{\partial^2\mathbf{r}}{\partial u^2}$, $\mathbf{r}_{12} = \dfrac{\partial^2\mathbf{r}}{\partial u\,\partial v} = \dfrac{\partial^2\mathbf{r}}{\partial v\partial u} = \mathbf{r}_{21}$, $\mathbf{r}_{22} = \dfrac{\partial^2\mathbf{r}}{\partial v^2}$

$$\mathbf{N} = \frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1\times\mathbf{r}_2|} = \frac{\mathbf{r}_1\times\mathbf{r}_2}{H} \qquad (8.2)$$

where $\mathbf{n}$ is the unit normal vector to the surface at the point $\mathbf{r} = \mathbf{r}(u,v)$. W denotes the resolved parts of the vectors $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{22}$ along the surface normal $\mathbf{N}$ by *L*, *M*, *N* respectively thus

$$L = \mathbf{N}.\mathbf{r}_{11}\ ;\quad M = \mathbf{N}.\mathbf{r}_{12}\ ;\quad N = \mathbf{N}.\mathbf{r}_{22}$$

$LN - M^2 = \mathbf{T}^2$ (say) where $\mathbf{T}^2$ is not necessarily positive. (8.3)

8.3. Second order magnitudes: The second derivatives of $\mathbf{r}$ with respect to u and v are denoted by

$$\mathbf{r}_{11} = \frac{\partial^2\mathbf{r}}{\partial u^2},\ \mathbf{r}_{12} = \frac{\partial^2\mathbf{r}}{\partial u\,\partial v},\ \mathbf{r}_{22} = \frac{\partial^2\mathbf{r}}{\partial v^2}.$$

The fundamental magnitudes of the second order are the resolved parts of these vectors in the direction of the normal to the surface. They will be denoted by *L*, *M*, *N*. Thus

$$L = \mathbf{n}.\mathbf{r}_{11}\ ,\ M = \mathbf{n}.\mathbf{r}_{12}\ ,\ N = \mathbf{n}.\mathbf{r}_{22}$$

It will be convenient to have a symbol for the quantity $LN - M^2$. We therefore write,

$$T^2 = LN - M^2$$

though this quantity is not necessarily positive. We may express *L*, *M*, *N* in terms of scalar triple products of vectors. For

$$[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{11}] = \mathbf{r}_1 \times \mathbf{r}_2 \bullet \mathbf{r}_{11} = H\mathbf{n} \bullet \mathbf{r}_{11} = HL$$

Similarly $$[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{12}] = \mathbf{r}_1 \times \mathbf{r}_2 \bullet \mathbf{r}_{12} = H\mathbf{n} \bullet \mathbf{r}_{12} = HM$$

and $$[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{22}] = \mathbf{r}_1 \times \mathbf{r}_2 \bullet \mathbf{r}_{22} = H\mathbf{n} \bullet \mathbf{r}_{22} = HN$$

It will be shown later that the second order magnitudes are intimately connected with the curvature of the surface. We may here observe in passing that they occur in the expression for the length of perpendicular to the tangent plane from a point on the surface in the neighborhood of the point of connection. Let **r** be the point of contact, *P* with parameter values u, v and **n** the unit normal there. The position vector of a neighboring point $Q(u + du, v + dv)$ on the surface has the value

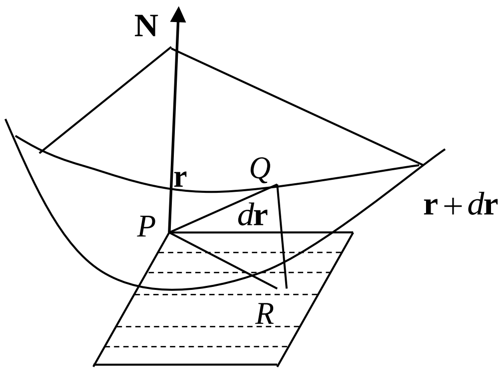

Figure 8.2

$$\mathbf{r} + (\mathbf{r}_1 du + \mathbf{r}_2 dv) + \frac{1}{2}(\mathbf{r}_{11} du^2 + 2\mathbf{r}_{12} du\, dv + \mathbf{r}_{22} dv^2) + \ldots.. \quad (8.4)$$

The length of the perpendicular from *Q* on the tangent plane at *P* is the projection of the vector *PQ* on the normal at *P*, and is therefore equal to

$$\mathbf{n} \bullet (\mathbf{r}_1 du + \mathbf{r}_2 dv) + \frac{1}{2}\mathbf{n} \bullet (\mathbf{r}_{11} du^2 + 2\mathbf{r}_{12} du\, dv + \mathbf{r}_{22} dv^2) + \ldots..$$

In this expression, the terms of the first order vanish since **n** is at right angles to $\mathbf{r}_1$ and $\mathbf{r}_2$. Hence the length of the perpendicular as far as terms of the second order is

$$\frac{1}{2}\mathbf{n} \bullet (Ldu^2 + 2Mdu\,dv + Ndv^2)$$

8.4. Geometrical interpretations of Second order magnitudes: Consider the surface represented by $\mathbf{x} = ue_1 + ve_2 + (u^2 - v^2)e_3$

Here $\mathbf{x}_u = e_1 + 2ue_3$, $\mathbf{x}_v = e_2 - 2ve_3$, $\mathbf{x}_{uu} = 2e_3$, $\mathbf{x}_{uv} = 0$, $\mathbf{x}_{vv} = -2e_3$,

$$\mathbf{N} = \frac{\mathbf{x_u} \times \mathbf{x_v}}{|\mathbf{x_u} \times \mathbf{x_v}|} = \left(4u^2 + 4v^2 + 1\right)^{-\frac{1}{2}}\left(-2ue_1 + 2ve_2 + e_3\right)$$

Thus the second fundamental coefficients are

$$L = x_{uu} \cdot \mathbf{N} = 2\left(4u^2 + 4v^2 + 1\right)^{-\frac{1}{2}},\ M = x_{uV} \cdot \mathbf{N} = 0,\ N = x_{vv} \cdot \mathbf{N} = -2\left(4u^2 + 4v^2 + 1\right)^{-\frac{1}{2}}$$

and the second fundamental form is

$$(Ldu^2 + 2Mdu\,dv + Ndv^2) = 2\left(4u^2 + 4v^2 + 1\right)^{-\frac{1}{2}}\left(du^2 - dv^2\right) \qquad (8.5)$$

Suppose P is a point on a surface of class ≥ 2, Q is a point in the neighborhood of P and $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P and Q. Let $d = PQ \cdot \mathbf{N}$ be the projection of PQ onto the unit normal $\mathbf{N}$ at P, as shown in the figure 8.3. Observe that d is positive or negative depending on whether Q is on one or the other side of the tangent plane at P and that $|d|$ is perpendicular distance from Q to the tangent plane at P. Now suppose P and Q are the points $\mathbf{x}(u,v)$ and $\mathbf{x}(u+du,\ v+dv)$ respectively.

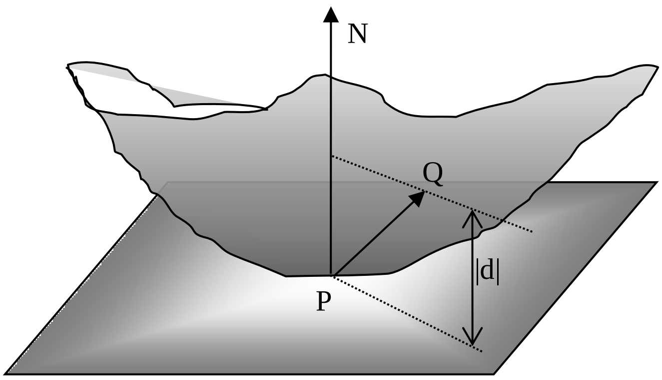

Figure 8.3

Taylor's theorem gives

$$\mathbf{x}(u+du,\ v+dv)=\mathbf{x}(u,\ v)+d\mathbf{x}+\frac{1}{2}d^2\mathbf{x}+o(d^2u+d^2v)$$

Thus $$d=PQ\cdot\mathbf{N}=\left(\mathbf{x}(u+du,\ v+dv)-\mathbf{x}(u,\ v)\right)\cdot\mathbf{N}$$

$$=\left[d\mathbf{x}+\frac{1}{2}d^2\mathbf{x}+o(d^2u+d^2v)\right]\cdot N$$

$$=\left[d\mathbf{x}\cdot\mathbf{N}+\frac{1}{2}d^2\mathbf{x}\cdot\mathbf{N}+o(d^2u+d^2v)\right] \tag{8.6}$$

But $d\mathbf{x}\cdot\mathbf{N}=0$,since $d\mathbf{x}$ is parallel to the tangent plane at P. Hence

$$d=\left[\frac{1}{2}d^2\mathbf{x}\cdot\mathbf{N}+o(d^2u+d^2v)\right]$$

$$d=\left[\frac{1}{2}(Ldu^2+2Mdu\,dv+Ndv^2)+o(d^2u+d^2v)\right] \tag{8.7}$$

Thus $(Ldu^2+2Mdu\,dv+Ndv^2)$ is the principal part of twice the projection of PQ onto N and $(Ldu^2+2Mdu\,dv+Ndv^2)$ is the principal part of twice the perpendicular distance from Q onto the tangent plane at P.

The function $\delta=\frac{1}{2}(Ldu^2+2Mdu\,dv+Ndv^2)$ is called the osculating paraboloid at P. The nature of this parboloid determines qualitatively the nature of the surface in the neighborhood of P. We distinguish four cases, depending upon the discriminant $LN-M^2$.

8.4.1 Elliptic case: A point is called an elliptic point if $LN-M^2>0$. In this case δ as a function of du and dv is an elliptic paraboloid as shown in the figure 8.3. Observe that δ maintains the same sign for all (du,dv). In the neighborhood of an elliptic point the surface lies on one side of the tangent plane at the point .

8.4.2 Hyperbolic case: A point is called a hyperbolic point if $LN-M^2<0$. In this case δ as a function of (du,dv) is a hyperbolic paraboloid. Here there are two distinct lines in the tangent plane through P which divide the tangent plane into four sections in which δ is alternately positive and negative. On the two lines, $\delta=0$. In the neighborhood of a hyperbolic point the surface lies on both sides of the tangent plane as in the figure.

8.4.3 Parabolic case: A point is called a parabolic point if $LN - M^2 = 0$ and $L^2 + M^2 + N^2 \neq 0$, i.e. if $LN - M^2 = 0$ and the coefficients L, M and N are not all zero. In this case δ as a function of (du, dv) is a parabolic cylinder. Here there is a single line in the tangent plane through P along which $\delta = 0$, otherwise δ maintains the same sign. It is to be noted that in the neighborhood of a parabolic point the surface itself may lie on both sides of the tangent plane.

8.4.4 Planar case: A point is called a planar point if $L = M = N = 0$. Here $\delta = 0$ for all (du, dv). In this case the degree of contact of the surface and the tangent plane is of higher order than in the preceding case.

Example 8.1 Calculate the fundamental magnitudes for the right helicoids given by

$$x = u\cos\phi,\ y = u\sin\phi,\ z = c\phi.$$

Solution. With u, ϕ as parameters, we have

$$\mathbf{r} = (u\cos\phi, u\sin\phi, c\phi),$$

$$\mathbf{r}_1 = (\cos\phi, \sin\phi, 0),$$

$$\mathbf{r}_2 = (-u\sin\phi, u\cos\phi, c)$$

Therefore

$$E = \mathbf{r}_1^2 = 1,\quad F = \mathbf{r}_1 \bullet \mathbf{r}_2 = 0,\quad G = \mathbf{r}_2^2 = u^2 + c^2,\quad H^2 = EG - F^2 = u^2 + c^2$$

Since $F = 0$ the parametric curves are orthogonal. The unit normal to the surface is

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = (c\sin\phi, -c\cos\phi, u)/H$$

Further $\mathbf{r}_{11} = (0, 0, 0)$

$$\mathbf{r}_{12} = (-\sin\phi, \cos\phi, 0)$$

$$\mathbf{r}_{22} = (-u\cos\phi, -\sin\phi, 0)$$

So that the second order magnitudes are

$$L = 0,\ M = -\frac{c}{H},\ N = 0.$$

Example 8.2 On the surface given by $x = a(u+v)$, $y = b(u-v)$, $z = uv$ the parametric curves are straight lines.

Solution. Further $\mathbf{r}_1 = (a, b, v)$

$$\mathbf{r}_2 = (a, -b, u)$$

and therefore $E = a^2 + b^2 + v^2$, $F = a^2 - b^2 + uv$, $G = a^2 + b^2 + u^2$

$$H^2 = 4a^2b^2 + a^2(u-v)^2 + b^2(u+v)^2.$$

The unit normal is $\mathbf{n} = (bu + bv, av - au, -2ab)/H$

Again $\mathbf{r}_{11} = (0,0,0)$,

$$\mathbf{r}_{12} = (0,0,1),$$

$$\mathbf{r}_{22} = (0,0,0)$$

And therefore $L = 0,\quad M = -2ab/H,\quad N = 0,\ T^2 = LN - M^2 = -4a^2b^2/H^2$.

8.5. Derivative of unit normal (n). Moreover, means of the fundamental magnitudes, we may express the derivatives of **n** in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$. Such an expression is clearly possible. For, since **n** is a vector of constant length, its first derivatives are perpendicular to **n** and therefore tangential to the surface. They are thus parallel to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$, and maybe express in terms of these.

We may proceed as follows. Differentiating the relation $\mathbf{n} \bullet \mathbf{r}_1 = 0$ with respect to u, we obtain

$$\mathbf{n} \bullet \mathbf{r}_1 + \mathbf{n} \bullet \mathbf{r}_{11} = 0$$

and therefore $\mathbf{n}_1 \bullet \mathbf{r}_1 = \mathbf{n}_1 \bullet \mathbf{r}_{11} = -L$

In the same manner, we find

$$\mathbf{n}_1 \bullet \mathbf{r}_2 = -\mathbf{n}_1 \bullet \mathbf{r}_{12} = -M$$

$$\mathbf{n}_2 \bullet \mathbf{r}_1 = -\mathbf{n} \bullet \mathbf{r}_{21} = -M$$

$$\mathbf{n}_2 \bullet \mathbf{r}_{21} = -\mathbf{n} \bullet \mathbf{r}_{22} = -N \qquad (8.8)$$

Now since $\mathbf{n}_1$ is perpendicular to $\mathbf{n}$ and therefore tangential to the surface, we may write

$$\mathbf{n}_1 = a\mathbf{r}_1 + b\mathbf{r}_2,$$

where a and b are to be determined. Forming the scalar products of each side with $\mathbf{r}_1$ and $\mathbf{r}_2$ successively, we have

$$-L = aE + bF,$$

$$-M = aF + bG$$

On solving the equations for *a* and *b*, and substituting the values, so obtained in the formula for $\mathbf{n}_1$, we find

$$H^2\mathbf{n}_1 = (FM - GL)\mathbf{r}_1 + (FL - EM)\mathbf{r}_2 \tag{8.9}$$

Similarly, it may be shown that

$$H^2\mathbf{n}_2 = (FN - GM)\mathbf{r}_1 + (FM - EN)\mathbf{r}_2 \tag{8.10}$$

It $\mathbf{r}_1$ and $\mathbf{r}_2$ be eliminated in succession from these two equations, we obtain an expression for $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{n}_1$ and $\mathbf{n}_2$. The reader will easily verify that

$$T^2\mathbf{r}_1 = (FM - EN)\mathbf{n}_1 + (EM - FL)\mathbf{n}_2$$

$$T^2\mathbf{n}_2 = (GM - FN)\mathbf{n}_1 + (FM - GL)\mathbf{n}_2 \tag{8.11}$$

These relations could also be proved independently by the same method as that employed in establishing (8.9).

From the equations (8.9) and (8.10) it follows immediately that

$$H^2\mathbf{n}_1 \times \mathbf{n}_1 = \{(FM - GL)(FM - EN) - (FL - EM)(FN - GM)\}\mathbf{r}_1 \times \mathbf{r}_2$$

$$= H^2T^2\mathbf{n}$$

so that $$H\mathbf{n}_1 \times \mathbf{n}_2 = T^2\mathbf{n} \tag{8.12}$$

Thus the scalar triple product

$$[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2] = \frac{T^2}{H}\mathbf{n} \bullet \mathbf{n} = \frac{T^2}{H}.$$

And as a further exercise the reader may easily verify the following relations which will be used later:

$$\left.\begin{aligned} H[\mathbf{n}, \mathbf{n}_1, \mathbf{r}_1] &= EM - FL \\ H[\mathbf{n}, \mathbf{n}_1, \mathbf{r}_2] &= FM - GL \\ H[\mathbf{n}, \mathbf{n}_2, \mathbf{r}_1] &= FN - FM \\ H[\mathbf{n}, \mathbf{n}_2, \mathbf{r}_2] &= FN - GM \end{aligned}\right\} \tag{8.13}$$

Example 8.3 Calculate the fundamental magnitudes for Monge's form of the surface $z = f(x, y)$.

Solution. Please remember, if $z = f(x, y)$ is the equation of the surface, then

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Now taking *x*, *y* as parameters, we have

$$\mathbf{r} = (x, y, f(x, y))$$

$$\mathbf{r}_1 = (1,0,p), \ \mathbf{r}_2 = (0,1,q), \ \mathbf{r}_{11} = (0,0,r), \ \mathbf{r}_{12} = (0,0,s), \ \mathbf{r}_{22} = (0,0,t)$$

Therefore $E = \mathbf{r}_1 \bullet \mathbf{r}_1 = 1 + p^2, \ F = \mathbf{r}_1 \bullet \mathbf{r}_2 = pq, \ G = \mathbf{r}_2 \bullet \mathbf{r}_2 = 1 + q^2$

$$H^2 = EG - F^2 = (1+p^2)(1+q^2) = p^2q^2 = 1 + p^2q^2.$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathbf{H}} = (-p, -q, 1)/H$$

$$L = \mathbf{N} \cdot \mathbf{r}_{11} = \frac{r}{H}, \ M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{s}{H}, \ N = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{t}{H}$$

$$\therefore \quad T^2 = LN - M^2 = \frac{rt - s^2}{H^2}.$$

Example 8.4 Calculate the fundamental magnitudes for the right helicoids given by $x = u\cos v$, $y = u\sin v$, $z = cv$.

Solution. With u, v as parameters, we have

$$\mathbf{r} = (u\cos v, u\sin v, cv)$$

If suffixes 1 and 2 represents partial differentiation with respect to u and v respectively, we have

$$\mathbf{r}_1 = (\cos v, \sin v, 0), \ \mathbf{r}_2 = (-u\sin v, u\cos v, c)$$

$$\mathbf{r}_{11} = (0,0,0), \ \mathbf{r}_{12} = (-\sin v, \cos v, 0), \ \mathbf{r}_{22} = (-u\cos v, -u\sin v, 0)$$

$$\therefore \quad E = \mathbf{r}_1^2 = \cos^2 v + \sin^2 v = 1, \ F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0, \ G = \mathbf{r}_2^2 = u^2 + c^2,$$

$$H^2 = EG - F^2 = u^2 + c^2$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathbf{H}} = \frac{(c\sin v, -c\cos v, u)}{H}$$

$$L = \mathbf{N} \cdot \mathbf{r}_{11} = 0, \ M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{c}{H}, \ N = \mathbf{N} \cdot \mathbf{r}_{22} = 0$$

$$\therefore \quad T^2 = LN - M^2 = \frac{c^2}{H^2}.$$

Since F=0, the parametric curves are orthogonal.

Example 8.5 Calculate the fundamental magnitudes for the right helicoids given by $x = u\cos v$, $y = u\sin v$, $z = f(v)$.

Solution. With u, v as parameters, we have

$$\mathbf{r} = (u\cos v, u\sin v, f(v))$$

$$\mathbf{r}_1 = (\cos v, \sin v, 0), \quad \mathbf{r}_2 = (-u\sin v, u\cos v, f') \qquad \text{where, } [f' = df/dv]$$

$$\mathbf{r}_{11} = (0,0,0), \quad \mathbf{r}_{12} = (-\sin v, \cos v, 0), \quad \mathbf{r}_{22} = (-u\cos v, -u\sin v, f'')$$

where dashes represent differential w.r.t. v

$$\therefore \quad E = \mathbf{r}_1^2 = \cos^2 v + \sin^2 v = 1, \quad F = \mathbf{r}_1 \cdot \mathbf{r}_2 = -u\cos v\sin v + u\cos v\sin v = 0$$

$$G = \mathbf{r}_2^2 = u^2\sin^2 v + u^2\cos^2 v + f'^2 = u^2 + f'^2,$$

$$H^2 = EG - F^2 = 1.(u^2 + f'^2) - 0 = u^2 + f'^2$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathbf{H}} = \frac{(f'\sin v, -f'\cos v, u)}{H}$$

$$L = \mathbf{N} \cdot \mathbf{r}_{11} = 0, \quad M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{1}{H}[-f'\sin^2 v - f'\cos^2 v + 0] = -\frac{f'}{H}$$

$$N = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{1}{H}[-uf'\cos v\sin v + uf'\sin v\cos v + uf''] = -\frac{uf''}{H}$$

$$\therefore \quad T^2 = LN - M^2 = -\frac{f'^2}{H^2}.$$

Since F=0, the parametric curves are orthogonal.

Example 8.6 Calculate the fundamental magnitudes for the surface of revolution $x = u\cos v$, $y = u\sin v$, $z = f(u)$ with u, v as parameters.

Solution. With u, v as parameters, we have

$$\mathbf{r} = (u\cos v, u\sin v, f(u))$$

$$\mathbf{r}_1 = (\cos v, \sin v, f'), \quad \mathbf{r}_2 = (-u\sin v, u\cos v, 0) \qquad \text{where, } [f' = df/du]$$

$$\mathbf{r}_{11} = (0,0,f''), \quad \mathbf{r}_{12} = (-\sin v, \cos v, 0), \quad \mathbf{r}_{22} = (-u\cos v, -u\sin v, 0)$$

$$\therefore \quad E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + f'^2, \quad F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0, \quad G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2,$$

$$H^2 = EG - F^2 = u^2(1 + f'^2)$$

$$\mathbf{N}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{\mathbf{H}}=\frac{(-u\cos vf',-u\sin vf',u)}{H}$$

$$L=\mathbf{N}\cdot\mathbf{r}_{11}=\frac{uf''}{H},\quad M=\mathbf{N}\cdot\mathbf{r}_{12}=0,\quad N=\mathbf{N}\cdot\mathbf{r}_{22}=\frac{u^2f'}{H}$$

$$\therefore\qquad T^2=LN-M^2=\frac{u^3f'f''}{H^2}.$$

Since F=0, the parametric curves are orthogonal.

