# M.Sc. MATHEMATICS

# MAL-523

# METHODS OF APPLIED MATHEMATICS

**DIRECTORATE OF DISTANCE EDUCATION**

**GURU JAMBHESHWAR UNIVERSITY OF SCIENCE AND TECHNOLOGY**

**HISAR-125001**

# MAL-523

# METHODS OF APPLIED MATHEMATICS

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## LESSON-1 FOURIER TRANSFORMS

**Integral Transform:** The integral transform of a function f(t) is defined by the equation

$g(s) = \int_a^b f(t)\, k(s, t)\, dt$, where k(s, t) is a known function of s and t, called the kernal of the transform; s is called the parameter of the transform and f(t) is called inverse transform of g(s).

Some of the well known transforms are given as under:

**(1) Laplace transform:-**

When $k(s, t) = e^{-st}$, we have Laplace transform of f(t) as: $g(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

we can also write

$$L[f(t)] = g(s) \text{ or } F(s) \text{ or } \bar{f}(s)$$

**(2) Fourier transform:-**

when $k(s, t) = \frac{1}{\sqrt{2\pi}} e^{-ist}$, we have th eFourier transform of f(t) as

$$F[f(t)] = g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

**(3) Mellin transform:-**

When $k(s, t) = t^{s-1}$, , we have Mellin transform of f(t) as: $M[f(t)] = g(s) = \int_0^\infty f(t)\, t^{s-1}\, dt$

**(4) Hankel transform** (Fourier-Bessel):-

When $k(s, t) = t\, J_n(st)$, , we have Hankel transform of f(t) as: $g(s) = \int_0^\infty f(t)\, t\, J_n(st)\, dt$

**Fourier Transform:-** If f(t) be a function defined on $(-\infty, \infty)$ and f be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$ , i.e., f(t) be a function s. t.

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

then the function

$$\bar{f}(s) = F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

is called Fourier transform (F.T.) of f(t).

Then $F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{ist}\, ds$ = f(t) is called inverse Fourier Transform of F(s).

**Remarks:**

(i) If $F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ist}\, dt$, then

$$F^{-1}[F(s)] = \int_{-\infty}^{\infty} F(s)\; e^{ist}\, ds$$

(ii) If $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$, then

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{ist}\, ds$$

(iii) If $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$, then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-ist}\, ds$$

**Fourier cosine and sine transform**

$F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t)\; \cos st\; dt$ is called Fourier cosine transform (FCT) and $F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t)\; \sin st\; dt$ is called Fourier sine transform (FST) of f(t).

The functions $f_c(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s)\; \cos st\; ds$

$$f_s(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s)\; \sin st\; ds$$

are called the Inverse Fourier cosine & inverse Fourier sine transform of $F_c(s)$ and $F_s(s)$, respectively.

**Properties of Fourier Transform**

**(1) Linearity Property**

If $f_1(t)$ and $f_2(t)$ are functions with Fourier Transform $F_1(s)$ and $F_2(s)$, respectively and $C_1$ & $C_2$ are constant, then Fourier Transform of $C_1 f_1(t) + C_2 f_2(t)$ is

$$F[c_1 f_1(t) + c_2 f_2(t)] = c_1 F[f_1(t)] + c_2 F[f_2(t)]$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

**Proof:** By definition

$$\text{L.H.S.} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} [c_1 f_1(t) + c_2 f_2(t)]\, dt$$

$$= \frac{c_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f_1(t)dt + \frac{c_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f_2(t)dt$$

$$= C_1 F[f_1(t)] + C_2 F[f_2(t)]$$

$$= C_1 F_1(s) + C_2 F_2(s)$$

**(2) Change of scale property or similarity theorem**

If 'a' is a real constant and $F(s) = F[f(t)]$ then

$$F[f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

**Proof :** For $a > 0$,

$$F[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-ist}\, dt$$

Put $at = x \Rightarrow dt = \frac{1}{a} dx$

$$\therefore \quad F[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\left(\frac{s}{a}\right)x} \frac{1}{a} dx$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\left(\frac{s}{a}\right)t}\, dt$$

$$\Rightarrow \qquad F[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

**For a < 0** :- $F[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at)e^{-ist}\, dt$

Put at = x $\Rightarrow$ dt = $\frac{dx}{a}$

$$\therefore \quad F[f(at)] = \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)e^{-i\frac{s}{a}x}\,dx$$

$$= \frac{-1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\frac{s}{a}t}\,dt$$

$$\Rightarrow \quad F[f(at)] = \frac{-1}{a}\,F\left(\frac{s}{a}\right)$$

Hence, $F[f(at)] = \frac{1}{|a|}\,F\left(\frac{s}{a}\right)$

**Particular case**:- If a = −1, then

$$F[f(-t)] = F(-s)$$

**(3) First shifting Property**

If F[f(t)] = F(s), then

$$F[f(t-u)] = e^{-ius}\ F[f(t)] = e^{-ius}\ F(s)$$

**Proof :-** $F[f(t-u)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-u)\,e^{-ist}\,dt$

Put t − u = v $\Rightarrow$ dt = dv

$$\Theta \quad F[f(t-u)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\ e^{-is(v+u)}\,dv$$

$$= e^{-isu}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-isv}\,dv$$

$$= e^{-isu}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt$$

$$= e^{-isu}\ F[f(t)].$$

**(4). Second shifting property**

If F(s) = F[f(t)], then

$$F[f(t)e^{iat}] = F(s-a)$$

**Proof :-** $F[f(t)\,e^{iat}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{iat}\,e^{-ist}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i(s-a)t}\,dt = F(s-a)$$

**(5) Symmetry property**

If $F(s) = F[f(t)]$, then $F[F(t)] = f(-s)$

**Proof :** We know that

$$\sqrt{2\pi}\,f(t) = \int_{-\infty}^{\infty} F(s)\,e^{ist}\,ds$$

changing t to – t, we have

$$\sqrt{2\pi}\,f(-t) = \int_{-\infty}^{\infty} F(s)e^{-ist}\,ds$$

Interchanging t & s, we get

$$\sqrt{2\pi}\,f(-s) = \int_{-\infty}^{\infty} F(t)e^{-ist}\,dt$$

$$\Rightarrow \quad f(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)\;e^{-ist}\,dt$$

$$\Rightarrow \quad f(-s) = F[F(t)]$$

**Example. 1.** Find Fourier transform of $f(t) = \begin{cases} e^{-\alpha t}, & t \geq 0,\ \alpha > 0 \\ 0, & t < 0 \end{cases}$ …(1)

**Solution :-** By definition, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt$

$$\Rightarrow \quad F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(t)e^{-ist}\,dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\alpha t}e^{-ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(\alpha+ist)t}\,dt \qquad \text{[using (1)]}$$

$$\Rightarrow \quad F(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(\alpha+is)}}{-(\alpha+is)}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[0+\frac{1}{\alpha+is}\right]=\frac{\alpha-is}{\sqrt{2\pi}(\alpha^2+s^2)}$$

**Example.2.** $f(t) = \begin{cases} 1 & , \quad |t| \le a \\ 0 & , \quad |t| > a \end{cases}$

**Solution** $F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(t)e^{-ist}\,dt + \int_{-a}^{a} f(t)e^{-ist}\,dt + \int_{a}^{\infty} f(t)e^{-ist}\,dt\right]$

$$= I_1 + I_2 + I_3$$

Let $I_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a} f(t)e^{-ist}\,dt$

Put $t = -u \Rightarrow dt = -du$

$\therefore \quad I_1 = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{a} f(-u)\,e^{isu}\,(-du)$

$$= \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} f(-u)e^{isu}\,du = 0$$

Similarly $I_3 = \int_{a}^{\infty} f(t)e^{-ist}\,dt = 0$

$\therefore \quad I_2 = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(t)e^{-ist}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} i.\,e^{-ist}\,dt \qquad [\Theta\ f(t) = 1]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ist}}{-is}\right]_{-a}^{a} = \frac{-1}{\sqrt{2\pi}}is \quad [e^{-ias} - e^{ias}]$$

$\Rightarrow \quad I_2 = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ias} - e^{ias}}{is}\right)$

$$= \frac{1}{\sqrt{2\pi}}.\frac{2\sin as}{is} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right), s \ne 0$$

when $s = 0$,

$$F(s) = \sqrt{\frac{2}{\pi}}\frac{a\cos as}{(1)} \quad \text{as} \quad s\to 0 \ \text{[By L-Hospital rule]}$$

$$= \sqrt{\frac{2}{\pi}}\ a.$$

**Example. 3 :-** $f(t) = \begin{cases} t, & |t| \le a \\ 0, & |t| > a \end{cases}$ …(1)

**Solution :-** $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{-a} f(t)e^{-ist}\,dt + \int_{-a}^{a} f(t)e^{-ist}\,dt + \int_{a}^{\infty} f(t)e^{-ist}\,dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} t\,e^{-ist}\,dt \qquad \text{[using (1)]}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(t\frac{e^{-ist}}{-is}\right)_{-a}^{a} - \int_{-a}^{a}(1)\frac{e^{-ist}}{-is}dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{-is}e^{-ias} + \frac{a}{-is}e^{ias} + \frac{1}{is(-is)}(e^{-ias} - e^{ias})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{-is}(e^{ias} + e^{-ias}) - \frac{1}{s^2}(e^{ias} - e^{-ias})\right]$$

$\Rightarrow$ $$F(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{2a\cos as}{-is} - \frac{1}{s^2}\text{si sin as}\right]$$

$$= \frac{2}{s^2\sqrt{2\pi}}\ [i\ as\cos as - i\sin as]$$

$\Rightarrow$ $$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{i}{s^2}[as\cos as - \sin as]$$

**Example. 4.** If F(s) = F[f(t)]. Find F.T. of f(t) cos at

**Solution :-** We know that

$$\cos at = \frac{1}{2}(e^{iat} + e^{-iat})$$

Then $F[f(t)\cos at] = \frac{1}{2}F[f(t)\,e^{iat}] + \frac{1}{2}F[f(t)\,e^{-iat}]$

or prove by definition [By linearity property]

$\Rightarrow \quad F[f(t)\cos at] = \frac{1}{2}F(s-a) + \frac{1}{2}F(s+a)$ [By using shifting property]

**Example. 5.** If $F_s(s)$ and $F_c(s)$ are FST and FCT of f(t) respectively, then

$$F_s[f(t)\cos at] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

$$F_c[f(t)\cos at] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

$$F_s[f(t)\sin at] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

$$F_c[f(t)\sin at] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$$

**Solution :-** (i) By definition of FST,

$$F_s[f(t)\cos at] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos at \sin st\, dt$$

$$= \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\, 2\cos at \sin st\, dt$$

Using $2\sin A\cos B = \sin(A+B) + \sin(A-B)$,

$$\Rightarrow \quad F_s[f(t)\cos at] = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)[\sin(s+a)t + \sin(s-a)t]\, dt$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(s+a)t\, dt + \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(s-a)t\, dt\right]$$

$$= \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

(ii) By definition FCT,

$$F_c[f(t)\cos at] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos at \cos st\, dt$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)[\cos(s+a)t + \cos(s-a)t]dt$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(s+a)t\ dt + \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(s-a)t\ dt\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(iii) By definition of FST,

$$F_s[f(t)\sin at] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin at \sin st\ dt$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\,[\cos(s-a) - \cos(s+a)]\ dt$$

$$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(s-a)dt - \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(s+a)dt\right]$$

$$= \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

(iv) By definition of FCT,

$$F_c[f(t)\sin at] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin at\cos st\ dt$$

$$= \frac{1}{2}\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\,[\sin(s+a)t - \sin(s-a)t]\ dt$$

$$[\text{using}\quad 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(s+a)t - \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(s-a)t\ dt\right]$$

$$= \frac{1}{2}[F_s(s+a) - F_s(s+a) - F_s(s-a)].$$

**Example. 6.** Find Fourier sine & cosine transform of $e^{-at}$, $a > 0$.

Let $\int_0^\infty e^{-at}\cos st\,dt = I_1$ …(1)

$$\int_0^\infty e^{-at} \sin st \; dt = I_2 \qquad \ldots(2)$$

Integrating (1) by parts

$$I_1 = \left[-\frac{1}{a}e^{-at}\cos st\right]_0^\infty - \int_0^\infty e^{-at} \sin st \, dt$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-at} \sin st \, dt$$

$$= \frac{1}{a} - \frac{s}{a} I_2 \qquad \ldots(3)$$

Integrating (2) by parts, we have

$$I_2 = \frac{s}{a} I_1 \qquad \ldots(4)$$

$$\left[\because \quad I_2 = \left[\frac{-1}{a}e^{-at} \sin st\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-at} \cos st \, dt\right]$$

solving (3) and (4) for $I_1$ & $I_2$,

$$I_1 = \frac{1}{a} - \frac{s}{a}.\frac{s}{a} I_1$$

$$\Rightarrow \quad I_1\left(1+\frac{s^2}{a^2}\right) = \frac{1}{a} \quad \Rightarrow \quad I_1 = \frac{1}{a}.\frac{a^2}{s^2+a^2}$$

$$\Rightarrow \quad I_1 = \frac{a}{s^2+a^2}$$

and $I_2 = \dfrac{s}{a}.\dfrac{a}{s^2+a^2} = \dfrac{s}{s^2+a^2}$

Hence $F_c[e^{-at}] = \sqrt{\dfrac{2}{\pi}}\; I_1 = \sqrt{\dfrac{2}{\pi}}\dfrac{a}{s^2+a^2}$ …(5)

and $F_s[e^{-at}] = \sqrt{\dfrac{2}{\pi}}\; I_2 = \sqrt{\dfrac{2}{\pi}}\dfrac{s}{s^2+a^2}$ …(6)

**Extensions :-** Differentiating both sides of (5) w.r.t. 'a', we find

$$F_c[-t\,e^{-at}] = \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)1 - a.2a}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \qquad -F_c[t\,e^{-at}] = \sqrt{\frac{2}{\pi}}\left[\frac{-a^2+s^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \qquad F_c[\,t\,e^{-at}] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

Differentiating both sides of (6) w.r.t. 'a', we get

$$F_s[-t\,e^{-at}] = \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)0-s.\,2a}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \qquad -F_s[\,t\,e^{-at}] = -\sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}$$

$$\Rightarrow \qquad F_s[\,t\,e^{-at}] = \sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}\,.$$

Put a = 1 in (5), (6), we get

$$F_s[e^{-t}] = \sqrt{\frac{2}{\pi}}\,\frac{s^2}{s^2+1} = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+1}$$

$$F_c[e^{-t}] = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{s^2+1}$$

**Results:**

$$\int e^{\alpha t}\cos\beta t\;dt = \frac{e^{\alpha t}}{\alpha^2+\beta^2}(\alpha\cos\beta t+\beta\sin\beta t)$$

$$\int e^{\alpha t}\sin\beta t\;dt = \frac{e^{\alpha t}}{\alpha^2+\beta^2}(\alpha\sin\beta\,t-\beta\cos\beta t)$$

**Example. 7 :-** Find F. T. of f(t) = $e^{-t^2/2}$

**Solution :-** By definition, $F\left[e^{-\frac{t^2}{2}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,e^{-ist}\,dt$

$$\Rightarrow \qquad F[e^{-t^2/2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+is)^2}.e^{\frac{-s^2}{2}}\;dt$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+is)^2}\;dt$$

Put $\frac{1}{\sqrt{2}}(t + is) = y \quad \Rightarrow \quad dt = \sqrt{2}\ \ dy$

$$\Rightarrow \qquad F[e^{-t^2/2}] = \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} . \sqrt{2}\ dy$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\ dy = \frac{e^{-s^2/2}}{\sqrt{\pi}} . \sqrt{\pi}$$

$$\Rightarrow \qquad F[e^{-t^2/2}] = e^{-s^2/2}, \quad F_c[e^{-t^2/2}] = e^{-s^2/2}$$

**Example. 8.** Find Fourier cosine transform of f(t) = $e^{-t^2}$

**Solution :-** $F_c[e^{-t^2}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2} \cos st\, dt = I$ …(1)

Differentiating w.r.t. 's', we get

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} t\ e^{-t^2}\ \sin st\, dt$$

$$= \frac{1}{2} . \sqrt{\frac{2}{\pi}} \int_0^{\infty} (-2t\, e^{-t^2})\ \sin\ st\ dt$$

Integrating by parts,

$$\frac{dI}{ds} = \frac{1}{2} . \sqrt{\frac{2}{\pi}} \left[ \sin ste^{-t^2} \Big|_0^{\infty} - s \int_0^{\infty} \cos st.\ e^{-t^2} dt \right]$$

$$= -\ \frac{s}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2}\ \cos\ st\, dt$$

$$\Rightarrow \qquad \frac{dI}{ds} = -\frac{s}{2} I \qquad \text{[using (1)]}$$

$$\Rightarrow \qquad \frac{dI}{I} = \frac{-s}{2} ds$$

Integrating, $\log I = -\frac{s^2}{4} + \log A$

$$\Rightarrow \qquad I = A\ e^{-s^2/4} \qquad \text{…(2)}$$

when s = 0, from (1), $I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2} \, dt$

$\Rightarrow \quad I = \sqrt{\frac{2}{\pi}} . \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$ …(3)

Also when s = 0, from (2) $\Rightarrow$ I = A …(4)

$\therefore$ from (3) & (4), we get $A = \frac{1}{\sqrt{2}}$

$\therefore$ (2) gives $I = \frac{1}{\sqrt{2}} e^{-s^2/4}$

**Extension** :- $F_c\,[e^{-a^2t^2}] = \frac{1}{\sqrt{2}} . \frac{1}{a} e^{-\frac{1}{4}\left(\frac{s}{a}\right)^2}$

$\Rightarrow \quad F_c[e^{-a^2t^2}] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}$ [using change of scale probability $F[f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

If $a = \frac{1}{\sqrt{2}}$, we have

$$F_c\,[e^{-t^2/2}] = e^{-s^2/2}$$

**Self-reciprocal function:**

A function f(t) with the property that F[f(t)] = f(s) is said to be self-reciprocal under Fourier transform, e.g. the function $e^{-t^2/2}$ is self-reciprocal function under F.T. The function $e^{-t^2/2}$ is also self reciprocal under F.C.T.

**To prove** :- $F_c\,[e^{-t^2/2}] = e^{-s^2/2}$

Let $\quad F_c[e^{-t^2/2}] = e^{-s^2/2}$

$\Rightarrow \quad \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} \cos st \, dt = e^{-s^2/2}$

Differentiating w.r.t. ‘s’ on both sides, we get

$$F_s[t\, e^{-t^2/2}] = s e^{-s^2/2}$$

Hence the function $t\, e^{-t^2/2}$ is self-reciprocal function under Fourier Sine Transform.

**Example. :-** We know that if $f(t) = \begin{cases} 1, & |t| \le a \\ 0, & |t| > a \end{cases}$

Then $F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}, s \neq 0$ …(1)

By definition of F. T., $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$

then $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-st}\, ds$

Put F(s) from (1), we get

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2\pi}{}} \frac{\sin sa}{s}\, e^{ist}\, ds$$

$$\Rightarrow \quad f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s}\, e^{ist}\, ds = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$

$$\text{L.H.S.} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} [\cos st + i \sin st]\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos st\, ds + \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \sin st\, ds$$

Since integrand in second integral is an odd function, so the integral is zero

$$\Rightarrow \quad \text{L.H.S.} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos st\, ds$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos st\, ds = \begin{cases} \pi, & |t| < a \\ 0, & |t| > a \\ \pi/2, & |t| = a \end{cases}$$

$$\text{or} \quad \int_{0}^{\infty} \frac{\sin sa \cos st}{s} ds = \begin{cases} \pi/2, & |t| < a \\ 0, & |t| > a \\ \pi/4, & |t| = a \end{cases} \qquad \text{…(2)}$$

Evaluate $\int_{0}^{\infty} \frac{\sin s}{s} ds$

**Solution :** Put t = 0, a = 1 in (2), we get

$$\int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} \text{ when } |t| < a.$$

**Relation between Laplace Transform and Fourier Ttransform**

Consider the function

$$f(t) = \begin{cases} e^{-xt}\ \varphi t, & t > 0 \\ 0 \quad , & t < 0 \end{cases}$$

Taking F.T. of f(t),

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}\ f(t)\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ist}\ e^{-xt}\ \phi(t)\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(x+is)t}\ \phi(t)\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-pt}\ \phi(t) \text{ where } p = x + i$$

$\Rightarrow$ $$F[f(t)] = \frac{1}{\sqrt{2\pi}} L[\phi(t)]$$

**F.T. of Derivatives**

If $F[f(t)] = F(s)$ & $f(t) \to 0$ as $t \to \pm\infty$, then

$$F[f'(t)] = i\,s\ F[f(t)] = i\,s\,F(s)$$

**Proof :-** By definition,

$$F[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t) e^{-ist}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{-ist}(t)\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} (-is) \int_{-\infty}^{\infty} f(t) e^{-ist} dt$$

$$= is\ .\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist}\ dt \qquad [\Theta\ f(t) \to 0 \text{ as } t \to \pm\infty]$$

$\Rightarrow$ $$F[f'(t)] = is\ F[f(t)] = is\ F(s)$$

Now

$$F[f''(t)] = is\,F[f'(t)] = (is)^2\,F[f(t)]$$

In general, we have

$$F[f^{(n)}(t)] = is\,F[f^{(n-1)}(t)]$$

$$= (i\,s)^n\,F[f(t)] = (is)^n\,F(s)$$

Find Fourier sine & cosine transform of f′(t), f″(t).

**Derivation :-** By definition

$$F_c[f'(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(t)\,\cos st\,dt$$

$$= \sqrt{\frac{2}{\pi}}[\cos st\ f(t)]_0^\infty + s\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt$$

Assuming $f(t) \to 0$ as $t\to\infty$,

$$\Rightarrow \quad F_c[f'(t)] = \sqrt{\frac{2}{\pi}}\ f(0) + s\,F_s\,[f(t)] \qquad \ldots(1)$$

$$F_s[f'(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(t)\ \sin st\,dt$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin st f(t)\right]_0^\infty - s\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt$$

Assuming $f(t) \to 0$ as $t\to\infty$,

$$\Rightarrow \quad F_s[f'(t)] = -\,s\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt$$

$$\Rightarrow \quad F_s[f'(t)] = -s\,F_c[f(t)] \qquad \ldots(2)$$

Now

$$F_c[f''(t)] = -\sqrt{\frac{2}{\pi}}\ f'(0) + s\,F_s\,[f'(t)] \quad \text{(using (1)}$$

$$= -f'(0)\sqrt{\frac{2}{\pi}} + s[-s\,F_c[f(t)]] \qquad \text{[By using (2)]}$$

$$\Rightarrow \quad F_c[f''(t)] = -f'(0)\sqrt{\frac{2}{\pi}} - s^2\,F_c[f(t)] \qquad \ldots(3)$$

and $F_s[f''(t)] = -s\,F_c[f'(t)]$ [using (2)]

$$= -\,s\left\{-\sqrt{\frac{2}{\pi}}f(0) + s\ F_s[f(t)]\right\} \qquad \text{[using (1)]}$$

$$\Rightarrow \qquad F_s[f''(t)] = \sqrt{\frac{2}{\pi}}\ s\ f(0) - s^2\ F_s[f(t)] \qquad \ldots(4)$$

**Theorem**:- If F[f(t)] = F(s), then

$$\frac{d^n}{ds^n}F[f(t)] = (-i)^n\ F[t^n\ f(t)],\ n = 1, 2, 3$$

**Proof:-** By definition,

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

Differentiate w.r.t. 's' under integral sign, we get

$$\frac{d}{ds}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial s}(e^{-ist})\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,(-i\,t)\, e^{-ist}\, dt$$

$$\Rightarrow \qquad \frac{d}{ds}F(s) = (-i)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\ f(t)e^{-ist}dt$$

$$\Rightarrow \qquad \frac{d}{ds}F(s) = (-i)\ F[t\ f(t)]$$

Now $\displaystyle\frac{d^2}{ds^2}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-i\,t)^2\ f(t)\ e^{-ist}\ dt$

$$= (-i)^2\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2\ f(t)\ e^{-ist}\ dt$$

$$\Rightarrow \qquad \frac{d^2}{ds^2}F(s) = (-i)^2\ F[t^2\ f(t)]$$

Generalising the result, we have

$$\frac{d^n}{ds^n}F[f(t)] = (-i)^n\ F[t^n\ f(t)],\ n = 1, 2, \ldots$$

**Theorem :-** If F[f(t)] = F(s)

and $\int_{-\infty}^{\infty} f(t)dt = F(0) = 0$

then $F\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{is}F(s)$

**Proof :** Consider $\phi(t) = \int_{-\infty}^{t} f(x)dx$

Then $\phi'(t) = f(t)$

Hence if $F[\phi(t)] = \Phi(s)$,

then $F[\phi'(t)] = F[f(t)] = is\,\Phi(s)$

$\Rightarrow \quad \Phi(s) = \frac{1}{is}F[f(t)]$

$\Rightarrow \quad \Phi(s) = \frac{F(s)}{is}$

$\Rightarrow \quad F[\phi(t)] = \frac{F(s)}{is}$

$\Rightarrow \quad F\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{is}F(s)$

**Convolution :-** Let $f_1(t)$ and $f_2(t)$ be two given functions, the convolution of $f_1(t)$ & $f_2(t)$ is defined by the function

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)\, f_2(t-x)dx$$

$$= f_1(t) * f_2(t)$$

**Special case :-**

$f_1(t) = 0$ for $t < 0$

$f_2(t) = 0$ for $t < 0$

Then

$$f(t) = f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi}}\int_{0}^{t} f_1(x)\, f_2(t-x)\, dx$$

**Proof :-** $f(t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f_1(x)f_2(t-x)dx + \int_{0}^{t} f_1(x)f_2(t-x)dx + \int_{t}^{\infty} f_1(x)f_2(t-x)dx\right]$

…(1)

Now $I_1 = 0$ $[\because f_1(t) = 0 \text{ for } t < 0]$ …(2)

and $I_3 = \frac{1}{\sqrt{2\pi}} \int_t^\infty f_1(x) f_2(t-x)dx$

Now when $t < 0 \Rightarrow f_2(t) = 0$

$\therefore$ when $t - x < 0 \Rightarrow f_2(t-x) = 0$

$\Rightarrow$ $f_2(t-x) = 0$ for $t < x$ or $x > t$

So $I_3 = 0$ $[\because f_2(t-x) = 0 \text{ for } x > t]$ …(3)

Using (2) & (3) in (1), we get $f(t) = \frac{1}{\sqrt{2\pi}} \int_1^t f_1(x)\, f_2(t-x)dx$

**Commutative Property:**

$f_1(t) * f_2(t) = f_2(t) * f_1(t)$

**Proof :-** By definition,

$$f_1 * f_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\ f_2(t-x)dx$$

put $t - x = y \Rightarrow dx = -dy$

when $x = -\infty$, $y = \infty$

and when $x = \infty$, $y = -\infty$

$$\Rightarrow \quad f_1 * f_2 = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f_1(t-y)\, f_2(y)\, (-dy)$$

$$\Rightarrow \quad f_1 * f_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(y)\, f_1(t-y)\, dy = f_2 * f_1$$

**Associativity property**

$(f_1 * f_2) * f_3 = f_1 *(f_2 * f_3)$ …(1)

Take $g(t) = f_1 * f_2$

$h(t) = f_2 * f_3$

Then (1) becomes, $g(t) * f_3 = f_1 * h(t)$

**Convolution Theorem (or Falting Theorem) on Fourier Transforms**

If $F[f_1(t)] = F_1(f)$

$F[f_2(t)] = F_2(s)$ i.e.

$$F[f_1(t) * f_2(t)] = F_1(s)\, F_2(s)$$

where

$$f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\, f_2(t - x)dx$$

**Proof :-** We have by definition of F.T.,

$$F[f_1(t) * f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f_1(t) + f_2(t)]\, e^{-ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) f_2(t - x) dx \right] e^{-ist}\, dt$$

[By using definition of convolution]

changing the order of integration, we get

$$F[f_1(t) * f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t - x) e^{-ist}\, dt \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t - x) e^{-is(t-x)} e^{-isx} dt \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(y) e^{-isy} dy \right] e^{-isx}\, dx$$

[By putting $t - x = y \Rightarrow dt = dy$]

$$\Rightarrow \quad F[f_1(t) * f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{-isx}\, F_2(s) dx$$

$$= F_2(s)\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\, e^{-isx}\, dx$$

$$\Rightarrow \quad F[f_1(t) * f_2(t)] = F_1(s)\, F_2(s)$$

∗ The convolution can be used to obtain the Fourier transform of the product of two functions.