Example 8.7 Calculate the fundamental magnitudes and the normal to the surface $2z=ax^2+2hxy+by^2$ taking x, y as parameters.

Solution. The position vector **r** of a current point on the surface is given by

$$\mathbf{r}=(x,y,\frac{1}{2}ax^2+hxy+\frac{1}{2}by^2)$$

$$\therefore\quad \mathbf{r}_1=(1,0,ax+hy),\quad \mathbf{r}_2=(0,1,hx+by)$$

$$\mathbf{r}_{11}=(0,0,a);\ \mathbf{r}_{12}=(0,0,h);\ \mathbf{r}_{22}=(0,0,b)$$

$$\mathbf{N}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{\mathbf{H}}=\frac{(-(ax+hy),-(hx+by),1)}{H}$$

$$\therefore\quad E=1+(ax+hy)^2,\ F=(ax+hy)(hx+by),\ G=1+(hx+by)^2,$$

$$H^2=EG-F^2=1+(ax+hy)^2+(hx+by)^2$$

$$L=\frac{a}{H},\quad M=\frac{h}{H},\quad N=\frac{b}{H},\quad T^2=LN-M^2=\frac{ab-h^2}{H^2}.$$

Example 8.8 For the paraboloid $\mathbf{r}=(u,v,u^2-v^2)$, find the metric.

Solution. Here, $\mathbf{r}=(u,v,u^2-v^2)$

$$\mathbf{r}_1=(1,0,2u),\quad \mathbf{r}_2=(0,1,-2v)$$

$$\therefore\quad E=\mathbf{r}_1\cdot\mathbf{r}_1=1+4u^2,\ F=\mathbf{r}_1\cdot\mathbf{r}_2=-4uv,\ G=\mathbf{r}_2\cdot\mathbf{r}_2=1+4v^2$$

Hence the metric $ds^2=Edu^2+2Fdu\,dv+Gdv^2$ becomes

$$ds^2=(1+4u^2)du^2-8uvdudv+(1+4v^2)dv^2$$

Example 8.9 State the second fundamental form and find it for Monge's form of the surface $z = f(x, y)$

.

Solution. Now proceeding as in example 1(above), we have

$$L = r/H,\ M = s/H,\ N = t/H$$

The second fundamental form namely

$$Ldu^2 + 2Mdu\,dv + Ndv^2$$

for the surface $z = f(x, y)$ is given by

$\frac{r}{H}dx^2 + \frac{2s}{H}dxdy + \frac{t}{H}dy^2$, where x, y are taken as parameters.

8.6 Formula for the curvature of a normal section in terms of fundamental magnitudes.

Solution. Let κ_n denote the curvature of the normal section, then κ_n is positive when the curve is concave on the side towards which **N** points out. Then we know

$$\frac{d\,\mathbf{t}}{ds} = \mathbf{r}'' - \kappa_n \mathbf{n} = \kappa_n \mathbf{N} \qquad \text{[since here } \mathbf{n}=\mathbf{N}]$$

$$\therefore \qquad \kappa_n = \mathbf{N}.\mathbf{r}'' \tag{8.14}$$

But we know $\mathbf{r}' = \mathbf{r}_1 u' + \mathbf{r}_2 v'$

Differentiating this relation w.r.t. 's' we get

$$\mathbf{r}'' = \mathbf{r}_1 u'' + \frac{d\mathbf{r}_1}{ds}u' + \mathbf{r}_2 v'' + \frac{d\mathbf{r}_2}{ds}v'$$

$$= \mathbf{r}_1 u'' + \left(\frac{\partial \mathbf{r}_1}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}_1}{\partial v}\frac{dv}{ds}\right)u' + \mathbf{r}_2 v'' + \left(\frac{\partial \mathbf{r}_2}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}_2}{\partial v}\frac{dv}{ds}\right)v'$$

$$= \mathbf{r}_1 u'' + (\mathbf{r}_{11}u' + \mathbf{r}_{12}v')u' + \mathbf{r}_2 v'' + (\mathbf{r}_{12}u' + \mathbf{r}_{22}v')v'$$

i.e. $$\mathbf{r}'' = \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11}u'^2 + 2\mathbf{r}_{12}v'u + \mathbf{r}_{22}v'^2 \tag{8.15}$$

Substituting this value of $\mathbf{r}''$ in ((8.14)) and using

$$\mathbf{N}.\mathbf{r}_{11} = L,\ \ \mathbf{N}.\mathbf{r}_{12} = M,\ \ \mathbf{N}.\mathbf{r}_{22} = N,\ \ \mathbf{N}.\mathbf{r}_1 = 0,\ \ \mathbf{N}.\mathbf{r}_2 = 0$$

We get

$$\kappa_n = \mathbf{N}.\mathbf{r}'' = Lu'^2 + 2Mu'v' + Nv'^2$$

or $$\kappa_n = \frac{Ldu^2 + 2M\,du\,dv + Ndv^2}{ds^2}\ ,\ \kappa_n = \frac{Ldu^2 + 2M\,du\,dv + Ndv^2}{Edu^2 + 2F\,du\,dv + Gdv^2} \qquad (8.16)$$

[using first fundamental theorem]

This gives the curvature of the normal section (usually called normal curvature) parallel to the direction (du, dv). Its reciprocal is called the **radius of normal curvature** and may be denoted by ρ_n.

Now we may define the normal curvature as follows:

Definition. Let a point *P* with position vector $\mathbf{r}(u,v)$ be on the surface $\mathbf{r} = \mathbf{r}(u,v)$. The normal curvature at *P* in the directions (du, dv) is equal to the curvature at *P* of the normal section at *P* parallel to the direction (du, dv).

Definition: Let a point *P* with position vector $\mathbf{r}(u,v)$ be on the surface $\mathbf{r} = \mathbf{r}(u,v)$. Consider a curve $\mathbf{r} = \mathbf{r}(s)$ through the point *P* laying on the surface. The component of the curvature vector $\mathbf{r}''$ along the normal to the surface is defined to be the normal curvature of the curve at point *P*. Therefore $\kappa_n = \mathbf{N}.\mathbf{r}''$.

8.7 Show that the definitions of normal curvature given above are equivalent.

Proof. Let **N** be the unit normal vector of the surface at *P*, then from the second definition

$$\kappa_n = \mathbf{N}.\mathbf{r}'' \qquad (8.17)$$

where $\mathbf{r}''$ is the curvature vector at *P*.

Again let κ be the curvature at *P* which contains the direction (du, dv).

$\therefore \quad \mathbf{r}'' = \kappa\mathbf{n} = \kappa\mathbf{N} \qquad [\because \mathbf{n} = \mathbf{N}]$

$\therefore \quad \mathbf{N}.\mathbf{r}'' = \kappa \qquad [\because \mathbf{N}.\mathbf{N} = 1] \qquad (8.18)$

$\therefore \quad \kappa = \kappa_n \qquad$ [using (8.17) and (8.18)]

Hence curvature at *P* of normal section at *P* which contains the direction (du, dv) is equal to the normal curvature at *P* in the same direction.

Thus in the future, the terms 'normal curvature' and 'curvature of normal section' will represent the same thing.

Note: We have seen $\mathbf{N}.\mathbf{r}'' = \kappa_n = \dfrac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}$

$$= L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

Now we know that the curves which have the same direction at the point *P* have the same value of their direction coefficients $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ at *P*. Also the values of second order fundamental magnitudes *L*, *M*, *N* are fixed at *P*. Hence all curves which have the same direction at *P*, have a fixed value of $\mathbf{N}.\mathbf{r}''$ which is equal to the normal curvature at *P* of any one of these curves.

8.8 Meusnier's Theorem

If κ_n and κ are the curvature of the normal and oblique sections through the same tangent line, and θ is the angle between these sections, then the relation between κ_n, κ and θ is given by

$$\kappa_n = \kappa \cos\theta.$$

Proof. Let the section be oblique and its curvature be denoted by κ. Since the section is oblique, its **n** is not parallel to **N** but will be parallel to the unit vector $\mathbf{r}''/\kappa$ $(\because \mathbf{r}'' = \kappa\mathbf{n})$. If θ is the angle of inclination of the oblique section with normal section touching the curve at the point under consideration, then θ is the angle between the normal of two sections *i.e.* it is the angle between the unit vector $\mathbf{r}''/\kappa$ and **N**. Thus

$$\cos\theta = \mathbf{N}.\mathbf{r}''/\kappa$$

$$= \frac{1}{\kappa}\frac{(Ldu^2 + 2du\,dv + Ndv^2)}{Edu^2 + 2Fdu\,dv + Gdv^2}$$

$$= \frac{\kappa_n}{\kappa}$$

or $$\kappa_n = \kappa\cos\theta. \tag{8.19}$$

Another statement of Meusnier's theorem

If θ is the angle between the principal normal to the curve on the surface and the surface normal at the point *P*, then

$$\kappa_n = \kappa\cos\theta$$

where κ is the curvature of the curve at P and κ_n is the normal curvature at P in the direction of the curve.

Proof. Let P be a point (u, v) on the surface $\mathbf{r} = \mathbf{r}(u, v)$. Suppose **n** and **N** denote the principal normal to the curve on the surface and surface normal at P respectively. Since θ is the angle between **n** and **N**, therefore

$$\cos\theta = \mathbf{n}.\mathbf{N} \tag{8.20}$$

Now suppose $\mathbf{r}''$ is the curvature vector of the given curve at P. Then

$$\mathbf{r}'' = \kappa\mathbf{n} \tag{8.21}$$

Taking the scalar product of both sides of (8.21) with **N**

$$\mathbf{r}''.\mathbf{N} = \kappa\mathbf{n}.\,\mathbf{N}$$

or $$\mathbf{r}''.\mathbf{N} = \kappa\cos\theta \qquad \text{[using ((8.19))]} \tag{8.22}$$

The values of $\mathbf{r}''.\mathbf{N}$ are fixed for all curves having the same direction at P and by definition, this value $\mathbf{r}''.\,\mathbf{N}$ is equal to the normal curvature κ_n in that direction (3).

$$\therefore \qquad \kappa_n = \kappa\cos\theta.$$

Note: We have $\kappa_n = \kappa$ if and only if $\theta = 0$. Thus the necessary and sufficient condition for the curvature of a curve at P to be equal to normal curvature at P in the direction of that curve is that the principal normal to the curve is along the surface normal at that point.

Example 8.10 Establish the formula

(i) $$\frac{\partial\mathbf{n}}{\partial u} \times \frac{\partial\mathbf{n}}{\partial v} = \frac{LN - M^2}{\sqrt{(EG - F^2)}}.$$

(ii) $$\kappa_n = \kappa\cos\theta.$$

where κ is the curvature of an oblique section C of a surface, κ_n is the normal curvature of the surface in the direction of C, and θ is the angle between the oblique section and the normal section.

Solution. (i) We are to prove

$$\mathbf{N}_1 \times \mathbf{N}_2 = \frac{LN - M^2}{\sqrt{(EG - F^2)}}\mathbf{N}$$

[$\because$ $\mathbf{n} = \mathbf{N}$, here **n** being surface normal] or $\qquad \mathbf{N}_1 \times \mathbf{N}_2 = \dfrac{T^2}{H}\mathbf{N}$.

Example 8.11 Find the curvature of a normal section of the right helicoids $x = u\cos v$, $y = u\sin v$, $z = cv$.

Solution. For the right helicoid given in the problem

$$E = 1,\ F = 0,\ G = u^2 + v^2;\ L = 0,\ M = -\frac{c}{H},\ N = 0$$

The curvature of the normal section κ_n is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdu\,dv + Gdv^2}$$

$$= \frac{-2cdu\,dv}{H[du^2 + (u^2 + v^2) + dv^2]} \qquad \text{[parameters being } u \text{ and } v\text{]}$$

Example 8.12 Find the curvature of a normal section of a helicoids

$x = u\cos v$, $y = u\sin v$, $z = f(u) + cv$.

Solution. Proceed as in example 8.11.

Example 8.13 Show that the curvature κ at any point P of the curve of intersection of two surfaces is given by

$$\kappa^2 \sin^2\theta = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\cos\theta$$

where κ_1, κ_2 are the normal curvature of the surfaces in the direction of the curve at P and θ is the angle between their normals at that point.

Solution. Let $\mathbf{N}$ and $\overline{\mathbf{N}}$ be the unit normals to the two surfaces at a point P of the curve, then as given in the question

$$\cos\theta = \mathbf{N} \bullet \overline{\mathbf{N}} \qquad (1)$$

Let $P(\mathbf{r})$ be the point on the curve of intersection of two surfaces, then

$$\frac{d\mathbf{t}}{ds} = \mathbf{r}'' = \kappa\,\mathbf{n} \qquad (2)$$

The vectors $\mathbf{N}$, $\overline{\mathbf{N}}$, $\mathbf{r}''$ all are coplanar as they are perpendicular to the vector $\mathbf{t}$ and hence they are expressible as a linear combination of the form

$$\mathbf{r}'' = a\mathbf{N} + b\overline{\mathbf{N}} \tag{3}$$

Taking the scalar product of (3) with **N** and $\overline{\mathbf{N}}$ using equation (1), we have

$$\kappa_1 = a + b\cos\theta \ , \ \ \kappa_2 = a\cos\theta + b$$

$$[\text{ using } \mathbf{N}\bullet\mathbf{r}'' = \kappa_1,\ \overline{\mathbf{N}}\bullet\mathbf{r}'' = \kappa_2\ ]$$

Solving for *a* and *b*, we get

$$a = (\kappa_1 - \kappa_2\cos\theta)/\sin^2\theta, \quad b = (\kappa_2 - \kappa_1\cos\theta)/\sin^2\theta$$

Substituting these values in equation (3), we get

$$\mathbf{r}''\sin^2\theta = (\kappa_1 - \kappa_2\cos\theta)\mathbf{N} + (\kappa_2 - \kappa_1\cos\theta)\overline{\mathbf{N}}$$

or $$\kappa\mathbf{n}\sin^2\theta = (\kappa_1 - \kappa_2\cos\theta)\mathbf{N} + (\kappa_2 - \kappa_1\cos\theta)\overline{\mathbf{N}} \quad \text{[using (2)]}$$

Hence, on squaring we get

$$\begin{aligned}\kappa^2\sin^4\theta &= (\kappa_1 - \kappa_2\cos\theta)^2 + (\kappa_2 - \kappa_1\cos\theta)^2 \\ &\quad + 2(\kappa_1 - \kappa_2\cos\theta)(\kappa_2 - \kappa_1\cos\theta) \\ &= (\kappa_1^2 + \kappa_2^2)(1 + \cos^2\theta) - 4\kappa_1\kappa_2\cos\theta \\ &\quad + 2[\kappa_1\kappa_2 - (\kappa_1^2 + \kappa_2^2)\cos\theta + \kappa_1\kappa_2\cos^2\theta]\cos\theta \\ &= (\kappa_1^2 + \kappa_2^2)(1 + \cos^2\theta - 2\cos^2\theta) - 2\kappa_1\kappa_2\cos\theta(1 - \cos^2\theta) \\ &= (\kappa_1^2 + \kappa_2^2)\sin^2\theta - 2\kappa_1\kappa_2\cos\theta\sin^2\theta\end{aligned}$$

i.e. $$\kappa^2\sin^2\theta = (\kappa_1^2 + \kappa_2^2) - 2\kappa_1\kappa_2\cos\theta.$$

Example 8.14 Show that the normal to the surface

$x = (u+v)/\sqrt{2}, \quad y = (u-v)/\sqrt{2}, \quad z = uv$ at a point (u, v) is described by the unit vector

$$\mathbf{n} = \frac{f(x, -y, -1)}{\sqrt{(1 + x^2 + y^2)}} = \frac{(u+v,\ u-v,\ -\sqrt{2})}{\sqrt{2}(1 + u^2 + v^2)}$$

Also, evaluate curvature at the origin for the normal section in any direction (du, dv) and show that the curvature is zero for the normal sections which have the same tangents at the parametric curves through the origin.

Solution. The equation of the surface with parameters (u, v) is given by

$$\mathbf{r} = \left(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}, uv \right)$$

$$\mathbf{r}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, v \right), \quad \mathbf{r}_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, u \right)$$

$$\mathbf{r}_{11} = (0,0,0), \quad \mathbf{r}_{12} = (0,0,1), \quad \mathbf{r}_{22} = (0,0,0)$$

$\because \qquad E = \mathbf{r}_1^2 = 1 + v^2, \quad F = \mathbf{r}_1 \bullet \mathbf{r}_2 = uv, \quad G = \mathbf{r}_2^2 = 1 + u^2$

and $\qquad H^2 = 1 + u^2 + v^2$

(unit normal to the surface)= $(\mathbf{r}_1 \times \mathbf{r}_2)/H$

$$\mathbf{r} = \frac{(u+v,\, u-v,\, -\sqrt{2})}{\sqrt{2}\sqrt{(1+u^2+v^2)}} = \frac{(x,-y,-1)}{\sqrt{(1+x^2+y^2)}}$$

$\therefore \qquad L = \mathbf{n}.\mathbf{r}_{11} = 0, \; M = \mathbf{n}.\mathbf{r}_{12} = \dfrac{-1}{\sqrt{1+u^2+v^2}}, \; N = \mathbf{n}.\mathbf{r}_{22} = 0$

$\therefore$ The curvature of the normal section is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdu\,dv + Gdv^2}$$

$$\kappa_n = \frac{-2du\,dv}{[(1+v^2)du^2 + 2uv\,du\,dv + (1+u^2)dv^2]\sqrt{(1+u^2+v^2)}}.$$

8.9 Orthogonal Trajectories

Definition: For any given family of curves on a surface here always exist a second family, called the orthogonal trajectories, and is such that at every point of the two curves, one from each family, are orthogonal.

Theorem 8.1 To find the differential equation of the orthogonal trajectories and to show that every family of curves on a surface possesses orthogonal trajectories.

Solution. Let $\mathbf{r} = (u,v)$ (1)

be the equation of the surface.

Again let $\qquad \phi = (u,v) = c$ (2)

be the equation of the given family of curves on (1). Hence ϕ has continuous derivatives ϕ_1 and ϕ_2 which do not vanish together.

From (2), we have

$$\phi_1 du + \phi_2 dv = 0 \tag{3}$$

Let (3) be equivalent to

$$Pdu + Qdv = 0 \tag{4}$$

$\therefore$ $\quad P = \lambda\phi_1,\ Q = \lambda\phi_2,\ \lambda \neq 0$

Also $\quad \dfrac{du}{dv} = \dfrac{-\phi_2}{\phi_1} = \dfrac{-Q}{P}$

or $$\frac{du}{-Q} = \frac{dv}{P} \tag{5}$$

Therefore (-Q, P) is the direction ratios of the tangent at any point (u,v) of a member of the family (2).

Let the direction ratios of the tangent at (u,v) of a member of the orthogonal trajectories of (2) be denoted by $(\partial u, \partial v)$. (du, dv) and $(\partial u, \partial v)$ are orthogonal if

$$E\, du\partial u + F(du\partial v + dv\partial u) + Gdv\partial v = 0$$

Using (5), it becomes

$$E(-Q)\partial u + F(-Q\partial v + P\partial u) + GP\,\partial v = 0$$

or $$(FP - EQ)\partial u + (GP - FQ)\partial v = 0 \tag{6}$$

which is the required differential equation of the orthogonal trajectories of the family of curves (2).

Since $(EG - F^2)$ is always positive and P and Q do not vanish together, therefore the coefficients of ∂u and ∂v in (6) are continuous. Hence (6) is integrable. Let the solution of (6) be

$$(f(u,v) = k \tag{7}$$

which is the equation of the orthogonal trajectories of the family of curves (2). This shows that ‘every family of curves on a surface possesses orthogonal trajectories’.

Theorem 8.2 For a surface, parameters can always be chosen so that the curves of a given family and their orthogonal trajectories become parametric curves.

Solution. Firstly proceed up to equation (7) of the above theorem (1). Now differentiating (7), the differential equation of orthogonal trajectories is

$$f_1 \partial u + f_2 \partial v = 0 \tag{8}$$

Relation (6) and (8) are equivalent. Therefore $FP \_ EQ = \mu f_1$ and $GP - FQ = \mu f_2$, where μ is a function of *u* and *v*, and does not vanish.

Now $$\frac{\partial(\phi, f)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{vmatrix} = \frac{1}{\lambda\mu} \begin{vmatrix} P & Q \\ FP \_ EQ & GP - FQ \end{vmatrix}$$

$$= \frac{1}{\lambda\mu}(EQ^2 - 2FPQ + GP^2) \neq 0$$

$\Rightarrow$ is independent of f.

Since $EQ^2 - 2FPQ + GP^2$ is positive definite and *P* and *Q* do not vanish together. Thus the proper transformation $U = \phi(u,v)$, $v = f(u,v)$ transforms the given family of curves and their orthogonal trajectories into the two families of parametric curves.

8.10 Double family of curves.

The quadratic differential equation of the form

$$Pdu^2 + 2Qdudv + Rdv^2 = 0 \tag{1}$$

Where *P*, *Q*, *R* are continuous functions of *u* and *v* and do not vanish together, representing two families of curves on the surface provided $Q^2 - PR > 0$. The equation (1) may be written as

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0$$

This is a quadratic equation in $\frac{du}{dv}$ and by solving this equation as a quadratic in $\frac{du}{dv}$, the separate differential equations for the two families are determined.

8.10.1 Condition for orthogonality

To find the condition that the quadratic differential equation (1) represents orthogonal families of curves or two orthogonal directions on the surface.

The differential equation of the family of curves is

$$Pdu^2 + 2Qdudv + Rdv^2 = 0$$

Let the two directions be $\dfrac{du}{dv}$ and $\dfrac{\partial u}{\partial v}$

Hence $$\frac{du}{dv} + \frac{\partial u}{\partial v} = -\frac{2Q}{P}, \quad \frac{du}{dv} \cdot \frac{\partial u}{\partial v} = \frac{R}{P}$$

or $$\frac{du\partial v + \partial u dv}{dv\partial v} = -\frac{2Q}{P}, \quad \frac{du\partial u}{dv\partial v} = \frac{R}{P}$$

if these directions are orthogonal, substituting the above values in equation (4), we get

$$E\, du\,\partial u + F(du\,\partial v + dv\,\partial u) + Gdv\,\partial v = 0$$

i.e. $$ER - 2FQ + GP = 0$$

Hence the two directions (or two families) given by $Pdu^2 + 2Qdudv + Rdv^2 = 0$, are orthogonal if and only if

$$ER - 2FQ + GP = 0 \qquad (2)$$

If P=0, R=0 the given differential equation (1) reduces to $dudv$=0, giving the two families of parametric curves and the condition (2) of orthogonality reduces to F=0.

Example 8.15 Show that if ψ is the angle at the point (u, v) between the two directions given by $Pdu^2 + 2Qdudv + Rdv^2 = 0$, then

$$\tan\psi = \frac{2H(Q^2 - PR)^{1/2}}{ER - 2FQ + GP}.$$

Hence or otherwise find the condition that the two directions are orthogonal.

Solution. We have

$$\frac{du}{dv} + \frac{\partial u}{\partial v} = -\frac{2Q}{P}, \quad \frac{du}{dv} \cdot \frac{\partial u}{\partial v} = \frac{R}{P}$$

and $$\tan\psi = \frac{|du\partial v - dv\partial u| H}{Edu\partial v + F(du\partial v + dv\partial u) + Gdv\partial v}$$

i.e. $$\tan\psi = \frac{\left|\dfrac{du}{dv} - \dfrac{\partial u}{\partial v}\right| H}{E\dfrac{du\partial v}{dv\partial v} + F\left(\dfrac{du}{dv} + \dfrac{\partial u}{\partial v}\right) + G}$$

$$= \frac{\left[\left(\frac{du}{dv} - \frac{\partial u}{\partial v}\right) - 4\frac{du\partial v}{dv\partial v}\right]^{1/2} H}{E\frac{R}{P} - 2\frac{FQ}{P} + G} \qquad \text{[from (1)]}$$

$$= \frac{2H[Q^2 - PR]^{1/2}}{ER - 2FQ + GP}.$$

The two directions are orthogonal if $\psi = \frac{1}{2}\pi$ *i.e.* the condition is

$$ER - 2FQ + GP = 0.$$

Example 8.16 Find the tangent of the angle between the two directions on the surface determined by the quadratic

$$Pdu^2 + Qdudv + Rdv^2 = 0$$

Hence the condition that the two directions are orthogonal is

$$ER - FQ + GP = 0.$$

Example 8.17 Show that the curves bisecting the angles between the parametric curves are given by

$$Edu^2 - Gdv^2 = 0.$$

Solution. If ψ is the angle between two curves, then

$$\cos\psi = \frac{Edu\partial u + F(du\partial u + dv\partial u) + Gdv\partial v}{ds\partial s}$$

Let $(\partial u, \partial v)$ refer to the parametric curves and (du, dv) the required curves.

If ψ_1 is the angle between the parametric curve *v*=constant and bisecting curve of direction (du, dv).

$$\cos\psi_1 = \frac{Edu\partial u + Fdv\partial u}{\sqrt{E}ds\partial s}$$

Similarly, If ψ_2 is the angle between the parametric curve *u*=constant and bisecting curve of direction (du, dv).

$$\cos\psi_2 = \frac{Fdu\partial v + Gdv\partial v}{\sqrt{G}ds\partial s}$$

If parametric curves are orthogonal F=0, and we must have $\psi_1 = \psi_2$ *i.e.* $\cos\psi_1 = \cos\psi_2$.

i.e. $$\frac{Edu\partial u}{\sqrt{E}ds\partial u} = \frac{Gdv\partial v}{\sqrt{G}ds\partial v} \quad \text{or} \quad \sqrt{E}du - \sqrt{G}dv = 0$$

the curve orthogonal to $\sqrt{E}du - \sqrt{G}dv = 0$ will also bisect one pair of angles between the orthogonal curves. The orthogonal curve is

$$\sqrt{E}du - \sqrt{G}dv = 0$$

Hence the required curves are given by the differential equation

$$\sqrt{E}du = \sqrt{G}dv = 0\,.$$

Example 8.18 Show that the curves

$$du^2 - (u^2 + a^2)dv^2 = 0$$

form an orthogonal system on the right helicoids $\mathbf{r} = (u\cos v, u\sin v, av)$.

Solution. The equation of the given right helicoids is

$$\mathbf{r} = (u\cos v, u\sin v, av)$$

$$\therefore \qquad E = 1,\ F = 0,\ G = u^2 + a^2$$

We know that the two families of curves given by the quadratic differential equation

$$Pdu^2 + 2Qdudv + Rdv^2 = 0 \qquad (1)$$

Form an orthogonal system if and only if

$$ER - 2FQ + GP = 0 \qquad (2)$$

Therefore comparing (1), with the given family of curves

$$du^2 - (u^2 + a^2)dv^2 = 0 \qquad (3)$$

We have $P = 1,\ Q = 0,\ R = -(u^2 + a^2)$

$$\therefore \quad ER - 2FQ + GP = 1.[-(u^2 + a^2)] - 2.0.0 + (u^2 + a^2).1 = 0$$

Hence condition (2) is satisfied for equation (3). Therefore the curves are given by (3) form an orthogonal system on the given surface.

Example 8.19 The metric of a surface is

$$v^2du^2 + u^2dv^2.$$

Find the equation of the family of curves orthogonal to the curves *uv*=constant, and find the metric referred to new parameters so that these two families are parametric.

Solution. The metric of the surface is

$$v^2du^2 + u^2dv^2 = ds^2 \qquad (1)$$

$\therefore$ Comparing (1) with the metric

$$E\,du^2 + 2Fdu\,dv + Gdv^2 = ds^2$$

We get $E = v^2$, $F = 0$, $G = u^2$

The equation of the given family of the curves is

$$uv=\text{constant} \qquad (2)$$

i.e. differential equation of (2) is

$$vdu + udv = 0 \qquad (3)$$

Comparing (3) with $Pdu + Qdv = 0$

We get $P = v,\; Q = u$

Hence orthogonal trajectories of (3) are

$$(0.v - v^2u)\partial u + (u^2v - 0.u)\partial v = 0$$

or $-uv^2\partial u + u^2v\partial v = 0$ or $\dfrac{\partial u}{u} - \dfrac{\partial v}{v} = 0$

On integrating, we get

$$\log\frac{u}{v}=\text{constant.}$$

i.e. $\dfrac{u}{v}$=constant. (4)

Equation (4) gives the required orthogonal trajectory.

Second Part. Now the two families are (i) the given family of curves which is given by *uv*=constant, and (ii) their orthogonal trajectories which are given by $\dfrac{u}{v}$=constant.

If these two families are taken as parametric curves, then the new parameters u', v' are given by

$$u' = \frac{u}{v}, \quad v' = uv \qquad (5)$$

Solving for u, v, we get

$$u^2 = u'v' ; \ v^2 = v'/u' .$$

$$\therefore \quad 2u\frac{\partial u}{\partial v} = v' \qquad \Rightarrow \quad \frac{\partial u}{\partial v} = \frac{v'}{2u}$$

Similarly $\quad \dfrac{\partial u}{\partial v'} = \dfrac{u'}{2u}, \quad \dfrac{\partial v}{\partial u'} = -\dfrac{v}{2vu'^2}$

$$\therefore \quad \frac{\partial v}{\partial v'} = \frac{1}{2vu'}$$

$$\mathbf{r}_1' = \frac{\partial \mathbf{r}}{\partial u'} = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial u'} = \mathbf{r}_1 \frac{v'}{2u} - \mathbf{r}_2 \frac{v'}{2u'^2 v}$$

$$\mathbf{r}_2' = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial v'} = \mathbf{r}_1 \frac{u'}{2u} + \mathbf{r}_2 \frac{1}{2vu'}$$

Now we shall calculate the new coefficients E', F', G'

$$E' = \mathbf{r}_1'.\mathbf{r}_1' = \left(\frac{v'}{2u}\mathbf{r}_1 - \frac{v'}{2vu'^2}\mathbf{r}_2\right).\left(\frac{v'}{2u}\mathbf{r}_1 - \frac{v'}{2vu'^2}\mathbf{r}_2\right)$$

$$= \frac{v'^2}{4u^2}\mathbf{r}_1^2 - \frac{v'^2}{2uvu'^2}\mathbf{r}_1.\mathbf{r}_2 + \frac{v'^2}{4v^2u'^4}\mathbf{r}_2^2$$

$$= \frac{v'^2}{4u^2}E + \frac{v'^2}{4v^2u'^4}G \qquad [\because \ \mathbf{r}_1.\mathbf{r}_2 = F = 0]$$

$$= \frac{v'^2v^2}{4u^2} + \frac{v'^2u^2}{4v^2u'^4}$$

$$= \frac{1}{4}\frac{v'^2}{u'^2} + \frac{1}{4}\frac{v'^2}{u'^2} \qquad \text{[using (5)]}$$

$F' = 0$ as new parametric curves are also orthogonal

$$G' = \mathbf{r}_2' \bullet \mathbf{r}_2' = \left(\frac{u'}{2u}\mathbf{r}_1 + \frac{1}{2vu'}\mathbf{r}_2\right)\bullet\left(\frac{u'}{2u}\mathbf{r}_1 + \frac{1}{2vu'}\mathbf{r}_2\right)$$

$$= \frac{u'^2}{4u^2}E + \frac{1}{4v^2u'^4}G \qquad [\because \ \mathbf{r}_1.\mathbf{r}_2 = F = 0]$$

$$=\frac{u'^2 v^2}{4u^2}+\frac{u^2}{4v^2u'^2}$$

$$=\frac{1}{4}+\frac{1}{4}=\frac{1}{2} \qquad \text{[using (5)]}$$

Hence the metric referred to new parameters u', v' is given by

$$ds'^2 = E'\,du'^2 + 2F'du'\,dv' + G'dv'^2$$

$$=\frac{1}{2}\frac{v'^2}{u'^2}du'^2+\frac{1}{2}dv'^2$$

In general

$$ds^2=\frac{1}{2}\frac{v^2}{u^2}du^2+\frac{1}{2}dv^2.$$

Example 8.20 Show that the parametric curves on the sphere given by

$x = a\sin u\cos v,\ y=\sin u\sin v,\ z = a\cos u$ form an orthogonal system. Determine the two families of curves that meet the curves v=constant at angles $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

Solution. The equation of the given sphere is

$$\mathbf{r} = a(\sin u\cos v, \sin u\sin v, \cos u) \qquad (1)$$

$$\mathbf{r}_1 = a(\cos u\cos v, \cos u\sin v, -\sin u)$$

$$\mathbf{r}_2 = a(-\sin u\sin v, \sin u\cos v, 0)$$

We have $E=\mathbf{r}_1^2 = a^2, \quad F=\mathbf{r}_1.\mathbf{r}_2=0, \quad G=\mathbf{r}_2^2 a^2\sin^2 u$

Here F=0, therefore the parametric curves on the sphere (1) are orthogonal.

Now $\quad H^2 = EG - F^2$ gives

$$H^2 = a^4\sin^2 u \quad i.e. \quad H = a^2\sin u.$$

The direction ratios of the curve v=constant are (1, 0).

Let the direction (du, dv) make an angle $\frac{1}{4}\pi$ with the direction (1, 0), we have

and $$\tan\psi = \frac{|du\partial v - dv\partial u|\,H}{Edu\partial v + F(du\partial v + dv\partial u) + Gdv\partial v} \qquad (2)$$

$\therefore \qquad \tan\dfrac{\pi}{4} = \dfrac{|du.0 - dv.1|\, a^2 \sin u}{a^2 du.1 + 0 + a^2 dv.0}$

or $\qquad = \dfrac{a^2 \sin u.dv}{a^2 du} \qquad$ or $\quad \operatorname{cosec} u\, du = dv$

on integrating, $\quad \log\tan\dfrac{1}{2}u = v + \log c$

or $\quad \tan\dfrac{1}{2}u = ce^{v} \qquad \Rightarrow \qquad e^{v}\tan\dfrac{1}{2}u = c \qquad (3)$

$\therefore$ family of curves which makes an angle $\dfrac{1}{4}\pi$ with the curve *v*=constant is given by (3).

$\therefore$ using formula (2), we get

$$\tan\frac{\pi}{4} = \frac{a^2 \sin u dv}{a^2 du} \qquad \text{or} \qquad -1 = \frac{\sin u dv}{du}$$

or $\quad -\operatorname{cosec} u\, du = dv$

On integrating, we get

$$-\log\tan\frac{1}{2}u = v - \log c$$

or $\quad \log\tan\dfrac{1}{2}u = \log c - v$

or $\quad \tan\dfrac{1}{2}u = ce^{-v} \qquad$ or $\qquad e^{-v}\tan\dfrac{1}{2}u = c$

This is the equation of the family of curves making an angle $\dfrac{3}{4}\pi$ with the curve *v*=constant.

Example 8.21 A helicoid is generated by the screw motion of a straight line that meets the axes at an angle α. Find the orthogonal trajectories of the generators. Find also the metric of the surface referred to the generators and their orthogonal trajectories as parametric curves.

Solution. The equation of the surface is given by

$$\mathbf{r} = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha + av)$$

where u, v are parameters and take all real values.

$\therefore \qquad \mathbf{r}_1 = (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha)$

$$\mathbf{r}_2 = (-u\sin\alpha\sin v, u\sin\alpha\cos v, a)$$

Hence, $E = 1, \quad F = a\cos\alpha, \quad G = a^2 + u^2\sin^2\alpha$

The generators are given by v=constant and have direction ratios (1, 0). If the direction (du, dv) is orthogonal to (1, 0), we have

$$Edu + Fdv = 0 \quad i.e. \quad du + a\cos\alpha dv = 0$$

On integrating, the orthogonal trajectories of the generators are given by

$$u + av\cos\alpha = \text{constant}.$$

To examines these trajectories, we note that u=0 for some values of v on every curve, so that every trajectory meets the axes of helicoids. There is no loss of generality, by taking the intersection, of a particular curve with the axes at the origin. Then $u = c - av\cos\alpha$ and the curve is given by

$\mathbf{r} = a\sin\alpha(-v\cos\alpha\cos v, -v\cos\alpha\sin v, v\sin\alpha); v$ being parameter. The parametric equations to the curve are

$$x = -av\sin\alpha\cos\alpha\cos v, \;\; y = -av\sin\alpha\cos\alpha\sin v, \quad z = av\sin^2\alpha$$

We clearly see that this curve is the intersection of the cone $x^2 + y^2 = z^2\cot^2\alpha$ and the cylinder whose cross-section by the xy-plane is the spiral $r = \dfrac{a}{2}\theta\sin 2\alpha$.

$$[\because z = u\cos\alpha + av = (-av\cos\alpha)\cos\alpha + av = av\sin^2\alpha \;\; at \; u = -av\cos\alpha \,]$$

Now let us consider a transformation

$$u' = u + av\cos\alpha, \; v' = v$$

which takes the generators and their orthogonal trajectories into parametric curves. The metric (with E, F, G as coefficients find above) is

$$ds^2 = du^2 + 2a\cos\alpha\, dudv + (a^2 + u^2\sin^2\alpha)dv^2$$

$$= (du + a\cos\alpha\, dv)^2 + \sin^2\alpha(a^2 + u^2)dv^2$$

with the above transformation, the metric become

$$ds'^2 = du'^2 + \sin^2\alpha\{a^2 + (u' - av'\cos\alpha)^2\}dv'^2$$

And thus the new coefficients are

$$E' = 1, \; F' = 0, \; G' = \sin^2\alpha\{a^2 + (u^2 - av'\cos\alpha)^2\}.$$

Example 8.22 A helicoid is generated by the screw motion of a straight line that meets the axes at an angle α. Find the orthogonal trajectories of the generators. Find also *E*, *F*, *G* of the surface referred to the generators and their orthogonal trajectories as parametric curves.

Example 8.23 On the paraboloid $x^2 - y^2 = z$, find the orthogonal trajectories of the section by the planes *z*-constant.

Solution. Let $x = u, \quad y = v$ so that $u^2 - v^2 = z$.

$\therefore$ Equation of the paraboloid may be written in vector form as

$$\mathbf{r} = (u,\ v,\ u^2 - v^2) \text{ so that given curves are } u^2 - v^2 = z.$$

$$\mathbf{r}_1 = (1,\ 0,\ 2u);\ \mathbf{r}_2 = (0, 1, -2v)$$

$$E = 1 + 4u^2, \quad F = -4uv, \quad G = 1 + 4v^2 \tag{1}$$

But the differential equations of the given curves are

$$udu - vdv = 0 \qquad \Rightarrow \qquad \frac{du}{v} = \frac{dv}{u}. \tag{2}$$

Thus the tangent at (u,v) has the direction ratios (v,u). We also note that $P = u$ and $Q = -v$ vanish together at the origin, this point must be excluded.

Let (du, dv) be orthogonal to the direction (v,u), we have

$$Evdu + F(vdu + udv) + Gudv = 0 \tag{3}$$

Substituting values from (1) in (3), we get

$$vdu + udv = 0$$

Integrating yield uv =constant, giving the orthogonal trajectories. Therefore they are the sections of the paraboloid by the hyperbolic cylinders xy =constants.

Example 8.24 Show that on a right helicoid, the family of curves orthogonal to the curves $u\cos v$ =constant is the family $(u^2 + a^2)\sin^2 v$ =constant.

Solution. Let the equation of right helicoids be

$$\mathbf{r} = (u\cos v, u\sin v, av)$$

then $\quad E = 1, \quad F = 0, \quad G = u^2 + a^2$

The family of given curves is

$u\cos v$=constant *i.e.* $\cos v du - u\sin v dv = 0$.

Hence the direction ratios of the tangent to the curve at the point (u, v) is $(u\sin v, \cos v)$. Let the required family of orthogonal curves be in the direction (du, dv). Hence for orthogonality of $(u\sin v, \cos v)$ and (du, dv), we have

$$Eu\ \sin v\, du + F(u\ \sin v\, du + \cos v dv) + G\cos v\, dv = 0$$

or $$u\ \sin v\, du + (u^2 + v^2)\cos v\, dv = 0$$

or $$\frac{2udu}{u^2 + v^2} + \frac{2\cos v dv}{\sin v} = 0$$

integrating, $\log(u^2 + v^2) + 2\log\sin v$=constant.

Which is the required family of orthogonal curves.

8.11 CHECK YOUR PROGRESS

SA1: On the right helicoids given by $x = u\cos v,\quad y = u\sin v,\quad z = cv$. Show that the parametric curves are circular helices and straight lines.

SA2: On the surface generated by the binomials of a twisted curve, show that the position vector **R** of the current point may be expressed as $\mathbf{R} = \mathbf{r} + u\,\mathbf{b}$, where **r** and **b** are the functions of *s*. Taking *u*, *s* as parameters, show that

$$E = 1,\quad F = 0,\quad G = 1 + \tau^2 u^2,\quad H^2 = 1 + \tau^2 u^2,\ \mathbf{N} = (\mathbf{n} + u\,\tau\,\mathbf{t})/H$$

$L = 0,\quad M = -\tau/H,\quad N = (\kappa + \kappa\tau^2 u^2 - \tau^2 u)/H$, where **n** is the unit principal normal to the curve.

SA3: If ψ is the angle between the two directions given by

$$Pdu^2 + 2Qdudv + Rdv^2 = 0$$

Show that $\tan\psi = \dfrac{H(Q^2 - 4PR)}{ER - FQ + GP}$.

Hence show that the two directions will be orthogonal if $ER - FQ + GP = 0$.

SA4: Show that the differential equation of the orthogonal trajectories of the family of curves given by $Pdu + Qdv = 0$ is $(EQ - FP)du + (FQ - GP)dv = 0$.

SA5: Show that the parametric curves are orthogonal on the surface given by

$$x = u\cos v, \quad y = u\sin v, \quad z = c\log\{u + \sqrt{(u^2 - c^2)}\}.$$

SA6: Obtain the formula $\kappa_n = \dfrac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdu\,dv + Gdv^2}$ for the curvature of the normal section of the surface and deduce Meusnier's theorem.

SA7: Define direction coefficient on a surface and obtain formulae for sine and cosine of the angle between two directions.

SA8: Define the normal curvature. State and prove Meusnier's theorem.

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CHAPTER-9

CURVES CONCERNING GEODESICS-I

Objectives: In continuation of the previous chapters in the current chapter the students will learn about principal directions, Joachimsthal's theorem, First and second curvatures, Euler's Theorem, Dupin's indicatrix, Surface in Monge's form, Surface of revolution, Conjugate directions, Conjugate systems, Asymptotic lines, Curvature and torsion, Isometric Parameters, Gauss's formulae for $\mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{22}$, Gauss characteristic equation, Mainardi-Codazzi relation.

9.1. Principal directions: Normals at consecutive points of a surface do not intersect; but at any point P there are two directions on the surface, at a right angle to each other, such that the normal at a consecutive point in either of these directions meets the normal at P. These are called principal directions at P. To prove this property, let $\mathbf{r}$ be the position vector of P and $\mathbf{n}$ the unit normal there. Let $\mathbf{r}+d\,\mathbf{r}$ be a consecutive point in the direction du, dv, and $\mathbf{n}+d\,\mathbf{n}$ the unit normal at this point. The normals will intersect if $\mathbf{n}$, $\mathbf{n}+d\,\mathbf{n}$ and $d\,\mathbf{r}$ are coplanar, that is to say, if $\mathbf{n}, d\,\mathbf{n}$, $d\,\mathbf{r}$ are coplanar. This will be so if their scalar triple product vanishes, so that

$$[\mathbf{n}, d\mathbf{n}, d\mathbf{r}] = 0 \tag{9.1}$$

This condition may be expanded in terms of du, dv. For

$$d\mathbf{n} = \mathbf{n}_1 du + \mathbf{n}_2 dv$$

$$d\,\mathbf{r} = \mathbf{r}_1 du + \mathbf{r}_2 dv$$

and the substitution of these values in (9.1), gives

$$[\mathbf{n}, \mathbf{n}_1, \mathbf{r}_1] du^2 + \{[\mathbf{n}, \mathbf{n}_1, \mathbf{r}_2] + [\mathbf{n}, \mathbf{n}_2, \mathbf{r}_1]\} du dv + [\mathbf{n}, \mathbf{n}_2, \mathbf{r}_2] dv^2 = 0$$

which, is equivalent to

$$(EM - FL) du^2 + (EN - GL) du dv + (FN - GM) dv^2 = 0 \tag{9.2}$$

This equation gives values of the ratio $du : dv$, and therefore two directions on the surface for which the required property holds. And these two directions are at right angles, for they satisfy the condition of orthogonality.