By definition of F.T., we have

$$F[f_1(t)\ f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) f_2(x) e^{-isx} dx$$

By using inverse F.T. of $f_2(x)$, we get

$$F[f_1(t)\ f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(s') e^{is'x} ds' \right] e^{-isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds' F_2(s') \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{-i(s-s')x} dx \right]$$

$\Rightarrow$ $$F[f_1(t)\ f_2(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_2(s') F_1(s - s') ds'$$

$= F_2(s) * F_1(s)$ [By using definition of convolution]

$\Rightarrow$ $F[f_1(t)\ f_2(t)] = F_2 * F_1$

$\Rightarrow$ $F[f_1(t)\ f_2(t)] = F_1 * F_2$

If $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-izt}\ dt$

Then inverse function $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(z)\, e^{-izt}\ dz$

**Example :-** Find F.T. of $f(t) = e^{-|t|}$ and verify the inverse transform.

**Sol.** Now $|t| = t$ for $t > 0$

$|t| = -t$ for $t < 0$

So

$$f(t) = \begin{cases} e^{t} & , \quad t < 0 \\ e^{-t} & , \quad t > 0 \end{cases}$$

$\therefore$ $$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-izt}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{0} e^{t(1-iz)} dt + \int_{0}^{\infty} e^{-t(1+iz)} dt \right]$$

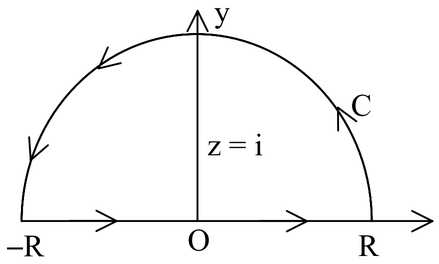
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-iz}+\frac{1}{1+iz}\right]=\frac{1}{\sqrt{2\pi}}\frac{2}{(1+z^2)}$$

We can invert the transform using contour integration (Residue theorem). First consider t > 0, then

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{2}{1+z^2}e^{izt}\,dz = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{izt}}{1+z^2}dz$$

$$= \frac{1}{\pi}\lim_{R\to\infty}\oint_C \frac{e^{izt}}{1+z^2}dz$$

where c is the contour shown in figure.

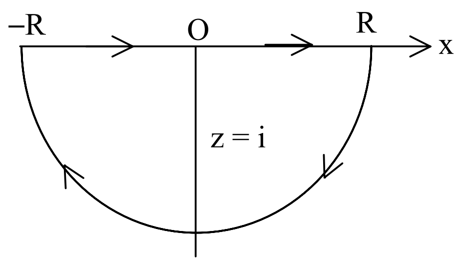



There is a simple pole at z = i and Res (z = i) = $\lim\limits_{z\to i}\frac{(z-i)e^{izt}}{(z-i)(z+i)} = \frac{1}{2i}e^{-t}$

Therefore by Residue theorem,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{izt}}{1+z^2}dz = \frac{1}{\pi}\,2\pi i\left(\frac{e^{-t}}{2i}\right) = e^{-t}$$

Consider t < 0, then we choose the contour with a semi-circular arc lying below the x-axis where c is the contour shown in figure.




Therefore there is a simple pole at z = –i.

Res (z = –) = $\lim\limits_{z\to -i}\frac{(z+i)e^{izt}}{(z+i)(z-i)} = \frac{e^{t}}{-2i}$

But in this case contour is in clockwise direction, hence the desired result is

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{izt}}{1+z^2}dz = \frac{-1}{\pi}2\pi i\left(\frac{-e^t}{2i}\right) = e^t$$

At t = 0, we can evaluate directly

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+z^2}dz = \frac{1}{\pi}\left(\tan^{-1}z\right)_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right]$$

$$= 1$$

**Example :** Find F.T. of $f(t) = \begin{cases}\sin wt, & 0 < t < \infty \\ 0 & , \quad \text{otherwise}\end{cases}$

or $\quad f(t) = \sin wt\, H(t)$

where

$$H(t) = \begin{cases}1, & t > 0 \\ 0, & t < 0\end{cases}$$

is a unit step function or Heaviside's unit step function.

**Sol.** Now $F(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\sin wt\, e^{-izt}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[\frac{e^{iwt}-e^{iwt}}{2i}\right]e^{-izt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{1}{2i}\int_0^{\infty}[e^{it(w-z)}-e^{it(w+z)}]\,dt$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{1}{2i}\left[\frac{e^{-it-w+z)}}{-i(-w+z)}\frac{-e^{-t(w+z)}}{-i(w+z)}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{1}{2i}\left[\frac{1}{i(z-w)}-\frac{1}{i(w+z)}\right]$$

$$= \frac{-1}{2i^2\sqrt{2\pi}}\left[\frac{-1}{z-w}+\frac{1}{z+w}\right] = \frac{1}{2\sqrt{2\pi}}\left[\frac{-z-w+z-w}{z^2-w^2}\right]$$

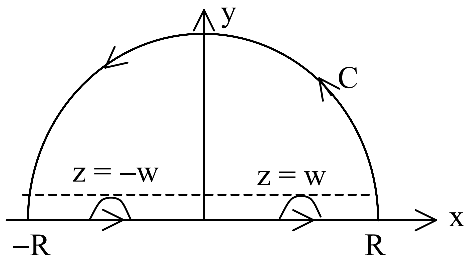
$$= \frac{-1}{\sqrt{2\pi}}\left(\frac{w}{z^2-w^2}\right)$$

Here $F(z) = \dfrac{-w}{\sqrt{2\pi}(z^2 - w^2)}$ is analytic in z-plane except at z = ± w

**If t > 0 :-** $f(t) = \dfrac{-1}{2\pi} \lim_{R\to\infty} \oint_C \dfrac{we^{izt}}{z^2 - w^2} dz$

where c is contour.

There are two simple pole z = w & z = w inside c.




where $f(t) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(z)e^{izt} \; dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-w}{\sqrt{2\pi}(z^2 - w^2)} \; e^{izt} dz$$

$$\Rightarrow \qquad f(t) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{we^{izt}}{z^2 - w^2} dz$$

$$\therefore \qquad \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{we^{izt}}{z^2 - w^2} dz = \frac{-1}{2\pi} \lim_{R\to\infty} \oint_C \frac{we^{izt}}{z^2 - w^2} dz$$

$$\text{Res } (z = w) = \lim_{z\to w} (z - w) \frac{we^{izt}}{(z - w)(z + w)} = \frac{we^{iwt}}{2w}$$

$$= \frac{e^{iwt}}{2}$$

$$\text{Res } (z = -w) = \lim_{z\to -w} (z + w) \frac{we^{izt}}{(z - w)(z + w)} = \frac{we^{-iwt}}{-2w}$$

$$= \frac{-1}{2} e^{-iwt}$$

Therefore by Residue theorem,

$$\frac{-1}{2\pi} \underset{R\to\infty}{\text{Lim}} \oint_C \frac{we^{izt}}{z^2 - w^2} dz = \frac{-1}{2\pi}(2\pi i) \text{ (Sum of residue}$$

$$= -i\left[\frac{e^{iwt} - e^{iwt}}{2}\right]$$

$\Rightarrow$ f(t) = –i(i sin wt) = sin wt

**If t < 0,** we choose the contour with a semi-circular arc below x-axis & since there are **no poles** inside the contour, the result is zero.

## LESSON-2 APPLICATIONS OF FOURIER TRANSFORMS

**Solution of Ordinary Differential Equation**

Consider the nth order differential equation

$$A_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + .... + a_1 \frac{dy}{dt} + ay = f(t)$$

Taking F.T. of both sides,

$$[a_n (i\,s)^n + a_{n-1} (i\,s)^{n-1} + \ldots + a_1(i\,s) + a]\, F[y(t)] = F[f(t)]$$

Let $F[f(t)] = G(s)$

$F[y(t)] = Y(s)$

$P(i\,s) = a_n (i\,s)^n + a_{n-1} (i\,s)^{n-1} + \ldots. + a_1(i\,s) + a$

is a polynomial in (i s).

Then we have $Y(s) = \dfrac{G(s)}{P(is)}$

Taking inverse F.T., we have the solution is given by

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(s)\, e^{ist}\, ds$$

**Example:-** Solve using F.T. technique

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-|t|} \qquad \ldots(1)$$

**Solution :-** Taking F.T. on both sides,

$$[(i\,z)^2 + 3(i\,z) + 2]\, F[y(t)] = F[e^{-|t|}]$$

$$\Rightarrow \quad P(i\,z)\, F[y(t)] = \frac{1}{\sqrt{2\pi}}\frac{2}{1+z^2}$$

$$\Rightarrow \quad Y(z) = \frac{1}{\sqrt{2\pi}}\frac{2}{(1+z^2).\dfrac{1}{P(iz)}} \quad \text{where } Y(z) = F[y(t)]$$

$$\Rightarrow \quad y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi}}\frac{e^{izt}\, dz}{(1+z^2)[(iz)^2 + 3iz + 2]}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{izt}\, dz}{(z^2+1)(-z^2+3iz+2)}$$

$$= \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{e^{izt}\,dz}{(z^2+1)(z^2-3iz-2)}$$

$$= \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{e^{izt}\,dz}{(z-1)(z+1)(z-i)(z-2i)}$$

$$= \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{e^{izt}\,dz}{(z+i)(z-i)^2(z-2i)}$$

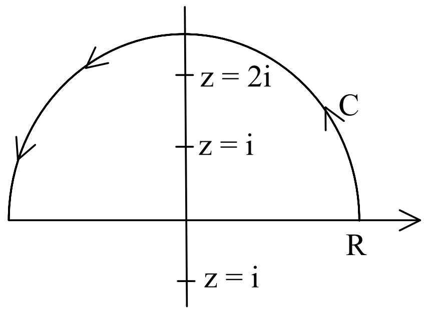
**For t > 0**

The singularity within the contour are a simple pole at z = 2i and a double pole at z = i.

$\therefore$ $$\text{Res. } (z = 2i) = \lim_{z\to 2i} \frac{(z-2i)e^{izt}}{(z+i)(z-i)^2(z-2i)}$$

$$= \underset{z\to 2i}{\text{Lt}} \frac{e^{izt}}{(z+i)(z-i)^2}$$

$$= \frac{e^{-2t}}{3i.i^2} = \frac{=1}{3i} e^{-2t}$$




$$\text{Res. } (z = i) = \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2 e^{izt}}{(z+i)(z-i)^2(z-2i)}$$

$$= \lim_{z\to i} \frac{d}{dz} \frac{e^{izt}}{(z+i)(z-2i)}$$

$\Rightarrow$ $$\text{Res. } (z = i) = \lim_{z\to i} \frac{(z+i)(z-2i)e^{izt}(i\,t) - e^{izt}(2z+i-2i)}{[(z+i)(z-2i)]^2}$$

$$= \underset{z\to i}{\text{Lt}} \frac{(z+i)(z-2i)(i\,t)e^{izt} - e^{izt}(2z-i)}{(z+i)^2(z-2i)^2}$$

$$= \frac{2i.(-i)(i\,t)e^{-t} - e^{-t}i}{4i^2.(-i)^2}$$

$$= \frac{2i\,te^{-t} - i\,e^{-t}}{4(-1)(-1)} = \frac{2i\,t\,e^{-t} - i\,e^{-t}}{4}$$

$\therefore$ $$y(t) = \frac{-1}{\pi}.2\pi i\left[\frac{+i}{3(1)}e^{-2t} + \frac{2i\,t\,e^{-t} - i\,e^{-t}}{4}\right]$$

$$= -2i\left[\frac{4ie^{-2t} + 6it\,e^{-t} - 3i\,e^{-t}}{12}\right]$$

$$= \frac{-i}{6}[6\,i\,t\,e^{-t} - 3i\,e^{-t} + 4\,i\,e^{-2t}]$$

$$= \frac{1}{6}(6\,t\,e^{-t} - 3\,e^{-t} + 4\,e^{-2t})$$

$$= \frac{2}{3}\,e^{-2t} + t\,e^{-t} - \frac{1}{2}\,e^{-t} \qquad \ldots(2)$$

**Verification :-** Put (2) in L.H.S. of DE. (1) we get

$$\text{L.H.S.} = \frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = \frac{d}{dt}\left[\frac{2}{3}e^{-2t}(-2) + e^{-t} - te^{-t} + \frac{1}{2}e^{-t}\right]$$

$$+\,3\left[\frac{-4}{3}e^{-2t} + e^{-t} - te^{-t} + \frac{1}{2}e^{-t}\right] + \frac{4}{3}e^{-2t} + 2\,t\,e^{-t} - e^{-t}$$

$$\text{L.H.S.} = \frac{-4}{3}e^{-2t}(-2)\frac{-3}{2}e^{-t} - e^{-t} + te^{-t} - 4e^{-2t} + \frac{9}{2}e^{-t} - 3te^{-t}$$

$$+\,\frac{4}{3}e^{-2t} + 2t\,e^{-t} - e^{-t} = 2e^{-t} - e^{-t} = e^{-t}$$

**For t < 0**

$$\text{Res. } (z = -i) = \lim_{z\to -i}\frac{(z+i)e^{izt}}{(z+i)(z-i)^2(z-2i)}$$

$$= \frac{e^t}{(-2i)^2(-3i)} = \frac{e^t}{4(-1)(-3i)} = \frac{e^t}{12i}$$

So $\quad y(t) = \frac{-1}{\pi}(-2\pi i)\frac{e^t}{12i} = \frac{e^t}{6} \qquad \ldots(3)$

**Verification** :- Put (3) in L.H.S. of (1),

$$\text{L.H.S.} = \frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = \frac{d}{dt}\left[\frac{e^t}{6}\right] + 3\frac{e^t}{6} + \frac{2e^t}{6}$$

$$= \frac{e^t}{6} + \frac{3e^t}{6} + \frac{2e^t}{6} = \frac{6e^t}{6} = e^t.$$ Hence verified.

**Example:-** Solve using F.T. techniques

$$\frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = H(t)\sin wt \qquad (1)$$

**Solution :-** Then we have on taking F.T. on both sides,

$$[(i\,z)^2 + 3\,i\,z + 2]\,F[y(t)] = \frac{1}{\sqrt{2\pi}}\frac{w}{w^2 - z^2}$$

Then $y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left(\frac{w}{w^2 - z^2}\right)\frac{e^{izt}dz}{(-z^2 + 3iz + 2)}$

$$\Rightarrow \qquad y(t) = \frac{w}{2\pi}\int_{-\infty}^{\infty}\frac{e^{izt}\,dt}{(z^2 - w^2)(z^2 - 3iz - 2)}$$

$$= \frac{w}{2\pi}\int_{-\infty}^{\infty}\frac{e^{izt}\,dt}{(z^2 - w^2)(z - i)(z - 2i)}$$

**For t > 0**

In this case singularity are at $z = \pm w$, $z = 2i$, $z = i$ and all these lies inside the upper half plane.

$$\text{Res.} \quad (z = 2i) = \lim_{z\to 2i}\left[\frac{(z - 2i)\,e^{izt}}{(z - w)(z + w)(z - i)(z - 2i)}\right]$$

$$= \frac{e^{-2t}}{(2i - w)(2i + w)i} = \frac{e^{-2t}}{-(4 + w^2)i^2} = \frac{i\,e^{-2t}}{w^2 + 4}$$

$$\text{Res.} \quad (z = i) = \lim_{z\to i}\left[\frac{(z - i)\,e^{izt}}{(zi-)(z - 2i)(z^2 - w^2)}\right]$$

$$= \frac{e^{-t}}{-i(-1 - w^2)} = \frac{e^{-t}i}{(-1 - w^2)} = \frac{e^{-t}}{i(w^2 + 1)}$$

$$\text{Res.} \quad (z = w) = \lim_{z\to w}\frac{(z - w)\,e^{izt}}{(z - w)(z + w)(z - i)(z - 2i)}$$

$$= \frac{e^{iwt}}{2w(w - i)(w - 2i)} = \frac{e^{iwt}}{2w(w^2 - 3iw - 2)}$$

$$\text{Res.} \quad (z = -w) = \lim_{z\to -w}\frac{(z + w)e^{izt}}{(z - w)(z + w)(z - i)(z - 2i)}$$

$$= \frac{e^{-iwt}}{(-2w)(-w-i)(-w-2i)} = \frac{e^{-iwt}}{-2w(w+i)(w+2i)}$$

$$\Rightarrow \quad \text{Res. } (z=-w) = \frac{e^{-iwt}}{-2w(w^2+3iw-2)} = \frac{-e^{-iwt}}{2w(w^2+3iw-2)}$$

$$\therefore\ y(t) = \frac{w}{2\pi}.2\pi i\left[\frac{ie^{-2t}}{w^2+4} + \frac{ie^{-t}}{-(w^2+1)} + \frac{e^{iwt}}{2w(w-i)(w-2i)} - \frac{e^{-iwt}}{2w(w+i)(w+2i)}\right]$$

$$= \frac{-we^{-2t}}{w^2+4} + \frac{we^{-t}}{w^2+1} + \frac{i}{2}\frac{e^{iwt}}{(w-i)(w-2i)} \quad \frac{-i}{2}\frac{e^{-iwt}}{(w+i)(w+2i)}$$

**For verification :-**

$$y'(t) = \frac{2we^{-2t}}{w^2+4} - \frac{we^{-t}}{w^2+1} - \frac{we^{-iwt}}{2(w-i)(w-2i)} - \frac{we^{-iwt}}{2(w+i)(w+2i)}$$

$$\Rightarrow \quad y''(t) = \frac{-4we^{-2t}}{w^2+4} + \frac{we^{-t}}{w^2+1}\frac{-iw^2e^{iwt}}{2(w-i)(w-2i)} + \frac{iw^2e^{-iwt}}{2(w+i)(w+2i)}$$

Then

$$y'' + 3y' + 2y = \frac{-4we^{-2t}}{w^2+4} + \frac{we^{-t}}{w^2+1} + \frac{iw^2e^{-iwt}}{2(w+i)(w+2i)}$$

$$\frac{-iw^2e^{iwt}}{2(w-i)(w-2i)} + \frac{6we^{-2t}}{w^2+4} - \frac{3we^{-t}}{w^2+1} - \frac{3we^{iwt}}{2(w-i)(w-2i)}$$

$$\frac{-3we^{-iwt}}{2(w+i)(w+2i)} - \frac{2we^{-2t}}{w^2+4} + \frac{2we^{-t}}{w^2+1} + \frac{2ie^{iwt}}{2(w-i)(w-2i)}\frac{-wie^{-iwt}}{2(w+i)(w+2i)}$$

$$\Rightarrow \quad y'' + 3y' + 2y = \frac{e^{-iwt}}{2(w+i)(w+2i)}\ [\,i\,w^2 - 3w - 2i]$$

$$+ \frac{e^{iwt}}{2(w-i)(w-2i)}[-i\,w^2 - 3w + 2i]$$

$$= \frac{ie^{-iwt}}{2(w+i)(w+2i)}(w^2+3iw-2)\frac{-ie^{iwt}(w^2-3iw-2)}{2(w-i)(w-2i)}$$

$$= \frac{-i}{2}(e^{iwt} - e^{-iwt}) = -i^2\left(\frac{e^{iwt}-e^{-iwt}}{2i}\right)$$

$\Rightarrow \quad y'' + 3y' + 2y = H(t) \sin wt$

Hence $y'' + 3y' + 2y = \sin wt. H(t)$ is verified for $t > 0$.

**for t < 0,** there is no pole inside the lower half plane so residue = 0.

$\therefore \quad y(t) = 0$

$\therefore \quad y'' + 3y' + 2y = H(t) \sin wt$ is verified.

For $t < 0$.

$\therefore \quad$ for $t < 0$ , $H(t) = 0$

So R.H.S. of (1) is zero.

And equal to L.H.S. Hence verify the result.

**Example:** Find FST of $\dfrac{e^{-at}}{t}$

**Solution :-** The sine transform of function $f(t) = \dfrac{e^{-at}}{t}$

$$I = F_s[f(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \sin st \, dt \qquad \ldots(1)$$

Differentiation w.r.t. s, we get

$$\frac{dI}{ds} = \frac{d}{ds} F_s(f(t)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} . t \cos st \, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-at}}{a^2 + s^2} (-a \cos st + s \sin st) \right]_0^\infty$$

$$\Rightarrow \quad \frac{d}{ds} F_s[f(t)] = \sqrt{\frac{2}{\pi}} . \frac{a}{a^2 + s^2} = \frac{dI}{ds}$$

Integrating we get,

$$\int \frac{-d}{ds} F_s[f(t)] = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2 + s^2} \, ds$$

$$\Rightarrow \quad I = F_s[f(t)] = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + A \qquad \ldots(2)$$

where A is constant of integration. For s = 0, we get from (2),

$$I = \sqrt{\frac{2}{\pi}}\,(0) + A \quad \Rightarrow \quad I = A \qquad \ldots(3)$$

when s = 0, from (1), $I = \sqrt{\frac{2}{\pi}}\int_0^\infty (0)\ dt = 0 \qquad \ldots(4)$

from (3) & (4), we get

$$A = 0$$

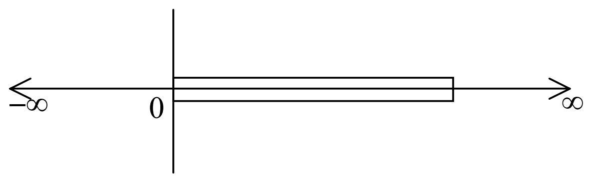
Hence the required FST of given function is

$$F_s\left[\frac{e^{-at}}{t}\right] = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right).$$

## Solution of partial differential equations (boundary value problems)

**Example: -** Determine the distribution of temperature in the semi-infinite medium, $x \geq 0$ when the end x = 0 is maintained at zero temperature & the initial distribution of temperature is f(x).

**Solution :-**




Heat equation is given by

$$\frac{\partial u(x,t)}{\partial t} = c^2\frac{\partial^2 u(x,t)}{\partial x^2},\ x > 0,\ t > 0 \qquad \ldots(1)$$

where u(x, t) is the distribution of temperature at any point x and time t. We want to determine the solution of (1) subject to initial condition u(x, 0) = f(x) …(2)

and the boundary condition u(0, t) = 0 …(3)

Since $(u)_{x=0}$ is given, we apply F.S.T.

Denote $F_s[u(x, t)] = \bar{u}_s(s,t) = \bar{u}_s$ and $F_c[u(x, t)] = \bar{u}_c$

Taking FST of both sides of (1), we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin st\ dx = c^2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\ dx$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty u \sin s\,x\, dx = c^2 \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty u \sin sx\, dx = \frac{d}{dt} F_s[u(x, t)]$$

$$= c^2\left\{\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\sin sx\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty s\frac{\partial u}{\partial x}\cos sx\, dx\right\}$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_s = 0 - c^2 s\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\cos sx\, dx \text{ if } \frac{\partial u}{\partial x} \to 0 \text{ as } x\to\infty$$

$$= -C^2 s \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x} \cos sx\, dx$$

$$= -c^2 s \sqrt{\frac{2}{\pi}}[u(x,t)\cos sx]_0^\infty - c^2 s^2 \sqrt{\frac{2}{\pi}}\int_0^\infty u \sin sx\, dx$$

$$\frac{d}{dt}\bar{u}_s = c^2\sqrt{\frac{2}{\pi}} s\, u(0,t) - c^2 s^2\, \bar{u}_s \text{; assuming } u\to 0 \text{ as } x\to\infty$$

$$\Rightarrow \quad \frac{d\bar{u}_s}{dt} = c^2\left[\sqrt{\frac{2}{\pi}} s\, u(0,t) - s^2 \bar{u}_s\right]$$

By using (3), $u(0, t) = 0$, we get

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_s + c^2 s^2\, \bar{u}_s = 0 \qquad \ldots(4)$$

Also taking FST of (2), we get

$$F_s[u(x, 0)] = F_s[f(x)]$$

$$\Rightarrow \quad \bar{u}_s(s, 0) = \bar{f}_s(s) \qquad \ldots(5)$$

from (4), we have $(D + C^2 s^2)\ \bar{u}_s = 0$

Auxiliary equation is $m + c^2 s^2 = 0$

$\Rightarrow \quad m = -c^2 s^2$

Solution of (4) is

$$\bar{u}_s(s,t) = \bar{u}_s = A\, e^{-c^2 s^2 t} \qquad \ldots(*)$$

To find A, we use (5), from (*), we have $\bar{u}_s(s,0) = A$

$\Rightarrow \quad A = \bar{f}_s(s) \quad$ [using (5)]

Hence solution is

$$\bar{u}_s(s,t) = \bar{f}_s(s)e^{-c^2s^2t}$$

Taking inverse FST, we get

$$F_s^{-1}[\bar{u}_s(s,t)] = F_s^{-1}[\bar{f}_s(s)e^{-c^2s^2t}]$$

$$\Rightarrow \quad u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_s \sin sx \, ds$$

$$\Rightarrow \quad u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t} \sin sx \, ds$$

This is the required solution of PDE.