It follows from the above that, displacement in a principal direction, $d\mathbf{n}$ is parallel to $d\mathbf{r}$. For $d\mathbf{r}$ is perpendicular to **n**, and $d\mathbf{n}$ is also perpendicular to **n** since **n** is a unit vector. But these three vectors are coplanar, and therefore $d\mathbf{n}$ is parallel to $d\mathbf{r}$. Thus, a principal direction $\mathbf{n}'$ is parallel to $\mathbf{r}'$, the dash denoting arc-rate of change.

A curve drawn on the surface, and possessing the property that the normals to the surface at consecutive points intersect, is called a line of curvature. It follows from the above that the direction of a line of curvature at any point is a principal direction at that point. Through each point on the surface pass two lines of curvature cutting each other at right angles, and on the surface, there are two systems of lines of curvature whose differential equation is (9.2). The point of intersection of consecutive normals along a line of curvature at *P* is called a center of curvature of the surface and its distance from *P*, measured in the direction of the unit normal **n**, is called a (principal) radius of curvature of the surface. The reciprocal of a principal radius of curvature is called a principal curvature. Thus at each point of the surface, there are two principal curvatures κ_a and κ_b, and these are the normal curvatures of the surface in the directions of the lines of curvature. They must not be confused with the curvatures of the lines of curvature. The principal normal of a line of curvature is not in general the normal to the surface. In other words, the osculating plane of a line of curvature does not, as a rule, give a normal section of the surface, but the curvature of a line of curvature is connected with the corresponding principal curvature as in Meunier's theorem.

The principal radii of curvature will be denoted by α, β. As these are the reciprocals of the principal curvature, we have

$$\alpha\kappa_a = 1,\ \kappa_b = 1$$

Those portions of the surface on which the two principal curvatures have the same sign are said to be synclastic. The surface of a sphere or an ellipsoid is synclastic. The surface of a hyperbolic paraboloid is anticlastic. The surface of a hyperbolic paraboloid is anticlastic at all points.

At any point of a surface, there are two centers of curvature, one for each principal direction. Both lie on the normal to the surface, for they are the centers of curvature of normal sections tangential to the lines of curvature. The locus of the centers of curvatures is a surface called the surface of centers, or the Centro-surface. It consists of two branches, one corresponding to each system of lines of curvature. The properties of the Centro-surface will be examined in a later chapter.

Theorem 1. Joachimsthal's theorem. If the curves of the intersection of two surfaces are a line of curvature on both, the surfaces cut at a constant angle. Conversely, if two surfaces cut at a constant angle, and the curve of intersection is a line of curvature on one of them, it is a line of curvature on the other also.

Solution. Let **t** be the unit tangent to the curve of intersection, and $\mathbf{n}, \overline{\mathbf{n}}$ the unit normals at the same point to the two surfaces. Then **t** is perpendicular to **n** and $\overline{\mathbf{n}}$, and therefore parallel to $\mathbf{n} \times \overline{\mathbf{n}}$. Further, if the curve is a line of curvature on both surfaces, **t** is parallel to $\mathbf{n}'$ and $\overline{\mathbf{n}}'$, the dashes are usual denoting arc-rate of change. Let θ be the inclination of the two normals.

Then $\cos\theta = \mathbf{n} \bullet \overline{\mathbf{n}}$, and

$$\frac{d}{ds}\cos\theta = \mathbf{n}' \bullet \overline{\mathbf{n}} + \mathbf{n} \bullet \overline{\mathbf{n}}'.$$

Put each of these terms vanishes because $\mathbf{n}'$ and $\overline{\mathbf{n}}'$ are both parallel to **t**. Thus $\cos\theta$ is constant, and the surfaces are cut at a constant angle.

Similarly if θ is constant, and the curve is a line of curvature on the first surface, all the terms of the above equation disappear except the last. Hence this must vanish also, showing that $\overline{\mathbf{n}}'$ is perpendicular to $\mathbf{n}$. But it is also perpendicular to $\overline{\mathbf{n}}$, because $\overline{\mathbf{n}}$ is a unit vector. Thus $\overline{\mathbf{n}}'$ is parallel to $\mathbf{n} \times \overline{\mathbf{n}}$ and therefore also for **t**. The curve of intersection is thus a line of curvature on the second surface also.

9.2. First and second curvatures:

To determine the principal curvatures at any point, we may proceed as follows. Let **r** be the position vector of the point, **n** the unit normal, and ρ a principal radius of curvature. Then the corresponding center of curvature is $\mathbf{r} + \rho\mathbf{n}$. For an infinitesimal displacement of the point along the line of curvature, we have therefore

$$d\mathbf{s} = (d\mathbf{r} + \rho d\mathbf{n}) + \mathbf{n} d\rho.$$

The vector in brackets is tangential to the surface, and consequently since $d\mathbf{s}$ has the direction of **n**.

$$0 = d\mathbf{r} + \rho d\mathbf{n} \tag{9.3}$$

or, if κ is the corresponding principal curvature,

$$0 = \kappa\, d\mathbf{r} + d\mathbf{n} \tag{9.3$'$}$$

This is the vector equivalent of Rodrigue's formula. It is of very great importance. Inserting the values of the differentials in terms of du and dv, we may write it

$$(\kappa\mathbf{r}_1+\mathbf{n}_1)du+(\kappa\mathbf{r}_2+\mathbf{n}_2)dv=0$$

Forming the scalar products of this with $\mathbf{r}_1$ and $\mathbf{r}_2$ successively, we have

$$\left.\begin{aligned}(\kappa E-L)du+(\kappa F-M)dv=0\\(\kappa F-M)du+(\kappa G-N)dv=0\end{aligned}\right\} \qquad (9.4)$$

These two equations determine the principal curvatures and the directions of the lines of curvatures.

or $$(\kappa E-L)(\kappa G-N)=(\kappa F-M)^2,$$

$$H^2\kappa^2-(EN-2FM+GL)\kappa+T^2=0 \qquad (9.5)$$

A quadratic, giving two values of κ as required.

The first curvature of the surface at any point may be defined as the sum of the principal curvatures. We will denote it by *J*.

Thus $$J=\kappa_a+\kappa_b.$$

Being the sum of the roots of the quadratic (9.5), it is given by

$$J=\frac{1}{H^2}(EN-2FM+GL) \qquad (9.6)$$

The second curvature or specific curvature, of the surface at any point is the product of the principal curvatures. It is also called the Gauss curvature and is denoted by *K*. It is equal to the product of the roots of (9.5), so that

$$K=\kappa_a\kappa_b=\frac{T^2}{H^2} \qquad (9.7)$$

when the principal curvatures have been determined from (9.5), the direction of the lines of curvatures is given by either of the equation (9.4). Thus corresponding to the principal curvature κ_a, the principal direction is given by

$$\frac{du}{dv}=-\left(\frac{\kappa_aF-M}{\kappa_aE-L}\right) \quad \text{or} \quad -\left(\frac{\kappa_aG-N}{\kappa_aF-M}\right),$$

similarly for the other principal direction.

The directions of the lines of the curvature may, of course, be found independently by eliminating κ from the equation (9.4). This leads to

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0 \qquad (9.8)$$

The same equation as (9.2) was found by a differential method. It may be remarked that this is also the equation giving the directions of maximum and minimum normal curvature at the point. For, the value of normal curvature, being

$$\kappa_n = \begin{cases} Ldu^2 + 2Mdudv + Ndv^2 \\ Edu^2 + 2Fdudv + Gdv^2 \end{cases} \qquad (9.9)$$

Some writers call *J* the mean curvature and *K* the total curvature. On this question, Is a function of the ratio of $du{:}dv$, and if its derivative with respect to this ratio is equated to zero, we obtain the same equation (9.8) as before. Thus the principal directions at a point are the directions of greatest and least normal curvature.

The equation (9.8) however, fails to determine these directions when the coefficients vanish identically, that is to say, when

$$E : F : G = L : M : N \qquad (9.10)$$

In this case, the normal curvature, as determined by (9.9), is independent of the ratio $du{:}dv$, and therefore has the same value for all, directions through the point. Such a point is called an *umbilic* on the surface.

If the amplitude of normal curvature, *A*, and the mean normal curvature, *B*, is defined by

$$A = \frac{1}{2}(\kappa_b - \kappa_a), \qquad B = \frac{1}{2}(\kappa_b + \kappa_a) \qquad (9.11)$$

It follows that

$$\kappa_a = B - A, \qquad \kappa_b = B + A \qquad (9.12)$$

Hence the second curvature may be expressed as

$$K = B^2 - A^2.$$

We may also mention in passing that when the first curvature vanishes at all points; the surface is called a minimal surface. The properties of such surfaces will be examined in a later chapter.

Example 9.1 Find the principal curvatures and the lines of curvature on the right helicoids

$x = u\cos\phi, \quad y = u\sin\phi, \quad z = c\phi$.

Solution. The fundamental magnitudes for this surface are

$$E = 1, \quad F = 0, \quad G = u^2 + c^2, \quad H^2 = u^2 + c^2$$

$$L = 0, \quad M = -\frac{c}{H}, \quad N = 0, \quad T^2 = -\frac{c^2}{H^2}.$$

The formula (9.5) for the principal curvatures then becomes

$$(u^2 + c^2)\kappa^2 - c^2 = 0$$

Whence $\quad \kappa = \pm\frac{c}{u^2 + c^2}$.

The first curvature is, therefore, zero so that the surface is minimal.

The second curvature is $\kappa = -\frac{c^2}{(u^2 + c^2)^2}$

The differential equation for lines of curvature becomes

$$-cdu^2 + (u^2 + c^2)cd\phi^2 = 0,$$

that is $\quad d\phi = \pm\frac{du}{\sqrt{u^2 + c^2}}$.

Example 9.2 Find the principal directions and the principal curvatures on the surface

$$x = a(u + v), \quad y = b(u - v), \quad z = uv.$$

Solution. We know that

$$a^2 + b^2 + v^2, \quad F = a^2 - b^2 + uv, \quad G = a^2 + b^2 + u^2,$$

$$H^2 = 4a^2b^2 + a^2(u - v)^2 + b^2(u + v)^2$$

and also $\quad L = 0, \quad M = -\frac{2ab}{H}, \quad N = 0, \quad T^2 = -\frac{4a^2b^2}{H^2}$.

The differential equation for the lines of curvature therefore gives

$$(a^2 + b^2 + v^2)du^2 - (a^2 + b^2 + u^2)dv^2 = 0$$

or $$\frac{du}{\sqrt{(a^2+b^2+u^2)}} = \pm\frac{dv}{\sqrt{(a^2+b^2+v^2)}}.$$

The equation for the principal curvatures becomes

$$H^4\kappa^2 - 4abH(a^2-b^2+uv)\kappa - 4a^2b^2 = 0$$

So the specific curvature is $K = -\dfrac{4a^2b^2}{H^4}$ and the first curvature is

$$J = 4ab(a^2-b^2+uv)/H^3.$$

Example 9.3 Find the principal curvatures etc. on the surface generated by the binormals of a twisted curve.

Solution. The position vector of the current point on the surface may be expressed

$$\mathbf{R} = \mathbf{r} + u\mathbf{b},$$

where **r** and **b** are functions of the arc-length *s*. Taking u, s as parameters, and using dashes as usual to denote *s*-derivatives of quantities belonging to the curve, we have

$$\mathbf{R}_1 = \mathbf{b}, \qquad \mathbf{R}_2 = \mathbf{t} - u\tau\bar{\mathbf{n}},$$

where $\bar{\mathbf{n}}$ is the unit principal normal to the curve. Hence

$$E = 1, \quad F = 0, \quad G = 1+\tau^2u^2, \quad H^2 = 1+\tau^2u^2,$$

And the unit normal to the surface is

$$\mathbf{n} = \frac{\mathbf{R}_1\times\mathbf{R}_2}{H} = \frac{\bar{\mathbf{n}}+\tau u\,\mathbf{t}}{H}.$$

Further $\quad \mathbf{R}_{11} = 0, \quad \mathbf{R}_{12} = -\tau\bar{\mathbf{n}},$

$$\mathbf{R}_{22} = (\kappa - u\tau')\bar{\mathbf{n}} + u\tau(\kappa\mathbf{t} - \tau\mathbf{b}),$$

and therefore $\quad L = 0, \qquad M = -\dfrac{\tau}{H},$

$$N = (\kappa + \kappa\tau^2u^2 - \tau'u)/H, \quad T^2 = -\frac{\tau^2}{H^2}$$

The equation for principal radii of curvature then becomes

$$(1+\tau^2u^2)^2 - \sqrt{1+\tau^2u^2}\,(\kappa+\kappa\tau^2u^2-\tau'u)\rho - \tau^2\rho^2 = 0$$

The Gauss curvature is therefore

$$K = -\frac{\tau^2}{(1+\tau^2u^2)^2},$$

and the first curvature

$$J = -\frac{\kappa + \kappa\tau^2u^2 - \tau' u}{(1+\tau^2u^2)^{3/2}}.$$

For points on the given curve, $u = 0$. At such points the Gauss curvature is $-\tau^2$, and the first curvature is κ.

The differential equation of the lines of curvature reduces to

$$\tau du^2 - (\kappa + \kappa\tau^2u^2 - \tau' u)duds - (1+\tau^2u^2)\tau\, ds^2 = 0.$$

9.3. Euler's Theorem: It is sometimes convenient to refer to the surface to its lines of curvature as parametric curves. If this is done the differential equation (9.2) for the lines of curvature becomes identical to the differential equation of the parametric curve, that is

$$dudv = 0.$$

Hence we must have

$$EM - FL = 0, \quad FN - GM = 0,$$

and $$EN - GL \neq 0.$$

From the first two relations, it follows that

$$\left.\begin{aligned}(EN - GL)M &= 0\\ (En - GL)\ F &= 0\end{aligned}\right\},$$

And therefore, since the coefficient of *F* and *M* does not vanish.

$$F = 0, \quad M = 0 \qquad (9.13)$$

These are the necessary and sufficient conditions that the parametric curves be lines of curvature. The condition $F = 0$ is that of orthogonality satisfied by all lines of curvature. The significance of the conditions $M = 0$ will appear shortly.

We may now prove Euler's theorem, expressing the normal curvature in any direction in terms of the principal curvatures at the point. Let the lines of curvature be taken as parametric curves, that $F = M = 0$. The principal curvature κ_a being the normal curvature for the direction $dv = 0$ is by (9.9)

$$\kappa_a = L/E,$$

Similarly, the principal curvature for the direction $du = 0$ is

$$\kappa_b = N/G.$$

Consider a normal section of the surface in the direction du, dv having an angle ψ with the principal direction $dv = 0$. Since $F = 0$ we have

$$\cos\psi = \sqrt{E}\frac{du}{ds},$$

$$\sin\psi = \sqrt{G}\frac{dv}{ds}.$$

The curvature κ_n of this normal section is (9.9)

$$\kappa_n = L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin^2\psi$$

that is $$\kappa_n = \kappa_a\cos^2\psi + \kappa_b\sin^2\psi \qquad (9.14)$$

this is Euler's theorem on normal curvature. An immediate and important consequence is the theorem, associated with the name Dupin, that the sum of the normal curvatures in two directions right angle is constant and equal to the sum of the principal curvature.

Then the surface is anticlastic in the neighborhood of the point considered, the principal curvatures have opposite signs, and the normal curvature, therefore, vanishes for the directions given by

$$\tan\psi = \pm\sqrt{-\kappa_a/\kappa_b}$$

$$= \pm\sqrt{-\frac{\beta}{\alpha}},$$

where α, β are the principal radii of curvature. But where the surface is synclastic, the curvature of any normal section has the +ve sign as the principal curvature that is to say sections are concave in the same direction. The surface in the neighborhood of the point then lies entirely on one side of the tangent plane at the point. The same result may also be deduced from the expression

$$\frac{1}{2}(Ldu^2 + 2Mdudv + Ndv^2),$$

for the length p of the perpendicular on the tangent plane from a point near the point of contact. For if K is positive, $LN - M^2$ is positive by (9.7), and therefore the above expression for p never changes sign with a variation of du/dv.

Example 9.4 If B is the mean normal curvature and A the amplitude, deduce from Euler's theorem that

$$\kappa_n = B - A\cos 2\psi,$$

$$\kappa_n - \kappa_a = 2A\sin^2\psi,$$

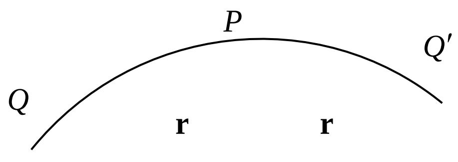
$$\kappa_b - \kappa_n = 2A\cos^2\psi.$$

9.4. Dupin's indicatrix Consider the section of the surface by a plane parallel and indefinitely close to the tangent plane at the point P. Suppose first that the surface is synclastic in the neighborhood of P. then near P it lies entirely on one side of the tangent plane. Let the plane be taken on this (concave) side of the surface, parallel to the tangent plane at P, and at an infinitesimal distance from it, whose measure is h in the direction of the unit normal **n**.

Thus h has the same sign as the principal radii of curvature, α and β. Consider also any normal plane QPQ' through P, cutting the former plane in QQ'. Then if ρ is the radius of curvature of this normal section, and $2r$ the length QQ', we have

$$r^2 = 2h\rho$$

to the first order. If ψ is the inclination of this normal section to the principal direction $dv = 0$.



ρ Figure 9.1

Euler's theorem gives

$$\frac{1}{\alpha}\cos^2\psi + \frac{1}{\beta}\sinh 2\psi = \frac{1}{\rho} = \frac{2h}{r^2}.$$

If then we write $\xi = r\cos\psi$ and $\eta = r\sin\psi$, we have

$$\frac{\xi^2}{\alpha}+\frac{\eta^2}{\beta}=2h.$$

Thus the section of the surface by the plane parallel to the tangent plane at *P*, and infinitely close to it, is similar and similarly situated to the ellipse

$$\frac{\xi^2}{|\alpha|}+\frac{\eta^2}{|\beta|}=1 \qquad (9.15)$$

Whose axes are tangents to the lines of curvature at *P*. this ellipse is called the indirectrix at the point *P*, and *P* is said to be an elliptic point. It is sometimes described as a point of positive curvature because the second curvature *K* is positive.

Next suppose that the Gauss curvature *K* is negative at *P*, so that the surface is anticlastic in the neighborhood. The principal radii α and β have opposite signs, and the surface lies partly on one side and partly on the other side of the tangent plane at *P*. Two planes parallel to this tangent plane, one on either side and equidistant from it, cut the surface in the conjugate hyperbolas

$$\frac{\xi^2}{\alpha}+\frac{\eta^2}{\beta}=\pm 1 \qquad (9.16)$$

which constitute the indicatrix at *P*. the point *P* is then called a hyperbolic point or a point of negative curvature. The normal curvature is zero in the direction of the asymptotes.

When *K* is zero at point *P*, it is called a parabolic point. One of the principal curvatures is zero, and the indicatrix is a pair of parallel straight lines.

9.5. The surface. Let $z = f(x,y)$ it frequently happens that the equation of the surface is given in Monge's form

i.e. $\qquad z = f(x,y)$.

Let x, y be taken as parameters and, with the usual notation for partial derivatives of z, let

$$z_1 = p, \quad z_2 = q, \quad z_{11} = r, \quad z_{12} = s, \quad z_{22} = t$$

Then if **r** is the position vector of a current point on the surface

$$\mathbf{r}_1 = (1, 0, p),$$

$$\mathbf{r}_2 = (1, 0, q),$$

And therefore

$$E = 1 + p^2, \quad F = pq, \quad G = 1 + q^2, \quad H^2 = 1 + p^2 + q^2 .$$

The inclination ω of the parametric curves is given by

$$\cos\omega = \frac{pq}{\sqrt{(1+p^2)(1+q^2)}} .$$

The unit normal to the surface is $\mathbf{n} = \dfrac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = (-p, -q, 1)/H$.

Further $\mathbf{r}_{11} = (0,0,r)$,

$$\mathbf{r}_{12} = (0,0,s),$$

$$\mathbf{r}_{22} = (0,0,t),$$

So that the second-order magnitudes are

$$L = \frac{r}{H}, \quad M = \frac{s}{H}, \quad N = \frac{t}{H}; \quad T^2 = \frac{rt - s^2}{H^2} = \frac{t - s^2}{1 + p^2 + q^2} .$$

The specific curvature is therefore

$$K = \frac{T^2}{H^2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2} .$$

And the first curvature is

$$J = \frac{1}{H^2}\{r(1+q^2) - 2pqs + t(1+p^2)\}.$$

The equation (9.5) for the principal curvature becomes

$$H^4\kappa^2 - H\{r(1+q^2) - 2pqs + t(1+p^2)\}\kappa + (rt - s^2) = 0 .$$

And the differential equation of the lines of curvature is

$$\{s(1+p^2) - rpq\}dx^2 + \{t(1+p^2) - r(1+q^2)\}dxdy + \{tpq - s(1+q^2)\}dy^2 = 0$$

Since a developable surface $rt - s^2$ is identically zero, it follows from the above values of K that the second curvature vanishes at all points of a developable surface; and conversely, if the specific curvature is identically zero, the surface is developable.

9.6. Surface of revolution A surface of revolution may be generated by the rotation of a plane curve about an axis in its plane. If this is taken as the axis of z and u denotes the perpendicular distance from it, the coordinate of a point of a surface may be expressed

$$x = u\cos\phi, \quad y = u\sin\phi, \quad z = f(u),$$

the longitude ϕ being the inclination of the axial plane through the given point to the zx -plane. The parametric curve v =constant are the "meridian lines" or the intersection of the surface by the axial planes; the curve u =constant are the "parallels" or intersections of the surface by planes perpendicular to the axis.

With u, ϕ as parameters, and **r** the position vector of a current point on the surface, we have

$$\mathbf{r}_1 = (\cos\phi, \sin\phi, f_1)$$

$$\mathbf{r}_2 = (-u\sin\phi, u\cos\phi, 0).$$

The first order magnitudes are therefore

$$E = (1 + f_1^2), \quad F = 0, G = u^2, \quad H^2 = u^2(1 + f_2^2).$$

Since $F = 0$ it follows that the parallels cut the meridians orthogonally. The unit normal to the surface is

$$\mathbf{n} = (-f_1 u\cos\phi, -f_1 u\sin\phi, u)/H$$

Further, $\mathbf{r}_{11} = (0, 0, f_u)$,

$$\mathbf{r}_{12} = (-\sin\phi, \cos\phi, 0),$$

$$\mathbf{r}_{22} = (-u\cos\phi, -u\sin\phi, 0),$$

So that the second-order magnitudes are

$$L = uf_{11}/H, \quad M = 0, \quad N = u^2 f_1/H, \quad T^2 = u^2 f_1 f_{11}/H^2.$$

Since F and M both vanish identically, the parametric curves are the lines of curvature.

The equation for the principal curvatures reduces to

$$u(1 + f_1^2)^2\kappa^2 - \sqrt{1 + f_1^2}\{uf_{11} + f_1(1 + f_1^2)\}\kappa + f_1 f_{11} = 0,$$

The roots of which are

$$\kappa_a = \frac{f_{11}}{(1 + f_1^2)^{3/2}} = \frac{d^2 f/du^2}{\left\{1 + (df/du)^2\right\}^{3/2}},$$

and $$\kappa_b = \frac{f_1}{u\sqrt{(1+f_1^2)}} = \frac{df/du}{u\sqrt{\{1+(df/du)^2\}}}.$$

The first of these is the curvature of the generating curve. The second is the reciprocal of the length of the normal intercepted between the curve and the axis of rotation. The Gauss curvature is given by

$$K = \frac{f_1 f_{11}}{u(1+f_1^2)^2},$$

and the first curvature by

$$J = \frac{uf_{11} + f_1(1+f_1^2)}{u(1+f_1^2)^{3/2}}.$$

9.7. Conjugate directions Conjugate directions at a given point P on the surface may be defined as follow. Let Q be a point on the surface adjacent to P, and let PR be the line of intersection of the tangent planes at P and Q. Then, as Q tends to coincidence with P, the limiting direction of PQ and PR are said to be conjugate directions at P. Thus the characteristic of the tangent plane, as the point of contact moves along a given curve, is the tangent lines in the direction conjugate to that of the curve at the point of contact. In other words, the tangent planes to the surface along with a curve C envelope a developable surface each of whose generators has the direction conjugate to that of C at their point of the intersection.

To find an analytical expression of the condition that two directions may be conjugate, let $\mathbf{n}$ be the unit normal at P where the parameters' values are u, v and $\mathbf{n} + d\mathbf{n}$ that at Q where values are $u+du, v+dv$. If R is adjacent to P, in the direction of the intersection of the tangent planes at P and Q, we may denote the vector PR by $\delta\mathbf{r}$ and the parameter values at R by $u+\delta u, v+\delta v$. Then since PR is parallel to the tangent planes at P and Q, and $\delta\mathbf{r}$ is perpendicular both to $\mathbf{n}$ and $\mathbf{n}+d\mathbf{n}$. Hence $\delta\mathbf{r}$ is perpendicular to $d\mathbf{n}$, so that

$$d\,\mathbf{n} \bullet \delta\mathbf{r} = 0,$$

and consequently

$$(\mathbf{n}_1 du + \mathbf{n}_2 dv) \bullet (\mathbf{r}_1 \delta u + \mathbf{r}_2 \delta v) = 0.$$

Expanding this product and remembering that

$$\mathbf{n}_1 \bullet \mathbf{r}_1 = -L, \quad \mathbf{n}_1 \bullet \mathbf{r}_2 = \mathbf{n}_2 \bullet \mathbf{r}_1 = -M, \quad \mathbf{n}_2 \bullet \mathbf{r}_2 = -N$$

We obtain the relation

$$L\,du\,\delta u + M\,(du\,\delta v + \delta u\,du) + N dv \delta v = 0 \qquad (9.17)$$

This is the necessary and sufficient condition that the direction $\delta u/\delta v$ be conjugate to the direction du/dv, and the symmetry of the relationship shows that the property is a reciprocal one. Moreover, the equation is linear in each of the ratios $du:dv$ and $\delta u:\delta v$, so that to a given direction there is one and only one conjugate direction.

The condition (9.17) that two directions be conjugate may be expressed

$$L\frac{du}{dv}\frac{\delta u}{\delta v} + M\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + N = 0 \qquad (9.17')$$

Hence the two directions are given by the equation

$$Pdu^2 + Qdudv + Rdv^2 = 0$$

Will be conjugate provide

$$L\left(\frac{R}{P}\right) + M\left(-\frac{Q}{P}\right) + N = 0$$

that is

$$LR - MQ + NP = 0 \qquad (9.18)$$

Now the parametric curves are given by $P = R = 0$ and $Q = 1$. Hence the directions of the parametric curves will be conjugated provided $M = 0$. We have seen that this condition is satisfied when the lines of curvature are taken as parametric curves. Hence the principal directions at a point of the surface are conjugate directions.

Let the lines of curvature be taken as parametric curves, so that $F = 0$ and $M = 0$. The directions du/dv and $\delta u/\delta v$ are inclined to the curve v =constant at angles θ, θ' such that

$$\tan\theta \tan\theta' = -\frac{L}{E}\frac{G}{N} = -\frac{\beta}{\alpha},$$

That is to say, provided they are parallel to conjugate diameters of the indicatrix.

9.8. Conjugate systems Consider the family of curves

$$\phi(u,v)=\text{constant}$$

The direction $\delta u/\delta v$ of a curve at any point is given by

$$\phi_1\delta u+\phi_2\delta=0.$$

The conjugate direction du/dv, in virtue of (17), is then determined by

$$(L\phi_2-M\phi_1)du+(M\phi_2-N\phi_1)dv=0 \qquad (9.19)$$

This is a differential equation of the first order and first degree, and therefore defines a one-parameter family of curves $\psi(u,v)$=constant. This and the family $\phi(u,v)$=constant are said to form a conjugate system. At a point of intersection of two curves, one from each family, their directions are conjugate.

$$\phi(u,v)=\text{constant},$$

$$\psi(u,v)=\text{constant},$$

we may determine the condition that they form a conjugate system. For, the directions of the two curves through a point u,v are given by

$$\left.\begin{aligned}\phi_1\delta u+\phi_2\delta v=0\\ \psi_1\delta u+\psi_2\delta v=0\end{aligned}\right\}$$

It then follows from $(9.17')$ that these directions will be conjugate if

$$L\phi_2\psi_2-M(\phi_1\psi_2+\phi_2\psi_1)+N\phi_1\psi_1=0 \qquad (9.20)$$

This is the necessary and sufficient condition that the two families of curves form a conjugate system. In particular, the parametric curves v=constant, u=constant will form a conjugate system if $M=0$. This agrees with the result found in the previous article. Thus, $M=0$ is the necessary and sufficient condition that the parametric curves form a conjugate system.

We have seen that when the lines of curvature are taken as parametric curves, both $F=0$ and $M=0$ are satisfied. Thus the lines of curvature form an orthogonal conjugate system. And they are the only orthogonal conjugate system. For, if such a system of curves exists, and we take them for parametric curves, $F=0$ and $M=0$. But this shows that the parametric curves are then lines of curvature. Hence the theorem.

Example 9.5 The parametric curves are conjugate on the following surfaces:

(i) A surface of revolution

$$x=u\cos\phi,\quad y=u\sin\phi,\quad z=f(u);$$

(ii) The surface generated by the tangents to a curve, on which

$\mathbf{R} = \mathbf{r} + u\,\mathbf{t}$ $(u, s$ parameters);

(iii) The surface

$x = \phi(u), \quad y = \psi(v), \quad z = f(u) + F(v)$;

(iv) The surface $z = f(x) + F(y)$, where x, y are parameters;

(v) $x = A(u-a)^m (v-a)^n, \quad y = B(u-b)^m (v-b)^n, \quad z = C(u-c)^m (v-c)^n$ where (A, B, C, a, b, c) are constants.

Example 9.6 Prove that at any point of the surface, the sum of the radius normal curvature in conjugate directions is constant.

9.9. Asymptotic lines The asymptotic directions at a point on the surface are the self-conjugate directions, and an asymptotic line is a curve whose direction at every point is self conjugate. Consequently, if in equation (9.17) connecting conjugate directions, we put $\delta u / \delta v$ equal to du/dv, we obtained the differential equation of the asymptotic lines on the surface

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \qquad (9.21)$$

Thus there are two asymptotic directions at a point. They are real and different when $M^2 - LM$ are positive, that is to say when the specific curvature is negative. They are imaginary when K are positive. They are identical when K is zero. In the last case, the surface is developable, and the single asymptotic line through a point is the generator.

Since the normal curvature in any direction is equal to

$$Lu'^2 + 2Mdu'dv' + Ndv'^2,$$

It vanishes for the asymptotic directions. These directions are therefore the directions of the asymptotes of the indicatrix, hence the name. They are at right angles when the indirectrix is a rectangular hyperbola, that is when the principal curvatures are equal and opposite. Thus the asymptotic lines are orthogonal when the surface is minimal.

The osculating plane at any point of an asymptotic line is the tangent plane to the surface. This may be provided as follows. Since the tangent **t** to the asymptotic line is perpendicular to the normal **n** to the surface, $\mathbf{n} \bullet \mathbf{t} = 0$. On differentiating this with respect to the arc length of the line, we have

$$\mathbf{n}' \bullet \mathbf{t} + \mathbf{n} \bullet (\kappa \bar{\mathbf{n}}) = 0,$$

where $\bar{\mathbf{n}}$ is the principal normal to the curve. Now the first term in this equation vanishes, because, **t** is perpendicular to the rate of change of the unit normal in the conjugate direction, and an asymptotic direction is self-conjugate. Thus $\mathbf{n}' \bullet \mathbf{t} = 0$ and the last equation becomes

$$\mathbf{n} \bullet \mathbf{n}' = 0.$$

Then since both **t** and $\bar{\mathbf{n}}$ are perpendicular to the normal, the osculating plane of the curve is tangential to the surface. The binormal is therefore normal to the surface, and we may take its direction so that

$$\mathbf{b} = \mathbf{n} \qquad (9.22)$$

Then the principal normal $\bar{\mathbf{n}}$ is given by

$$\bar{\mathbf{n}} = \mathbf{n} \times \mathbf{t}.$$

If the parametric curves be asymptotic lines the differential equation (9.21) is identical to the differential equation of the parametric curves

$$du dv = 0.$$

Hence the necessary and sufficient conditions that the parametric curves be asymptotic lines are

$$L = 0, \quad N = 0, \quad M \neq 0.$$

In this case, the differential equation of the lines of curvature becomes

$$E du^2 - G dv^2 = 0$$

and the equation of the principal curvatures is

$$H^2 \kappa^2 + 2FM\kappa - M^2 = 0$$

so that $$K = -\frac{M^2}{H^2}, \qquad J = -\frac{2FM}{H^2} \qquad (9.23)$$

9.10. Curvature and torsion We have seen that the unit binormal to an asymptotic line is the unit normal to the surface or $\mathbf{b} = \mathbf{n}$. The torsion τ is found by differentiating the relation with respect to the arc-length *s*, thus obtaining

$$-\tau \bar{\mathbf{n}} = \mathbf{n}',$$

where $\bar{\mathbf{n}} = \mathbf{n} \times \mathbf{r}'$ is the principal normal to the curve. Forming the scalar product of each side with $\bar{\mathbf{n}}$, we have

$$-\tau = \mathbf{n} \times \mathbf{r}' \bullet \mathbf{n}'$$

so that $$\tau = [\mathbf{n}, \mathbf{n}', \mathbf{r}'] \qquad (9.24)$$

which is one formula for torsion.

The scalar triple product in this formula is of the same formula as that occurring in principal directions, the vanishing of which gives the differential equation of the lines of curvature. The expression (9.24) may then be expanded exactly as in the principal direction, giving the torsion of an asymptotic line

$$\tau = \frac{1}{H}\{(EM - FL)u'^2 + ((EN - GL)u'v'(FN - GM)v'^2\}.$$

Suppose now that the asymptotic lines are taken as parametric curves. Then $L = N = 0$, and this formula becomes

$$\tau = \frac{M}{H}(Eu'^2 - Gv'^2).$$

Hence for the asymptotic line $dv = 0$, we have

$$\tau = \frac{M}{H}E\left(\frac{du}{ds}\right)^2 = -\frac{M}{H} = \sqrt{-K} \qquad (9.25)$$

in virtue of (9.23). Similarly, for the asymptotic line $du = 0$, the torsion is

$$\tau = \frac{M}{H}G\left(\frac{dv}{ds}\right)^2 = -\frac{M}{H} = \sqrt{-K} \qquad (9.25')$$

Thus the torsions of the two asymptotic lines through a point are equal in magnitude and opposite in sign, and the square of either is the negative of the specific curvature. This theorem is due to Beltrami and Enneper.

To find the curvature κ of an asymptotic line, differentiate the unit tangent $\mathbf{t} = \mathbf{r}'$ with respect to the arc-length *s*. Then

$$\kappa \bar{\mathbf{n}} = \mathbf{r}''.$$

Forming the scalar product of each side with the unit vector $\bar{\mathbf{n}} = \mathbf{n} \times \mathbf{r}'$, we have the result

$$\kappa = [\mathbf{n}, \mathbf{r}', \mathbf{r}''] \qquad (9.26)$$

Example 9.7 On the surface $z = f(x, y)$ the asymptotic lines are $rdx^2 + 2sdxdy + tdy^2 = 0$, and their torsion are $\pm\sqrt{s^2 - rt}/(1 + p^2 + q^2)$.

Example 9.8 On the surface of revolution, the asymptotic lines are $f_{11}du^2 + uf_1 d\phi^2 = 0$, write down the value of their torsions.

Example 9.9 Find the asymptotic lines, and their torsions, on the surface generated by the binormals lines to a twisted curve.

Example 9.10 Find the asymptotic lines on the surface $z = y\sin x$.

9.11. Isometric Parameters

Suppose that in terms of the parameters u, v the square of the linear element of the surface has the form

$$ds^2 = \lambda(du^2 + dv^2) \tag{9.27}$$

where λ is a function of u, v or a constant. Then the parametric curves are orthogonal because $F = 0$. Further the length of elements of the parametric curves are $\sqrt{\lambda}du$ and $\sqrt{\lambda}dv$, and these are equal if $du = dv$. Thus the parametric curves corresponding to the values $u, u + du, v, v + dv$ bound a small square provided $du = dv$. In this way, the surface may be mapped out into small squares through parametric curves, the sides of anyone square corresponding to equal increments in u and v.

More generally, if the square of the linear element has the form

$$ds^2 = \lambda(Udu^2 + Vdv^2) \tag{9.28}$$

where U is the function of u only and V a function of v only, we may change the parameters to ϕ, ψ by the transformation

$$d\phi = \sqrt{U}du, \quad d\psi = \sqrt{V}dv.$$

This does not alter the parametric curves; for the curve u =constant is identical to the curves ϕ =constant, and similarly, the curves v =constant are also the curves ψ =constant. The equation (9.28)

$$ds^2 = \lambda(d\phi^2 + d\psi^2) \tag{9.29}$$

which is of the same form as (9.27). Whenever the square of the linear element has the form (9.28) so that, without alternation of the parametric curve, it may be reduced to the form (9.27), the parametric curves are called isometric lines, and the parameters isometric parameters. Sometimes the terms isothermal or isothermic are used.

In the form (9.27) the fundamental magnitudes E and G are equal; but in the more general form (9.28), they are such that

$$\frac{E}{G} = \frac{U}{V} \qquad (9.30)$$

and therefore $$\frac{\partial^2}{\partial u \partial v} \log \frac{E}{G} = 0 \qquad (9.31)$$

Either of these equations, in conjunction with $F = 0$, expresses the condition that the parametric variable may be isometric. For, if it is satisfied, ds^2 has the form (9.28) and may therefore be reduced to the form (9.27).

A simple example of isometric curves is afforded by the meridians and parallels on a surface of revolution. With the usual notation

$$x = u\cos\phi, \quad y = u\sin\phi, \quad z = f(u),$$

$$E = 1 + f_1^2, \quad F = 0, \quad G = u^2,$$

we have, $$ds^2 = (1 + f_1^2)du^2 + u^2 d\phi^2$$

$$u^2\left(\frac{1 + f_1^2}{u^2} du^2 + d\phi^2\right) \qquad (9.32)$$

which is of the form (9.28). The parametric curves are the meridians constant and the parallel u =constant. If we make the transformation

$$d\psi = \frac{1}{u}\sqrt{1 + f_1^2}\, du,$$

and curves ψ =constant are the same as the parallels, and the square of the linear element becomes

$$ds^2 = u(d\psi^2 + d\phi^2),$$

which is of the form (9.27). Thus the meridians and the parallels of the surface of revolution are isometric lines.

Example 9.11 Show that a system of confocal ellipses and hyperbolas are isometric lines in the plane.

Example 9.12 Determine $f(v)$ so that on the right conoid $x = u\cos v, \quad y = u\sin v, \quad z = f(v)$, parametric curves may be isometric lines.

Example 9.13 Find the surface of revolution for which $ds^2 = du^2 + (a^2 - u^2)dv^2$.

Example 9.14 Find the asymptotic lines of the cylindroids

$$x = u\cos v, \quad y = u\sin v, \quad z = m\sin 2v.$$

Example 9.15 On the surface $x = 3u(1+v^2) - u^2, \quad y = 3v(1+u^2) - v^2, \quad z = 3(u^2 - v^2)$,

the asymptotic lines are $u \pm v$ =constant.

Example 9.16 On the paraboloid $2z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$, the asymptotic lines are $\frac{x}{a} \pm \frac{y}{b}$ =constant.

Example 9.17 Find the lines of curvature and the principal curvatures on the cylindroids $z(x^2 + y^2) = 2mxy$.

Example 9.18 If a plane cuts a surface everywhere at the same angle, the section is a line of curvature on the surface.

Example 9.19 Along a line of curvature of a conicoid, one principal radius varies as the cube of the other.

Example 9.20 Find the principal curvatures and the lines of curvature on the surface

$$z^2(x^2 + y^2) = c^2.$$

Example 9.21 Find the asymptotic lines and the lines of curvature on the catenoid of revolution $u = c\cosh\frac{z}{c}$.

9.12. Gauss's formulae for $\mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{22}$. The second derivatives of $\mathbf{r}$ with respect to the parameters may be expressed in terms of $\mathbf{n}, \mathbf{r}_1$, and $\mathbf{r}_2$. Remembering that the L, M, N are the resolved parts of $\mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{22}$ normal to the surface, we may write

$$\left.\begin{aligned} \mathbf{r}_{11} &= L\mathbf{n} + l\mathbf{r}_1 + \lambda\mathbf{r}_2 \\ \mathbf{r}_{12} &= M\mathbf{n} + m\mathbf{r}_1 + \mu\mathbf{r}_2 \\ \mathbf{r}_{22} &= N\mathbf{n} + n\mathbf{r}_1 + \upsilon\mathbf{r}_2 \end{aligned}\right\} \qquad (9.33)$$

and the values of the coefficients $l, m, n, \lambda, \mu, \upsilon$ may be found as follows. Since

$$\mathbf{r}_1 \bullet \mathbf{r}_{11} = \frac{1}{2}\frac{\partial \mathbf{r}_1^2}{\partial u} = \frac{1}{2}E_1,$$

and $$\mathbf{r}_2 \bullet \mathbf{r}_{11} = \frac{\partial}{\partial u}(\mathbf{r}_1 \bullet \mathbf{r}_2) - \frac{1}{2}\frac{\partial}{\partial u}\mathbf{r}_1^2 = F_1 - \frac{1}{2}E_2,$$

we find from the first of (9.33), on forming the scalar product of each side with $\mathbf{r}_1$ and $\mathbf{r}_2$ successively,

$$\left.\begin{aligned} \frac{1}{2}E_1 &= lE + \lambda F \\ F_1 - \frac{1}{2}E_2 &= lF + \lambda G \end{aligned}\right\}.$$

Solving these for l and λ, we have

$$\left.\begin{aligned} l &= \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2) \\ \lambda &= \frac{1}{2H^2}(2EF_1 - EE_2 - FE_1) \end{aligned}\right\} \qquad (9.34)$$

Again since $\mathbf{r}_1 \bullet \mathbf{r}_{12} = \frac{1}{2}E_2$ and $\mathbf{r}_2 \bullet \mathbf{r}_{12} = \frac{1}{2}G_1$, we find from the second of (9.33), on forming the scalar product of each side with $\mathbf{r}_1$ and $\mathbf{r}_2$ successively,

$$\left.\begin{aligned} \frac{1}{2}E_2 &= mE + \mu F \\ \frac{1}{2}G_1 &= mF + \mu G \end{aligned}\right\}.$$

Solving these for m and μ, we have

$$\left.\begin{aligned} m &= \frac{1}{2H^2}(GE_2 - 2FG_1) \\ \mu &= \frac{1}{2H^2}(EG_1 - FE_2) \end{aligned}\right\} \qquad (9.35)$$

Clearly, using the relations $\mathbf{r}_1 \bullet \mathbf{r}_{22} = F_2 - \frac{1}{2}G_1$ and $\mathbf{r}_2 \bullet \mathbf{r}_{22} = \frac{1}{2}G_2$, and from the third of (9.33),

$$\left.\begin{aligned}\eta &= \frac{1}{2H^2}(2GF_2 - GG_1 + FG_2)\\ \lambda &= \frac{1}{2H^2}(EG_2 - 2FF_2 + FG_1)\end{aligned}\right\} \qquad (9.36)$$

using formulae (9.33), with the values of coefficients given by (9.34), and (9.36), are the equivalent of Gauss's formulae for $\mathbf{r}_{11}, \mathbf{r}_{12}, \mathbf{r}_{22}$, may be referred to under this name.

and the parametric curves are orthogonal, the values of the coefficients are greatly simplified. For, in this case, $F = 0$ and $= EG$, so that

$$\left.\begin{aligned}\mathbf{r}_{11} &= L\mathbf{n} + \frac{E_1}{2E}\mathbf{r}_1 - \frac{E_2}{2G}\mathbf{r}_2\\ \mathbf{r}_{12} &= M\mathbf{n} + \frac{E_2}{2E}\mathbf{r}_1 + \frac{G_1}{2G}\mathbf{r}_2\\ \mathbf{r}_{22} &= N\mathbf{n} + \frac{G_1}{2E}\mathbf{r}_1 + \frac{G_2}{2G}\mathbf{r}_2\end{aligned}\right\} \qquad (A)$$

Hence unit vectors parallel to $\mathbf{r}_1$ and $\mathbf{r}_2$, we have

$$\mathbf{a} = \frac{\mathbf{r}_1}{\sqrt{E}}, \quad \mathbf{b} = \frac{\mathbf{r}_2}{\sqrt{G}},$$

a, **b**, **n** form a right-handed system of unit vectors, mutually perpendicular. From these formulae, we deduce immediately that

$$\left.\begin{aligned}\frac{\partial \mathbf{a}}{\partial u} &= \frac{L}{\sqrt{E}}\mathbf{n} - \frac{E_2}{2H}\mathbf{b}\\ \frac{\partial \mathbf{a}}{\partial v} &= \frac{M}{\sqrt{E}}\mathbf{n} + \frac{G_1}{2H}\mathbf{b}\\ \frac{\partial \mathbf{b}}{\partial u} &= \frac{M}{\sqrt{G}}\mathbf{n} + \frac{E_2}{2H}\mathbf{a}\\ \frac{\partial \mathbf{b}}{\partial v} &= \frac{N}{\sqrt{G}}\mathbf{n} - \frac{G_1}{2H}\mathbf{a}\end{aligned}\right\} \qquad (B)$$

The derivative of **a** is perpendicular to **a**, and the derivative of **b** is perpendicular to **b** since **a** and **b** are vectors of constant (unit).