**Example:-** The temperature u in the semi-infinite rod $0 \le x < \infty$ (or $x \ge 0$) is determined by equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

subject to conditions

(i) $u = 0$ when $t = 0$, $x \ge 0$

i.e. $u(x, 0) = 0$

(ii) $\frac{\partial u}{\partial x} = -\mu$ (a constant) when $x = 0$, $t > 0$

i.e. $\frac{\partial u}{\partial x}(0, t) = -\mu$

or $\quad u_x(0, t) = -\mu$

**Solution :-** Since $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ is given,

So taking Fourier cosine transform of both sides of (1),

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos sx \, dx = K\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx \, dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c(s,t) = K\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\cos sx\right]_0^\infty + K\, s\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\sin sx \, dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c = -K\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} + Ks\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\sin sx\, dx$$

$$\left[\text{By assuming } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c = K\sqrt{\frac{2}{\pi}}\,\mu + Ks\sqrt{\frac{2}{\pi}}\left[u\sin sx\right]_0^\infty - Ks\sqrt{\frac{2}{\pi}}\int_0^\infty us\cos sx\, dx$$

$$= K\sqrt{\frac{2}{\pi}}\mu - Ks^2\bar{u}_c \text{ if } u\to 0 \text{ as } x\to\infty$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c + Ks^2\bar{u}_c = K\mu\sqrt{\frac{2}{\pi}} \qquad \ldots(2)$$

$$\bar{u}_c = \sqrt{\frac{2}{\pi}}\frac{\mu}{s^2}(1 - e^{-Ks^2t})$$

$$u(x, t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos sx}{s^2}(1 - e^{-Ks^2t})\, ds$$

This is linear DE of Ist order.

$$\text{I.F. } e^{\int ks^2 dt} = e^{s^2kt}$$

Solution of (2) is $\bar{u}_c.e^{s^2kt} = \int K.\mu\sqrt{\frac{2}{\pi}}e^{s^2kt}\, dt + A$

$$\Rightarrow \quad \bar{u}_c e^{s^2kt} = A + K\mu\sqrt{\frac{2}{\pi}}\frac{e^{s^2kt}}{s^2k} = A + \frac{\mu}{s^2}\sqrt{\frac{2}{\pi}}e^{s^2kt}$$

$$\Rightarrow \quad \bar{u}_c = Ae^{-s^2kt} + \sqrt{\frac{2}{\pi}}\frac{\mu}{s^2} \qquad \ldots(3)$$

Put t = 0 $\Rightarrow$ $\bar{u}_c(s,0) = A + \sqrt{\frac{2}{\pi}}\frac{\mu}{s^2}$ $\qquad \ldots(4)$

from condition (i),

$$\bar{u}_c(s, 0) = \int_0^\infty u(x,0)\cos sx\, dx = 0 \qquad \ldots(5)$$

(4), (5) $\Rightarrow A = -\sqrt{\frac{2}{\pi}}\frac{\mu}{s^2}$

$\therefore \quad (3) \Rightarrow \bar{u}_c = \sqrt{\frac{2}{\pi}}\frac{\mu}{s^2}(1 - e^{-k^2 st})$

**Finite Fourier Transform :-**

**Finite Fourier Since Transform :-** Let f(x) denote a function that is sectionally continuous over some finite interval (0, λ) of the variable x. The finite fourier sine transform of f(x) on the interval is defined as

$$f_s(s) = \int_0^{\lambda} f(x)\sin\frac{s\pi x}{\lambda}\,dx \text{ where s is an integer}$$

**Inversion formula for sine transform**

$$f(x) = \frac{2}{\lambda}\sum_{s=1}^{\infty} f_s(s)\sin\frac{s\pi x}{\lambda} \text{ for the interval } (0, \lambda).$$

If (0, π) is the interval for $f_s(s)$,

$$f(x) = \frac{2}{\pi}\sum_{s=1}^{\infty} f_s(s)\ \sin sx$$

**Finite FCT :-** Let f(x) denote a function that is sectionally continuous over some finite interval (0, λ) of the variable x. The finite Fourier cosine transform of f(x) on the interval is defined as

$$f_c(s) = \int_0^{\lambda} f(x)\cos\frac{s\pi x}{\lambda}\,dx \text{ where s is an integer,}$$

If (0, π) is the interval,

$$f_c(s) = \int_0^{\pi} f(x)\cos sx\,dx$$

**Inversion formula for FCT**

$$f(x) = \frac{1}{\lambda}f_c(0) + \frac{2}{\lambda}\sum_{s=1}^{\infty} f_c(s)\cos\frac{s\pi x}{\lambda}$$

where $f_c(0) = \int_0^{\lambda} f(x)\,dx$

If f(x) = 1, (0, π), then

$$f_s(s) = \int_0^{\pi} \sin sx\,dx = \left(\frac{-\cos sx}{s}\right)_0^{\pi}$$

$$= \frac{1}{s}[-\cos s\pi + \cos 0] = \frac{1}{s}[1-(-1)^2]$$

and $f_c(s) = \int_0^\pi \cos sx \, dx = \left(\frac{\sin sx}{s}\right)_0^\pi = 0$ if s = 1, 2, 3,….

If s = 0, then

$$f_c(s) = \int_0^\pi 1.\, dx = \pi$$

Find Finite FST and finite FCT of $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$, u(x, t) for $0 < x < \lambda$, $t > 0$.

**Solution :-** By definition, finite FST of $\frac{\partial u}{\partial x}$ is

$$f_s(s) = \int_0^\lambda \frac{\partial u}{\partial x} \sin \frac{s\pi x}{\lambda} \, dx$$

$$= \sin \frac{s\pi x}{\lambda} u(x,t)\Big|_0^\lambda - \frac{s\pi}{\lambda}\int_0^\lambda u(x,t)\cos\frac{s\pi x}{\lambda} dx$$

$$\Rightarrow \quad f_s(s) = F_s\left[\frac{\partial u}{\partial x}\right] = \frac{-s\pi}{\lambda}\int_0^\lambda u(x, t) \cos \frac{s\pi x}{\lambda} dx$$

$$\Rightarrow \quad f_s(s) = F_s\left[\frac{\partial u}{\partial x}\right] = \frac{-s\pi}{\lambda} F_c[u(x,t)] \qquad \ldots(1)$$

$$F_c\left[\frac{\partial u}{\partial x}\right] = \int_0^\lambda \frac{\partial u}{\partial x}\cos\frac{s\pi x}{\lambda} dx = \cos\frac{s\pi x}{\lambda} u(x,t)\Big|_0^\lambda + \int_0^\lambda \frac{s\pi}{\lambda} u(x,t)\sin\frac{s\pi x}{\lambda} dx$$

$$= \frac{s\pi}{\lambda} F_s[u(x, t)] - [u(0, t) - u(\lambda, t) \cos s\pi] \qquad \ldots(2)$$

To calculate the finite FST & finite FCT of $\frac{\partial^2 u}{\partial x^2}$,

Replace u by $\frac{\partial u}{\partial x}$ in (1) and (2), we get

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{-s\pi}{\lambda} F_c\left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{-s\pi}{\lambda}\left[\frac{s\pi}{\lambda}F_s(u)\right] + \frac{s\pi}{\lambda}\{u(0,t) - u(\lambda,t)\cos s\pi\}$$

$$= \frac{-s^2\pi^2}{\lambda^2}F_s[u] + \frac{s\pi}{\lambda}[u(0, t) - u(\lambda, t) \cos s\pi] \qquad \ldots(3)$$

Similarly $F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{+s\pi}{\lambda}F_s\left[\frac{\partial u}{\partial x}\right] - [u_x(0, t) - u_x(\lambda, t) \cos s\pi]$

$$= \frac{s\pi}{\lambda}\left(\frac{-s\pi}{\lambda}\right)F_c[u(x, t)] - \{u_x(0, t) - u_x(\lambda, t) \cos s\pi]$$

$$= \frac{-s^2\pi^2}{\lambda^2}F_c[u] - \{u_x(0, t) - u_x(\lambda, t) \cos s\pi\} \qquad \ldots(4)$$

**Example:-** Use finite Fourier transform to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

$u(0, t) = 0, u(4, t) = 0$

$u(x, 0) = 2x$ when $0 < x < 4, t > 0$.

**Solution :-** $(u)_{x = 0}$ is given, we apply FST.

Here $\lambda = 4$.

Taking Finite FST of (1), we get

$$\int_0^4 \frac{\partial u}{\partial t}\sin\left(\frac{s\pi x}{4}\right)dx = \int_0^4 \frac{\partial^2 u}{dx^2}\sin\left(\frac{s\pi x}{4}\right)dx$$

Writing $\bar{u}_s = F_s[u(x, t)]$, we get

$$\frac{d}{dt}\bar{u}_s = \frac{-s^2\pi^2}{16}\bar{u}_s + \frac{s\pi}{4}(0, -0)$$

[Using equation (3) in previous article and given condition]

$$\Rightarrow \qquad \frac{d}{dt}\bar{u}_s + \frac{s^2\pi^2}{16}\bar{u}_s = 0$$

$$\Rightarrow \qquad \bar{u}_s(s,t) = E\, e^{\frac{-s^2\pi^2 t}{16}} \qquad \ldots(*)$$

To find A, we will take finite FST of u(x, 0) = 2x, we get

$$\bar{u}_s(s,0) = \int_0^4 2x \sin\left(\frac{s\pi x}{4}\right) dx$$

$$= 2\left[x\left(-\cos\frac{s\pi x}{4}\right).\frac{4}{s\pi}\right]_0^4 + 2.\frac{4}{s\pi}\int_0^4 \cos\frac{\pi s x}{4} dx$$

$$= 2(-1)^{s+1}\ \frac{16}{s\pi} + \frac{8}{s\pi}\left[\sin\left(\frac{s\pi x}{4}\right).\frac{4}{s\pi}\right]_0^4$$

$$= 2(-1)^{s+1}.\ \frac{16}{s\pi} + \frac{8}{s\pi}(0)$$

$$= (-1)^{s+1}\ \frac{32}{s\pi} = -\frac{32}{s\pi}\cos s\pi$$

from $(*)$, $\bar{u}_s(s,\,0) = A$

$$\therefore \qquad A = \frac{-32}{s\pi}\cos s\pi$$

Hence $\bar{u}_s(s,t) = -\frac{32}{s\pi}\cos s\pi e^{\frac{-s^2\pi^2}{16}t}$

Taking Inverse Finite Fourier sine transform, we get

$$u(x,\,t) = \frac{2}{4}\sum_{s=1}^{\infty}\left(\frac{-32}{s\pi}\cos s\pi\right)e^{\frac{-s^2\pi^2}{16}t}\ \sin\left(\frac{s\pi x}{4}\right)$$

$$\therefore \qquad u(x,\,t) = \frac{-16}{\pi}\sum_{s=1}^{\infty}\frac{\cos s\pi}{s}\ \sin\left(\frac{s\pi x}{4}\right)e^{\frac{-s^2\pi^2 t}{16}}$$

**Example :-** Solve the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

subject to conditions

(i) $u(0,\,t) = 0$

(ii) $u(x,\,0) = e^{-x}$

(iii) $u(x,\,t)$ is bounded when $x > 0,\ t > 0$

**Solution :-** Since $(u)_{x=0}$ is given, so taking FST, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\ dx = \sqrt{\frac{2}{\pi}}\int_0^\infty 2\frac{\partial^2 u}{\partial x^2}\sin sx\ dx$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty u\sin sx\ dx = \sqrt{\frac{2}{\pi}}\int_0^\infty 2\frac{\partial^2 u}{\partial x^2}\sin sx\ dx$$

$$\Rightarrow \quad \frac{d}{dt}\sqrt{\frac{2}{\pi}}\int_0^\infty u\sin sx\ dx = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\ dx$$

$$\Rightarrow \quad \frac{d}{dt}\overline{u}_s = 2\sqrt{\frac{2}{\pi}}\left[\left(\sin sx\frac{\partial u}{\partial x}\right)_0^\infty - s\int_0^\infty \cos sx\frac{\partial u}{\partial x}dx\right]$$

$$= 2\sqrt{\frac{2}{\pi}}(0) - 2\sqrt{\frac{2}{\pi}}s\int_0^\infty \cos sx\ \frac{\partial u}{\partial x}dx$$

$$\left[\text{assuming } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$\Rightarrow \quad \frac{d}{dt}\overline{u}_s = 2\sqrt{\frac{2}{\pi}}\left[-s(u(x,t)\cos sx)_0^\infty - s^2\int_0^\infty \sin sx\, u(x,t)dx\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[-s^2\int_0^\infty u(x,t)\sin sx dx\right] \qquad [\text{assuming } u(x, t) \to 0 \text{ as } x\to\infty]$$

$$= -2s^2\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin sx\ dx$$

$$\Rightarrow \quad \frac{d}{dt}\overline{u}_s = -2s^2\,\overline{u}_s$$

$$\Rightarrow \quad \frac{d}{dt}\overline{u}_s + 2s^2\,\overline{u}_s = 0 \qquad \ldots(2)$$

A.E. is $D + 2s^2 = 0$

Solution is $\overline{u}_s(s,t) = Ae^{-2s^2t}$ …(3)

To find A, taking FST of condition

(ii) $u(x, 0) = e^{-x}$

Taking FST, we get

$$\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx \; dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2} \text{ at } t = 0 \qquad \ldots(4)$$

$$\left[\because \text{using} \int e^{ax}\sin bx \; dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) \text{ taking } a = -1, b = s\right.$$

and then taking lim, we get $\left.\sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}\right]$

Putting t = 0 in (3), we get

$$\bar{u}_s(s,0) = A \qquad \ldots(5)$$

from (4) and (5), we get

$$A = \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)$$

Putting value of A in (3), we get

$$\bar{u}_s(s,t) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)e^{-2s^2t}$$

Taking inverse FST, we get

$$u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)e^{-2s^2t} \; \sin sx \; ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}e^{-2s^2t} \sin sx \; ds \text{ which is required solution.}$$

**Example:** Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

subject to conditions

(i) $u_x(0, t) = 0$

(ii) $u(x, 0) = \begin{cases} x, & 0 < x \le 1 \\ 0, & x > 1 \end{cases}$

(iii) u(x, t) is bounded

**Solution :-** Since $(u_x)_{x=0}$ is given, so taking FCT of (1), we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\, dx$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty u\cos sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c = \sqrt{\frac{2}{\pi}}\left[\left(\frac{\partial u}{\partial x}\cos sx\right)_0^\infty + s\int_0^\infty \sin sx\frac{\partial u}{\partial x}dx\right]$$

[$\because$ $u \to 0$ as $x \to \infty$ $\Rightarrow u_x = 0$ as $x \to \infty$ and also $u_x(0, t) = 0$]

$$\frac{d}{dt}\bar{u}_c\sqrt{\frac{2}{\pi}}[(0-0) + s[\sin sx u(x,t)]_0^\infty - s^2\int_0^\infty u(x,t)\cos sx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c = \sqrt{\frac{2}{\pi}}(-s^2)\int_0^\infty \cos sx\, u(x,t)\, dx$$

$$= (-s^2)\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c = -s^2\bar{u}_c$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_c + s^2\bar{u}_c = 0 \qquad \ldots(2)$$

A.E. is $D + s^2 = 0$

$\therefore$ solution is $\bar{u}_c(s,t) = Ae^{-s^2 t}$ …(3)

Now to find A, taking FCT of condition (ii)

$$\bar{u}_c(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 x\cos sx\, dx \qquad [\because u(x, 0) = 0, x > 1]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[x\frac{\sin sx}{s}\right]_0^1 - \int_0^1 \frac{\sin sx}{s}dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{\sin s}{s} + \left[\frac{\cos sx}{s^2}\right]_0^1\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right] \text{ at } t = 0 \qquad \ldots(4)$$

Putting t = 0 in (3), we get

$$\bar{u}_c(s, 0) = A \quad \Rightarrow A = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right]$$

from (3) and (4), we get

$$\bar{u}_c(s,t) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right]e^{-s^2 t}$$

Taking inverse FCT, we get

$$u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_c(s,t) \cos sx \, ds$$

$$\Rightarrow \qquad u(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t} \cos sx \, ds$$

which is required solution.

**Example:-** Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; y > 0 \qquad \ldots(1)$$

Subject to condition

(i) $u(0, t) = 0$

(ii) $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ when t = 0

(iii) u(x, t) is bounded.

**Solution :-** Since $(u)_{x=0}$ is given, so taking FST of (1), we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\Rightarrow \qquad \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty u \sin sx \, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\Rightarrow \quad \frac{d}{dt}\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin sx\, dx = \sqrt{\frac{2}{\pi}}\left[\sin sx\frac{\partial u}{\partial x}\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty s \cos sx\frac{\partial u}{\partial x}dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_s = (0) - \sqrt{\frac{2}{\pi}}s\int_0^\infty \frac{\partial u}{\partial x} \cos sx\, dx \quad \left[\text{assuming } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$= -s\sqrt{\frac{2}{\pi}}[\cos sx.u(x,t)]_0^\infty + s\sqrt{\frac{2}{\pi}}\int_0^\infty (-s \sin sx)u(x,t)dx$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_s = s\sqrt{\frac{2}{\pi}}u(0,t) - s^2\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin sx\, dx$$

$$= 0 - s^2\, \bar{u}_s \qquad [\because u(0, t) = 0 \text{ given}]$$

$$\Rightarrow \quad \frac{d}{dt}\bar{u}_s + s^2\,\bar{u}_s = 0 \qquad \ldots(2)$$

A.E. is $D + s^2 = 0$

$\therefore$ Solution is $\bar{u}_s(s, t) = Ae^{-s^2 t}$ …(3)

To find A, we take FST of condition (ii),

$$\bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\ \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 (1) \sin sx\, dx \qquad [\because u(x, 0) = 0 \text{ for } x \geq 1]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos sx}{s}\right]_0^1$$

$$\Rightarrow \quad \bar{u}_s(s,0) = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s} + \frac{1}{s}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \qquad \ldots(4)$$

From (3), we have $\bar{u}_s(s,0) = A$ …(5)

$$A = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)$$

$$\therefore \quad (3) \Rightarrow \bar{u}_s(s, t) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)e^{-s^2 t}$$

Taking inverse FST, we get

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left( \frac{1 - \cos s}{s} \right) e^{-s^2 t} \sin sx \, ds$$

$$\Rightarrow \qquad u(x, t) = \frac{2}{\pi} \int_0^\infty \left( \frac{1 - \cos s}{s} \right) \sin sx \; e^{-s^2 t} ds$$

which is required solution.

# LESSON 3 CURVILINEAR CO-ORDINATES

**Transformation of coordinates**

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of ($u_1$, $u_2$, $u_3$) so that

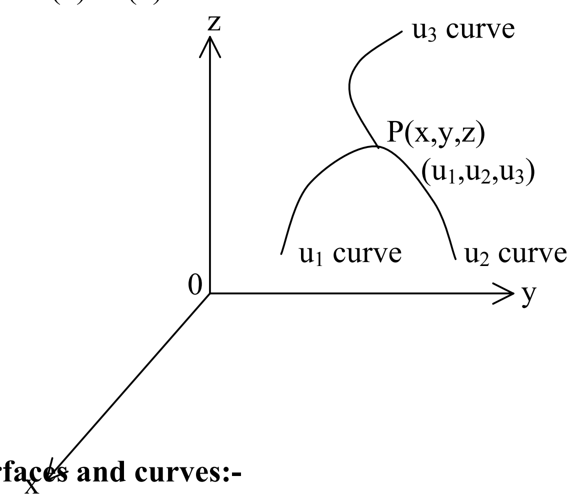
$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3 \end{aligned}\right\} \quad \ldots(1)$$

Suppose (1) can be solved for $u_1$, $u_2$, $u_3$ in terms of x, y, z i.e.

$$\left.\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned}\right\} \quad \ldots(2)$$

Here correspondence between (x, y, z) and ($u_1$, $u_2$, $u_3$) is **unique** i.e. if to each point P(x, y, z) of some region R, there corresponds one & only one triad ($u_1$, $u_2$, $u_3$), then $u_1$, $u_2$, $u_3$) are said to be **curvilinear coordinates** of the point P.

The set of equations (1) & (2) define a transformation of co-ordinates.




**Co-ordinate surfaces and curves:-**

The surfaces

$u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$

where $c_1$, $c_2$, $c_3$ are constants i.e. surfaces whose equations are

$u_1 = u_1(x, y, z) = c_1$

$u_2 = u_2(x, y, z) = c_2$

$u_3 = u_3(x, y, z) = c_3$

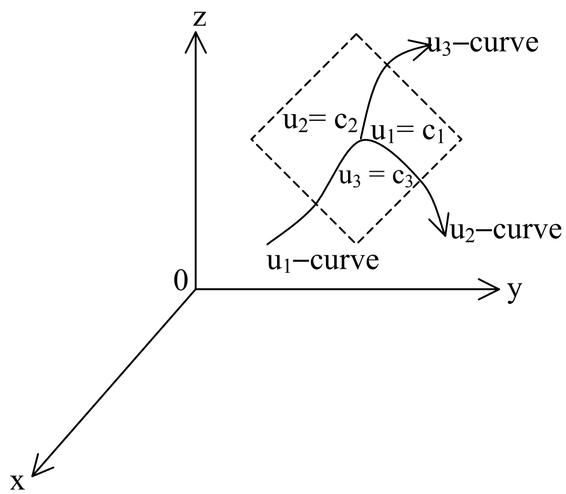
are called **co-ordinate surfaces** and each pair of these surfaces intersect in curves called **co-ordinate** curves or lines.

So $u_1 = c_1$ and $u_2 = c_2$ gives $u_3$–curve

Similarly $u_2 = c_2$ and $u_3 = c_3$ gives $u_1$ – curve

and $u_1 = c_1$ and $u_3 = c_3$ gives $u_2$– curve

So if $u_2 = c_2$ and $u_3 = c_3$ and if $u_1$ is the only variables then point P describe a curve known as $u_1$ curve which is a function of $u_1$.




**Orthgonal Curvilinear Co-ordinates**

If the three co-ordinate surfaces intersect at right angles (orthogonal), the curvilinear co-ordinate system is called orthogonal i.e. co-ordinates ($u_1$, $u_2$, $u_3$) are said to be orthogonal curvilinear co-ordinates.

Then $u_1$, $u_2$, $u_3$ co-ordinate curves of a curvilinear system are analogous to the x, y, z – coordinates axes of rectangular system.

**Unit vectors in curvilinear system**

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

be position vector of P, then (1) can be written as

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

A tangent vector to the $u_1$-curve at P(for which $u_2$ & $u_3$ are constant) is $\dfrac{\partial \vec{r}}{\partial u_1}$

Then, A unit tangent vector in this direction is

$$\hat{e}_1 = \frac{\partial \vec{r} / \partial u_1}{\left| \frac{\partial \vec{r}}{\partial u_1} \right|}$$

or $$\frac{\partial \vec{r}}{\partial u_1} = \hat{e}_1 \left| \frac{\partial \vec{r}}{\partial u_1} \right|$$

$$\Rightarrow \quad \frac{\partial \vec{r}}{\partial u_1} = h_1 \hat{e}_1 \text{ where } h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right|$$

Similarly if $\hat{e}_2$ and $\hat{e}_3$ are unit tangent vectors to the $u_2$ – & $u_3$– curves at P respectively, then

$$\frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{e}_2, \quad \frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{e}_3$$

where $h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right|, \ h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$

The quantities $h_1$, $h_2$, $h_3$ are called **scale factors.**

Also, condition for the orthogonality of co-ordinate surfaces are

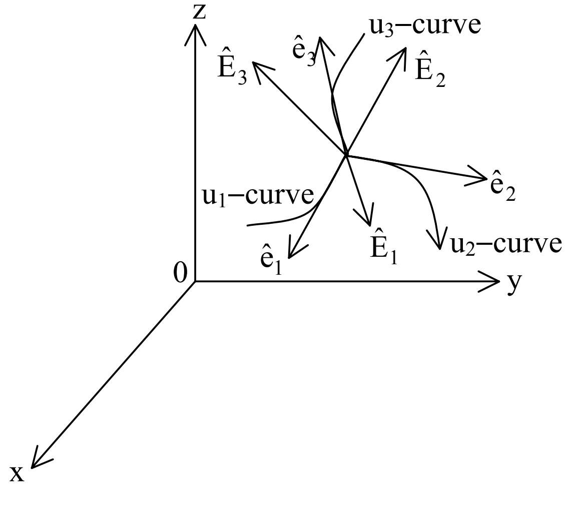
$$\frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} = 0, \qquad \frac{\partial \vec{r}}{\partial u_2} \cdot \frac{\partial \vec{r}}{\partial u_3} = 0, \quad \frac{\partial \vec{r}}{\partial u_3} \cdot \frac{\partial \vec{r}}{\partial u_1} = 0$$

Also the vectors $\nabla u_1$, $\nabla u_2$, $\nabla u_3$ at P are directed along the normal to the co-ordinate surfaces

$u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ respectively.

So unit-vectors in these directions are given by

$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|}, \quad \hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|}, \quad \hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$



Thus, at each point P of a curvilinear system, there are, in general, two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the coordinate curves and $\hat{E}_1. \hat{E}_2, \hat{E}_3$ normal to the co-ordinate surfaces. The sets become identical if & only if the curvilinear coordinate system is orthogonal.

**Arc length and Volume element**

From $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$$\Rightarrow \quad d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \qquad \ldots(1)$$

$$= h_1\, du_1\, \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

Then, the differential of arc length ds is determined from

$$(ds)^2 = d\vec{r}.d\vec{r}$$

For orthogonal system,

$$\hat{e}_1.\hat{e}_2 = \hat{e}_2.\hat{e}_3 = \hat{e}_3.\hat{e}_1 = 0$$

or $$\frac{\partial \vec{r}}{\partial u_1}.\frac{\partial \vec{r}}{\partial u_2} = \frac{\partial \vec{r}}{\partial u_2}.\frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial u_3}.\frac{\partial \vec{r}}{\partial u_1} = 0 \qquad \ldots(2)$$

using (∗)

and $$(ds)^2 = \left(\frac{\partial \vec{r}}{\partial u_1}.\frac{\partial \vec{r}}{\partial u_1}\right)(du_1)^2 + \left(\frac{\partial \vec{r}}{\partial u_2}.\frac{\partial \vec{r}}{\partial u_2}\right)(du_2)^2 + \left(\frac{\partial \vec{r}}{\partial u_3}.\frac{\partial \vec{r}}{\partial u_3}\right)(du_3)^2 + 2\frac{\partial \vec{r}}{\partial u_1}.\frac{\partial \vec{r}}{\partial u_2} du_1\, du_2 + 2\frac{\partial \vec{r}}{\partial u_2}.\frac{\partial \vec{r}}{\partial u_3} du_2\, du_3 + 2\frac{\partial \vec{r}}{\partial u_3}.\frac{\partial \vec{r}}{\partial u_1} du_1\, du_3$$

$$\Rightarrow \quad (ds)^2 = \left(\frac{\partial \vec{r}}{\partial u_1}.\frac{\partial \vec{r}}{\partial u_1}\right)(du_1)^2 + \left(\frac{\partial \vec{r}}{\partial u_2}.\frac{\partial \vec{r}}{\partial u_2}\right)(du_2)^2 + \left(\frac{\partial \vec{r}}{\partial u_3}.\frac{\partial \vec{r}}{\partial u_3}\right)(du_3)^2$$

$$= \left(\frac{\partial \vec{r}}{\partial u_1}\right)^2 (du_1)^2 + \left(\frac{\partial \vec{r}}{\partial u_2}\right)^2 (du_2)^2 + \left(\frac{\partial \vec{r}}{\partial u_3}\right)^2 (du_3)^2 \quad \text{[using (2)]}$$

$\Rightarrow \quad (ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$

where $h_1 = \left|\dfrac{\partial \vec{r}}{\partial u_1}\right| = \dfrac{1}{|\nabla u_1|}$

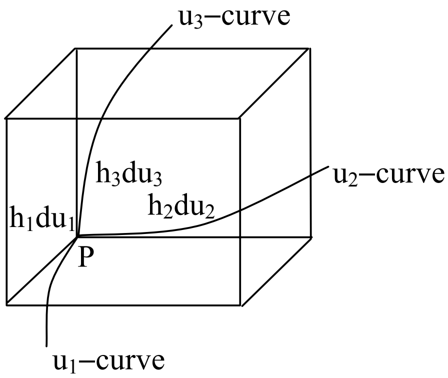
$h_2 = \left|\dfrac{\partial \vec{r}}{\partial u_2}\right| = \dfrac{1}{|\nabla u_2|}$

$h_3 = \left|\dfrac{\partial \vec{r}}{\partial u_3}\right| = \dfrac{1}{|\nabla u_3|}$

Now if $\vec{r} = \vec{r}\,(x, y, z)$

Then $\dfrac{\partial u_1}{\partial \vec{r}} = \dfrac{\partial u_1}{\partial x}\hat{i} + \dfrac{\partial u_1}{\partial y}\hat{j} + \dfrac{\partial u_1}{\partial z}\hat{k} = \nabla u_1$

$\Rightarrow \quad \dfrac{\partial \vec{r}}{\partial u_1} = \dfrac{1}{\nabla u_1}$




**Length along $u_1$-curve**

For this $u_2$ & $u_3$ are constant.