9.13. Gauss characteristic equation The six fundamental magnitudes E, F, G, L, M, N are not functionally independent but are connected by three differential relations. One of these, due to Gauss is an expression for $LN - M^2$ in terms of E, F, G and their derivatives of the first two orders. It may be deduced from the formulae of the preceding article. For, in virtue of these,

$$\mathbf{r}_{11} \bullet \mathbf{r}_{22} = LN + \ln E + (l\upsilon + \lambda n)F + \lambda\upsilon G,$$

and $$\mathbf{r}_{12}^2 = M^2 + m^2 E + 2m\mu F + \mu^2 G.$$

It is also easily verified that

$$\mathbf{r}_{12}^2 - \mathbf{r}_{11} \bullet \mathbf{r}_{22} = \frac{1}{2}(E_{22} + G_{11} - 2F_{12}).$$

Adding the first and third, and subtracting the second, we obtain the required formula, which may be written

$$\begin{aligned} LN - M^2 = \frac{1}{2}(2F_{12} - E_{22} - G_{11}) + (m^2 E + 2m\mu F + \mu^2 G) \\ -\{\ln E + (l\upsilon + \lambda n)F + \lambda\upsilon G\} \end{aligned} \qquad (9.37)$$

This is the Gauss characteristics equation. It is sometimes expressed in the alternative form

$$LN - M^2 = \frac{1}{2}H\frac{\partial}{\partial u}\left\{\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right\} + \frac{1}{2}H\frac{\partial}{\partial v}\left\{\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right\} \quad (9.38)$$

This equation shows that the specific curvature K, which is equal to $(LN - M^2)/H^2$, is expressible in terms of the fundamental magnitude E, F, G and their derivatives of the first two orders. In this respect, it differs from the first curvature.

Cor. Surface which has the same first-order magnitudes E, F, G (irrespective of the second-order magnitudes L, M, N) has the same specific curvature.

9.14. Mainardi-Codazzi relation

In addition to the Gauss characteristics, there are two other independent relations between the fundamental magnitudes and their derivatives. If in the identity,

$$\frac{\partial}{\partial \upsilon}\mathbf{r}_{11} = \frac{\partial}{\partial u}\mathbf{r}_{12}$$

We substitute the values i=of $\mathbf{r}_{11}$ and $\mathbf{r}_{12}$,

$$\begin{aligned} &L_2\mathbf{n} + l_2\mathbf{r}_1 + \lambda_2\mathbf{r}_2 + L\mathbf{n}_2 + l\mathbf{r}_{12} + \lambda\mathbf{r}_{22} \\ &= M_1\mathbf{n} + m_1\mathbf{r}_1 + \mu_1\mathbf{r}_2 + M\mathbf{n}_1 + m\mathbf{r}_{11} + \mu\mathbf{r}_{12} \end{aligned}$$

If in this, we substitute again from (9.33) the values of the second derivatives of $\mathbf{r}_1$ and also for $\mathbf{n}_1$ and $\mathbf{n}_2$, we obtain a vector identity, expressed in terms of the non-coplanar vectors $\mathbf{n}, \mathbf{r}_1, \mathbf{r}_2$. We may then equate coefficients of like vectors on the two sides, and obtain three scalar equations. By equating coefficients of **n**, we have

$$L_2 + lM + \lambda N = M_1 + mL + \mu M \,,$$

i.e. $$L_2 - M_1 = mL - (l - \mu)M - \lambda N \qquad (9.39)$$

Similarly from the identity $\frac{\partial}{\partial \upsilon}\mathbf{r}_{12} = \frac{\partial}{\partial u}\mathbf{r}_{22}$, on substituting from (9.33) the values of $\mathbf{r}_{12}$ and $\mathbf{r}_{22}$, we obtain the relation

$$M_2\mathbf{n} + m_2\mathbf{r}_1 + \mu_2\mathbf{r}_2 + M\mathbf{n}_2 + m\mathbf{r}_{12} + \mu\mathbf{r}_{22}$$
$$= N_1\mathbf{n} + n_1\mathbf{r}_1 + \upsilon_1\mathbf{r}_2 + N\mathbf{n}_1 + n\mathbf{r}_{11} + \upsilon\mathbf{r}_{12}$$

Substituting again for second derivatives of **r** and also for $\mathbf{n}_1$ and $\mathbf{n}_2$, in terms of $\mathbf{n}, \mathbf{r}_1, \mathbf{r}_2$ and equating coefficients of **n** on the two sides of the identity, we have

$$M_2 + mM + \mu N = N_1 + nL + \upsilon M \,,$$

i.e. $$M_2 - N_1 = nL - (m - \upsilon)M - \mu N \qquad (9.40)$$

the formulae (9.39) and (9.40) are frequently called the codazzi equations. But as Mainardi gave similar results twelve-year earlier than Codazzi, the more justly termed the Mainardi-Codazzi relations. Four other formulae are obtained by equating coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$ in the two identities: but they are not independent. They are all deducible from (9.39) and (9.40) with the aid of the Guass characteristics equation.

9.15. Alternative expression The above relations may be expressed in a different form, which is sometimes more useful. By differentiating the relation $H^2 = EG - F^2$ with respect to the parameters, it is easy to verify that

$$H_1 = H(l + \mu),$$

and $$H_2 = H(m + \upsilon).$$

Therefore $$\frac{\partial}{\partial u}\left(\frac{N}{H}\right) = \frac{N_1}{H} - \frac{N}{H^2}H_1$$

$$= \frac{N_1}{H} - \frac{N}{H^2}(l+\mu),$$

and similarly, $$\frac{\partial}{\partial v}\left(\frac{M}{H}\right) = \frac{M_2}{H} - \frac{N}{H^2}H_2$$

$$= \frac{M_2}{H} - \frac{M}{H^2}(m+\upsilon).$$

Consequently

$$\frac{\partial}{\partial v}\left(\frac{M}{H}\right) - \frac{\partial}{\partial u}\left(\frac{N}{H}\right) = \frac{1}{H}(M_2 - N_1) - \frac{M}{H}(m+v) + \frac{N}{H}(l+\mu)$$

$$= (nL - 2mM + lN)/H \qquad (9.41)$$

In virtue of (9.40). Similarly, it may be proved that

$$\frac{\partial}{\partial u}\left(\frac{M}{H}\right) - \frac{\partial}{\partial v}\left(\frac{L}{H}\right) = (\upsilon L - 2\mu M + \lambda N)/H \qquad (9.42)$$

The equations (9.41) and (9.42) are alternative forms of the Mainardi-Codazzi relations.

We have seen that if functions E, F, G, L, M, N constitute the fundamental magnitudes of a surface, they are connected by the three differential equations and Mainardi-Codazzi relations. Conversely, Bonnet has proved the theorem: *When six fundamental magnitudes are given, satisfying the Gauss characteristic equation and Mainardi-Codazzi relation, they determine a surface uniquely, except as to position and orientation in space.* The proof of the theorem is beyond the scope of this book, and we shall not have occasion to use it.

9.16 CHECK YOUR PROGRESS

SA1: For the equation of the principal curvatures and the differential equation of the lines of curvatures, for the surfaces

(i) $2z = \frac{x^2}{a} + \frac{y^2}{b}2$ (ii) $3z = ax^3 + by^3$ (iii) $z = c\tan^{-1}\frac{y}{x}$.

SA2: The indicatrix at every point of the helicoids $z = c\tan^{-1}\frac{y}{x}$ is a rectangular hyperbola.

SA3: The indicatrix at a point of the surface $z = f(x, y)$ is a rectangular hyperbola if $(1+p^2)t - 2pqs + (1+q^2)r = 0$.

SA4: At a point of intersection of the paraboloid $xy = cz$ with the hyperboloid $x^2 + y^2 - z^2 + c^2 = 0$ the principal radii of the paraboloid are $z^2(1 \pm \sqrt{2})/c$.

SA5: Show that the other four relations, similar to the Mainardi-Codazzi relations, obtainable by equating coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$ in the proof of Mainardi- Codazzi relations, are equivalent to

$$FK = m_1 - l_2 + m\mu - n\lambda,$$

$$FK = \mu_2 - \upsilon_1 + m\mu - n\lambda,$$

$$EK = \lambda_2 - \mu_1 + ml\mu - m\lambda + \lambda\upsilon - \mu^2$$

$$GK = n_1 - m_2 + \ln - m^2 + m\upsilon - n\mu.$$

SA6: Prove that these formulae may be deduced from the Gauss characteristic equation and the Mainardi-Codazzi relations.

SA7: Prove that the relations

$$\frac{\partial}{\partial v}\left(\frac{H\lambda}{E}\right) - \frac{\partial}{\partial u}\left(\frac{H\mu}{E}\right) = HK,$$

$$\frac{\partial}{\partial u}\left(\frac{Hn}{G}\right) - \frac{\partial}{\partial v}\left(\frac{Hm}{G}\right) = HK.$$

SA8: If ω is the angle between the parametric curves, proves that

$$-\omega_{12} = \frac{\partial}{\partial u}\left(\frac{H\mu}{E}\right) + \frac{\partial}{\partial v}\left(\frac{Hm}{G}\right) + HK$$

$$- = \frac{\partial}{\partial u}\left(\frac{Hn}{G}\right) + \frac{\partial}{\partial v}\left(\frac{H\lambda}{E}\right) - HK.$$

SA9: If the asymptotic lines are taken as parametric curves, show that the Mainardi- Codazzi relations become

$$\frac{M_1}{M} = (l - \mu), \qquad \frac{M_2}{M} = (\upsilon - m)$$

Hence deduce that

$$2l = \frac{H_1}{H} + \frac{M_1}{M}, \qquad 2\mu = \frac{H_1}{H} - \frac{M_1}{M},$$

$$2m = \frac{H_2}{H} - \frac{M_2}{M}, \qquad 2\upsilon = \frac{H_2}{H} + \frac{M_2}{M}.$$

SA10: If the surface of revolution is a minimal surface,

$$u\frac{d^2 f}{du^2} + \frac{df}{du}\left\{1+\left(\frac{df}{du}\right)^2\right\} = 0.$$

Hence show that the only real minimal surface of revolution is that formed by the revolution of a catenary about its directrix.

SA11: On the surface formed by the revolution of a parabola about directrix, one principal curvature is double the other.

SA12: The moment about the origin of the unit normal **n** at a point **r** of the surface is $\mathbf{m} = \mathbf{r} \times \mathbf{n}$. Prove that the differential equation of the lines of curvature is

$$d\mathbf{m} \bullet d\mathbf{n} = 0.$$

SA13: Find equations for the principal radii, the lines of curvature, and the first and second curvatures of the following surfaces:

(i) the conoid $x = u\cos\theta,\ \ y = u\sin\theta,\ \ z = f(\theta)$;

(ii) the catenoid $x = u\cos\theta,\ \ y = u\sin\theta,\ \ z = a\log(u + \sqrt{u^2 - c^2})$;

(iii) the cylindroids $z(x^2 + y^2) = 2mxy$;

(iv) the surface $2z = ax^2 + 2hxy + by^2$;

(v) the surface $x = 3u(1+v^2) - u^3,\ \ y = 3v(1+u^2) - v^3,\ \ z = 3(u^2 - v^2)$;

(vi) the surface $\frac{x}{a} = \frac{1+uv}{u+v},\ \ \frac{y}{b} = \frac{u-v}{u+v},\ \ \frac{z}{c} = \frac{1-uv}{u+v}$;

(vii) the surface $xyz = a^3$.

SA14: The lines of curvature of the paraboloid $xy = az$ lie on the surfaces

$$\sinh^{-1}\frac{x}{a} \pm \sinh^{-1}\frac{y}{a} = \text{constant}.$$

SA15: Show that the surface $4a^2z^2 = (x^2 - 2a^2)(y^2 - 2a^2)$ has a line of umbilics laying on the sphere $x^2 + y^2 + z^2 = 4a^2$.

SA16: On the surface generated by the tangents to a twisted curve, the current point is $\mathbf{R} = \mathbf{r} + u\mathbf{t}$, taken u, s as parameters, proves that

$$E = 1, \quad F = 1, \quad G = 1 + u^2\kappa^2, \quad H^2 = u^2\kappa^2,$$

$$L = 0, \quad M = 0, \quad N = u\kappa\tau, \quad T^2 = 0,$$

$$K = 0, \quad J = \frac{\tau}{u\kappa}, \quad \kappa_a = 0, \quad \kappa_b = \frac{\tau}{u\kappa}.$$

The lines of curvature are s =constant, $u + s$ =constant.

SA17: Show that the equation of the indicatrix, referred to the tangents to the parametric curves as (oblique) axes, is $\frac{L}{E}\xi^2 + \frac{2M}{\sqrt{EG}}\xi\eta + \frac{N}{G}\eta^2 = 1$.

SA18: Find the equation of the helicoids generated by a circle of radius α, whose plane passes through the axis; and determine the lines of curvature on the surface.

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CHAPTER-10

CURVES CONCERNING GEODESICS-II

Objectives: In continuation of the previous chapters in the current chapter the **s**tudents will learn about Geodesic and its properties, the Equation of Geodesic, Surface of revolution, Torsion of Geodesic, Bonnet's Theorem, Joachimsthal's theorem, Geodesic curvature, Geodesic parallels, Geodesic polar coordinates, Beltram's theorem, Geodesic triangle, Geodesic ellipses and hyperbolas, Liouville surfaces,

10.1. Geodesic property A geodesic line, or briefly a geodesic, on a surface, may be defined as a curve whose osculating plane at each point contains the normal to the surface at that point. It follows that the principal normal to the geodesic coincides with the normal to the surface, and we agree to take it also in the same sense. The curvature of a geodesic is therefore the normal curvature of the surface in the direction of the curve and has the value

$$\kappa = Lu'^2 + 2Mu'v' + Nv'^2 \tag{10.1}$$

and the dashes denoting derivatives with respect to the arc-length s of the curve.

Moreover, of all plane sections through a given tangent line to the surface, the normal section has the least curvature, by Meunier's theorem. Therefore of all sections through two consecutive points P, Q on the surface, the normal section makes the length of the arc PQ a minimum. But this is the arc of the geodesic through P, Q. Hence a geodesic is sometimes defined as the path of the shortest distance on the surface between two given points on it. Starting with the definition, we may reverse the argument, and deduce the property that the principal normal to the geodesic coincides with the normal to the surface. The same may be done by the calculus of variation, or by statistical considerations in the following manner. The path of the shortest distance between two given points on the surface is the curve along which a flexible string would lie, on the (smooth) convex side of the surface, tightly stretched between the two points. Now the only forces on an element of the string are the tensions at its extremities and the reaction normal to the surface. But the tensions are in the osculating plane of the element, and therefore so also is the reaction by the condition of equilibrium. Thus the normal to the surface coincides with the principal normal to the curve.

10.2. Equations of geodesics From the defining property of geodesics, and the Serret-Frenet formulae, it follows that

$$\mathbf{r}'' = \kappa\mathbf{n} \qquad (10.2)$$

which may be expanded.

$$\mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11} u'^2 + 2\mathbf{r}_{12} u'v' + \mathbf{r}_{22} v'^2 = \kappa\mathbf{n}.$$

Forming the scalar product of each side with $\mathbf{r}_1$ and $\mathbf{r}_2$ successively, we have

$$\left.\begin{aligned} &Eu'' + Fv'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + (F_2 - \frac{1}{2}G_1)v'^2 = 0 \\ &Fu'' + Gv'' + \frac{1}{2}(F_1 - \frac{1}{2}E_2)u'^2 + G_1 u'v' + \frac{1}{2}G_2 v'^2 = 0 \end{aligned}\right\} \qquad (10.3)$$

These are the general differential equations of geodesic on a surface. They are equivalent to the equations

$$\left.\begin{aligned} &\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_1 u'^2 + 2F_1 u'v' + G_1 v'^2) \\ &\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}(E_2 u'^2 + 2F_2 u'v' + G_2 v'^2) \end{aligned}\right\} \qquad (10.4)$$

A third form, which is sometimes more convenient, may be found by solving (10.3) for u'' and v'', thus obtaining

$$\left.\begin{aligned} &u'' + lu'^2 + 2mu'v' + nv'^2 = 0 \\ &v'' + \lambda u'^2 + v'' + 2\mu u'v' + \upsilon v'^2 = 0 \end{aligned}\right\} \qquad (10.5)$$

where l, λ etc. are the coefficients.

A curve on the surface is, however, determined by a single relation between the parameters. Hence the above pair of differential equations may be replaced by a single relation between u, v. If for example, we take the equations (10.5), multiply the first by $\frac{dv}{du}\left(\frac{ds}{du}\right)^2$, the second by $\left(\frac{ds}{du}\right)^2$, and subtract, we obtained the single differential equation of geodesics in the form

$$\frac{d^2v}{du^2}=n\left(\frac{dv}{du}\right)^2+(2m-\upsilon)\left(\frac{dv}{du}\right)^2+(l-2\mu)\frac{dv}{du}-\lambda \qquad (10.6)$$

Now from the theory of differential equations, it follows that there exists a unique integral v on this equation that takes a given value v_0 when $u=u_0$, and whose derivative dv/du also takes a given value when $u=u_0$. Thus through each point of the surface, there passes a single geodesic in each direction. Unlike lines of curvature and asymptotic lines, geodesics are not determined uniquely or in pairs at a point by the nature of the surface. Through any point pass, an infinite number of geodesics involve only the magnitudes of the first order E,F,G and their derivatives. Hence if the surface is deformed without stretching or tearing, so that the length *ds* of each arc element is unaltered, the geodesics remains geodesics on the deformed surface. In particular, when the developable surface is developed into a plane, the geodesics on the surface become straight lines on the plane. This agrees with the fact that a straight line is the path of the shortest distance between two given points on the plane.

From (10.6), it follows that the parametric curves v=constant will be geodesics if $\lambda=0$. Similarly the curve u=constant will be geodesics if $\mathbf{n}$=constant. Hence if the parametric curves are orthogonal (F=0), the curve v=constant will be geodesics provided *E* is a function of *u* only, and the curve u=constant will be geodesics if *G* is a function of *v* only.

Example 10.1 On the right helicoids given by

$$x=u\cos\phi,\quad y=u\sin\phi,\quad z=c\phi$$

Solution. We have seen that $E=1, \quad F=0, \quad G=u^2+c^2, \quad H^2=u^2+c^2$

Therefore the coefficients are

$$l=0, \quad m=0, \quad n=-u$$

$$\lambda=0, \quad \mu=u/(u^2+c^2), \quad \upsilon=0$$

The equation for the geodesics becomes

$$\left.\begin{aligned}&u''-u\phi'^2=0\\&(u^2+c^2)\phi''+2uu'\phi'=0\end{aligned}\right\}$$

From the second of these, it follows that

$$(u^2+c^2)\frac{d\phi}{ds}=\text{constant}=h \text{ (say)}.$$

But on any arc on the surface

$$ds^2=du^2+(u^2+c^2)d\phi^2.$$

Hence for the arc of geodesic,

$$(u^2+c^2)d\phi^2=h^2du^2+(u^2+c^2)h^2d\phi^2$$

and therefore $\dfrac{du}{d\phi}=\pm\dfrac{1}{h}\sqrt{(u^2+c^2)(u^2+c^2-h^2)}$. This is the first integral of the differential equation of geodesics. The complete integral may be found in terms of elliptic functions.

Example 10.2 When the equation of the surface is given in Monge's form $z=f(x,y)$ then we have seen that, with x,y as parameters,

$$E=1+p^2,\quad F=pq,\quad G=1+q^2,\quad H^2=1+p^2+q^2.$$

Solution. We have $l=\dfrac{pr}{h^2},\quad m=\dfrac{ps}{H^2},\quad n=\dfrac{pt}{h^2}$

$$\lambda=\frac{qr}{H^2},\quad \mu=\frac{qs}{H^2},\quad \upsilon=\frac{qt}{H^2}$$

The equation of geodesics then takes the form

$$(1+p^2+q^2)\frac{d^2y}{dx^2}=pt\left(\frac{dy}{dx}\right)^3+(2ps-qt)\left(\frac{dy}{dx}\right)^2+(pr-2qs)\frac{dy}{dx}-rq$$

$$=\left(p\frac{dy}{dx}-q\right)\left\{t\left(\frac{dy}{dx}\right)^2+2s\frac{dy}{dx}+r\right\}.$$

10.3. Surface of revolution

On the surface of revolution

$$x=u\cos\phi,\quad y=u\sin\phi,\quad z=f(u),$$

We have seen that with u,ϕ as parameters

$$E=1+f_1^2,\quad F=0,\quad G=u^2,\quad H^2=u^2(1+f_1^2).$$

$$\lambda=0,\quad \mu=\frac{1}{u},\quad \upsilon=0$$

The second equations for geodesics then take the form

$$\frac{d^2\phi}{ds^2}+\frac{2}{u}\frac{du}{ds}\frac{d\phi}{ds}=0$$

On multiplying by u^2 the equation becomes exact, and has its integral

$$u^2\frac{d\phi}{ds}=h \tag{10.7}$$

where h is a constant. Or, if ψ is the angle at which the geodesic cuts the meridian, we may write this result

$$u\sin\psi=h \tag{10.7$'$}$$

a theorem due to Clairaut. This is the first integral of the equation of geodesics, involving one arbitrary constant h. To obtain the complete integral, we observe that, for any arc on the surface,

$$ds^2=(1+f_1^{\,2})du^2+u^2d\phi^2$$

And therefore by (10.7), for the arc of geodesic,

$$u^4d\phi^2=h^2(1+f_1^{\,2})du^2+h^2u^2d\phi^2$$

So that $$d\phi=\pm\frac{h}{u}\sqrt{\frac{1+f_1^{\,2}}{u^2-h^2}}du.$$

Thus, $$\phi=C\pm h\int\frac{1}{u}\sqrt{\frac{1+f_1^{\,2}}{u^2-h^2}}du \tag{10.8}$$

involving the two arbitrary constants C and h is the complete integral of the equation of geodesics on a surface of revolution.

Cor. It follows from $(10.7')$ that h is the minimum distance from the axes of a point on the geodesic, and is attained where the geodesic cuts a meridian at right angles.

Example 10.3 The geodesics on a circular cylinder are helices. For from $(10.7')$, since u is constant, ψ is constant. Hus the geodesics cut the generators at a constant angle, and are therefore helices.

Example 10.4 In the case of a right circular cone of semi-vertical angle α, show that the equation (10.8) for geodesics is equivalent to

$$u=h\sec(\phi\sin\alpha+\beta)$$

where h and β are constants.

Example 10.5 The perpendicular from the vertex of a right circular cone, to a tangent to a given geodesic, is of constant length.

10.4. Torsion of Geodesic If $\mathbf{r}$ is a point on the geodesic, $\mathbf{r}'$ is the unit tangent and the principal normal is the unit normal $\mathbf{n}$ to the surface. Hence the unit binormal is

$$\mathbf{b} = \mathbf{r}' \times \mathbf{n}$$

Differentiating with respect to arc length gives for the torsion of the geodesic

$$-\tau\,\mathbf{n} = \mathbf{r}'' \times \mathbf{n} + \mathbf{r}' \times \mathbf{n}'.$$

The first term in the second member is zero because $\mathbf{r}''$ is parallel to $\mathbf{n}$. Hence

$$\tau\,\mathbf{n} = \mathbf{n}' \times \mathbf{r}' \tag{10.9}$$

and therefore

$$\tau = [\mathbf{n}, \mathbf{n}'\mathbf{r}'] \tag{10.10}$$

This expression for the torsion of a geodesic is identical to that found for the torsion of an asymptotic line. The geodesic which touches a curve at any point is often called its geodesic tangent at that point. Hence the torsion of an asymptotic line is equal to the torsion of its geodesic tangent.
Further, the expression $[\mathbf{n}, \mathbf{n}', \mathbf{r}']$ vanishes for a principal direction. Hence the torsion of a geodesic vanishes where it touches a line curve of curvature. It is also from (10.10) that if a geodesic is a plane curve, it is a line of curvature; and, conversely, if a geodesic is a line of curvature it is also a plane curve.

The triple product $[\mathbf{n}, \mathbf{n}', \mathbf{r}']$ may be expanded, by writing $\mathbf{n}' = \mathbf{n}_1 u' + \mathbf{n}_2 v'$ and $\mathbf{r}' = \mathbf{r}_1 u' + \mathbf{r}_2 v'$. The formula for the torsion of a geodesic then becomes

$$\tau = \frac{1}{H}\{(EM - FL)u'^2 + (EN - GL)u'v' + (FN - GM)v'^2\} \tag{10.11}$$

This may be expanded in terms of the inclination of the geodesic to the principal directions. Let the lines of curvature be taken as parametric curves then

$$F = M = 0, \qquad H^2 = EG,$$

and the last formula becomes

$$\tau = \sqrt{EG}\,u'v'\left(\frac{N}{G} - \frac{L}{E}\right).$$

But, if ψ is the inclination of the geodesic to the line of curvature v =constant,

$$\sqrt{E}u' = \cos\psi\,, \qquad \sqrt{G}v' = \sin\psi\,.$$

Also, the principal curvatures are

$$\kappa_a = L/E\,, \quad \kappa_b = N/G\,.$$

Hence the formula for the torsion of the geodesic becomes

$$\tau = \cos\psi \sin\psi(\kappa_b - \kappa_a) \tag{10.12}$$

From this, it follows that two geodesics at right angles have their torsions equal in magnitude but opposite in sign. Further, besides vanishing in the principal directions, the torsion of a geodesic vanishes at an umbilic. And of all geodesics through a given point, those which bisect the angles between the lines of curvature have the greatest torsion.

The curvature of a geodesic is the normal curvature in its direction. Its value, as given by Euler's theorem, is therefore

$$\kappa = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi \tag{10.13}$$

Example 10.6 If κ, τ are the curvature and the torsion of a geodesic, prove that

$$\tau^2 = (\kappa - \kappa_a)(\kappa_b - \kappa)\,.$$

Also if the surface is developable $(\kappa_a = 0)$, show that

$$\kappa = \tau \tan\psi\,.$$

Example 10.7 Deduce from equation (12) that the torsions of the two asymptotic lines at a point are equal in magnitude and opposite in sign.

Example 10.8 Prove that the torsion of a geodesic is equal to

$$\frac{1}{H}\begin{vmatrix} Eu' + Fv' & Fu' + Gv' \\ Lu' + Mv' & Mu' + Nv' \end{vmatrix}.$$

Example 10.9 Prove that, with the notation of the above article for a geodesic,

$$\kappa\cos\psi - \tau\sin\psi = \kappa_a\cos\psi\,,$$

$$\kappa\sin\psi + \tau\cos\psi = \kappa_b\sin\psi\,.$$

10.5. Bonnet's Theorem Let C be any curve drawn on the surface, $\mathbf{r}'$ its unit tangent, $\bar{\mathbf{n}}$ its principal normal, τ its torsion, and W the torsion of the geodesics which touches it at the point considered. We define the normal angle ϖ of the curves as the angle from $\bar{\mathbf{n}}$ to the normal $\mathbf{n}$ to the surface, in the positive sense

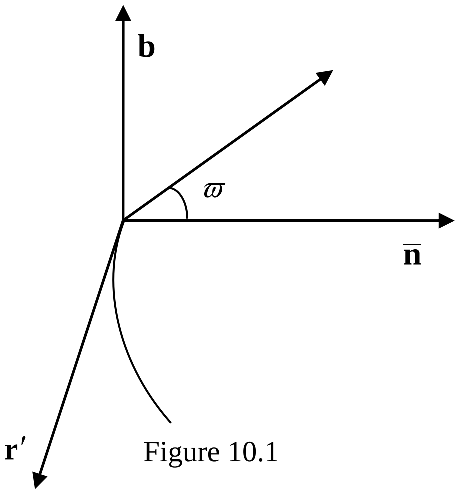

Figure 10.1

for a rotation about $\mathbf{r}'$. Thus ϖ is positive if the rotation from $\bar{\mathbf{n}}$ to $\mathbf{n}$ is in the sense from $\bar{\mathbf{n}}$ to the binormal **b**; negative if in the opposite sense. Then at any point of the curve, these quantities are connected by the relation

$$\frac{d\varpi}{ds} + \tau = W \qquad (10.14)$$

This may be proved in the following manner. By (10.9) of the previous article, we have, $W\mathbf{n} = \mathbf{n}' \times \mathbf{r}'$. The unit binormal to the curve is $\mathbf{b} = \mathbf{r}' \times \bar{\mathbf{n}}$,

$$\cos\varpi = \bar{\mathbf{n}} \bullet \mathbf{n}, \qquad \sin\varpi = \mathbf{b} \bullet \mathbf{n}$$

Differentiating this equation, we have

$$\cos\varpi \frac{d\varpi}{ds} = \mathbf{b}' \bullet \mathbf{n} + \mathbf{b} \bullet \mathbf{n}'$$

$$= -\tau\bar{\mathbf{n}} \bullet \mathbf{n} + \mathbf{r}' \times \bar{\mathbf{n}} \bullet \mathbf{n}$$

$$= -\tau\cos\varpi + W\mathbf{n} \bullet \bar{\mathbf{n}}$$

$$= (-\tau + W)\cos\varpi$$

Hence the formula $\dfrac{d\varpi}{ds} + \tau = W$, Expressing a result due to Bonnet. Since W is the torsion of the geodesic tangent, it follows that the quantity $\left(\frac{d\varpi}{ds}+\tau\right)$ has the same value for all curves touching at the point considered. The formula also shows that ϖ' is the torsion of the geodesic tangent relative to the curve C, or that $-\varpi'$ is that of C relative to the geodesic tangent.

10.6. Joachimsthal's theorem We have seen that the torsion W of the geodesic tangent to a line of curvature vanishes at the point of contact. If then a curve C on the surface is both a plane curve and a line of curvature, $\tau=0$ and $W=0$; and therefore, in virtue of (10.14), $\varpi'=0$. Consequently, its plane cuts the surface at a constant angle. Conversely, if a plane cuts a surface at a constant angle, the curve of intersection has zero torsion, so that $\tau=0$ and $\varpi'=0$. Therefore in virtue of (10.14), W vanishes identically, showing that the curve is a line of curvature. Similarly if ϖ is constant and the curve is a line of curvature, τ must vanish, and the curve is plane. Hence if a curve on a surface has two of the following properties, it also has the third: (a) it is a line of curvature, (b) it is a plane curvature, (c) its normal angle is constant.

Moreover, if the curve of the intersection of two surfaces is a line of curvature on each, the surface is cut at a constant angle. Let ϖ and ϖ_0 be the normal angles of the curve for the two surfaces. Then since the torsion W of the geodesic tangent vanishes on both surfaces,

$$\frac{d\varpi}{ds}+\tau=0, \qquad \frac{d\varpi_0}{ds}+\tau=0$$

Hence $$\frac{d}{ds}(\varpi-\varpi_0)=0,$$

So that $$\varpi-\varpi_0 = \text{constant}.$$

Thus the surfaces are cut at a constant angle. Similarly, if two surfaces cut at a constant angle, and the curve of intersection is a line of curvature on one, it is a line of curvature on the other also. For since

$$\varpi-\varpi_0 = \text{constant}.$$

It follows that $$\frac{d\varpi}{ds}=\frac{d\varpi_0}{ds}.$$

Hence by (10.14), if W and W_0 are the torsions of the geodesic

$$W-\tau = W_0 - \tau$$

So that $\quad W = W_0$

If then W vanishes, so does W_0, showing that the curve is a line of curvature on the second surface also.

Further, we can prove the theorems for spherical lines of curvature, similar to those proved above for plane lines of curvature. Geodesics on a sphere are great circles, and therefore plane curves. Their torsion W_0 , therefore, vanishes identically. Hence for any curve on a sphere, if ϖ_0 is its normal angle,

$$\frac{d\varpi_0}{ds} + \tau = 0,$$

Suppose then that a surface is cut by a sphere in a line of curvature. Then since the torsion W of the geodesic tangent to a line of curvature is zero, we have on this surface also

$$\frac{d\varpi}{ds} + \tau = 0$$

From these two equations, it follows that

$$\frac{d}{ds}(\varpi - \varpi_0) = 0,$$

So that $\quad \varpi - \varpi_0$ =constant.

Hence if the curve of the intersection of the sphere and another surface is a line of curvature on the latter, the two surfaces cut at a constant angle.

Conversely, if a sphere cuts a surface at a constant angle, the curve of intersection is a line of curvature on the surface. For

$$\frac{d\varpi}{ds} = \frac{d\varpi_0}{ds}$$

and therefore $\quad \tau = \tau - W$.

Thus W vanishes identically, and the curve is a line of curvature.

10.7. Geodesic curvature Consider any curve *C* drawn on a surface. We define the geodesic curvature of the curve at a point *P* as its curvature relative to the geodesic which touches it at *P*.

Now the vector curvature of the curve is $\mathbf{r}''$, and the resolved part of this in the direction of the normal to the surface is $\mathbf{n} \bullet \mathbf{r}''$ or κ_n by Meunier's theorem. But the vector curvature of the geodesic is normal to the surface, and its magnitude is also κ_n. That is to say, the curvature of the geodesic is the normal resolved part of the vector curvature of *C*. hence the curvature of *C* relative to the geodesics its resolved part tangential to the surface. This tangential resolute is sometimes called the tangential curvature of *C*, but more frequently its geodesic curvature. As a vector, it is given by

$$\mathbf{r}'' - \mathbf{n} \bullet \mathbf{r}''\mathbf{n} \quad \text{or} \quad \mathbf{r}'' - \kappa_n \mathbf{n} \tag{10.15}$$

Its magnitude must be regarded as positive when the deviation of *C* from the geodesic tangent is in the positive sense for a rotation about the normal to the surface. Thus we must take the resolved part of the vector curvature $\mathbf{r}''$ in the direction of the unit vector $\mathbf{n} \times \mathbf{r}'$. Denoting it by κ_g, we have

$$\kappa_g = [\mathbf{n}, \mathbf{r}', \mathbf{r}''] \tag{10.16}$$

Then $$[\mathbf{n}, \mathbf{r}', \mathbf{r}''] = \frac{1}{H}(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}' \bullet \mathbf{r}''$$

$$= \frac{1}{H}(\mathbf{r}_1 \bullet \mathbf{r}'\mathbf{r}_2 - \mathbf{r}_2 \bullet \mathbf{r}'\mathbf{r}_1) \bullet \mathbf{r}''$$

So that $$\kappa_g = \frac{1}{H}(\mathbf{r}_1 \bullet \mathbf{r}'\mathbf{r}_2 - \mathbf{r}_2 \bullet \mathbf{r}'\mathbf{r}_1) \bullet \mathbf{r}'' \tag{10.17}$$

It is also clear from the above argument, that, if κ is the curvature of curve *C*, and ϖ its normal angle,

while $$\left.\begin{aligned} \kappa_g &= \kappa \sin \varpi \\ \kappa_n &= \kappa \cos \varpi \end{aligned}\right\} \tag{10.18}$$

Hence $$\left.\begin{aligned} \kappa^2 &= \kappa_g^2 + \kappa_n^2 \\ \kappa_g &= \kappa_n \tan \varpi \end{aligned}\right\} \tag{10.19}$$

All these expressions for κ_g vanish when *C* is a geodesic. For then $\mathbf{r}''$ is parallel to **n**, and therefore perpendicular to $\mathbf{r}_1$ and $\mathbf{r}_2$, while ϖ is zero. This means simply that the curvature of a geodesic relative to itself is zero.

It will be noticed that the expression $[\mathbf{n}, \mathbf{r}', \mathbf{r}'']$ for the geodesic curvature is the same as that found for the curvature of an asymptotic line. This is because the osculating plane for an asymptotic line is the tangent plane to the surface, while the curvature of the geodesic tangent, being the normal curvature in the asymptotic direction, is zero. Thus the curvature of an asymptotic line is equal to its geodesic curvature.

10.8. In Other formulas for κ_g From (10.16) and (10.17) we may deduce an expansion for the geodesic curvature in terms of u', u'' etc. For instance, on substitution of the values of $\mathbf{r}'$ and $\mathbf{r}''$ in terms of these, (10.17) becomes

$$\begin{aligned}\kappa_g &= \frac{1}{H}(Eu' + Fv')\{Fu'' + Gv'' + (F_1 - \frac{1}{2}E_2)u'^2 + G_1 u'v' + \frac{1}{2}G_2 v'^2\} \\ &\quad - \frac{1}{H}(Fu' + Gv')\{Eu'' + Fv'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + (F_2 - \frac{1}{2}G_1)v'^2\}\end{aligned}$$

which may also be written

$$\kappa_g = Hu'(v'' + \lambda u'^2 + 2\mu u'v' + \upsilon v'^2) - Hv'(u + lu'^2 + 2mu'v' + nv'^2)$$

Each part of which vanishes for a geodesic, in virtue of (10.5). In particular, for the parametric curve v =constant. We have $v' = v'' = 0$ and the geodesic curvature κ_{gu} of this curve is therefore equal to $Hu'\lambda u'^2$, which may be written

$$\kappa_{gu} = H\lambda E^{-3/2}.$$

Similarly, the geodesic curvature κ_{gv} of the curve u =constant has the value

$$\kappa_{gv} = -HnG^{-3/2}.$$

When the parametric curves are orthogonal, these become

$$\kappa_{gu} = -\frac{E_2}{2E\sqrt{G}}, \qquad \kappa_{gv} = -\frac{E_1}{2G\sqrt{E}}.$$

From these formulae, we may deduce the results, that the curves v =constant will be geodesics provided $\lambda = 0$, and the curve u =constant provided $n = 0$. When the parametric curves are orthogonal, these conditions are $E_2 = 0$ and $G_1 = 0$, so that the curve v =constant will be geodesics if E is a function of u only; and the curve u =constant if G is a function of v only.

Another formula for the geodesic curvature of a curve may be found in terms of the arc rate of increase of its inclination of the curve to the parametric curves. Let θ be the inclination of the curve to the parametric curves v =constant, measured in the positive sense. Then since

$$Eu' + Fv' = \sqrt{E}\cos\theta .$$

We have on differentiation

$$\frac{d}{ds}(Eu' + Fv') = \frac{d\sqrt{E}}{ds}\cos\theta - \sqrt{E}\sin\theta\frac{d\theta}{ds}$$

$$= \frac{1}{2E}(E_1u' + E_2v')(Eu' + Fv') - Hv'\frac{d\theta}{ds}$$

Now, if the curve is geodesic, the first member of this equation is equal to

$$\frac{1}{2}(E_1u'^2 + 2F_1u'v' + G_1v'^2).$$

On substitution of this value, we find for a geodesic

$$H\frac{d\theta}{ds} = -\frac{H^2\lambda}{E}u' - \frac{H^2\mu}{E}v' .$$

Thus the rate of increase of the inclination of a geodesic to the parametric curves v =constant is given by

$$\frac{d\theta}{ds} = -\frac{H}{E}(\lambda u' + \mu v') .$$

Now the geodesic curvature of a curve *C* is tangential to the surface, and its magnitude is the arc rate of deviation of *C* from its geodesic tangent. This is equal to the difference in the values of $d\theta/ds$ the curve and for its geodesic tangent. But its value for the geodesic has just been found. Hence if $d\theta/ds$ denotes its value for the curve *C*, the geodesic curvature of *C* is given by

$$\kappa_g = \frac{d\theta}{ds} + \frac{H}{E}(\lambda u + \mu v) \qquad (10.21)$$

Or, if ζ is the inclination of the parametric curve u =constant to curve *C*, we may write this

$$\kappa_g = \frac{d\theta}{ds} + \frac{\lambda\sqrt{G}}{E}\sin\zeta + \frac{\mu}{\sqrt{E}}\sin\theta \qquad (10.22)$$

In the particular case when the parametric curves are orthogonal, $\sin\zeta = \cos\theta$. Also, the coefficient of $\sin\zeta$ becomes equal to the geodesic curvature of the curve v =constant and the coefficient of $\sin\theta$ to that of the curve u =constant. Denoting these by κ_{gu} and κ_{gv} respectively, we have Liouville's formula

$$\kappa_g = \frac{d\theta}{ds} + \kappa_{gu}\cos\theta + \kappa_{gv}\sin\theta \qquad (10.23)$$

Example 10.10 Bonnet's formula for the geodesic curvature of the curve $\phi(u,v)$.