$u_2 = c,\ u_3 = c$

$\Rightarrow \quad du_2 = 0,\ du_3 = 0$

If $ds_1 \rightarrow$ differential of length along $u_1$-curve $(ds_1)^2 = h_1^2(du_1)^2$

$\Rightarrow \quad ds_1 = h_1\, du_1$

Similarly length along $u_2$-curve is

$ds_2 = h_2\, du_2$

and length along $u_3$-curve is

$ds_3 = h_3\, du_3$

**The volume element**

For an orthogonal curvilinear coordinate system is given by

$dV = |(h_1\, du_1\, \hat{e}_1).(h_2 du_2 \hat{e}_2)(h_3 du_3 \hat{e}_3)|$ $[\Theta \text{ volume} = |\hat{a}.(\hat{b}\times\hat{c})|$

$\Rightarrow$ $dv = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

**Differential operators in terms of orthogonal curvilinear coordinates ($u_1$, $u_2$, $u_3$)**

**GRADIENT**

Let $\phi \to$ scalar point function

and $\vec{f} = f_1\hat{a} + f_2\hat{b} + f_3\hat{c}$

$= f_1\, \hat{e}_1 + f_2\, \hat{e}_2 + f_3\, \hat{e}_3$

i.e. $\phi = \phi(u_1, u_2, u_3)$

$= \phi[u_1(x, y, z), u_2(x, y, z), u_3(x, y, z)]$

$$\Rightarrow \quad \frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial\varphi}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial\varphi}{\partial u_3}\frac{\partial u_3}{\partial x} \qquad \ldots(1)$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial u_1}\frac{\partial u_1}{\partial y} + \frac{\partial\varphi}{\partial u_2}\frac{\partial u_2}{\partial y} + \frac{\partial\varphi}{\partial u_3}\frac{\partial u_3}{\partial y} \qquad \ldots(2)$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial u_1}\frac{\partial u_1}{\partial z} + \frac{\partial\varphi}{\partial u_2}\frac{\partial u_2}{\partial z} + \frac{\partial\varphi}{\partial u_3}\frac{\partial u_3}{\partial z} \qquad \ldots(3)$$

Operating $\hat{i} \times (1) + \hat{j} \times (2) + \hat{k} \times (3)$, we get

$$\nabla\phi = \text{grad}\ \phi = \frac{\partial\varphi}{\partial u_1}\nabla u_1 + \frac{\partial\varphi}{\partial u_2}\nabla u_2 + \frac{\partial\varphi}{\partial u_3}\partial u_3$$

But $\hat{e}_1 = h_1\nabla u_1$, $\hat{e}_2 = h_2\nabla u_2$, $\hat{e}_3 = h_3\nabla u_3$

$$\Rightarrow \quad \text{grad}\ \phi = \nabla\phi = \frac{\hat{e}_1}{h_1}\frac{\partial\varphi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\varphi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\varphi}{\partial u_3}$$

**Example:-** If ($u_1$, $u_2$, $u_3$) are orthogonal coordinate then prove that

(i) $|\nabla u_p| = hp^{-1}$, $p = 1, 2, 3$

(ii) $\hat{e}_p = \hat{E}_p$

**Proof :-** (i) let $\phi = u_1$, then

$$\nabla u_1 = \frac{1}{h_1}\frac{\partial u_1}{\partial u_1}\hat{e}_1 = \frac{\hat{e}_1}{h_1}$$

$$|\nabla u_1| = \frac{|\hat{e}_1|}{|h_1|} = \frac{1}{h_1} = h_1^{-1}$$

Similarly $\phi = u_2$, then

$$\nabla u_2 = \frac{1}{h_2}\frac{\partial u_2}{\partial u_2}\hat{e}_2 = \frac{\hat{e}_2}{h_2}$$

$$\Rightarrow \quad |\nabla u_2| = \frac{|\hat{e}_2|}{|h_2|} = \frac{1}{h_2} = h_2^{-1}$$

Similarly if $\phi = u_3$, then $|\nabla u_3| = h_3^{-1}$

(ii) By definition, $\hat{E}_p = \dfrac{\nabla u_p}{|\nabla u_p|}$

$$\Rightarrow \quad \hat{E}_p = h_p \nabla u_p = \hat{e}_p$$

**Results :-**

I. div $(\phi \hat{f}) = \nabla.(\phi \hat{f}) = \phi$ div f + $\hat{f}$ . Grad $\phi$

$= \phi\, \nabla.\, \hat{f} + \hat{f} \,.\, \nabla\phi$

II. div $(\hat{f} \times \vec{g}) = \nabla.\,(\hat{f} \times \vec{g})$ = curl $\hat{f}$ . $\vec{g}$ – curl $\vec{g}.\hat{f}$

III. Curl grad $\phi = \nabla \times \nabla\, \phi = 0$

div curl $\hat{f} = \nabla.\nabla \times \hat{f} = 0$

IV. Curl $(\phi \hat{f}) = \nabla \times (\phi\, \hat{f})$ = grad $\phi \times \hat{f} + \phi$ Curl $\hat{f}$

**DIVERGENCE**

Consider a vector function

$$\vec{f}(u_1, u_2, u_3) = f_1\, \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$$

where $(u_1, u_2, u_3)$ are orthogonal curvilinear coordinates.

$$\vec{f} = f_1(\hat{e}_2 \times \hat{e}_3) + f_2(\hat{e}_3 \times \hat{e}_1) + f_3(\hat{e}_1 \times \hat{e}_2)$$

using $\hat{e}_1 = h_1\, \nabla u_1$

$\hat{e}_2 = h_2\, \nabla u_2$

$\hat{e}_3 = h_3\, \nabla u_3$

$$\vec{f} = f_1\, h_2\, h_3(\nabla u_2 \times \nabla u_3) + f_2\, h_3\, h_1\, (\nabla u_3 \times \nabla u_1) + f_3\, h_1\, h_2(\nabla u_1 \times \nabla u_2)$$

$\therefore$ dif $f = \nabla . \vec{f} = \nabla.[f_1 h_2 h_3(\nabla u_2 \times \nabla u_3)]$

$+ \nabla . [f_2 h_3 h_1 (\nabla u_3 \times \nabla u_1)]$

$+ \nabla. [f_3 h_1 h_2 (\nabla u_1 \times \nabla u_2)]$ …(*)

Taking first art,

$\nabla . [f_1 h_2 h_3(\nabla u_2 \times \nabla u_3)] = f_1 h_2 h_3 \nabla . (\nabla u_2 \times \nabla u_3) + (\nabla u_2 \times \nabla u_3). \nabla(f_1h_2h_3)$

[By using $\nabla. (\phi \vec{f}) = \phi \nabla. \vec{f} + \vec{f} . \nabla\phi$]

$\Rightarrow$ $\nabla.[f_1 h_2 h_3(\nabla u_2 \times \nabla u_3)] = f_1 h_2 h_3 \nabla.(\nabla u_2 \times \nabla u_3) + (\nabla u_2 \times \nabla u_3).$

$\nabla(f_1 h_2h_3)$ …(1)

$\Rightarrow$ $\nabla.[(\nabla u_2 \times \nabla u_3)] = \text{curl } \nabla u_2 . \nabla u_3 - \text{curl } \nabla u_3 . \nabla u_2$

$= (\text{curl grad } u_2). (\nabla u_3) - (\text{curl grad } u_3). \nabla u_2$

$= 0 - 0 = 0$

$\therefore$ from (1), we get

$\nabla . [f_1 h_2 h_3 (\nabla u_2 \times \nabla u_3)] = (\nabla u_2 \times \nabla u_3). \nabla(f_1 h_2 h_3)$

$$= (\nabla u_2 \times \nabla u_3). \left[\frac{\partial}{\partial u_1}(f_1 h_2 h_3) \nabla u_1 + \frac{\partial}{\partial u_2}(f_1 h_2 h_3) \nabla u_2 + \frac{\partial}{\partial u_3}(f_1 h_2 h_3)\nabla u_3\right]$$

$$= \frac{\hat{e}_2 \times \hat{e}_3}{h_2 h_3}\left[\frac{\partial}{\partial u_1}(f_1h_2h_3)\frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial u_2}(f_1h_2h_3)\frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(f_1h_2h_3)\frac{\hat{e}_3}{h_3}\right]$$

$$\left[\because \nabla u_1 = \frac{\hat{e}_1}{h_1}, \nabla u_2 = \frac{\hat{e}_2}{h_2}, \nabla u_3 = \frac{\hat{e}_3}{h_3}\right]$$

$$= \frac{\hat{e}_1}{h_2 h_3}.\left[\frac{\partial}{\partial u_1}(f_1h_2h_3)\frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial u_2}(f_1h_2h_3)\frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(f_1h_2h_3)\frac{\hat{e}_3}{h_3}\right]$$

$$= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(f_1h_2h_3)$$

$\Rightarrow$ $\nabla.[f_1 h_2 h_3(\nabla u_2 \times \nabla u_3)] = \dfrac{1}{h_1h_2h_3}.\dfrac{\partial}{\partial u_1}(f_1 h_2 h_3)$

Similarly $\nabla.[f_2 h_3 h_1(\nabla u_3 \times \Delta u_1)] = \dfrac{1}{h_1h_2h_3}\dfrac{\partial}{\partial u_2}(f_2 h_3 h_1)$

and $\nabla.[f_3\, h_1\, h_2(\nabla u_1 \times \Delta u_2)] = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}(f_3\, h_1\, h_2)$

So from (∗), we get

$$\therefore\ \nabla\ .\ \vec{f} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(f_1 h_2 h_3) + \frac{\partial}{\partial u_2}(f_2 h_3 h_1) + \frac{\partial}{\partial u_3}(f_3 h_1 h_2)\right]$$

**CURL**

Consider $\vec{f} = f_1\,\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$

$\Rightarrow \quad \vec{f} = f_1 h_1\ \nabla u_1 + f_2\, h_2\ \nabla\ u_2 + f_3\, h_3\ \nabla u_3 \qquad [\Theta\ h_1\ \nabla\ u_1 = \hat{e}_1]$

then Curl $\vec{f} = \nabla \times \vec{f} = \nabla \times (h_1\ f_1\ \Delta u_1)$

$+ \nabla \times (h_2\ f_2\ \nabla u_2)$

$+ \times (h_3\ f_3\ \nabla u_3)$ …(1)

Taking the first art, we have by using property

Curl $(\phi\ \vec{F}) = \nabla\phi \times \vec{F} + \phi\ \nabla\times\vec{F}$

we get

$\nabla \times(h_1\ f_1\ \nabla u_1) = \nabla(h_1\ f_1) \times \nabla u_1 + h_1\ f_1\ (\nabla\times \nabla u_1)$

$= \nabla(h_1\ f_1) \times \Delta u_1 \quad [\Theta\ \ \Delta\times \Delta u_1 = \text{curl grad } u_1 = 0]$

$$\Rightarrow \quad \nabla\times(h_1 f_1\ \nabla u_1) = \left[\frac{\partial}{\partial u_1}(f_1 h_1)\nabla u_1 + \frac{\partial}{\partial u_2}(f_1 h_1)\nabla u_2 + \frac{\partial}{\partial u_3}(f_1 h_1)\nabla u_3 \times \nabla u_1\right]$$

[Θ By definition of gradient of a function]

$$\Rightarrow \quad \nabla\times[h_1 f_1\ \nabla u_1] = \left[\frac{\partial}{\partial u_1}(f_1 h_1)\frac{\hat{e}_1}{h_1} + (f_1 h_1)\frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(f_1 h_1)\frac{\hat{e}_3}{h_3} \times \frac{\hat{e}_1}{h_1}\right]$$

$$\Rightarrow \quad \nabla\times(h_1 f_1\ \nabla u_1) = \frac{\hat{e}_2}{h_1 h_3}\frac{\partial}{\partial u_3}(f_1 h_1) - \frac{\hat{e}_3}{h_2 h_1}.\frac{\partial}{\partial u_2}(f_1 h_1)$$

Similarly $\nabla\times(h_1 f_2\ \nabla u_2) = \frac{\hat{e}_3}{h_2 h_1}\frac{\partial}{\partial u_1}(f_2 h_2)\frac{-\hat{e}_1}{h_3 h_2}\frac{\partial}{\partial u_3}(f_2 h_2)$

and $\quad \nabla\times\ (h_3\ f_3\ \nabla u_3) = \frac{\hat{e}_1}{h_3 h_2}\frac{\partial}{\partial u_2}(f_3 h_3) - \frac{\hat{e}_2}{h_1 h_3}\frac{\partial}{\partial u_1}(f_3\, h_3)$

∴ from (1),

$$\text{Curl } \vec{f} = \frac{\hat{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(f_3 h_3) - \frac{\partial}{\partial u_3}(f_2 h_2)\right]$$

$$+ \frac{\hat{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(f_1 h_1) - \frac{\partial}{\partial u_1}(f_3 h_3)\right]$$

$$+ \frac{\hat{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(f_2 h_2) - \frac{\partial}{\partial u_2}(f_1 h_1)\right]$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

**LAPLACIAN OF SCALAR POINT FUNCTION ($\nabla^2 \phi$)**

Now

$$\nabla^2\phi = \nabla.(\nabla \phi)$$

$$\Rightarrow \quad \nabla^2\phi = \nabla.\left[\frac{\partial\varphi}{\partial u_1}\nabla u_1 \frac{\partial\varphi}{\partial u_2}\nabla u_2 + \frac{\partial\varphi}{\partial u_3}\nabla u_3\right]$$

$$= \nabla.\left[\frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\hat{e}_3\right]$$

We know that

$$\nabla.\vec{f} = \frac{1}{h_2 h_1 h_3}\left[\frac{\partial}{\partial u_1}(f_1 h_2 h_3) + \frac{\partial}{\partial u_2}(f_2 h_3 h_1) + \frac{\partial}{\partial u_3}(f_3 h_1 h_2)\right]$$

Here put $f_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}$,

$$f_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2},\ f_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$$

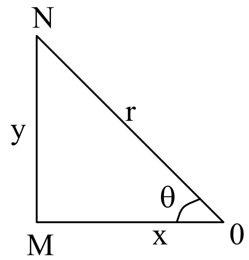
Therefore

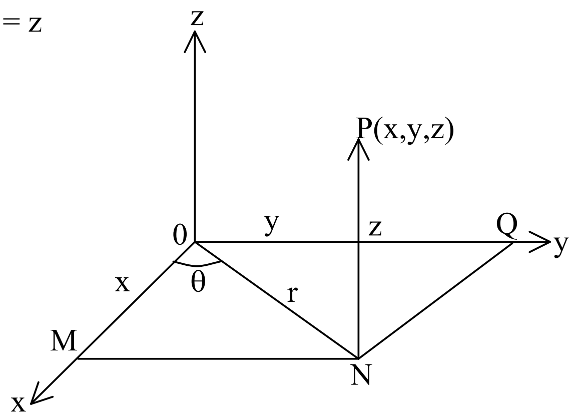
$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)\right]$$

**CYLINDRICAL POLAR COORDINATES (r, θ, z)**

Let P is a point having Cartesian co-ordinates (x, y, z)

$OM = x, \quad MN = y(= OQ), PN = z$




$x = OM = ON \cos\theta$

$\Rightarrow \quad x = r\cos\theta$ …(1)

& $\quad y = MN = N \sin\theta$

$\Rightarrow \quad y = r \text{ sion } \theta$ …(2)

and $\quad z = z$ …(3)

so we have,

$$r = \sqrt{x^2 + y^2}$$

and $\quad \theta = \tan^{-1} \dfrac{y}{x}$

Determine the transformation from cylindrical to rectangular coordinates :- Operating equation $(1)^2 + (2)^2$, we get

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$(2)/(1) \Rightarrow \dfrac{y}{x} = \tan\theta \qquad \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

and $\quad z = z.$

$(ds)^2 = (dx)^2 + (dy)^2 \ (dz)^2$ …(4)

Now $\quad dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$dz = dz$

$\therefore \quad (4) \Rightarrow$

$\therefore \quad (ds)^2 = (dr \cos\theta - r \sin\theta\, d\theta)^2 + (\sin\theta\, dr + r \cos\theta d\theta^2 + (dz)^2$

$\qquad = (dr)^2 (\sin^2\theta + \cos^2\theta) + (r\, d\theta)^2 (\sin^2\theta + \cos^2\theta) + (dz)^2$

$\Rightarrow \quad (ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2$

Comparing it with $(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$

We get,

$h_1 = 1,\ h_2 = r,\ h_3 = 1,\ u_1 = r,\ u_2 = \theta, \quad u_3 = z$

Take $\hat{e}_1 = \hat{e}_r,\ \hat{e}_2 = \hat{e}_\theta,\ \hat{e}_3 = \hat{e}_z$

Using these, we have

$$\text{grad}\quad \Phi = \frac{\partial \Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \Phi}{\partial z}\hat{e}_z$$

where $\Phi$ is a scalar point function

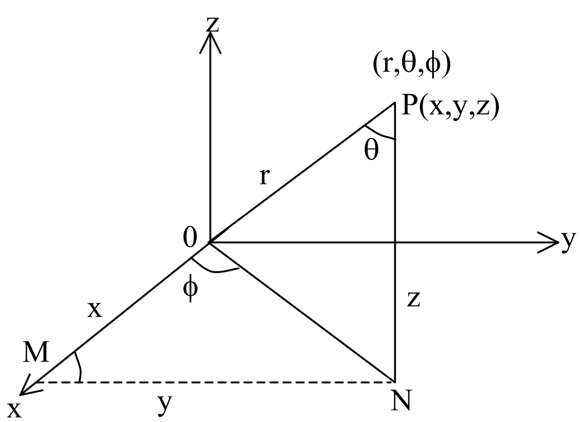
$$\text{div } \vec{f} = \frac{1}{r}\left[\frac{\partial}{\partial r}(f_1 r) + \frac{\partial}{\partial \theta}(f_2) + \frac{\partial}{\partial z}(f_3 r)\right]$$

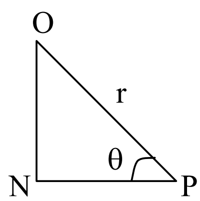
$$\text{Curl } \vec{f} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f_1 & f_2 r & f_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial \Phi}{\partial z}\right)\right]$$

$$= \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

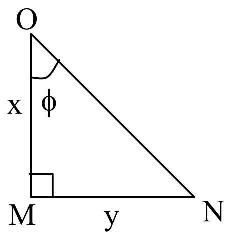
**SPHERICAL POLAR COORDINATES (r, θ, ϕ)**



OM = x, MN = y, PN = z, OP = r z = PN = r cos θ ON = r sinθ

In Δ OMN, x = OM = ON cos ϕ




⇒ x = r sinθ cosϕ

y = MN = ON sinϕ ⇒ y = r sin θ sin ϕ

Now $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ …(1)

$dx = \sin\theta \cos\phi\, dr + r \cos\theta \cos\phi\, d\theta - r \sin\theta \sin\phi\, d\phi$

$dy = \sin\theta \sin\phi\, dr + r \cos\theta \sin\phi\, d\theta + r \sin\theta \cos\phi\, d\phi$

$dz = \cos\theta\, dr - r \sin\theta\, d\theta$ …(2)

Put the value of (2) in (1) and collecting the coefficients of $(dr)^2$, $(d\theta)^2$, $(d\phi)^2$, we get

$(ds)^2 = (dr)^2 (1) + (d\theta)^2 (r^2) + (d\phi)^2 (r^2 \sin^2\theta)$

Comparing it with,

$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$

we get

$h_1 = 1,\ h_2 = r,\ h_3 = r \sin\theta$

$u_1 = r,\ u_2 = \theta,\ u_3 = \phi$

$\hat{e}_1 = \hat{e}_r,\ \hat{e}_2 = \hat{e}_\theta,\ \hat{e}_3 = \hat{e}_\varphi$

So grad $\Phi = \hat{e}_r \frac{\partial\Phi}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}$

$$\text{div } \vec{f} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(f_1 r^2 \sin\theta) + \frac{\partial}{\partial\theta}(f_2 r \sin\theta + \frac{\partial}{\partial\varphi}(f_3 r)\right]$$

$$\text{Curl } \vec{f} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{(1)}\frac{\partial\Phi}{\partial r}\right)\right]$$

$$\text{Curl } \vec{f} = \frac{1}{r^2 \sin\theta}\, \vec{f} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ f_1 & f_2 r & f_3 r \sin\theta \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial \Phi}{\partial \varphi}\right)\right]$$

$$= \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

**Example:-** Prove that cylindrical coordinate system is orthogonal.

**Solution :-** The position vector of any point in cylindrical coordinate is

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\therefore \quad \vec{r} = r\cos\theta\ \hat{i} + r\ \sin\theta\,\hat{j} + z\,\hat{k}$$

$$\vec{r} = \vec{r}(s)\ \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s} \text{ are unit vector}$$

The tangent vector to the r, θ and z curves are given by

$$\frac{\partial \vec{r}}{\partial r}, \frac{\partial \vec{r}}{\partial \theta}, \frac{\partial \vec{r}}{\partial z} \text{ respectively.}$$

where $\frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{k}$$

Therefore the unit-vector in these tangent directions are

$$\hat{e}_1 = \hat{e}_r = \frac{\partial \vec{r}}{\partial r}\Big/\left|\frac{\partial r}{\partial r}\right|$$

$$\Rightarrow \quad \hat{e}_r = \frac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta\ \hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_2 = \hat{e}_\theta = \frac{\partial \vec{r}/\partial\theta}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}}$$

$$\Rightarrow \qquad \hat{e}_\theta = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{r} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_3 = \hat{e}_z = \frac{\partial\vec{r}/\partial z}{|\partial\vec{r}/\partial z|} = \hat{k}$$

Now $\hat{e}_r.\hat{e}_\theta = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$$\hat{e}_r.\hat{e}_z = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).\hat{k} = 0$$

$$\hat{e}_\theta.\hat{e}_z = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}).\hat{k} = 0$$

$$\hat{e}_r.\hat{e}_r = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$$\hat{e}_\theta.\hat{e}_\theta = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}).(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= \sin^2\theta + \cos^2\theta = 1$$

$$\hat{e}_z.\hat{e}_z = \hat{k}.\hat{k} = 1$$

This shows that $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ are mutually $\perp_r$ and therefore the coordinate system is orthogonal.

**Example:-** (a) Find the unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ of a spherical co-ordinate system in terms of $\hat{i}, \hat{j}, \hat{k}$.

**Solution :-** The position vector of any point in spherical coordinate is

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$= (r\sin\theta\cos\phi)\,\hat{i} + (r\sin\theta\sin\phi)\,\hat{j} + (r\cos\theta)\,\hat{k}$$

We want to find $\frac{\partial\vec{R}}{\partial r}, \frac{\partial\vec{R}}{\partial\theta}, \frac{\partial\vec{R}}{\partial\varphi}$

Now $\frac{\partial\vec{R}}{\partial r} = \sin\theta\cos\phi\ \hat{i}\ \sin\theta\ \sin\phi\ \hat{j} + \cos\theta\ \hat{k}$

So $\qquad \hat{e}_r = \frac{\partial\vec{R}/\partial r}{|\partial\vec{R}/\partial r|} = \frac{\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta}}$

$$\Rightarrow \qquad \hat{e}_r = \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\,\hat{k}$$

Also $\quad \frac{\partial\vec{R}}{\partial\theta} = r\cos\theta\cos\phi\ \hat{i} + r\cos\theta\sin\phi\ \hat{j} - r\sin\theta\ \hat{k}$

So $\hat{e}_\theta = \dfrac{\partial\vec{R}/\partial\theta}{|\partial\vec{R}/\partial\theta|} = \dfrac{r\cos\theta\cos\varphi\,\hat{i} + r\cos\theta\sin\varphi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + r^2\sin^2\theta}}$

$$= \frac{r\cos\theta\cos\varphi\,\hat{i} + r\cos\theta\sin\varphi\,\hat{j} - r\sin\theta\,\hat{k}}{r}$$

$\Rightarrow \quad \hat{e}_\theta = \cos\theta\cos\phi\;\hat{i} + \cos\theta\sin\phi\;\hat{j} - \sin\theta\;\hat{k}$

Also $\dfrac{\partial\vec{R}}{\partial\varphi} = -r\sin\theta\sin\phi\;\hat{i} + r\sin\theta\cos\phi\;\hat{j} + \theta.\hat{k}$

So $\hat{e}_\varphi = \dfrac{\partial\vec{R}/\partial\varphi}{|\partial\vec{R}/\partial\varphi|} = \dfrac{-r\sin\theta\sin\varphi\;\hat{i} + r\sin\theta\cos\varphi\,\hat{j} + \theta.\hat{k}}{\sqrt{r^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi)}}$

$$= \frac{-r\sin\theta\sin\varphi\;\hat{i} + r\sin\theta\cos\varphi\,\hat{j}}{r\sin\theta}$$

$\Rightarrow \quad \hat{e}_\varphi = -\sin\phi\;\hat{i} + \cos\phi\;\hat{j}$

**Example:-** Prove that a spherical coordinate system is orthogonal.

**Solution :-** $\hat{e}_r.\hat{e}_\theta = (\sin\theta\cos\phi\;\hat{i} + \sin\theta\sin\phi\;\hat{j} + \cos\theta\;\hat{k}).$

$(\cos\theta\cos\phi\;\hat{i} + \cos\theta\sin\phi\;\hat{j} - \sin\theta\;\hat{k})$

$= \cos^2\phi\sin\theta\cos\theta + \sin^2\phi\sin\theta\cos\theta - \sin\theta\cos\theta$

$= \sin\theta\cos\theta(\sin^2\phi + \cos^2\phi) - \sin\theta\cos\theta$

$\Rightarrow \quad \hat{e}_r.\hat{e}_\theta = 0$

$\hat{e}_\theta.\hat{e}_\varphi = (\cos\theta\cos\phi\;\hat{i} + \cos\theta\sin\phi\;\hat{j} - \sin\theta\;\hat{k}).(-\sin\phi\;\hat{i} + \cos\phi\,\hat{j})$

$= -\sin\phi\cos\phi\cos\theta + \cos\theta\cos\phi\sin\phi - 0$

$\Rightarrow \quad \hat{e}_\theta.\hat{e}_\varphi = 0$

Also $\hat{e}_\varphi.\hat{e}_r = (-\sin\phi\;\hat{i} + \cos\phi\;\hat{j})(\sin\theta\cos\phi\;\hat{i} + \sin\theta\sin\phi\;\hat{j} + \cos\theta\;\hat{k})$

$= -\sin\theta\sin\phi\cos\phi + \sin\theta\sin\phi\cos\phi + 0$

$\Rightarrow \quad \hat{e}_\varphi.\hat{e}_r = 0$

and $\hat{e}_r.\hat{e}_r = (\sin\theta\cos\phi\;\hat{i} + \sin\theta\sin\phi\;\hat{j} + \cos\theta\;\hat{k})(\sin\theta\cos\phi\;\hat{i}$

$+ \sin\theta\sin\phi\;\hat{j} + \cos\theta\;\hat{k})$

$= \cos^2\phi\sin^2\theta + \sin^2\phi + \cos^2\theta$

$= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta$

$\Rightarrow \quad \hat{e}_r.\hat{e}_r = 1$

and $\hat{e}_\theta.\hat{e}_\theta = (\cos\theta \cos\phi\ \hat{i} + \cos\theta \sin\phi\ \hat{j} - \sin\theta\ \hat{k})$

$(\cos\theta \cos\phi\ \hat{i} + \cos\theta \sin\phi\ \hat{j} - \sin\theta\,\hat{k})$

$= \cos^2\phi \cos^2\theta + \cos^2\theta \sin^2\phi + \sin^2\theta$

$= \cos^2\theta\ (\sin^2\phi + \cos^2\phi) + \sin^2\theta$

$\Rightarrow \quad \hat{e}_\theta.\hat{e}_\theta = 1$

and $\quad \hat{e}_\varphi.\hat{e}_\varphi = (-\sin\phi\ \hat{i} + \cos\phi\ \hat{j}) . (-\sin\phi\ \hat{i} + \cos\phi\ \hat{j})$

$= \sin^2\phi + \cos^2\phi = 1$

$\Rightarrow \quad \hat{e}_\varphi.\hat{e}_\varphi = 1$

This shows that $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ are mutually and therefore the coordinate system is orthogonal.