Solution. By differentiation, we have

$$\phi_1 u' + \phi_2 v' = 0 \qquad (1)$$

So that $$\frac{u'}{\phi_2} = \frac{v'}{-\phi_1} = \frac{1}{\theta},$$

where $$\theta = \sqrt{E}\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2$$

again differentiating (1), we find

$$\phi_1 u'' + \phi_2 v'' + \phi_{11}u'^2 + 2\phi_{12}u'v' + \phi_{22}v'^2 = 0$$

which may be written

$$\theta(uv'' - v'u'') + \phi_{11}u^2 + 2\phi_{12}u'v + \phi_{22}v'^2 = 0$$

By means of these relations, we find that

$$\frac{\partial}{\partial u}\left(\frac{F\phi_2 - G\phi_1}{\theta}\right) + \frac{\partial}{\partial v}\left(\frac{F\phi_1 - E\phi_2}{\theta}\right)$$

$$= H^2u'^2(v'' + \lambda u'^2 + 2\mu u'v' + vv'^2) - H^2v'(u'' + lu'^2 + 2mu'v' + nv'^2)$$

$$= H\kappa_g$$

Hence Bonnet's formula for the geodesic curvature

$$\kappa_g = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{F\phi_2 - G\phi_1}{\theta}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{F\phi_1 - E\phi_2}{\theta}\right) \qquad (2)$$

From this result, we may deduce the geodesic curvature of a curve of a family defined by the differentiation equation

$$Pdu + Qdv = 0 \qquad (3)$$

For, on comparing this equation with (2), we see that the required value

$$\kappa_g = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FQ-GP}{\sqrt{EQ^2-2FPQ+GP^2}}\right)+\frac{1}{H}\frac{\partial}{\partial v}\left(\frac{FP-EQ}{\sqrt{EQ^2-2FPQ+GP^2}}\right).$$

10.9. Geodesic parallels Let a single infinity family of geodesics on the surface is taken as parametric curves v =constant, and their orthogonal trajectories as the curve u =constant. Then $F = 0$, the square of the linear element has the form

$$ds^2 = Edu^2 + Gdv^2.$$

Further, since the curve v =constant, E is a function of u alone. Hence, if we take $\int \sqrt{E}du$ as a new parameter of u, we have

$$ds^2 = du^2 + Gdv^2 \tag{10.24}$$

Which is called the geodesic form ds^2. Since E is now equal to unity, the length of an element of the arc of a geodesic is du; and the length of a geodesic intercepted between the two trajectories $u = a$ and $u = b$ is

$$\int_a^b du = b - a.$$

This is the same for all geodesics of the family and is called the geodesic distance between the two curves. On account of this property of minimum length characteristic of the arc of a geodesic joining two points on it. Consider, for example, the two points P, Q in which a geodesic is cut by the parallel $u = a$, $u = b$. The length of the arc of geodesic joining the two points is $(b-a)$. For any other curve joining them the length of the arc is

$$\int_P^Q ds = \int_P^Q \sqrt{du^2 + Gdv^2} > \int_a^b du,$$

Since G is positive. Thus the distance is least in the case of the geodesic. With the above choice of parameters, many results take a simpler form. Since G is positive it may be replaced by D^2, so that

$$ds^2 = du^2 + D^2dv^2 \tag{10.25}$$

Then since $F = 0$ and $E = 1$, we have $H^2 = G$, so that

$$l=0, \quad m=0, \quad n=-\frac{1}{2}G_1=-DD_1 \ ,$$

$$\lambda=0, \quad \mu=\frac{1}{2}\frac{G_1}{G}=\frac{D_1}{D}, \quad \nu=\frac{1}{2}\frac{G_2}{G}=\frac{D_2}{D}.$$

The Gauss characteristic equation becomes

$$LN-M^2=-\frac{1}{2}G_{11}+\frac{G_1^2}{4G}=-\sqrt{G}\,\frac{\partial^2\sqrt{G}}{\partial u^2},$$

and therefore the specific curvature is

$$K=\frac{LN-M^2}{G}=-\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2},$$

Or $$K=-\frac{1}{D}\frac{\partial^2 D}{\partial u^2} \tag{10.26}$$

The first curvature is

$$J=L+\frac{N}{G}.$$

The general equation (10.4) of geodesics becomes

$$\left.\begin{aligned} u''-DD_1 v'^2 &= 0 \\ \frac{d}{ds}(D^2 v')-DD_2 v'^2 &= 0 \end{aligned}\right\} \tag{10.27}$$

And, the single equation (10.6) gives

$$D\frac{d^2v}{du^2}+D^2D_1\left(\frac{dv}{du}\right)^2+D_2\left(\frac{dv}{du}\right)^2+2D_1\frac{dv}{du}=0 .$$

Example 10.11 (Beltrami's theorem). Consider a single infinite family o geodesic, out by a curve C whose direction at any point P is conjugate to that of the geodesic through P. the tangent to the geodesics at the point of C generates a developable surface, and are tangents to its edge of regression. Beltram's theorem is that the center of geodesic curvature at P, of that orthogonal trajectory of the geodesics which passes through the point, is the corresponding point on the edge of regression.

Solution. Let the geodesics be taken as the curve v =constant, and their orthogonal trajectories as the curves u =constant. Then the square of the linear element has the geodesic form

$$ds^2 = du^2 + Gdv^2$$

The geodesic curvature of the parametric curve u =constant is

$$\kappa_g = \frac{1}{2G}\frac{\partial G}{\partial u}.$$

This is measured in the sense of rotation from $\mathbf{r}_1$ to $\mathbf{r}_2$ .hence the distance ρ from *P* to the center of geodesic curvature, measured in the direction $\mathbf{r}_1$, is given by

$$\frac{1}{\rho} = \frac{1}{2G}\frac{\partial G}{\partial u}.$$

Let $\mathbf{r}$ be the positive vector of the point *P* on the curve *C*, $\mathbf{R}$ that of the corresponding point *Q* on the edge of regression, r the distance *PQ*, also measured in the direction $\mathbf{r}_1$. Then, since $E = 1$,

$$\mathbf{R} = \mathbf{r} + r\mathbf{r}_1.$$

Along with *C*, the quantities are functions of the arc-length *s* of the curve. Hence, on differentiation,

$$\mathbf{R}' = (\mathbf{r}_1 u' + \mathbf{r}_2 v') + r'\mathbf{r}_1 + r(\mathbf{r}_{11} u' + \mathbf{r}_{12} v').$$

But, because the generators are tangents to the edge of regression, $\mathbf{R}'$ are parallel to $\mathbf{r}_1$ and therefore parallel to $\mathbf{r}_2$. In forming the scalar product with $\mathbf{r}_2$, we have

$$0 = Gv' + r\mathbf{r}_2 \bullet \mathbf{r}_{12} v' = v'\left(G + \frac{1}{2}\frac{\partial G}{\partial u}\right),$$

the other term vanishing in virtue of relations $F = 0$ and $E = 1$. Hence, since v' is not zero,

$$\frac{1}{r} = -\frac{1}{2G}\frac{\partial G}{\partial u}$$

showing that $r = \rho$. Therefore the point *Q* on the edge of regression is the center of geodesic curvature of the orthogonal trajectory of the geodesic.

10.10. Geodesic polar coordinates An important particular case of the preceding is that in which the geodesic v =constant is the singly infinite family of geodesics through a fixed point *O*, called the pole. Their orthogonal trajectories are the geodesic parallel u =constant and we suppose u chosen so that the $E = 1$. If we take the infinitesimal trajectory at the pole as the curve $u = 0$, u is the geodesic distance of a point from the pole. Hence the name geodesic circles given to the parallels u =constant when the geodesic is concurrent. We may take v as the inclination of the geodesic at *O* to a fixed geodesic of reference *OA*. The position of any point *P* on the surface is determined by the geodesic through *O* on which it lies, and its distance u from *O* along the geodesic. These parameters u, v are called the geodesic polar coordinates of *P*. they are analogous to plane polar coordinates. On a curve *C* drawn on the surface, let *P* and *Q* be consecutive points (u, v) and $(u+du, v+dv)$. Then dv is the angle at *O* between the geodesics *OP* and *OQ*.

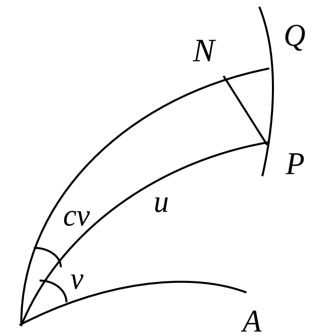

Figure 10.2

Let *PN* be an element of the geodesic circle through *P*, cutting *OQ* at *N*. Then *ON*=*OP* therefore *NQ*=*du*. And since the angle at *N* is a right angle,

$$NP^2 + du^2 = PQ^2 = ds^2$$

$$= du^2 + D^2 dv^2,$$

showing that $PN = Ddv$.

Hence if ψ is the angle *NPQ*, at which the geodesic cuts curves *C*,

$$\sin\psi = D\frac{dv}{ds}, \quad \cos\psi = \frac{du}{ds}, \quad \tan\psi = D\frac{dv}{du}.$$

And we may also notice that the area of the element of sur-bounded by the geodesic $v, v+dv$ and the geodesic circle $u+du$ is

$$dS = Ddudv.$$

If the curve *C* is itself a geodesic, we may write the first equation (10.27) for geodesic in the form

$$\frac{d}{ds}(\cos\psi) - D_1 \sin\psi \frac{dv}{ds} = 0,$$

or $$\sin\psi d\psi + D_1 \sin\psi dv = 0.$$

Hence for a geodesic, $d\psi = -D_1 dv$ (10.28)

It is also important to notice that the pole D_1 has v unity. To see this we consider a small geodesic circle distance from the pole. The element of the geodesic from the pole to the circle is practically straight, and the element of the geodesic is therefore udv to the first order. Thus near the origin

D=u + term of higher-order,

And therefore, at the pole, $D_1 = 1$.

10.11. Geodesic triangle If dS is the area of an element to the surface at a point where the specific curvature is K, we have KdS the second curvature of the element, and $\int\int KdS$ taking any portion of the surface is the whole second curvature of the portion. We shall now prove a theorem, due to Gauss, on the second curvature of the curvilinear triangle ABC bounded by geodesics. Such a triangle is called a geodesic triangle, and Gauss's theorem may be stated: the whole second curvature of a geodesic triangle is equal to the excess of the sum of the angles of the triangle over two right angles.

Let us choose geodesic polar coordinates with vertex A as the pole. Then the specific curvature is $K = -\frac{1}{D}\frac{K\partial^2 D}{\partial u^2}$ and the area of the element of the surface is $Ddudv$. Consequently, the whole second curvature Ω of the geodesic triangle is

$$\Omega = \int\int KdS = -\int\int \frac{\partial^2 D}{\partial u^2} dudv.$$

Integrate first with respect to u from the pole A to side BC.

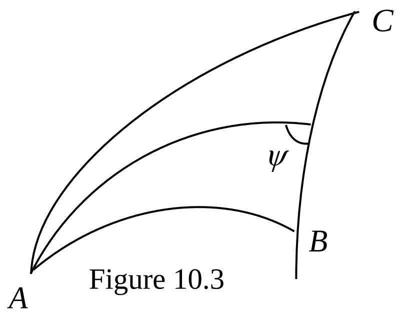

Figure 10.3

Then since at the pole D_1 is equal to unity, we find on integration

$$\Omega = \int (1 - D_1) dv,$$

where the integration with respect to v is along the side *BC*. But we have seen that, for a geodesic

$$-D_1 dv = d\psi.$$

Hence our formula may be written

$$\Omega = \int dv + \int d\psi.$$

Now the first integral, taken from *B* to *C*, is equal to angle *A* of the triangle. Also

$$\int d\psi = C - (\pi - B).$$

Hence the whole second curvature of the triangle is given by

$$\Omega = A + B + C - \pi \qquad (10.29)$$

as required. The specific curvature is positive, zero, or negative according to as the surface is synclastic, developable, or anticlastic. Consequently $A+B+C$ is greater than π for a synclastic surface, equal to π for a developable, and less than π for an anticlastic surface. When the surface is a sphere Gauss's theorem is identical to Girard's theorem on the area of a spherical triangle.

10.12. Theorem on parallels An arbitrarily chosen family of curves, $\phi(u,v)$=constant, does not, in general, constitute a system of geodesic parallels. So that they may do so, the function $\phi(u,v)$ must satisfy a certain condition, which may be found as follows. If the family of curves $\phi(u,v)$=constant is geodesic parallel to the family of geodesic $\psi(u,v)$=constant, the square of the linear element can be expressed in the geodesic form

$$ds^2 = ed\phi^2 + D^2 d\psi^2$$

where *e* is a function of ϕ only, and *D* is a function of ϕ and ψ.

Equating two expressions ds^2, we have the identity

$$Edu^2 + 2Fdudv + Gdv^2 = e(\phi_1 du + \phi_2 dv)^2 + D^2(\psi_1 du + \psi_2 dv)^2$$

and therefore

$$E = e\phi_1^2 + D^2\psi_2^2,$$

$$F = e\phi_1\phi_2 + D^2\psi_1\psi_2,$$

$$G = e\phi_2^2 + D^2\psi_2^2.$$

Consequently, eliminating ψ_1 and ψ_2, we must have

$$(E - e\phi_1^2)(G - \phi_2^2) - (F - e\phi_1\phi_2)^2 = 0 \qquad (*)$$

which is equivalent to

$$\frac{1}{H^2}(G\phi_1^2 - 2F\phi_1\phi_2 + E\phi_2^2) = \frac{1}{e} \qquad (10.30)$$

Thus so that the family of curves $\phi(u,v)$=constant may be a family of geodesic parallels,

$$(G\phi_1^2 - 2F\phi_1\phi_2 + E\phi_2^2)/H^2$$

must be a function of ϕ only, or a constant. The condition is also sufficient. For

$$ds^2 - ed\phi^2 = (E - e\phi_1^2)du^2 + 2(F - e\phi_1\phi_2)dudv + (G - e\phi_2^2)dv^2$$

and this regarded as a function of du and dv, is a perfect square, in virtue of $(*)$ being satisfied. We can therefore write it as $D^2 d\psi^2$, so that

$$ds^2 = ed\phi^2 + D^2 d\psi^2,$$

Proving the sufficiency of the condition. So $\phi = 0$, it is necessary and sufficient that $e = 1$, that is

$$G\phi_1^2 - 2F\phi_1\phi_2 + E\phi_2^2 = H^2 \qquad (10.30')$$

10.13. Geodesic ellipses and hyperbolas Let two independent systems of geodesic parallels be taken as parametric curves, and let the parametric variables be chosen so that u and v are the actual geodesic distances of the point (u,v) from the particular curves $u = 0$ and $v = 0$ (or form the poles in case the parallels are geodesic circles). Then since the curves u =constant and v =constant are geodesic parallels for which $e = 1$, we have

$$E = G = H^2.$$

Hence, if ω is the angle between the parametric curves, it follows that

$$E = G = \frac{1}{\sin^2\omega}, \quad F = \frac{\cos\omega}{\sin^2\omega},$$

so that the square of the linear element is

$$ds^2 = \frac{du^2 + 2\cos\omega\, dudv + dv^2}{\sin^2\omega} \qquad (10.31)$$

And conversely, when the linear element is of this form, the parametric curves are systems of geodesic parallels.

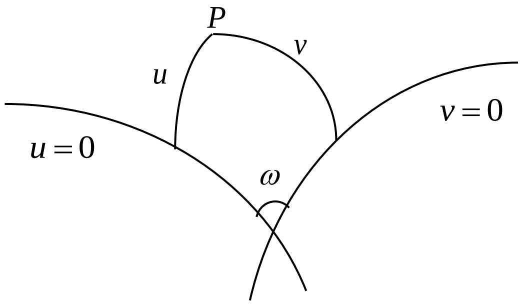

Figure 10.4

With this choice of parameters the locus of a point for which $u+v$=constant is called a geodesic ellipse. Similarly the locus of a point for which $u-v$=constant is a geodesic hyperbola. If we put

$$\bar{u}=\frac{1}{2}(u+v), \quad \bar{v}=\frac{1}{2}(u-v) \tag{10.32}$$

The above expression ds^2 becomes

$$ds^2=\frac{d\bar{u}^2}{\sin^2\frac{\omega}{2}}+\frac{d\bar{v}^2}{\cos^2\frac{\omega}{2}} \tag{10.33}$$

Showing that the curves $\bar{u}$=constant and $\bar{v}$=constant are orthogonal. But these are geodesic ellipses and hyperbolas. Hence a system of geodesic ellipses and the corresponding system of geodesic hyperbolas are orthogonal. Conversely, whenever ds^2 is of the form (10.33), the substitution (10.32) reduces it to the form (10.31), showing that the parametric curves in (10.33) are geodesic ellipses and hyperbolas.

Further, if θ is the inclination of the curve $\bar{v}$=constant to the curve v=constant, it follows that

$$\cos\theta=\cos\frac{\omega}{2}, \quad \sin\theta=\sin\frac{\omega}{2},$$

and therefore $\theta=\frac{\omega}{2}$.

Thus the geodesic ellipses and hyperbolas bisect the angles between the corresponding systems of geodesic parallels.

10.14. Liouville surfaces Surfaces for which the linear element is reducible to the form

$$ds^2 = (U+V)(Pdu^2 + Qdv^2) \qquad (10.34)$$

in which U, P are functions of u alone, and V, Q are functions of v alone, were first studied by Liouville, and are called after him. The parametric curves constitute an isometric system. It is also easy to show that they are a system of geodesic ellipses and hyperbolas. For if we change the parametric variables by the substitution

$$\sqrt{P}du = \frac{d\bar{u}}{\sqrt{U}}, \quad \sqrt{Q}dv = \frac{d\bar{v}}{\sqrt{V}},$$

The parametric curves are unaltered, and the linear element takes the form

$$ds^2 = (U+V)\left(\frac{d\bar{u}^2}{U} + \frac{d\bar{v}^2}{V}\right).$$

But this is of the form (10.33), where

$$\sin^2\frac{\omega}{2} = \frac{U}{U+V2}, \quad \cos^2\frac{\omega}{2}\frac{V}{U+V}.$$

Hence the parametric curves are geodesic ellipses and hyperbolas.

Liouville also showed that, when ds^2 has the form (10.34), a first integral of the differential equation of geodesics is given by

$$U\sin^2\theta - V\cos^2\theta = \text{constant} \qquad (10.35)$$

Where θ is the inclination of the geodesic to the parametric curve v =constant. To prove this, we observe that $F = 0$, while

$$E = (U+V)P, \qquad G = (U+V)Q,$$

so that
$$E_1 = U_1P + (U+V)P_1, \qquad G1 = U_1Q,$$

$$E_2 = V_2P, \qquad G_2 = V_2Q + (U+V)Q.$$

Taking the general equations (10.4) of the geodesics, multiplying the first by $-2u'V$, and the second by $2v'U$, and adding, we may arrange the result in the form

$$\frac{d}{ds}(UGv^2 - VEu'^2) = u'^2 v'\{(U+V)E_2 - V_2E\} - u'v'^2\{(U+V)G_1 - U_1G\}$$

Now the second member vanishes identically in virtue of the preceding relations. Hence

$$UGv'^2 - VEu'^2 = \text{constant},$$

which is equivalent to

$$U\sin^2\theta - V\cos^2\theta = \text{constant, as required.}$$

Example 10.11 If on the geodesic through a point *O*, points be taken at equal geodesic distances from *O*, the locus of the points is an orthogonal trajectory of the geodesics.

Solution. Let the geodesic through the pole *O* be taken as the curve v =constant, and let u denote the geodesic distance measured from the pole. We have to show that the parametric curves are orthogonal. Since the element of the arc of a geodesic is du, it follows that $E = 1$. Also since the curve v =constant are geodesic, $\lambda = 0$. Hence $F_1 = 0$, so that F is a function of v alone. Now at the pole, $\mathbf{r}_2$ is zero, and therefore $\mathbf{r}_1 \bullet \mathbf{r}_2 = F$ is at the pole. But F is independent of u, and therefore it vanishes along any geodesic. Thus F vanishes identically, and the parametric curves are orthogonal.

10.15 CHECK YOUR PROGRESS

SA1: Deduce the geodesic curvatures of the parametric curves.

SA2: A curve *C* touches the parametric curve v =constant. Find its curvature relative to the parametric curve at the point of contact.

SA3: Find the geodesic curvature of a parallel on a surface of revolution.

SA4: Deduce the geodesic curvature of the curve v = constant and u =constant.

SA5: When the curves of an orthogonal system have constant geodesic curvature, the system is isometric.

SA6: If the curve of one family of an isometric system has constant geodesic curvature, so also have the curves of the other family.

SA7: Straight lines on a surface are the only asymptotic lines that are geodesics.

SA8: Find the geodesics of an ellipsoid of revolution.

SA9: If two families of geodesics out at a constant angle, the surface is developable.

SA10: A curve is drawn on a cone, semi-vertical angle α, to cut the generators at a constant angle β. Prove that the torsion of its geodesic tangent is $\sin\beta\cos\beta/(R\tan\alpha)$, where R is the distance from the vertex.

SA11: Prove that any curve is a geodesic on the surface generated by its binormals, and an asymptotic line on the surface generated by its principal normals.

SA12: Find the geodesics on the catenoid of revolution $u = c\cosh\frac{z}{c}$.

SA13: if a geodesic on a surface of revolution cuts the meridian at a constant angle, the surface is a right cylinder.

SA14: If the principal normals of a curve intersect a fixed line, the curve is a geodesic on a surface of revolution, and the fixed line is the axis of the surface.

SA15: A curve for which κ/τ is constant is a geodesic on a cylinder, and a curve for which $\frac{d}{ds}(\kappa/\tau)$ is constant is a geodesic on a cone.

SA16: Show that the family of curves given by the differential equation $Pdu + Qdv = 0$ will constitute a system of geodesic parallel provided

$$\frac{\partial}{\partial u}\left(\frac{HQ}{\sqrt{EQ^2 - 2FPQ + GP^2}}\right) = \frac{\partial}{\partial v}\left(\frac{HP}{\sqrt{EQ^2 - 2FPQ + GP^2}}\right).$$

SA17: If on geodesics which cut a given curve *C* orthogonally, points be taken at an equal geodesic distance from *C*, the locus of the points is an orthogonal trajectory of the geodesics.

SA18: Necessary and sufficient conditions that a system of geodesics coordinates be polar are that $\sqrt{G}$ vanish with u, and $\partial\sqrt{G}/\partial u = 1$ when $u = 0$.

SA19: Two points *A*, *B* on the surface are joined by a fixed curve C_0 and a variable curve *C*, enclosing between them a portion of the surface of the constant area. Prove that the length of *C* is least when its geodesic curvature is constant.

SA20: If in the previous example the length of *C* is constant, prove that the area enclosed is greatest when the geodesic curvature of *C* is constant.

SA21: If the tangent to a geodesic is inclined at a constant angle to a fixed direction, the normal to the surface along the geodesic is everywhere perpendicular to the fixed direction.

SA22: Two surfaces touch each other along a curve. If the curve is geodesic on one surface, it is geodesic on other surfaces also.

SA23: The ratio of the curvature to the torsion of a geodesic on a developable surface is equal to the tangent of the inclination of the curve to the corresponding generating line.

SA24: If the geodesic on a developable surface is a plane curve, it is one of the generators, or else the surface is a cylinder.

SA25: If a geodesic on a surface lies on a sphere, the radius of curvature of the geodesic is equal to the perpendicular from the centre of the sphere on the tangent plane to the surface.

SA26: The locus of the centre of geodesic curvature of a line of curvature is an evolute of the latter.

SA27: The orthogonal trajectories of the helices on a helicoid are geodesic.

SA28: If the curve $x = f(u)\cos u, \quad y = f(u)\sin u, \quad z = -\frac{1}{c}\int f^2(u)du$ is given a helicoid's motion of pitch $2\pi c$ about the z-axis, the various positions of the curve are orthogonal trajectories of the helices, and also geodesic on the surface.

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