**Example:-** Represent the vector $\vec{A} = 2y\,\hat{i} - z\ \hat{j} + 3x\ \hat{k}$ …(1)

in spherical coordinates

**Solution :-** Here $x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$ …(2)

and $\quad \hat{e}_r = \sin\theta \cos\phi\ \hat{i} + \sin\theta \sin\phi\ \hat{j} + \cos\theta\ \hat{k}$

$\hat{e}_\theta = \cos\theta \cos\phi\ \hat{i} + \cos\theta \sin\phi\ \hat{j} - \sin\theta\ \hat{k}$

$\hat{e}_\varphi = -\sin\phi\ \hat{i} + \cos\phi\ \hat{j}$ …(3)

Solving (3), we get

$\hat{i} = \hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi$

$\hat{j} = \hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta \sin\phi \cos\theta + \hat{e}_\varphi \cos\phi$

$\hat{k} = \hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta$ …(4)

Put (2) & (4) in (1), we get

$\vec{A} = 2r \sin\theta \sin\phi(\hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi)$

$- r \cos\theta\ (\hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta \sin\phi \cos\theta + \hat{e}_\varphi \cos\theta)$

$+ 3\ r \sin\theta \cos\phi\ (\hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta)$

$\Rightarrow \quad \vec{A} = \hat{e}_r\,(2r \sin^2\theta \cos^2\phi - r \sin\theta \cos\theta \sin\phi + 3\ r \sin\theta \cos\phi \cos\theta)$

$+\ \hat{e}_\theta\,(2r \sin\theta \cos\theta \sin\phi \cos\phi - r \cos^2\theta \sin\phi - 3r \sin^2\theta \cos\phi)$

$+\ \hat{e}_\varphi\,(-2r \sin\theta \sin\phi - r \cos\theta \cos\phi)$

$\Rightarrow \quad \vec{A} = \hat{e}_r\ 2r \sin\theta\ (\sin\theta \cos^2\phi + \cos\theta \sin\phi)$

$+\ \hat{e}_\theta\, r(2\sin\theta \cos\theta \sin\phi \cos\phi - r \cos^2\theta \sin\phi - 3r \sin^2\theta \cos\phi)$

$-\ \hat{e}_\varphi\ r(2 \sin\theta \sin^2\phi + \cos\theta \cos\phi)$

$\Rightarrow \quad \vec{A} = A_r\, \hat{e}_r + A_\theta \hat{e}_\theta + A_\varphi\, \hat{e}_\varphi$

**Example :-** Prove that for cylindrical coordinate system (r, θ, z),

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_r$$

$$\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r$$

**Solution :-**We have $\hat{e}_r = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$ …(1)

$\hat{e}_\theta = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$ …(2)

$\therefore \quad \frac{d}{dt}\hat{e}_r = (-\sin\theta)\dot{\theta}\hat{i} + (\cos\theta)\dot{\theta}\hat{j}$

$\Rightarrow \quad \frac{d}{dt}\hat{e}_r = (-\sin\theta\ \hat{i} + \cos\theta\ \hat{j})\ \dot{\theta} = \dot{\theta}\hat{e}_\theta$ [using (2)]

Also $\frac{d}{dt}\hat{e}_\theta = (-\cos\theta)\dot{\theta}\hat{i} + (-\sin\theta)\dot{\theta}\hat{j}$

$= -(\cos\theta\ \hat{i} + \sin\theta\,\hat{j})\dot{\theta}$

$\Rightarrow \quad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r$ [using (1)]

**Example:-** Express the velocity $\vec{v}$ and acceleration $\vec{a}$ of a particle in cylindrical coordinates.

**Solution :-** Position vector of a particle P in rectangular coordinates

$\vec{r} = x\ \hat{i} + y\ \hat{j} + z\ \hat{k}$

vector $\vec{r}$ in cylindrical coordinate system is

$\vec{r} = r \cos\theta(\cos\theta\ \hat{e}_r - \sin\theta\ \hat{e}_\theta)$

$$+ r\sin\theta(\sin\theta\;\hat e_r + \cos\theta\;\hat e_\theta) + z\,\hat e_z$$

$$\Rightarrow \quad \vec r = r\cos^2\theta\;\hat e_r - r\cos\theta\sin\theta\;\hat e_\theta + r\sin^2\theta\;\hat e_r + r\sin\theta\cos\theta\;\hat e_\theta + z\,\hat e_z$$

$$\Rightarrow \quad \vec r = r\hat e_r + z\,\hat e_z$$

Then velocity $\vec v = \dfrac{d\vec r}{dt}$

$$\Rightarrow \quad \vec v = \frac{dr}{dt}\hat e_r + r\frac{d}{dt}\hat e_r + \frac{dz}{dt}\hat e_z + z\frac{d}{dt}\hat e_z$$

$$\vec v = \dot r\hat e_r + r\dot\theta\hat e_\theta + \dot z\hat e_z \qquad \left[\because \;\; \frac{d}{dt}\hat e_z = 0\right] \qquad \ldots(1)$$

Differentiating (1) again, we obtain acceleration $\vec a = \dfrac{d\vec v}{dt} = \dfrac{d^2\vec r}{dt^2}$

$$\Rightarrow \quad \vec a = \frac{d}{dt}(\dot r\hat e_r + r\dot\theta\hat e_\theta + z\,\hat e_z)$$

$$= \ddot r\hat e_r + \dot r\frac{d}{dt}\hat e_r + \dot r\dot\theta\hat e_r + r\ddot\theta\hat e_\theta + r\dot\theta\frac{d}{dt}\hat e_\theta\;\ddot z\hat e_z + \dot z\frac{d}{dt}\hat e_z$$

using $\dfrac{d}{dt}\hat e_r = \dot\theta\hat e_\theta, \quad \dfrac{d}{dt}\hat e_\theta = -\dot\theta\hat e_r, \; \dfrac{d}{dt}\hat e_z = 0,$

we get

$$\vec a = \ddot r\hat e_r + \dot r\dot\theta\hat e_\theta + \dot r\dot\theta\hat e_\theta + \dot r\dot\theta\hat e_\theta + r\ddot\theta\hat e_\theta + r\dot\theta(-\dot\theta\hat e_r) + \ddot z\hat e_z$$

$$\Rightarrow \quad \vec a = (\ddot r - r\dot\theta^2)\hat e_r + (r\ddot\theta + 2\dot r\dot\theta)\hat e_\theta + \ddot z\hat e_z$$

**Example: -** Prove that in spherical coordinates (r, θ, ϕ)

$$\frac{d}{dt}\hat e_r = \hat e_{\dot r} = \dot\theta\hat e_\theta + \sin\theta\,\dot\varphi\hat e_\varphi$$

$$\frac{d}{dt}\hat e_\theta = \hat e_{\dot\theta} = -\dot\theta\hat e_r + \cos\theta\,\dot\varphi\hat e_\varphi$$

$$\frac{d}{dt}\hat e_\varphi = \hat e_{\dot\varphi} = -\sin\theta\,\dot\varphi\hat e_r - \cos\theta\,\dot\varphi\hat e_\theta \qquad \ldots(*)$$

**Proof: -** Now $\hat e_r = \sin\theta\cos\varphi i + \sin\theta\sin\varphi\hat j + \cos\theta\,\hat k$ …(1)

$$\therefore \quad \frac{d}{dt}\hat e_r = \cos\theta\cos\varphi\,\dot\theta\hat i + \sin\theta(-\sin\varphi)\dot\varphi\hat i$$

$$+ \cos\theta\cos\varphi\,\hat i\sin\theta\cos\varphi\,\dot\varphi\hat j - (\sin\theta)\dot\theta\hat k$$

$$= \dot\theta(\cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k})$$

$$+\sin\theta\,\dot\varphi(-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}) \qquad \ldots(2)$$

using $\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}$ …(3)

$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$ …(4)

Put (3) & (4) in (2), we get

$$\frac{d}{dt}\hat{e}_r = \dot\theta\,\hat{e}_\theta + \sin\theta\,\dot\varphi\,\hat{e}_\varphi$$

Also from(3),

$$\frac{d}{dt}\hat{e}_\theta = -\sin\theta\cos\varphi\,\dot\theta\,\hat{i} - \cos\theta\sin\varphi\,\dot\varphi\,\hat{i}$$

$$-\sin\theta\sin\varphi\,\dot\theta\,\hat{i} - \cos\theta\sin\varphi\,\dot\varphi\,\hat{j} - \cos\theta\,\hat{k}\,\dot\theta$$

$$= \dot\theta(\sin\theta\cos\varphi\,\hat{i} + in\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}) + \cos\theta\,\dot\varphi(-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})$$

$$\Rightarrow \quad \frac{d}{dt}\hat{e}_\theta = -\dot\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\varphi\,\dot\varphi$$

Also from (4) , $\frac{d}{dt}\hat{e}_\varphi = -\cos\varphi\,\dot\varphi\,\hat{i} - \sin\varphi\,\dot\varphi - \sin\varphi\,\dot\varphi\,\hat{j}$ …(5)

Taking R.H.S of equation (*)

R.H.S = $-\sin\theta\,\dot\varphi\,\hat{e}_r - \cos gq\,\dot\varphi\,\hat{e}_\theta$

$$= -\sin\theta\,\dot\varphi(\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k})$$

$$-\cos\theta\,\dot\varphi(\cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}) \qquad \text{[using(1) \& (3)]}$$

$$= -\sin^2\cos\phi\,\dot\varphi\,\hat{i} - \dot\varphi\sin^2\theta\sin\varphi\,\hat{j} - \dot\varphi\sin\theta\cos\theta\,\hat{k}$$

$$-\cos^2\theta\cos\phi\,\dot\varphi\,\hat{i} - \cos^2\theta\sin\phi\,\dot\varphi\,\hat{j} + \sin\theta\cos\theta\,\dot\varphi\,\hat{k}$$

$$= -\dot\varphi\,\hat{i}\cos\varphi(\sin^2\theta + \cos^2\theta) - \dot\varphi\,\hat{j}\sin\phi\,(\sin^2\theta + \cos^2\theta)$$

$$= -\dot\varphi\cos\phi\,\hat{i} - \sin\varphi\,\hat{j} = \text{L.H.S.} \qquad \text{[from (5)]}$$

$\Rightarrow$ L.H.S. = R.H.S.

So $\frac{d}{dt}\hat{e}_\varphi = -\sin\theta\,\dot\varphi\,\hat{e}_r - \cos\theta\,\dot\varphi\,\hat{e}_\theta$

**Example:-** Express the velocity $\vec{v}$ and acceleration $\vec{a}$ of a particle in spherical coordinates.

**Solution :** Position vector of a particle P in rectangular coordinates

$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ vector $\vec{r}$ in spherical coordinate system is

$$\vec{r} = r\cos\phi\,\sin\theta\,(\hat{e}_r \sin\theta\,\cos\varphi + \hat{e}_\theta \cos\theta\cos\varphi - \hat{e}_\varphi \sin\phi)$$

$$+\, r\sin\theta\,\sin\phi\,(\hat{e}_r \sin\theta\sin\varphi + \hat{e}_\theta \cos\theta\sin\varphi + \hat{e}_\varphi \cos\varphi)$$

$$+\, r\cos\theta\,(\hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta)$$

$$\Rightarrow \quad \vec{r} = r\,\hat{e}_r(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)$$

$$+\, r\,\hat{e}_\theta(\sin\theta\cos\theta\cos^2\varphi + \sin\theta\sin^2\varphi\cos\theta - \cos\theta\sin\theta)$$

$$+\, r\,\hat{e}_\varphi(-\cos\varphi\sin\theta\sin\varphi + \sin\theta\sin\varphi\cos\varphi)$$

$$\Rightarrow \quad \vec{r} = r\,\hat{e}_r\,[\sin^2\theta\,(\sin^2\phi + \cos^2\phi) + \cos^2\theta]$$

$$+\, r\,\hat{e}_\theta\,[\sin\theta\,\cos\theta\,(\sin^2\phi + \cos^2\phi) - \cos\theta\,\sin\theta]$$

$$= r\,\hat{e}_r\,(\sin^2\theta + \cos^2\theta) + r\,\hat{e}_\theta\,(\sin\theta\,\cos\theta - \sin\theta\,\cos\theta)$$

$$\Rightarrow \quad \vec{r} = r\,\hat{e}_r(1) = r\,\hat{e}_r$$

Then velocity $\vec{v}$ is $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\hat{e}_r)$

$$\Rightarrow \quad \vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r)$$

$$= \dot{r}\hat{e}_r + r(\dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\phi}\hat{e}_\varphi) \qquad \text{[from previous eg.]}$$

$$= \dot{r}\hat{e}_r + r\,\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\hat{e}_\varphi \qquad \ldots(1)$$

Differentiating (1) again, we obtain acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\Rightarrow \quad \vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\,\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\varphi)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{e}_\theta)$$

$$+\, \dot{r}\sin\theta\,\dot{\phi}\hat{e}_\varphi + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{e}_\varphi + r\sin\theta\,\ddot{\phi}\hat{e}_\varphi + r\sin\theta\,\dot{\phi}\frac{d}{dt}(\hat{e}_\varphi)$$

$$= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\phi}\hat{e}_\varphi) + \dot{r}\dot{\theta}\hat{e}_\theta + r\,\ddot{\theta}\hat{e}_\theta$$

$$+\, r\dot{\theta}(-\dot{\theta}\hat{e}_r + \cos\theta\,\dot{\phi}\hat{e}_\varphi) + \dot{r}\sin\theta\,\dot{\phi}\hat{e}_\varphi$$

$$+ r\cos\theta\,\dot{\varphi}\,\hat{e}_{\varphi}\,\dot{\theta} + r\sin\theta\,\ddot{\varphi}\,\hat{e}_{\varphi}$$

$$+ r\sin\theta\,\dot{\varphi}(-\sin\theta\,\dot{\varphi}\,\hat{e}_r - \cos\theta\,\dot{\varphi}\,\hat{e}_{\theta})$$

$$\Rightarrow \quad \vec{a} = \hat{e}_r(\ddot{r} - r\dot{\theta}^2 - \dot{\varphi}^2 r\sin^2\theta)$$

$$+ \hat{e}_{\theta}[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2]$$

$$+ \hat{e}_{\varphi}[2\dot{r}\dot{\varphi}\sin\theta + 2r\,\dot{\theta}\dot{\varphi}\cos\theta + r\sin\theta\,\ddot{\varphi}]$$

$$\Rightarrow \quad \vec{a} = \hat{e}_r(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2)$$

$$+ \hat{e}_{\theta}\left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\,\dot{\varphi}^2\right]$$

$$+ \hat{e}_{\varphi}\left[\frac{1}{r\sin\theta}\frac{d}{dt}(r^2 sn^2\theta\,\dot{\varphi})\right] \qquad \ldots(2)$$

$$\left[\because \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(r^2\ddot{\theta} + 2r\,\dot{r}\dot{\theta}) = r\ddot{\theta} + 2\dot{r}\dot{\theta} \text{ and } \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\,\dot{\varphi})\right.$$

$$= \frac{1}{r\sin\theta}[r^2\sin^2\theta\,\ddot{\varphi} + 2r\,\dot{r}\sin^2\theta\,\dot{\varphi} + \dot{\varphi}r^2\,2\sin\theta(\cos\theta)\dot{\theta}]$$

$$\left. = r\sin\theta\,\ddot{\varphi} + 2\,\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta \right]$$

So (2) is required expression for acceleration of a particle in spherical coordinate.

**Example :** $\frac{d}{dt}\hat{e}_{\varphi} = \hat{e}_{\dot{\varphi}} = -\sin\theta\,\dot{\varphi}\,\hat{e}_r - \cos\theta\,\dot{\varphi}\,\hat{e}_{\theta}$

**Solution** :- Now $\hat{e}_{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

$$\Rightarrow \quad \frac{d}{dt}\hat{e}_{\varphi} = -\cos\varphi\,\dot{\varphi}\,\hat{i} - \sin\varphi\,\dot{\varphi}\,\hat{j} \qquad \ldots(5)$$

Put values of $\hat{i}$ and $\hat{j}$ in(5), we get

$$\frac{d}{dt}\hat{e}_{\varphi} = -\cos\varphi\,\dot{\varphi}(\hat{e}_r\sin\theta\cos\varphi + \hat{e}_{\varphi}\cos\theta\cos\varphi - \hat{e}_{\varphi}\sin\varphi)$$

$$- \sin\phi\,\dot{\varphi}(\hat{e}_r\sin\theta\sin\varphi + \hat{e}_{\theta}.\cos\theta\sin\varphi + \hat{e}_{\varphi}\cos\varphi)$$

$$= -\sin\theta\,\dot{\varphi}\,\hat{e}_r(\cos^2\varphi + \sin^2\varphi) - \cos\theta\,\dot{\varphi}\,\hat{e}_{\theta}(\cos^2\varphi + \sin^2\varphi)$$

$$+ \hat{e}_{\varphi}\cos\varphi\sin\varphi\,\dot{\varphi} - \hat{e}_{\varphi}\cos\varphi\sin\varphi\,\dot{\varphi}$$

$= -\sin\theta\,\dot{\phi}\,\hat{e}_r - \cos\theta\,\dot{\phi}\,\hat{e}_\theta$. Hence Proved.

**Example:-** Represent the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

**Solution :-** In cylindrical coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \; z = z \qquad \ldots(1)$$

and $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta \qquad \ldots(2)$$

$$\hat{e}_z = \hat{k} \qquad \ldots(3)$$

The position vector of a particle P in rectangular coordinates is

$$\vec{A} = z\hat{i} + (-2x)\hat{j} + y\hat{k}$$

$$\Rightarrow \quad \vec{A} = z\hat{i} - 2r\cos\theta\,\hat{j} + r\sin\theta\,\hat{k} \qquad \ldots(4)$$

Solve (2) for $\hat{i}$ and $\hat{j}$, we get on operating $\hat{e}_r \times \sin\theta + \hat{e}_\theta \times \cos\theta$

$$= \cos\theta\sin\theta\,\hat{i} + \sin^2\theta\,\hat{j} - \sin\theta\cos\theta\,\hat{i} + \cos^2\theta\,\hat{j}$$

$$\Rightarrow \quad \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta = (\sin^2\theta + \cos^2\theta)j = \hat{j} \qquad \ldots(5)$$

Similarly $\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta = \cos^2\theta\,\hat{i} + \sin\theta\cos\theta\,\hat{j}$

$$-(-\sin^2\theta\,\hat{i} + \cos\theta\sin\theta\,\hat{j})$$

$$\Rightarrow \quad \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta = (\sin^2\theta + \cos^2\theta)\hat{i} + \sin\theta\cos\theta\,\hat{j} - \sin\theta\cos\theta\,\hat{j}$$

$$\Rightarrow \quad \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta = \hat{i} \qquad \ldots(6)$$

Put the value of $\hat{i}, \hat{j}, \hat{k}$ from (3), (5), (6) in equation (4), we get

$$\vec{A} = z(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta) - 2r\cos\theta(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta) + r\sin\theta\,\hat{e}_z$$

$$= \hat{e}_r(z\cos\theta - 2r\cos\theta\sin\theta)$$

$$-\hat{e}_\theta(z\sin\theta + 2r\cos^2\theta) + \hat{e}_z(r\sin\theta)$$

$$\Rightarrow \quad \vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_z\hat{e}_z \qquad \ldots(1)$$

where

$$A_r = z\cos\theta - 2r\cos\theta\sin\theta$$

$$A_\theta = -z\sin\theta - 2r\cos\theta\sin\theta$$

$$A_z = r\sin\theta$$

Equation (1) is required vector $\vec{A}$ in cylindrical coordinates.

**Example:-** Represent the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in spherical coordinates (r, θ, ϕ)

**Solution :-** In spherical coordinates,

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta \qquad \ldots(2)$$

and

$$\hat{e}_r = \sin\theta \cos\phi\ \hat{i} + \sin\theta \sin\phi\ \hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\ \hat{i} + \cos\theta \sin\phi\ \hat{j} - \sin\theta\ \hat{k}$$

$$\hat{e}_\varphi = -\sin\phi\ \hat{i} + \cos\phi\ \hat{j} \qquad \ldots(3)$$

Solving system (3) for $\hat{i}$, $\hat{j}$, $\hat{k}$ by Cramer's rule, for this we have

$$D = \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{vmatrix}$$

$$= \sin\theta \cos\phi(\sin\theta \cos\phi) + \sin\theta \sin\phi\ (\sin\theta \sin\phi)$$
$$+ \cos\theta(\cos\theta \cos^2\phi + \cos\theta \sin^2\phi)$$
$$= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta\ (\cos^2\phi + \sin^2\phi)$$
$$= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta\ (1)$$
$$= \sin^2\theta + \cos^2\theta = 1$$

$\Rightarrow \qquad D = 1$

and

$$D_1 = \begin{vmatrix} \hat{e}_r & \sin\theta\sin\varphi & \cos\theta \\ \hat{e}_\theta & \cos\theta\sin\varphi & -\sin\theta \\ \hat{e}_\varphi & \cos\varphi & 0 \end{vmatrix}$$

$$= \hat{e}_r(\sin\theta \cos\phi) + \sin\theta \sin\phi\ (-\sin\theta\ \hat{e}_\varphi)$$
$$+ \cos\theta\ (\hat{e}_\theta \cos\varphi - \hat{e}_\varphi \cos\theta\sin\varphi)$$
$$= \hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta\ \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi(\sin^2\theta + \cos^2\theta)$$

$\Rightarrow \qquad D_1 = \hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi$

$$D_2 = \begin{vmatrix} \sin\theta\cos\varphi & \hat{e}_r & \cos\theta \\ \cos\theta\cos\varphi & \hat{e}_\theta & -\sin\theta \\ -\sin\varphi & \hat{e}_\varphi & 0 \end{vmatrix}$$

$= \sin\theta \cos \phi(\sin\theta\ \hat{e}_\varphi) + \hat{e}_r (\sin\theta \sin\phi - 0)$

$+ \cos \theta( \hat{e}_\varphi \cos\theta \cos\theta + \hat{e}_\theta \sin \phi)$

$= \sin\theta \sin\phi\ \hat{e}_r + \cos\theta \sin\phi\ \hat{e}_\theta + \cos\phi\ \hat{e}_\varphi (\sin^2\theta + \cos^2\theta)$

$\Rightarrow \quad D_2 = \hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta\ \cos\theta \sin\phi + \cos\phi\ \hat{e}_\varphi$

$$D_3 = \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \hat{e}_r \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & \hat{e}_\theta \\ -\sin\varphi & \cos\varphi & \hat{e}_\varphi \end{vmatrix}$$

$= \sin\theta \cos\phi\ (\cos\theta \sin\phi\ \hat{e}_\varphi - \cos\phi\ \hat{e}_\theta)$

$+ \sin\theta \sin\phi(-\sin\phi\ \hat{e}_\theta - \hat{e}_\varphi \cos\theta \cos\phi)$

$+ \hat{e}_r (\cos\theta \cos^2\phi + \cos\theta \sin^2\phi)$

$= \hat{e}_r \cos\theta(\sin^2\phi + \cos^2\phi) +$

$\hat{e}_\theta (-\sin\theta\cos^2\phi - \sin\theta \sin^2\phi)$

$+ \hat{e}_\varphi (\sin\theta \cos\theta \sin\phi \cos\phi - \sin\theta \cos\theta \sin\phi \cos\phi)$

$\Rightarrow \quad D_3 = \hat{e}_r \cos\theta - \sin\theta\ \hat{e}_\theta$

Then $\hat{i} = \dfrac{D_1}{D} = \hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi$

$\hat{j} = \dfrac{D_2}{D} = \hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta \cos\theta \sin\phi + \cos\phi\ \hat{e}_\varphi$

$\hat{k} = \dfrac{D_3}{D} = \hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta$

from (1), (2) and (4), we get

$\vec{A}(r\cos\theta)(\hat{e}_r \sin\theta\cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\varphi \sin\phi)$

$- 2r \sin\theta \cos\phi\ (\hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta \cos\theta \sin\phi + \hat{e}_\varphi \cos\phi)$

$+ r \sin\theta \sin\phi\ (\hat{e}_r \cos\theta - \hat{e}_\theta\ \sin\theta)$

$\Rightarrow \quad \vec{A} = r\,\hat{e}_r\,(\cos\theta \sin\theta \cos\phi - 2 \sin^2\theta \cos\phi \sin\phi + \sin\theta \cos\theta \sin\phi)$

$\quad + r\ \hat{e}_\theta\,(\cos^2\theta \cos\phi - 2 \sin\theta \cos\theta \sin\phi \cos\phi)$

$\quad + r\ \hat{e}_\varphi\,(-\cos\theta \sin\phi - 2 \sin\theta \cos^2\phi)$

$= r\ \hat{e}_r[\sin\theta\cos\theta\,(\cos\phi + \sin\phi) - 2 \sin^2\theta \cos\phi \sin\phi]$

$\quad + r\ \hat{e}_\theta\,(\cos^2\theta \cos\phi - 2 \sin\theta \cos\theta \sin\phi \cos\phi)$

$\quad + r\ \hat{e}_\varphi\,(-\cos\theta \sin\phi - 2 \sin\theta \cos^2\phi)$

$\Rightarrow \quad \vec{A} = A_r\ \hat{e}_r + A_\theta\ \hat{e}_\theta + A_\phi\ \hat{e}_\varphi$

which is required expression.

**Example:-** Find the squre of the element of arc length in cylindrical & spherical coordinates

**Solution: -** In cylindrical coordinates,

$x = r \cos\theta,\ y = r \sin\theta,\ z = z$

Now $\quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \qquad \ldots(1)$

as $\quad dx = dr \cos\theta - r \sin\theta\ d\theta$

$dy = dr \sin\theta + r \cos\theta\ d\theta \qquad \ldots(2)$

$dz = dz$

Put (2) in (1), we get

$(ds)^2 = (\cos\theta\ dr - r \sin\theta\ dz\theta)^2$

$\quad +(\sin\theta\ dr + r \cos\theta\ d\theta)^2 + (dz)^2$

$= (dr)^2 (\sin^2\theta + \cos^2\theta) + (d\theta)^2 (r^2 \sin^2\theta + r^2 \cos^2\theta)$

$\quad -2r \sin\theta \cos\theta\ dr\ d\theta + 2r \sin\theta \cos\theta\ dr\ d\theta + (dz)^2$

$\Rightarrow \quad (ds)^2 = (dr)^2 + r^2 (d\theta)^2 + (dz)^2 = h_1^2 (dr)^2 + h_2^2 (d\theta)^2 + h_3^2 (dz)^2$

In spherical coordinates $(r, \theta, \phi)$

$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\theta \qquad \ldots(1)$

as $\quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

using (1), we get

$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\ (d\phi)^2$

Camparing it with,

$(ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$, we get

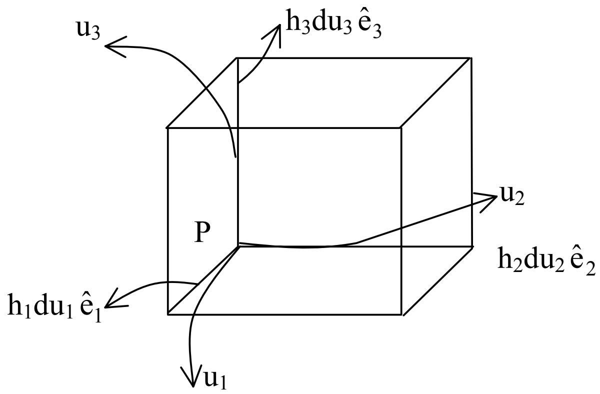
$h_1 = 1, h_2 = r_1, h_3 = r \sin \theta$

$u_1 = r, u_2 = \theta, u_3 = \phi$

we get $(ds)^2 = h_1^2 (dr)^2 + h_2^2 (d\theta)^2 + h_3^2 (d\phi)^2$

**Example : -** Find expression for the elements of area in orthogonal curvilinear coordinates.

**Solution: -**




Now area element is given by

$dA_1 = |h_2 du_2\ \hat{e}_2 \times h_3\ du_3\ \hat{e}_3|$

$= h_2\ h_3\ |\hat{e}_2 \times \hat{e}_3|\ du_2\ du_3 = h_2\ h_3\ |\hat{e}_1|\ du_2\ du_3$

$= h_2\ h_3\ du_2\ du_3$ $\qquad [\Theta |\hat{e}_1| = 1]$

$dA_2 = |h_3\ du_3\ \hat{e}_3 \times h_1\ du_1\ \hat{e}_1|$

$= h_3\ h_1 |\hat{e}_3 \times \hat{e}_1|\ du_3\ du_1 = |\hat{e}_2|\ du_3\ du{=}1$ $\qquad [\Theta |\hat{e}_2| = 1]$

$dA_3 = |\ h_1\ du_1\ \hat{e}_1 \times h_2 du_2 \hat{e}_2|$

$= h_1\ h_2\ |\ \hat{e}_1 \times \hat{e}_2|\ du_1\ du_2$

$= h_1\ h_2\ |\hat{e}_1 \times\ \hat{e}_2|\ du_1\ du_2$

$= h_1\ h_2\ |\hat{e}_3|\ du_1\ du_2 = h_1\ h_2\ du_1\ du_2$ $\qquad [\Theta |\hat{e}_3| = 1|$

**Example:-** Find the volume element dv in cylenderical & spherical coordinate system.

**Solution: -** We know that volume element dV in orthogonal curvilinear coordinates is

$dV = h_1\, h_2\, h_3\, du_1 du_2\, du_3$ $\qquad \ldots(1)$

In cylindrical coordinates $(r, \theta, \phi)$

$h_1 = 1, h_2 = r\ , h_3 = 1$

$u_1 = r, u_2 = \theta, u_3 = z$

So (1) $\Rightarrow$ dV = r dr dθ dz

In spherical coordinates (r, θ , ϕ)

$h_1 = 1, h_2 = r, h_3 = r \sin\theta$

$u_1 = r, u_2 = \theta, u_3 = \phi$

$\therefore$ So (1) $\Rightarrow$ dV $r^2 \sin^2$ dr dθ dϕ

**Example:-** If $u_1$, $u_2$, $u_3$ are general coordinates, show that $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ & $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocail system of vectors.

**Solution: -** We know that if $\phi = \phi(x, y, z)$

$$d\phi = \frac{\partial \varphi}{dx}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$$= \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}\right).(d x\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow \quad d\phi = \nabla\phi . d\vec{r}$$

Replacing ϕ with ∇ up, we get

$$du_p = \nabla u_p \,.\, d\vec{r} \qquad \ldots(1)$$

$$\Rightarrow \quad du_1 = \nabla u_1 \,.\, d\vec{r}$$

$$du_2 = \nabla u_2 \,.\, d\vec{r}$$

$$du_3 = \nabla u_3 \,.\, d\vec{r}$$

Now $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$$\Rightarrow \quad d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3 \qquad \ldots(2)$$

Taking dot product with $\nabla u_1$, we get

$$\nabla u_1 . d\vec{r} = \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_1}\right)du_1 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_2}\right)du_2 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_3}\right)du_3 \qquad \ldots(3)$$

$$\Rightarrow \quad du_1 = \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_1}\right)du_1 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_2}\right)du_2 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_3}\right) \qquad \text{[using (1)]}$$

Comparing like coefficients on both sides,

$$\nabla u_1 . \frac{\partial \vec{r}}{\partial u_1} = 1, \quad \nabla u_1 . \frac{\partial \vec{r}}{\partial u_2} = 0, \quad \nabla u_1 . \frac{\partial \vec{r}}{\partial u_3} = 0$$

Similarly taking dot product of (2) with $\nabla u_2$ and $\nabla u_3$, we get

$$\nabla u_2, \frac{\partial \vec{r}}{\partial u_2} = 1,\ \nabla u_2 . \frac{\partial \vec{r}}{\partial u_1} = 0\ \nabla u_2 . \frac{\partial \vec{r}}{\partial u_3} = 0$$

$$\nabla u_3 . \frac{\partial \vec{r}}{\partial u_3} = 1,$$

So we get the required result i.e.

$$\nabla u_p . \frac{\partial \vec{r}}{\partial u_q} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}$$

Where (p, q) = (1, 2, 3)

**Example: -** Prove that $\hat{e}_1 = h_2 h_3\ \nabla u_2 \times \nabla u_3$ with similar equation for $\hat{e}_2$ and $\hat{e}_3$ where $u_1$, $u_2$, $u_3$ are orthogonal coordinates.

**Solution: -** We know that

$$\nabla u_1 = \frac{\hat{e}_1}{h_1}, \quad \nabla u_2 = \frac{\hat{e}_2}{h_2}, \quad \nabla u_3 = \frac{\hat{e}_3}{h_3}$$

Then $\nabla u_2 \times \nabla u_3 = \dfrac{\hat{e}_2 \times \hat{e}_3}{h_2 h_3} = \dfrac{\hat{e}_1}{h_2 h_3}$

$\Rightarrow \quad \hat{e}_1 = h_2 h_3 \nabla u_2 \times \nabla u_2 \times \nabla u_3$

Similarly $\quad \nabla u_3 \times \nabla u_1 = \dfrac{\hat{e}_3 k \hat{e}_1}{h_3 h_1} = \dfrac{\hat{e}_2}{h_3 h_1}$

$\Rightarrow \quad \hat{e}_2 = h_3 h_1 \nabla u_3 \times \nabla u_1$

and $\quad \nabla u_1 \times \nabla u_2 = \dfrac{\hat{e}_1 \times \hat{e}_2}{h_1 h_2} = \dfrac{\hat{e}_3}{h_1 h_2}$

$\Rightarrow \quad \hat{e}_3 = h_1 h_2 \nabla u_1 \times \nabla u_2$ Hence proved.

**Example: -** If $u_1$, $u_2$, $u_3$ are orthogonal curvilinear coordinates, show that the jacobian of x, y, z w.r.t. $u_1$, $u_2$, $u_3$ is

$$J\left(\frac{x,y,z}{u_1u_2u_3}\right)=\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}=\begin{vmatrix}\frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1}\\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2}\\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3}\end{vmatrix}$$

$$= h_1h_2h_3$$

**Proof: -** we know that $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\vec{B} = (B_1, B_2, B_3), \qquad \vec{C} = C(C_1, C_2, C_3)$$

Then

$$\vec{A}.(\vec{B}\times\vec{C}) = \vec{A}.\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).[\hat{i}(B_2C_3-B_3C_2) + \hat{j}(B_3C_1-B_1C_3 + \hat{k}(B_1C_2-B_2C_1)]$$

$\Rightarrow \quad \vec{A}.(\vec{B}\times\vec{C}) = A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_1 - B_2C_1)$

$$\Rightarrow \quad \vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

Therefore

$$\begin{vmatrix}\frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1}\\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2}\\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3}\end{vmatrix} = \left(\frac{\partial x}{\partial u_1}\hat{i} + \frac{\partial y}{\partial u_1}\hat{y} + \frac{\partial z}{\partial u_1}\hat{k}\right).\left[\left(\frac{\partial x}{\partial u_2}\hat{i} + \frac{\partial y}{\partial u_2}\hat{j} + \frac{\partial z}{\partial u_2}\hat{k}\right)\right.$$

$$\left.\times\left(\frac{\partial x}{\partial u_3}\hat{i} + \frac{\partial y}{\partial u_3}\hat{j} + \frac{\partial z}{\partial u_3}\hat{k}\right)\right]]$$

$$= \left(\frac{\partial\vec{r}}{\partial u_1}\right).\left(\frac{\partial\vec{r}}{\partial u_2}\times\frac{\partial\vec{r}}{\partial u_3}\right)$$

$$= h_1\,\vec{e}_1.(h_2\vec{e}_2 \times h_3\vec{e}_3)$$

$$= h_1\vec{e}_1.h_2h_3(\vec{e}_2\times\vec{e}_3) = h_1\,\vec{e}_1.h_2h_3\vec{e}_1$$

$$= h_1 h_2 h_3 \ \vec{e}_1.\vec{e}_1$$

$$= h_1 h_2 h_3 \ \vec{e}_1.\vec{e}_1$$

$$= h_1 h_2 h_3 \qquad\qquad [\because \ \vec{e}_1.\vec{e}_1 = 1]$$

Hence proved.

**Contravariant Components of $\vec{A}$ and covariant components of $\vec{A}$**

$\vec{A}$ in terms of unit base vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ or $\hat{E}_1, \hat{E}_2, \hat{E}_3$ can be written as

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = a_1 \hat{E}_1 + a_2 \hat{E}_2 + a_3 \hat{E}_3$$

Where $A_1$, $A_2$, $A_3$ & $a_1$, $a_2$, $a_3$ are components of $\vec{A}$ in each that $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ and $\nabla u_1$, $\nabla u_2$, $\nabla u_3$ constitite reciprocal system of vectors. We can also represent $\vec{A}$ in terms of base vectors $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ or $\nabla u_1$, $\nabla u_2$, $\nabla u_3$ which are called **unitary base vectors** & are not unit vectors.

In general,

$$\vec{A} = C_1 \frac{\partial \vec{r}}{\partial u_1} + C_2 \frac{\partial \vec{r}}{\partial u_2} + C_3 \frac{\partial \vec{r}}{\partial u_3}$$

$$\Rightarrow \qquad \vec{A} = C_1 \vec{\alpha}_1 + C_2 \vec{\alpha}_2 + C_3 \vec{\alpha}_3 \qquad \text{where } \vec{\alpha}_p = \frac{\partial \vec{r}}{\partial u_p}, p = 1,2,3$$

$$\& \qquad \vec{A} = C_1 \nabla u_1 + C_2 \nabla u_2 + C_3 \nabla u_3$$

$$= C_1 \vec{\beta}_1 + C_2 \vec{\beta}_2 + C_3 \vec{\beta}_3 \qquad \text{where } \vec{\beta}_p = \nabla u_p, \ p = 1, 2, 3$$

where $C_1$, $C_2$, $C_3$ are called **contravariant component** of $\vec{A}$ and $C_1$, $C_2$, $C_3$ are called **covariant components of** $\vec{A}$.

**Example: -** Let $\vec{A}$ be a given vector defined w.r.t. two gereral curvilinear coordinates system ($u_1$, $u_2$, $u_3$)& $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Find the relation between the contravariant components of the vector in the two coordinate system.

(Find relation between $C_p$ and $\overline{C}_p$)

**Solution: -** Suppose the transformation equation from rectangular (x, y, z) system to the two general curvilinear coordinations systems ($u_1$, $u_2$, $u_3$) and $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ are given by

$$x = x_1(u_1, u_2, u_3), \qquad y = y_1(u_1, u_2, u_3), \qquad z = z_1(u_1, u_2, u_3)$$

$$x = x_2(\bar{u}_1, \bar{u}_2, \bar{u}_3), \qquad y = y_2(\bar{u}_1, \bar{u}_2, \bar{u}_3), \qquad z = z_2(\bar{u}_1, \bar{u}_2, \bar{u}_3) \qquad \ldots(1)$$

Then

I a transformation directly from $(u_1, u_2, u_3)$ system to $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ system is defined by

$$u_1 = u_1(\bar{u}_1, \bar{u}_2, \bar{u}_3), \qquad u_2 = u_2(\bar{u}_1, \bar{u}_2, \bar{u}_3), \qquad u_3 = u_3(\bar{u}_1, \bar{u}_2, \bar{u}_3) \qquad \ldots(i)$$

$$\bar{u}_1 = \bar{u}_1(u_1, u_2, u_3), \qquad \bar{u}_2 = \bar{u}_2(u_1, u_2, u_3), \qquad \bar{u}_3 = \bar{u}_3(u_1, u_2, u_3) \qquad \ldots(2)$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then use (1)

$$\vec{r} = \vec{r}(u_1, u_2, u_3), \qquad \vec{r} = \vec{r}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3$$

$$= \vec{\alpha}_1 du_1 + \vec{\alpha}_2 du_2 + \vec{\alpha}_3 du_3 \qquad \ldots(3)$$

& $$d\vec{r} = \frac{\partial \vec{r}}{\partial \bar{u}_1} d\bar{u}_1 + \frac{\partial \vec{r}}{\partial \bar{u}_2} d\bar{u}_2 + \frac{\partial \vec{r}}{\partial \bar{u}_3} d\bar{u}_3$$

$$= \vec{\bar{\alpha}}_1 d\bar{u}_1 + \vec{\bar{\alpha}}_2 d\bar{u}_2 + \vec{\bar{\alpha}} du_3 \qquad \ldots(4)$$

where $\vec{\alpha}_p = \dfrac{\partial \vec{r}}{\partial u_p}, \qquad \vec{\bar{\alpha}}_p = \dfrac{\partial \vec{r}}{\partial \bar{u}_p}$, p = 1, 2, 3

from (3) & (4), we get

$$\vec{\alpha} du_1 + \vec{\alpha}_2 du_2 + \vec{\alpha}_3 du_3 = \vec{\bar{\alpha}}_1 d\bar{u}_1 + \vec{\bar{\alpha}}_2 d\bar{u}_2 + \vec{\bar{\alpha}} du_3 \qquad \ldots(5)$$

from 2(i) since $u_p = u_p(\bar{u}_1, \bar{u}_2, \bar{u}_3)$

$\Rightarrow$ $$du_1 = \frac{\partial u_1}{\partial \bar{u}_1} d\bar{u}_1 + \frac{\partial u_1}{d\bar{u}_2} d\bar{u}_2 + \frac{\partial u_1}{d\bar{u}_3} d\bar{u}_3$$

$$du_2 = \frac{\partial u_2}{\partial \bar{u}_1} d\bar{u}_1 + \frac{\partial u_2}{d\bar{u}_2} d\bar{u}_2 + \frac{\partial u_{21}}{d\bar{u}_3} d\bar{u}_3$$

$$du_3 = \frac{\partial u_3}{\partial \bar{u}_1} d\bar{u}_1 + \frac{\partial u_3}{d\bar{u}_2} d\bar{u}_2 + \frac{\partial u_3}{d\bar{u}_3} d\bar{u}_3$$

using these in L.H.S of (5) and equating coefficient of $d\bar{u}_1, d\bar{u}_2, d\bar{u}_3$, we get

$$\vec{\alpha}_1 du_1 + \vec{\alpha}_2 du_2 + \vec{\alpha}_3 du_3 = \left( \frac{\partial u_1}{\partial \bar{u}_1} d\bar{u}_1 + \frac{\partial u_1}{d\bar{u}_2} d\bar{u}_2 + \frac{\partial u_1}{d\bar{u}_3} d\bar{u}_3 \right) \bar{\alpha}_1$$

$$+\left(\frac{\partial u_2}{\partial \bar{u}_1}d\bar{u}_1+\frac{\partial u_2}{d\bar{u}_2}d\bar{u}_2+\frac{\partial u_{21}}{d\bar{u}_3}d\bar{u}_3\right)\vec{\alpha}_2$$

$$+\left(\frac{\partial u_3}{\partial \bar{u}_1}d\bar{u}_1+\frac{\partial u_3}{d\bar{u}_2}d\bar{u}_2+\frac{\partial u_3}{d\bar{u}_3}d\bar{u}_3\right)\vec{\alpha}_3$$

$$=\vec{\bar{\alpha}}_1 d\bar{u}_1+\vec{\bar{\alpha}}_2 d\bar{u}_2+\vec{\bar{\alpha}}_3 d\bar{u}_3$$

We get $\vec{\bar{\alpha}}_1=\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_1}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_1}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_1}$

$$\vec{\bar{\alpha}}_2=\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_2}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_2}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_2}$$

$$\vec{\bar{\alpha}}_3=\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_3}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_3}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_3} \qquad \ldots(6)$$

Also $\vec{A}=\vec{\alpha}_1C_1+\vec{\alpha}_2C_2+\vec{\alpha}_3C_3$

$$=\vec{\bar{\alpha}}_1\bar{C}_1+\vec{\bar{\alpha}}_2\bar{C}_2+\vec{\bar{\alpha}}_3\bar{C}_3 \qquad \ldots(7)$$

where $C_1$, $C_2$, $C_3$ and $\bar{C}_1, \bar{C}_2, \bar{C}_3$ are contravariant components of $\vec{A}$ in the two systems ($u_1$, $u_2$, $u_3$) and $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Substituting (6) in (7),

$$C_1\vec{\alpha}_1+C_2\vec{\alpha}_2+C_3\vec{\alpha}_3=\bar{C}_1\left(\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_1}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_1}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_1}\right)$$

$$+\bar{C}_2\left(\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_2}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_2}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_2}\right)$$

$$+\bar{C}_3\left(\vec{\alpha}_1\frac{\partial u_1}{\partial \bar{u}_3}+\vec{\alpha}_2\frac{\partial u_2}{\partial \bar{u}_3}+\vec{\alpha}_3\frac{\partial u_3}{\partial \bar{u}_3}\right)$$

$$=\vec{\alpha}_1\left(\bar{C}_1\frac{\partial u_1}{\partial \bar{u}_1}+\bar{C}_2\frac{\partial u_1}{\partial \bar{u}_2}+\bar{C}_3\frac{\partial u_1}{\partial \bar{u}_3}\right)$$

$$+\vec{\alpha}_2\left(\bar{C}_1\frac{\partial u_2}{\partial \bar{u}_1}+\bar{C}_2\frac{\partial u_2}{\partial \bar{u}_2}+\bar{C}_3\frac{\partial u_2}{\partial \bar{u}_3}\right)$$

$$+\vec{\alpha}_3\left(\bar{C}_1\frac{\partial u_3}{\partial \bar{u}_1}+\bar{C}_2\frac{\partial u_3}{\partial \bar{u}_2}+\bar{C}_3\frac{\partial u_3}{\partial \bar{u}_3}\right)$$

Equating coefficients of $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$, we get

$$C_1 = \overline{C}_1 \frac{\partial u_1}{\partial \overline{u}_1} + \overline{C}_2 \frac{\partial u_1}{\partial \overline{u}_2} + \overline{C}_3 \frac{\partial u_1}{\partial \overline{u}_3}$$

$$C_2 = \overline{C}_1 \frac{\partial u_2}{\partial \overline{u}_1} + \overline{C}_2 \frac{\partial u_2}{\partial \overline{u}_2} + \overline{C}_3 \frac{\partial u_2}{\partial \overline{u}_3}$$

$$C_3 = \overline{C}_1 \frac{\partial u_3}{\partial \overline{u}_1} + \overline{C}_2 \frac{\partial u_3}{\partial \overline{u}_2} + \overline{C}_3 \frac{\partial u_3}{\partial \overline{u}_3} \qquad \ldots(8)$$

or $$C_p = \overline{C}_1 \frac{\partial u_p}{\partial \overline{u}_1} + \overline{C}_2 \frac{\partial u_p}{\partial \overline{u}_2} + \overline{C}_3 \frac{\partial u_p}{\partial \overline{u}_3},\ p = 1, 2, 3 \qquad \ldots(9)$$

or $$C_p = \sum_{q=1}^{3} \overline{C}_q \frac{\partial u_p}{\partial u_q},\ p = 1, 2, 3 \qquad \ldots(10)$$

Similarly by interchanging the coordinates,

We can get

$$\overline{C}_p = \sum_{q=1}^{3} C_q \frac{\partial \overline{u}_p}{\partial u_q}, p = 1, ,2\ 3 \qquad \ldots(11)$$

Equation (8), (9), (10), (11) gives the relation between contravariant components of vector in two coordinates systems.

**Example: -** Let $\vec{A}$ be given vector defined w.r.t. two curvilinear coordinates system ($u_1$, $u_2$, $u_3$) and $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$. Find the relation between the covariant components of the vectors in the two co ord system.

**Solution: -** Let the covariant component of $\vec{A}$ in the system ($u_1$, $u_2$, $u_3$) and $(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ are $C_1$, $C_2$, $C_3$ and $\overline{C}_1, \overline{C}_2, \overline{C}_3$ respectively

$$\therefore \quad \vec{A} = C_1\nabla u_1 + C_2\nabla u_2 + C_3\nabla u_3 = \overline{C}_1\nabla\overline{u}_1 + \overline{C}_2\nabla\overline{u}_2 + \overline{C}_3\nabla\overline{u}_3 \qquad \ldots(1)$$

Since $\overline{u}_p = \overline{u}_p(u_1, u_2, u_3)$, p=1, 2, 3

$$\frac{\partial \overline{u}_p}{\partial x} = \frac{\partial \overline{u}_p}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \overline{u}_p}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \overline{u}_p}{\partial u_3}\frac{\partial u_3}{\partial x}$$

$$\frac{\partial \overline{u}_p}{\partial y} = \frac{\partial \overline{u}_p}{\partial u_1}\frac{\partial u_1}{\partial y} + \frac{\partial \overline{u}_p}{\partial u_2}\frac{\partial u_2}{\partial y} + \frac{\partial \overline{u}_p}{\partial u_3}\frac{\partial u_3}{\partial y}$$

$$\frac{\partial \overline{u}_p}{\partial z} = \frac{\partial \overline{u}_p}{\partial u_1}\frac{\partial u_1}{\partial z} + \frac{\partial \overline{u}_p}{\partial u_2}\frac{\partial u_2}{\partial z} + \frac{\partial \overline{u}_p}{\partial u_3}\frac{\partial u_3}{\partial z} \qquad \ldots(2)$$

Also $C_1\nabla u_1 + C_2\nabla u_2 + C_3\nabla u_3 = C_1\left(\frac{\partial u_1}{\partial x}\hat{i} + \frac{\partial u_1}{\partial y}\hat{j} + \frac{\partial u_1}{\partial z}\hat{k}\right)$

$$+ C_2\left(\frac{\partial u_2}{\partial x}\hat{i} + \frac{\partial u_2}{\partial y}\hat{j} + \frac{\partial u_2}{\partial z}\hat{k}\right)$$

$$+ C_3\left(\frac{\partial u_3}{\partial x}\hat{i} + \frac{\partial u_3}{\partial y}\hat{j} + \frac{\partial u_3}{\partial z}\hat{k}\right)$$

$$= \left(C_1\frac{\partial u_1}{\partial x} + C_2\frac{\partial u_2}{\partial x} + C_3\frac{\partial u_3}{\partial x}\right)\hat{i}$$

$$+ \left(C_1\frac{\partial u_1}{\partial y} + C_2\frac{\partial u_2}{\partial y} + C_3\frac{\partial u_3}{\partial y}\right)\hat{j}$$

$$+ \left(C_1\frac{\partial u_1}{\partial z} + C_2\frac{\partial u_2}{\partial z} + C_3\frac{\partial u_3}{\partial z}\right)\hat{k} \qquad \ldots(3)$$

and $\overline{C}_1\nabla\overline{u}_1 + C_2\nabla\overline{u}_2 + \overline{C}_3\nabla\overline{u}_3 = \left(\overline{C}_1\frac{\partial\overline{u}_1}{\partial x} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial x} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial x}\right)\hat{i}$

$$+ \left(\overline{C}_1\frac{\partial\overline{u}_1}{\partial y} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial y} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial y}\right)\hat{j}$$

$$+ \left(\overline{C}_1\frac{\partial\overline{u}_1}{\partial z} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial z} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial z}\right)\hat{k} \qquad \ldots(4)$$

Equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ in (3) & (4),

We get

$$C_1\frac{\partial u_1}{\partial x} + C_2\frac{\partial u_2}{\partial x} + C_3\frac{\partial u_3}{\partial x} = \overline{C}_1\frac{\partial\overline{u}_1}{\partial x} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial x} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial x}$$

$$C_1\frac{\partial u_1}{\partial y} + C_2\frac{\partial u_2}{\partial y} + C_3\frac{\partial u_3}{\partial y} = \overline{C}_1\frac{\partial\overline{u}_1}{\partial y} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial y} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial y}$$

$$C_1\frac{\partial u_1}{\partial z} + C_2\frac{\partial u_2}{\partial z} + C_3\frac{\partial u_3}{\partial z} = \overline{C}_1\frac{\partial\overline{u}_1}{\partial z} + \overline{C}_2\frac{\partial\overline{u}_2}{\partial z} + \overline{C}_3\frac{\partial\overline{u}_3}{\partial z} \qquad \ldots(5)$$

Substituting equation (2) with p = 1, 2, 3 on R.H.S in the corresponding equation of (5) and equating coefficients of $\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x}$ or $\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, \frac{\partial u_3}{\partial y}$ or $\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, \frac{\partial u_3}{\partial y}$ on each side,

We can get,

Taking first equation of (5),

$$C_1 \frac{\partial u_1}{\partial x} + C_2 \frac{\partial u_2}{\partial x} + C_3 \frac{\partial u_3}{\partial x} = \overline{C}_1\left[\frac{\partial \overline{u}_1}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \overline{u}_1}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \overline{u}_1}{\partial u_3}\frac{\partial u_3}{\partial x}\right]$$

$$+ \overline{C}_1\left[\frac{\partial \overline{u}_2}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \overline{u}_2}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \overline{u}_2}{\partial u_3}\frac{\partial u_3}{\partial x}\right]$$

$$+ \overline{C}_1\left[\frac{\partial \overline{u}_3}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \overline{u}_3}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \overline{u}_3}{\partial u_3}\frac{\partial u_3}{\partial x}\right]$$

Here equating coefficients of $\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x}$

We get

$$C_1 = \overline{C}_1 \frac{\partial \overline{u}_1}{\partial u_1} + \overline{C}_2 \frac{\partial \overline{u}_2}{\partial u_1} + \overline{C}_3 \frac{\partial \overline{u}_3}{\partial u_1}$$

$$C_2 = \overline{C}_1 \frac{\partial \overline{u}_1}{\partial u_2} + \overline{C}_2 \frac{\partial \overline{u}_2}{\partial u_2} + \overline{C}_3 \frac{\partial \overline{u}_3}{\partial u_2}$$

$$C_3 = \overline{C}_1 \frac{\partial \overline{u}_1}{\partial u_3} + \overline{C}_2 \frac{\partial \overline{u}_2}{\partial u_3} + \overline{C}_3 \frac{\partial \overline{u}_3}{\partial u_3} \qquad \ldots(6)$$

or $$C_p = \overline{C}_1 \frac{\partial \overline{u}_1}{\partial u_p} + \overline{C}_2 \frac{\partial \overline{u}_2}{\partial u_p} + \overline{C}_3 \frac{\partial \overline{u}_3}{\partial u_p}, \ p = 1, 2, 3$$

or $$C_q = \sum_{q=1}^{3} \overline{C}_q \frac{\partial \overline{u}_q}{\partial u_p}, \ p = 1, 2, 3$$

Similarly, we cam show that

$$\overline{C}_p = \sum_{q=1}^{3} C_q \frac{\partial u_q}{\partial u_q}, \ p = 1, 2, 3 \qquad \ldots(7)$$

Equation (6) and (7) are required relation.

**Example: -** Show that square of element of arc length in general curvilinear coordinate can by

$$ds^2 = \sum_{q=1}^{3} \sum_{q=1}^{3} g_{pq} du_p du_q$$

**Solution: -** $(ds)^2$ m = $d\vec{r} = d\vec{r}.d\vec{r} = \left(\frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3}\right)$

$$\cdot\left(\frac{\partial \vec{r}}{\partial u_1}du_1+\frac{\partial \vec{r}}{\partial u_2}du_2+\frac{\partial \vec{r}}{\partial u_3}\right)$$

$$(ds)^2 = (\vec{\alpha}_1 du_1+\vec{\alpha}_2 du_2+\vec{\alpha}_3 du_3).(\vec{\alpha}_1 du_1+\vec{\alpha}_2 du_2+\vec{\alpha}_3 du_3)$$

where $\vec{\alpha}_p = \frac{\partial \vec{r}}{\partial u_p}$

$$= \vec{\alpha}_1.\vec{\alpha}_1 du_1^2+\vec{\alpha}_1.\vec{\alpha}_2 du_1 du_2+\vec{\alpha}_1.\vec{\alpha}_3 du_1 du_3$$

$$+\ \vec{\alpha}_2.\vec{\alpha}_1 du_2 du_1+\vec{\alpha}_2.\vec{\alpha}_2 du_2^2+\vec{\alpha}_2.\vec{\alpha}_3 du_2 du_3$$

$$+\ \vec{\alpha}_3.\vec{\alpha}_1 du_3 du_1+\vec{\alpha}_3.\vec{\alpha}_2 du_3 du_2+\vec{\alpha}_3.\vec{\alpha}_3 du_2 du_3^2$$

$$(ds)^2 = \sum_{q=1}^{3}\sum_{q=1}^{3} g_{pq} du_p du_q, \qquad g_{pq} = \vec{\alpha}_p.\vec{\alpha}_q$$

Thos is called **fundamental quadratic** form or **Metric form.** The quantities $g_{pq}$ are called **metric coefficients** and these are symmetric i.e. = $g_{pq} = g_{qp}$

If $g_{pq} = 0$ $p \neq q$ , the coordinate system is orthogonal.

and in this case $g_{11} = h_1^2$, $g_{22} = h_2^2$, $g_{33} = h_3^2$

Here also

$$\vec{\alpha}_1 = \frac{\partial \vec{r}}{\partial u_1} = h_1\hat{e}_1,\ \vec{\alpha}_2 = \frac{\partial \vec{r}}{\partial u_2}h_2\hat{e}_2,\ \vec{\alpha}_3 = \frac{\partial \vec{r}}{\partial u_3}h_3\hat{e}_3$$

**Example: - (a)** Prove that in general coordinaye ($u_1$, $u_2$, $u_3$)

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \left(\frac{\partial \vec{r}}{\partial u_1}.\frac{\partial \vec{r}}{\partial u_2}.\frac{\partial \vec{r}}{\partial u_3}\right)^2$$

where $g_{pq}$ are coefficients of $du_p\, du_q$ in $ds^2$

**Solution: -** we know that

$$g_{pq} = \vec{\alpha}_p.\vec{\alpha}_q = \frac{\partial \vec{r}}{\partial h_p}.\frac{\partial \vec{r}}{\partial u_q}$$

$$= \frac{\partial x}{\partial u_p}\frac{\partial x}{\partial u_q}+\frac{\partial y}{\partial u_p}\frac{\partial y}{\partial u_q}+\frac{\partial z}{\partial u_p}\frac{\partial z}{\partial u_q},\ p\ = 1, 2, 3$$

$$\left(\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial \vec{r}}{\partial u_2}\cdot\frac{\partial \vec{r}}{\partial u_3}\right)^2 = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix}$$

$$= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = g$$

**Example (b)** show that the volume element in general coordinate is $\sqrt{g}\ du_1\, du_2\, du_3$.

**Solution: -** The volume element is given by

$$dV = \left|\left(\frac{\partial \vec{r}}{\partial u_1}du_1\right)\cdot\left(\frac{\partial \vec{r}}{\partial u_2}du_2\right)\times\left(\frac{\partial \vec{r}}{\partial u_3}du_3\right)\right|$$

$$= \left|\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial \vec{r}}{\partial u_2}\times\frac{\partial \vec{r}}{\partial u_3}\right| du_1\, du_2\, du_3$$

$$= \sqrt{g}\ du_1\, du_2\, du_3$$

# LESSON 4 RANDOM VARIABLE AND MATHEMATICAL EXPECTATION

**Sample space**

The set points representing the possible outcomes of an experiment is called the sample space of the experiment.

**Example: -** (1) In tossing one coin, the sample space is S = {H, T}

(2) Two coins are tossed,

S = {HH, HT, TH, TT}

(3) In throw a dice,

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
M
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

Total outcome = 36

**Rondom Variable**

A random variable is a real valued function defined on a sample space. When an experiment is performed, several outcomes are possible corresponding to each outcome, a number can be associatd.

**Example: -** If two coins are tossed, the possible outcomes are TT, TH, HT, HH

Let X $\rightarrow$ denotes the number of heads

The number associated with the outcome are : TT, TH, HT, HH

No. of heads, X : 0, 1, 1, 2

The variable X is said to be random variable.

and may be defined as

“ Let S be the sample space associated with a given experiment. A real valued function defined on S & taking values in R( $-\infty, \infty$) ( real no.) is called a **random variable** (or chance variable or Stochastic variable or variate).

**Discrete and Continouse Sample Space**

A sample space that consists of a finite number or an infinite sequence, of points is called a **discrete sample space** and that consints of one or more intervals of points is called a cotinouse sample space.

**Discrete and Continouses Random Variable**

A random variable defined on a discrete sample space is called **discrete r.v.** or If a r.v. takes at most a countable no. values, it is called Discrete r.v.

A r.v. is said to be **Continuous** if it can takes al possible value between certain limits or certain interval.

**Example: -**

**1)** If X represents the sum of points on two dice, X is a discrete r.v.

**2)** If X represents the height or weight of students in a class , then it is a continuous r.v.

**3)** If X represents the amount of rainfall, it is a continuous r.v.

**Density function (d.f) or probability density function(p.d.f)**

A function associated with discrete r.v.

X s.t. f(x) = prob[X = x] is called density function of X.

**Example: -** In tossing twocoins,

Outcomes = {HH, TH, HT, TT}

X = [0, 1, 2 ]

$f(0) = P[X = 0] = \frac{1}{4}$

f(1) = P[X = 1] = 2/4 = ½

f(2) = P[X = 2] = ¼

**Example: -** In throwing two dice, the sample space of sum of points on two dice is

S = [1, 2, 3,……,12]

$f(2) = P[X = 2] = (1, 1) = \frac{1}{36} = f(12)\ (6,6)$

$f(3) = \frac{2}{36} = f(11)$ [Θ (2, 1), (1, 2)]

$f(4) = f(10) = \frac{3}{36}$

$f(5) = f(9) = \frac{4}{36}$

$f(6) = f(8) = \frac{5}{36}$

$f(7) = P[X = 7] = \frac{6}{36}$

Also $\sum f(x_i) = 1$

$X = x_i = 2, 3, \ldots\ldots 12$

**Distribution function**

For a r.v. X, the function $F(x) = P(X \leq x)$ is called the distribution function of X or **Cumulative distribution.**

Since $F(x) = P(X \leq x)$, then f(x) the p.d.f.

We have

$$F(x) = \sum_{t \leq x} f(t)$$

If $P(X \leq 2)$, then

$$\sum_{x \leq 2} f(x) = f(0) + f(1) + f(2)$$

**Example: -** A r.v. X has the following distribution

| X : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| f(x) : | k | 3k | 5k | 8k | 9k | 11k | 12k | 14k | 17k |

(1) Find k, As $\sum f(x) = 1$

$\Rightarrow \quad k + 3k + \ldots. + 17k = 1$

$\Rightarrow \quad 80k = 1 \quad \Rightarrow \quad k = \frac{1}{80}$

(2) Find $P(X < 2)$

$P(X = 0) + P(X = 1) = P(0) + P(1)$

$= k + 3k = 4k = \frac{4}{80} = \frac{1}{20}$

$P(X < 3) = P(0) + P(1) + P(2) = \frac{9}{80}$

(3) $P(X \geq 3) = P(3) + P(4) + \ldots + P(8) = \frac{71}{80}$

Also $P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{9}{80} = \frac{71}{80}$

(4) $P(0 < X < 5) = P(1) + P(2) + P(3) + P(4) = 25/80$

Distribution function F(x) is obtained

$$F(x) : \quad \frac{1}{80} \quad \frac{4}{80} \quad \frac{9}{80} \quad \frac{17}{80} \quad \frac{26}{80} \quad \frac{37}{80} \quad \frac{49}{80} \quad \frac{63}{80} \quad \frac{80}{80} = 1$$

**Joint density function**

Let X, Y be two r.v. , Joint d.f. gives the probability that X will assume Y will taken a value y

i.e. $f(x, y) = P(X = x, Y = y)$

**Example: -** 52 cards and 2 cards are drawn

$X \rightarrow$ no. of spade in the 1st draw

$Y \rightarrow$ no. of spade in the 2nd draw

Without replacing the 1st card drawn.

$$f(0, 0) = P[X = 0, Y = 0] = \frac{39}{52} \times \frac{38}{51}$$

$$f(0, 1) = P[X = 0, Y = 1] = \frac{39}{52} \times \frac{13}{51}$$

| X \ Y | 0 | 1 | Total |
|---|---|---|---|
| 0 | 19/34 | 13/68 | 51/68 |
| 1 | 13/68 | 1/17 | 17/68 |
| Total | 51/68 | 17/68 | 1 |

$$f(1, 0) = P[X = 1, Y = 0] = \frac{13}{52} \times \frac{39}{51}$$

$$f(1, 1) = P[X = 1, Y = 1] = \frac{13}{52} \times \frac{12}{51}$$

Let A & B be two events, then

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

**Conditional density function**

f(y/x) is defined as

f(y/x) = P[Y = y | X = x]

gives distribution of Y when X is fixed

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

and $$f(x/y) = \frac{f(x, y)}{f(x)}$$

**Marginal density function**

Here $f(x, y) = f(y/x).f(x)$ …(1)

Condidional distance Y = y when X = x is fixed.

$$\sum_y f(y/x) = 1$$

Summing over all possible values of y on bith sides of (1), we get

$$\sum_y f(x, y) = \sum_y f(y/x)\ f(x)$$

$$\Rightarrow \quad \sum_y f(x, y) = f(x)$$

This f(x) is known as Marginal density function of X.

Similarly $g(y) = \sum_x f(x, y)$

This gives Marginal function of Y.

**Example: -** 2 white & 4 black balls find probability of having two white ball.

$$X, Y = \begin{cases} 0 & \text{for black ball} \\ 1 & \text{for white black} \end{cases}$$

**Solution: -** f(0, 0) = f(0) f(0/0)

$$= \frac{4}{6}.\frac{3}{5} = \frac{6}{15}$$

$$f(0, 1) = f(0)\ f(1/0) = \frac{4}{6}.\frac{2}{5}$$

$$f(1,0) = f(1)\ f(0/1) = \frac{2}{6}.\frac{4}{5}$$

$$f(1, 1) = f(1)\ f(1/1) = \frac{2}{6}.\frac{1}{5}$$

Marginal density function of X is

$$f(x) = \sum_y f(x, Y)$$

$$\Rightarrow \quad f(0) = \sum_y f(0, y) = f(0, 0) + f(0, 1)$$

$$\Rightarrow \quad f0) = \frac{6}{15} + \frac{4}{15} = \frac{10}{15} = \frac{2}{3}$$

$$\& \quad f(1) = \sum_y f(1, y)$$

$$f(1, 0) + f(1, 1) = \frac{8}{30} + \frac{2}{30} = \frac{10}{30}$$

$$f(1) = \frac{1}{3} \qquad \ldots(2)$$

The conditional density function of Y for fixed x can be obtained from (1),

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

$$\therefore \quad f(0/1) = \frac{f(1,0)}{f(1)} = \frac{4/15}{1/3} = \frac{4}{5} \qquad \text{[using (1)]}$$

$$f(1/1) = \frac{f(1,1)}{f(1)} = \frac{1/15}{1/3} = \frac{1}{5}$$

**Continouse r.v. :-** $-\infty < X < \infty$

**Density function**

A d.f. for a continuouse r.v. X is a function f(x) that possesses the following properties

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\ dx = 1$

(iii) $\int_a^b f(x)\ dx = P[a < X < b]$ where $a < b$

**Distribution function**

$$F(x) = \int_{-\infty}^{\infty} f(x)\ dx = P[X \leq x]$$

Provided integral exists.

**Marginal density function**

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\ dy \rightarrow \text{Marginal density function of X}$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \rightarrow \text{Marginal density function of Y.}$$

**Independent Random Variable**

Two r.v. X and Y are said to be independent if f(x, y) = f(x) f(y)

**Example: -** Let the joint d.f. f(x, y) of r.v.'s X and Y be

$$f(x, y) = \begin{cases} k(xy + e^x) & , \quad 0 < x, y < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

1) Determine k
2) Examine whether X&Y are independent.

**Solution: -** 1) For d.f. $\iint f(x, y)\, dx\ dy = 1$

$$\therefore \qquad k \int_0^1 \int_0^1 (xy + e^x)\ dx\ dy = 1$$

$$\Rightarrow \qquad k \int_0^1 \left( \frac{x^2}{2} y + e^x \right)_0^1 dy = 1$$

$$\Rightarrow \qquad k \int_0^1 \left( \frac{y}{2} + e - 1 \right) dy = 1$$

$$\Rightarrow \qquad k \left[ \frac{y^2}{4} + ey - y \right]_0^1 = 1$$

$\Rightarrow \quad k\left[\frac{1}{4}+e-1\right]=1$

$\Rightarrow \quad k\left(e-\frac{3}{4}\right)=1$

$\Rightarrow \quad k = \frac{4}{4e-3}$

**(2)** $\quad f(x) = \int_0^1 f(x,y)\ dy$

$f(y) = \int_0^1 f(x,y)\,dx$

check whether f(x, y) = f(x) f(y)

Now $\quad f(x) = \int_0^1 k(xy+e^x)\ dy$

$$= k\left[\frac{xy^2}{2}+e^x y\right]_0^1$$

$$= k\left[\frac{x}{2}+e^x\right]$$

$$f(y) = k\int_0^1 (xy+e^x)\ dx$$

$$= k\left[\frac{x^2}{2}y+e^x\right]_0^1$$

$$= k\left[\frac{y}{2}+e-1\right]$$

So $\quad f(x,y) \neq f(x)\, f(y)$ so they are not independent.

**Example: -** $f(x, y) = \begin{cases} 2 & , \quad 0<x<1, 0<y<x \\ 0 & , \quad \text{otherwise} \end{cases}$

1) Find marginal d.f. of X and Y
2) Find conditional d.f. of Y given X = x

   Find conditional d.f. of X given Y = y

3) Check whether X&Y are independent or not?

**Solution: -**(1) $f(x) = \int_0^x f(x,y)\ dy$

$= \int_0^x 2\ dy = (2y)_0^x = 2x,\ 0 < x < 1$

$f(y) = \int_y^1 f(x,y)\ dx = 2\int_y^1 dx = (2y)_y^1$

$= 2(1 - y),\ 0 < y < 1$

So $f(x, y) \neq f(x)\ f(y)$

$\Rightarrow$ X and Y are not independent.

(2) $f(Y/X = x) = \dfrac{f(x,y)}{f(x)} = \dfrac{2}{2x} = \dfrac{1}{x}, \qquad 0 < x < 1$

$f(X/Y = y) = \dfrac{f(x,y)}{f(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}, \qquad 0 < y < 1$

**Example: -** A continoues r.v. X has p.d.f. $f(x) = 3x^2,\ 0 \le x \le 1$. Find a abd b s.t.

(1) $P(X \le a) = P(X \ge a)$

(2) $P(X \le b) = 0.05$

**Solution: -** Since $P(X \le a) = P(X > a)$

So, each must be equal to ½ because total probability is always one i.e. $\int_0^a + \int_a^1 = 1$

$P(X \le a) = \dfrac{1}{2} \Rightarrow \int_0^a f(x)\ dx = \dfrac{1}{2}$

$\Rightarrow \quad 3\int_0^a x^2\ dx = \dfrac{1}{2} \Rightarrow a = \left(\dfrac{x^3}{3}\right)_0^a = \dfrac{1}{2}$

$\Rightarrow \quad a^3 = \dfrac{1}{2} \Rightarrow a = \left(\dfrac{1}{2}\right)^{1/3}$

Also $\int_a^1 f(x)\ dx = \int_a^1 3x^2\ dx = \dfrac{1}{2}$

$\Rightarrow \quad 3\left(\frac{x^3}{3}\right)_a^1 = \frac{1}{2} \Rightarrow 1 - a^3 = \frac{1}{2}$

$\Rightarrow \quad a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}$

(2) $\quad P(X > b) = 0.05 \Rightarrow \int_b^1 f(x)\ dx = 0.05$

$\Rightarrow \quad 3\int_b^1 x^2\ dx = 0.5 \qquad \Rightarrow \quad 3\left(\frac{x^3}{3}\right)_b^1 = 0.05$

$\Rightarrow \quad (1 - b^3) = \frac{1}{20} \qquad \Rightarrow \quad b^3 = \frac{19}{20}$

$\Rightarrow \quad b = \left(\frac{19}{20}\right)^{1/3}$

**Example: -** If X be a continuous r. v. with

$$f(x) = \begin{cases} ax & , \quad 0 \le x \le 1 \\ a & , \quad 1 \le x \le 2 \\ -ax + 3a, & \quad 2 \le x \le 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(1) Find constant a

(2) Find $P(X \le 1.5)$

**Solution: -** (1) $\int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^{\infty} f(x)dx = 1$

$\Rightarrow \quad 0 + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + 0 = 1$

$\Rightarrow \quad \int_0^1 ax\ dx + \int_1^2 a\ dx + \int_2^3 (3a - ax)\ dx = 1$

$\Rightarrow \quad a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + a\left(3x - \frac{x^2}{2}\right)_2^3 = 1$

$\Rightarrow \quad \frac{a}{2} + a + a + \left(9 - \frac{9}{2} - 6 + 2\right) = 1$

$\Rightarrow \quad \frac{3a}{2} + \frac{a}{2} = 1$

$\Rightarrow \quad a = \frac{1}{2}$

(2) Now $P(X \le 1.5) = \int_{-\infty}^{1.5} f(x)\,dx$

$= \int_{-\infty}^{0} f\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$

$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{3/2} a\,dx$

$= a\left(\frac{1}{2}\right) + a(x)_{1}^{3/2} = \frac{a}{2} + a\left(\frac{3}{2} - 1\right)$

**Example: -** From the given bivariate probability distribution

(1) Obtain marginal distribution of X & Y.

(2) The conditional distribution of X given Y = 2

| | −1 | 0 | 1 | $\sum_x f(x,y) = f(y)$ |
|---|---|---|---|---|
| 0 | 1/15 | 2/15 | 1/15 | **4/15** |
| 1 | 3/15 | 2/15 | 1/15 | **6/15** |
| 2 | 2/15 | 1/15 | 2/15 | **5/15** |
| **f(x) =** $\sum_y f(x,y)$ | **6/15** | **5/15** | **4/15** | |

**Solution**: (1) Marginal distribution of X

From above table $f(x) = \sum_y f(x,y)$

Therefore $f(-1) = P(X = -1) = 6/15$

$f(0) = P(X = 0) = 5/15$

$f(1) = P(X = 1) = 4/15$

Marginal distribution of Y

From above table $f(y) = \sum_{x} f(x, y)$

Therefore $P(Y = 0) = 4/15$

$P(Y = 1) = 6/15$

$P(Y = 2) = 5/15$

(2) Conditional distribution of X given Y = 2, we get

$$P(X = x/y = 2) = \frac{P(X = x, Y = 2)}{P(Y = 2)}$$

For $X = -1$, $P(X = -1/Y = 2) = \frac{2/15}{5/15} = \frac{2}{5}$

Similarly $P(X = 0/Y = 2) = \frac{1/15}{5/15} = \frac{1}{5}$

$P(X = 1/Y = 2) = \frac{2/15}{5/15} = \frac{2}{5}$

**MATHEMATICAL EXPECTATION:-**

Let X be a r.v. with p.d.f. f(x), then its mathematical expectation (or its mean value) is defined as

$$E(X) = \sum x f(x)$$

It X assumes values $x_1, x_2 \ldots\ldots x_n \ldots.$

With probalities $f(x_1), f(x_2) \ldots\ldots f(x_n) \ldots\ldots$

then $E(X) = \sum_{i=1}^{\infty} x_i f(x_i)$

Also E(X) = Mean of distribution and $\sum f(x_i) = 1$

The expected value or Mathematical expectation of the function g(x) of discrete r.v. X, whose p.d.f. is f(x) is given by

$$E[g(X)] = \sum_{i=1}^{\infty} g(x_i) f(x_i)$$

and $f(x_i) = P(X = x_i)$

**Example: -** When 3 coins are tossed and

X$\rightarrow$ represents the number of heads is a r. v., then total outcomes are

{HHH, HTH, HHT, THH, THT, TTH, HTT, TTT}

Here X can take values  X = 0, 1, 2, 3

with  $f(x) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$

Then  $E(X) = \sum x f(x) = 0.\frac{1}{8} + 1.\frac{3}{8} + 2.\frac{3}{8} + 3.\frac{1}{8}$

$$= \frac{12}{8} = \frac{3}{2} \times 1 \text{ Rs.} = \text{Rs. } 1.50$$

If $g(x) = x^2$

Then $g(x) = 0^2, 1^2, 2^2, 3^2$

$\therefore$  $E\,[g(x)] = \sum g(x_i) f(x_i)$

$$= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \text{Rs. } 3.0$$

**Mathematical Expectation for Continuous r.v.**

Let X be a continuous r.v. with p.d.f. f(x), then its mathematical expectation is

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

For function g(x),  $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

**Theorem**: - If C is a finite real number & if E(X) exists, then

**Proof: -** for continuous r.v.

$$\text{Let } E\,(CX) = \int_{-\infty}^{\infty} C\, x\, f(x)\, dx$$

$$= C \int_{-\infty}^{\infty} x f(x)\, dx = C\, E(X)$$

for discrete r.v.;

$$E(CX) = \sum C x f(x) = C \sum x f(x)$$

$$= CE(X)$$

**Result**: - E[a + CX] = E(a) + CE(X) = a + CE(X)

**Proof**: - Now E(a) = $\sum af(x) = a\sum f(x)$

$= a.1 = a$ … (1)

for continuous r.v., E(a) = $\int_{-\infty}^{\infty} af(x)\,dx$

$\Rightarrow \quad E(a) = a\int_{-\infty}^{\infty} f(x)\,dx = a.1 = a$

Now By definition, E[a + CX] = $\sum (a + Cx)f(x)$

$\Rightarrow \quad E[a + CX] = \sum[af(x) + Cxf(x)] = \sum af(x) + \sum Cxf(x)$

$= \sum af(x) + CE(X) = E(a) + CE(X)$

$\Rightarrow \quad E(a + CX) = a + CE(X)$ …using (1)

**Theorem**: - The expectation of the sum of two r.v.´s is equal to the sum of their expectations, i.e., E(X + Y) = E(X) + E(Y)

**Proof**: - For discrete case

Let X & Y be two discrete r.v., $f(x_i, y_i)$ is the joint p.d.f. of X and Y. then (X + Y) is also a r.v.

$f(x_I, y_i) = P(X = x_i, Y = y_j)$

Now by definition,

$$E(X + Y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$$

$$= \sum_i \sum_j x_i f(x_i, y_j) + \sum_i \sum_j y_j f(x_i, y_j)$$

$$= \sum_i x_i \sum_j (x_i, y_j) + \sum_j y_j \sum_i f(x_i, y_j)$$

$$= \sum_i x_i f(x_i) + \sum_j y_j f(y_j)$$

$\Rightarrow \quad E(X + Y) = E(X) + E(Y)$

For continuous r.v.

$$E(X + Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f(x, y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[ \int_{-\infty}^{\infty} f(x, y)\, dy \right] dx + \int_{-\infty}^{\infty} y \left[ \int_{-\infty}^{\infty} f(x, y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} xf(x)\, dx + \int_{-\infty}^{\infty} yf(y)\, dy$$

$\Rightarrow$ E(X + Y) = E(X) + E(Y)

In general,

$E(X_1 + X_2 + \ldots\ldots + X_n) = E(X_1) + E(X_2) + \ldots\ldots + E(X_n)$

**Theorem: -** If $Y = a_1X_1 + a_2X_2 + \ldots..+a_nX_n$, where a's are constants, then

$E(Y) = a_1E(X_1) + a_2E(X_2) +\ldots..+ a_nE(X_n)$

**Proof: -** As $E(Y) = E(a_1X_1 + a_2X_2+\ldots...+ a_nX_n)$

$= E(a_1X_1) + E(a_2X_2) + \ldots\ldots+ E(a_nX_n)$

$= a_1E(X_1) + a_2E(X_2) + \ldots\ldots+ a_nE(X_n)$

$[\because E(aX) = aE(X)]$

**Theorem: -** If X is a continuous r.v. and a and b are constants, then E(aX + b) = a E(X) + b, provided all expectation exists.

**Proof: -** By definition, we have

$$E(aX + B) = \int_{-\infty}^{\infty} (ax + b)\, f(x)dx$$

$$= \int_{-\infty}^{\infty} axf(x)\, dx + \int_{-\infty}^{\infty} bf(x)\, dx$$

$$= a \int_{-\infty}^{\infty} xf(x)\, dx + b \int_{-\infty}^{\infty} f(x)\, dx$$

$\Rightarrow$ E(aX + b) = aE(X) + b $\left[\because \int_{-\infty}^{\infty} f(x)dx = 1\right]$

**Case1: -** If b = 0, E (aX) = aE(X)

**Case2: -** If a = 1, $b = -\overline{X} = -E(X)$, we get $E(X-\overline{X}) = 0$

**Example: -** If f(x, y) = $e^{-(x+y)}$ is the joint p.d.f. of r.v. X and Y, then find P = (1 < X < 2, 0 < Y < 2)

**Solution: -** First we check for:-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\text{L.H.S} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x+y)}dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x}e^{-y}dx\,dy$$

$$= \int_{-\infty}^{\infty}\left[-e^{-x}e^{-y}\right]_{-\infty}^{\infty} dy$$

$$= \int_{-\infty}^{\infty} P(1 < X < 2, 0 < Y < 2)$$

$$= \int_{1}^{2}\int_{0}^{2} e^{-(x+y)}dy\,dx$$

$$= \int_{1}^{2}\int_{0}^{2} e^{-x}e^{-y}dy\,dx$$

$$= \int_{1}^{2} e^{-x}\,dx\int_{0}^{2} e^{-y}\,dy = (1-e^{-2})\int_{1}^{2} e^{-x}\,dx$$

$$= (1 - e^{-2})\,(e^{-1} - e^{-2}) = \left(1-\frac{1}{e^2}\right)\left(\frac{1}{e}-\frac{1}{e^2}\right)$$

$$= \left(\frac{e^2-1}{e^2}\right)\left(\frac{e-1}{e^2}\right)$$

$$= \frac{(e^2-1)(e-1)}{e^4} = \frac{(e^3-e^2-e+1)}{e^4}$$

**Example: -** Let $f(x_1, x_2) = \begin{cases} C(x_1x_2 + e^{x_1}),\ 0 < (x_1, x_2) < 1 \\ 0 \qquad\qquad ,\ \text{otherwise} \end{cases}$

(i) Determine C

(ii) Examine whether $X_1$ and $X_2$ are independent or not.

**Solution: -** For density function $F(x_1, x_2)$ we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$\Rightarrow$ $$C\int_0^1 \left[\frac{x_1^2 x_2}{2} + e^{x_1}\right]_0^1 dx_2 = 1$$

$\Rightarrow$ $$C\int_0^1 \left[\frac{1}{2}x_2 + e - 1\right] dx_2 = 1$$

$\Rightarrow$ $$C\left[\frac{1}{2}\times\frac{x_2^2}{2} + ex_2 - x_2\right]_0^1 = 1$$

$\Rightarrow$ $$C\left[\frac{1}{4} + e - 1\right] = 1$$

$\Rightarrow$ $C\,[1 + 4e - 4] = 4$

or $$C = \frac{4}{4e-3}$$

**Theorem: -** Show that the expectation of the product of independent r.v. is equal to product of their expectations. i.e. E(XY) = E(X).E(Y)

**Proof:** - Let X and Y are two independent random variables, then

X: $x_1, \quad x_2, \ldots\ldots, x_n$

With d.f: - $f(x_1), \; f(x_2), \ldots f(x_n)$

and Y : $y_1, \quad y_2, \ldots\ldots, \; y_m$

with d.f: $f(y_1), f(y_2), \ldots\ldots f(y_m)$

$\therefore$ $E(X) = \sum_{i=1}^{n} x_i f(x_i)$, $\qquad E(Y) = \sum_{j=1}^{m} y_j f(y_j)$

Let $f(x_i,y_j)$ is the joint p.d.f. of X & Y.

and Since X & Y are independent, so

$f(x_i, y_j) = f(x_i)\, f(y_j)$

Now $$E(XY) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j\, f(x_i, y_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j f(x_i) f(y_j)$$

$$= \sum_i x_i f(x_i) \sum_j y_j f(y_j)$$

E(XY) = E(X) E(Y)

Similarly for continuous r.v.: – f(x, y) = f(x) f(y) , since X and Y are independent.

Now $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx\, dy$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x) f(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \int_{-\infty}^{\infty} y f(y)\, dy = E(X)\, E(Y)$$

In general, we get

$$E(X_1 X_2 X_3 \ldots.. X_n) = E(X_1)\, E(X_2) \ldots\ldots E(X_n)$$

**Example: -** Let X represents the number on the face of dice then

| | | | | | | |
|---|---|---|---|---|---|---|
| X: | 1 | 2 | 3 | 4 | 5 | 6 |
| f(x): | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

Now $E(X) = \sum x f(x)$

$$= \frac{1}{6}(1 + 2 + \ldots\ldots. + 6) = \frac{21}{6} = \frac{7}{2}$$

And when X is sum of points when two dies are thrown, i.e.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| f(x): | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

$\therefore$ $E(X) = \sum x f(x) = \frac{1}{36}[2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12]$

$\Rightarrow$ $E(X) = \frac{1}{36}.252 = 7$

**Example: -** Let X be the profit that a person makes in a business. He may earn Rs. 2800 with a probability 0.5, he may lose Rs 5500 with probability 0.3 and he may neither earn nor lose with a probability 0.2. Calculate E(X)

**Solution: -** Here P(X = 2800) = 0.5

P(X = −5500) = 0.3

P(X = 0) = 0.2

Then $E(X) = \sum xf(x)$

$= 2800(0.5) + (-5500)(0.3) + (0)(0.2)$

$= 1400 - 1650 = -250$, he may lose Rs 250.

**Example: -** A and B in turns throw an ordinary dice for a price of Rs. 44. The first to throw a "six" wins. If A has first throw, what is his expectation?. Also calculate B's expectation

**Solution: -** The problem of getting a "six" on dice is $p(x) = \frac{1}{6}$

A has $1^{st}$ throw, so he can win in the $1^{st}$, $3^{rd}$, $5^{th}$ ……

Hence A′ chance (probability) of winning is

$$= \frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} \ldots\ldots$$

$$= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\ldots\right]$$

$$= \frac{1}{6}.\frac{1}{1-\left(\frac{5}{6}\right)^2} = \frac{1}{6}.\frac{36}{11} = \frac{6}{11}$$

∴ Amount of A = $44 \times \frac{6}{11}$ = Rs. 24

Similarly B can win in $2^{nd}$, $4^{th}$, $6^{th}$,…..

Hence B′(chance) of winning are

$$= \frac{5}{6}.\frac{1}{6} + \frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{1}{6} + \ldots\ldots$$

$$= \frac{5}{6}.\frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{6}.\frac{5}{6}.\frac{36}{11} = \frac{5}{11}$$

$\therefore$ Amount of B = $\frac{5}{11} \times 49$ = Rs. 20

**Example: -** A bag contains a coin of value M and a number of other coins whose aggregate value is m. a person draws one at a time till he draws the coin M, find the value of his expectation.

**Solution: -** Let there be K other coins each of value m/K, so that their aggregate value is m. he may draw the coin M at 1st draw or 2nd or 3rd or…..or (K+1)th draw with probability $\frac{1}{K+1}, \frac{1}{K+1}, \frac{1}{K+1}, \ldots\ldots$

$$\Theta \quad \frac{1}{K+1}, \left[1 - \frac{1}{K-1}\right]\frac{1}{K}, \left[1 - \frac{1}{K+1}\right]\left[1 - \frac{1}{K}\right]\frac{1}{K-1}, \ldots\ldots.$$

$$= \frac{1}{K+1}, \frac{1}{K+1}, \frac{1}{K+1}, \ldots\ldots\ldots.$$

The corresponding amount drawn X is

$$M, \frac{m}{K} + M, \frac{2m}{K} + M, \ldots\ldots., \frac{(k-1)m}{K} + M, \frac{Km}{k} + M$$

$$\therefore \quad E(X) = \frac{1}{K+1}\left[M + \frac{m}{K} + M + \frac{2m}{K} + M + \ldots\ldots. + \frac{Km}{K} + M\right]$$

$$= \frac{1}{K+1}\left[M + (K+1) + \frac{m}{K}(1 + 2 + 3 + \ldots\ldots + K)\right]$$

$$= M + \frac{1}{K+1}.\frac{m}{K}\frac{K(K+1)}{2} = M + \frac{m}{2}$$

# LESSON 5 MOMENTS AND MOMENT GENERATING FUNCTIONS

**Moments: -** Let X is a r. v., then E $[X^r]$, if exists is called the $r^{th}$ moment of X about origin and is denoted by $\mu'_r$

i.e $\mu'_r = E[X^r]$ … (1)

and about some point 'a', it is defined as

$\mu'_r(a) = E[(X-a)^r]$

and $r^{th}$ moment about mean is

$\mu_r = E[(X - E(X)^r] = E[(X-\mu)^r]$ … (2)

**Moments about Mean are called Central Moments.**

In case of discrete r.v.:-

$\mu'_r = E[X^r] = \sum x^r f(x)$

& $\mu_r = E[(X-\mu)^r] = \sum_{i=1}^{n} (x_i - \mu)^r f(x_i)$

Where $\mu = E(X)$

and $f(x_i) = P(X = x_i)$

when r = 1 from (1), we get

$\mu'_r = E(X) = \sum x f(x)$ = Mean

and from (2),

$\mu_1 = E(X-\mu) = E(X) - E(\mu)$

$\Rightarrow$ $\mu_1 = \mu - \mu = 0$ $[\because E(\mu) = \mu, E(X) = \mu]$

Moments about Mean ($\mu_r$) in terms of moments about any point 'a':-

Let $X \rightarrow$ r. v.,

$E(X) = \overline{X} = \mu$

then $\mu_r = E[X-\mu]^r = E[X - a - \mu + a]^r$

$\Rightarrow$ $\mu^r = E(X - a - d)^r$

Where $d = \mu - a$

or $\mu_r = E[(X-a)^r - {}^rC_1 d(X-a)^{r-1} + \ldots .. + {}^rC_{r-1}(-1)^{r-1} d^{r-1}(X-a) + d^r(-1)^r]$

$\Rightarrow$ $\mu_r = \mu'_r(a) - {}^rC_1 d \mu'_{r-1}(a) + \ldots\ldots + {}^rC_{r-1}(-1)^{r-1} d^{r-1} E(X-a) + d^r(-1)^r$

Now $\mu_1 = 0$ $\quad$ $[\mu_1 = \mu'_1(a) - d = \mu'_1(a) - \mu'_1(a) = 0]$

and $\quad \mu_2 = \mu'_2(a) - 2d\ \mu'_1(a) + d^2$

$\Rightarrow \quad \mu_2 = \mu'_2 - \mu'^2_1$ $\quad$ $[\Theta\ d = E(X) - a = E(X - a),\ d = \mu'_1(a)]$

This $\mu_2$ is called Variance. Similarly,

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$$

and $\quad \mu_4 = \mu'_4 - 4\mu'_3 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$

If X is a continuous r.v.

$$\mu'_r = \int_{-\infty}^{\infty} (x - a)^r f(x)\,dx$$

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x)\,dx$$

and $\quad E(X) = \mu'_1 \rightarrow$ Mean

Also $\quad \mu_2 = E(X - \mu)^2$. This $E(X - \mu)^2$ is called Variance and is denoted by $\sigma^2$.

**Covariance between X and Y**

$$\text{Cov}(X, Y) = E[(X - \overline{X})(Y - \overline{Y})]$$

$$= E[(X - E(X))(Y - E(y))]$$

**Example: -** Find $E(X)$, $E(X^2)$, $E(X - E(X))^2$ from the following:-

| X: | 8 | 12 | 16 | 20 | 2 |
|---|---|---|---|---|---|
| f(x): | $\frac{1}{8}$ | $\frac{1}{6}$ | $\frac{3}{8}$ | $\frac{1}{4}$ | $\frac{1}{12}$ |

**Solution: -** $E(\mathbf{X}) = \sum x f(x)$

$$= 8 \times \frac{1}{8} + 12 \times \frac{1}{6} + \ldots\ldots + 24 \times \frac{1}{12}$$

$\Rightarrow \quad E(X) = 16 =$ Mean

**an**d $\quad E(X^2) = \sum x^2 f(x) = 64 \times \frac{1}{8} + 144 \times \frac{1}{6} + 256 \times \frac{3}{8} + \ldots\ldots + 576 \times \frac{1}{12}$

$\Rightarrow \quad E(X^2) = 276$

$\mu_2 = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$ $\quad$ $[\Theta \mu_2 = \mu'_2 - \mu'^2_1]$

$\Rightarrow \quad \mu_2 = 276 - (16)^2 = 20$

This $\mu_2 = 20$ is the variance.

**Example: -** Find E(X), E($X^2$), $E(X - E(X))^2$

| X: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| f(x): | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

**Solution: -** $E(X) = \sum xf(x) = \frac{1}{36}$ [2+6+12+20 + 30 + 42 + 40 + 36 + 30 + 22 + 12]

$= 7 = \text{Mean}$

$E(X^2) = \sum Ex^2 f(X) = \frac{1}{36}$ [4+18+48+10+305+49+64+81+100+121+144]

$= \frac{1}{36}(1974) = \frac{329}{6}$

$\mu_2 = E\,[X - E(X)]^2 = E(X^2) - [E(X)]^2$

$= \frac{1974}{36} - 49 = \frac{329}{6} - 49$

$= \frac{329 - 294}{6} = \frac{35}{6}$

Let m represents median, then

$P(X < m) = P(X > m)$

In this eg, m = 7

mode = 7

Also mean = 7

So it is a very good distribution.

**Example: -** For distribution**,**

$$f(x) = \begin{cases} \frac{1}{16}(3\_x^2 & , \quad -3 < x \le -1 \\ \frac{1}{16}(6 - 2x^2) & , \quad -1 < x \le 1 \\ \frac{1}{16}(3 - x)^2 & , \quad 1 < x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Check whether it represents a probability distribution or not. Also find its mean and variance.

**Solution: -** First we prove that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \text{or} \qquad \int_{-3}^{3} f(x)\,dx = 1$$

Now $\displaystyle\int_{-\infty}^{\infty} f\,dx = \left(\int_{-\infty}^{-3} + \int_{-3}^{-1} + \int_{-1}^{1} + \int_{1}^{3} + \int_{3}^{\infty}\right) f\,dx$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(dx) = \int_{-3}^{-1} \frac{1}{16}(3+x^2)dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)dx + \int_{1}^{3} \frac{1}{16}(3-x^2)dx$$

$$= \frac{1}{16}\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \frac{1}{16}\left[x^6 - \frac{2x^3}{3}\right]_{-1}^{1} + \frac{1}{16}\left[\frac{(3-x)^3}{-3}\right]_{1}^{3}$$

$$= \frac{1}{48}[8-0] + \frac{1}{16}\left[6 - \frac{2}{3} + 6 - \frac{2}{3}\right] - \frac{1}{48}[0-8]$$

$$= \frac{1}{6} + \frac{1}{16}\left(12 - \frac{4}{3}\right) + \frac{1}{6}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = \frac{1}{3} + \frac{1}{16}.\frac{32}{3} = \frac{1}{3} + \frac{2}{3} = 1$$

Also Mean = E(X) = $\displaystyle\int_{-\infty}^{\infty} x f(x)\,dx$

$$\Rightarrow \quad E(X) = \int_{-3}^{3} x f(x) dx$$

$$= \int_{-3}^{-1} \frac{x}{16}(3+x)^2 dx + \int_{-1}^{1} \frac{x}{16}(6-2x^2)dx + \int_{1}^{3} \frac{x}{16}(3-x)^2 dx$$

$$= \frac{1}{16}\left[x\frac{(3+x)^3}{3}\right]_{-3}^{-1} - \int_{-3}^{-1} \frac{1}{16}(1)\frac{(3+x)^3}{3}dx + 0$$

$$+ \frac{1}{16}\left\{\left[\frac{x(3-x)^3}{-3}\right]_{1}^{3} - \int_{1}^{3}(1)\frac{(3-x)^3}{-3}dx\right\}$$

[Integrand in second integral is an odd function of x, so its value is zero]

$$= \frac{1}{48}[(-1)8+0] - \frac{1}{48}\left[\frac{(3+x)^4}{4}\right]_{-3}^{-1} - \frac{1}{48}[0-1{,}8] + \frac{1}{48}\left[\frac{(3-x)^4}{-4}\right]_1^3$$

$$= -\frac{1}{6} - \frac{1}{48}.\frac{1}{4}(16-0) + \frac{1}{6} - \frac{1}{48}.\frac{1}{4}(0-16)$$

$$\Rightarrow \quad E(X) = -\frac{1}{48}.\frac{1}{4}.16 + \frac{1}{48}.\frac{1}{4}.16 = -\frac{1}{12} + \frac{1}{12} = 0$$

Also Variance = V(X) = $E(X - E(X))^2 = E(X^2) - [E(X)]^2$

Now

$$E(X^2) = \int_{-3}^{3} x^2 f(x)dx$$

$$= \int_{-3}^{-1} \frac{x^2}{16}(3+x)^2 dx + \int_{-1}^{1} \frac{x^2}{16}(6-2x^2)dx + \int_{1}^{3} \frac{x^2}{16}(3-x)^2 dx$$

$$\Rightarrow \quad E(X^2) = \int_{-3}^{-1} \frac{x^2}{16}(9+x^2+6x)dx + \int_{-1}^{1} \frac{1}{16}(6x^2-2x^4)dx + \int_{1}^{3} \frac{x^2}{16}(9+x^2-6x)x$$

$$= \frac{1}{16}\left[\frac{9x^3}{3} + \frac{x^5}{5} + \frac{6x^4}{4}\right]_{-3}^{-1} + \frac{1}{16}\left[\frac{6x^3}{3} - \frac{2x^5}{5}\right]_{-1}^{1}$$

$$+ \frac{1}{16}\left[\frac{9x^3}{3} + \frac{x^5}{5} - \frac{6x^4}{4}\right]_1^3$$

$$= \frac{1}{16}\left(-3 - \frac{1}{5} + \frac{3}{2} + 81 + \frac{243}{5} - \frac{3}{2}\times 81\right) + \frac{1}{16}\left(2 - \frac{2}{5} + 2 - \frac{2}{5}\right)$$

$$+ \frac{1}{16}\left[81 + \frac{243}{5} - \frac{3}{2}\times 81 - 3 - \frac{1}{5}\times 81\right]$$

$$= 2.\ \frac{1}{16}\left[-3 - \frac{1}{5} + \frac{3}{2} + 81 + \frac{243}{5} - \frac{243}{2}\right] + \frac{1}{16}\left(4 - \frac{4}{5}\right)$$

$$= \frac{1}{8}\left[78 - \frac{240}{2} + \frac{242}{5}\right] + \frac{1}{16}.\frac{16}{5}$$

$$= \frac{1}{8}\left[-42 + \frac{242}{5}\right] + \frac{1}{5}$$

$\Rightarrow \quad E(X^2) = \frac{1}{8}.\frac{32}{5} + \frac{1}{5} = \frac{4}{5} + \frac{1}{5} = 1$

Hence Variance = $E(X^2) - [E(X)]^2$

$= 1 - 0^2 = 1$

**Example: -** Show that the value of cov(X, Y) for probability distribution

$$f(x, y) = \begin{cases} \frac{1}{9} e^{-y/3}, & 0 \le x \le y \le \infty \\ 0 \quad , & \text{otherwise} \end{cases} \quad \text{is 9.}$$

**Solution: -** We Know that Cov.(X,Y) = E(XY) − E(X) E(Y)

Thus we have to find E(X), E(Y) and E(XY)

But $\quad E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$E(Y) = \int_{-\infty}^{\infty} y f(y) dy$

$\therefore \quad$ First we have to find f(x) and f(y).

Now marginal density function of X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} f(x, y) dy + \int_{x}^{\infty} f(x, y) dy$$

$$= \int_{x}^{\infty} \frac{1}{9} e^{-y/3} dy = \left[\frac{1}{9} e^{-y/3}(-3)\right]_{x}^{\infty}$$

$\Rightarrow \quad g(x) = \frac{1}{3} e^{-y/3} \quad ,0 \le x \le \infty$

Similarly Marginal density function of Y will be

$$h(y) = \int_{0}^{y} \frac{1}{9} e^{-y/3} dx = \frac{y}{9} e^{-y/3}, \; 0 \le y \le \infty$$

$\therefore \quad E(X) = \int_{0}^{\infty} x f(x) dx$

$$= \int_{0}^{\infty} x . \frac{1}{3} e^{-x/3} dx = \frac{1}{3} \int_{0}^{\infty} x \; e^{-x/3} \, dx$$

$$= \frac{1}{3}\left[\left[x.e^{-x/3}(-3)\right]_0^{\infty} + 3\int_0^{\infty} e^{-x/3}\,dx\right]$$

$$= \frac{1}{3}\left[0 + 3\left[e^{-x/3}(-3)\right]\right]_0^{\infty} = 3 \qquad \ldots (1)$$

$$E(Y) = \int_0^{\infty} y\,h(y)dy = \frac{1}{9}\int_0^{\infty} y^2 e^{-y/3}dy$$

$$= \frac{1}{9}\left[\left\{y^2 e^{-y/3}(-3)\right\}_0^{\infty} + 6\int_0^{\infty} ye^{-t/3}dy\right]$$

$$= \frac{2}{3}\left[\left[ye^{-y/3}(-3)\right]_0^{\infty} + 3\int_0^{\infty} e^{-y/3}dy\right]$$

$$= \frac{2}{3}\times 3\left[e^{-y/3}(-3)\right]_0^{\infty} = 2 \times 3 = 6 \qquad \ldots (2)$$

$$E(XY) = \int_{x=0}^{\infty}\int_{y=x}^{\infty} xyf(x, y)\,dx\,dy$$

$$= \int_{x=0}^{\infty}\int_{y=x}^{\infty} xy.\frac{1}{9}e^{-y/3}dx\,dy$$

$$= \frac{1}{9}\int_{x=0}^{\infty} x\left[\int_{y=x}^{\infty} ye^{-y/3}dy\right]dx$$

$$= \frac{1}{9}\int_{x=0}^{\infty} x\left\{\left[ye^{-y/3}(-3)\right]_x^{\infty} + 3\int_x^{\infty} e^{-y/3}dy\right\}dx$$

$$= \frac{1}{9}\int_{x=0}^{\infty} x\left(3xe^{-x/3} + 3(3)e^{-x/3}\right)dx$$

$$= \frac{1}{9}\int_0^{\infty}\left[3x^2e^{-x/3} + 9e^{-x/3}x\right]dx$$

$$= \frac{1}{3}\left[\int_0^{\infty} x_I^2 e_{II}^{-x/3}\,dx + 3\int_0^{\infty} e_{II}^{-x/3} x_I\,dx\right]$$

$$= \frac{1}{3}[54 + 3 \times 9]$$

$$= \frac{54+27}{3} = \frac{81}{3} = 27$$

$\therefore$ Cov(X, Y) = E(XY) – E(X)E(Y)

= 27 – 3 × 6 = 27 – 18 = 9

Hence proved

**Example: -** If $f(x) = \frac{1}{a}\left[1 - \frac{|x-b|}{a}\right], |x-b| < a$. Find mean and variance.

**Solution: -** $f(x) = \frac{1}{a}\left[1 - \frac{|x-b|}{a}\right]$

Now $|x - b| < a \quad \Rightarrow -a < x - b < a$

$\Rightarrow \quad b - a < x < a + b$

$\therefore \quad f(x) = \frac{1}{a}\left[1 + \frac{(x-b)}{a}\right]$ for $b - a < x < b$ $\therefore |x - b| = -(x - b)$ for $(b - a) < x < b$

Similarly

$f(x) = \frac{1}{a}\left[1 - \frac{(x-b)}{a}\right]$ for $b < x < b + a$ as $|x - b| = x - b$ for $b < x < b + a$

$\therefore$ Mean = E(X) = $\int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{b-a}^{b} x.\frac{1}{a}\left[1 + \frac{(x-b)}{a}\right]dx + \int_{b}^{b+a} x.\frac{1}{a}\left[1 - \frac{(x-b)}{a}\right]dx$$

$$= \frac{1}{a}\int_{b-a}^{b}\left(x + \frac{x^2 - bx}{a}\right)dx + \frac{1}{a}\int_{b}^{b+a}\left(x - \frac{x^2}{a} + \frac{bx}{a}\right)dx$$

$$= \frac{1}{a}\left[\frac{x^2}{2} + \frac{x^3}{3a} - \frac{bx^2}{2a}\right]_{b-a}^{b} + \frac{1}{a}\left[\frac{x^2}{2} - \frac{x^3}{3a} + \frac{bx^2}{2a}\right]_{b+a}^{b}$$

$$= \frac{1}{a}\frac{b^2}{2} + \frac{b^3}{3a^2} - \frac{b^3}{2a^2} - \left(\frac{b^2 + a^2 - 2ab}{2a}\right) - \frac{(b-a)^2}{3a^2} + \frac{b}{2a^2}(b-a)^2$$

$$+ \frac{1}{a}\left(\frac{b^2 + a^2 + 2ab}{2}\right) - \frac{(b+a)^3}{3a^2} + \frac{b(b+a)^2}{2a^2} - \frac{b^2}{2a} + \frac{b^3}{3a^2} - \frac{b^3}{2a^2}$$

$$= \frac{b^2}{2a} + \frac{b^3}{3a^2} - \frac{b^3}{2a^2} - \frac{b^2}{2a} - \frac{a^2}{2a} + \frac{2ab}{2a} + \frac{b^2}{2a} + \frac{a^2}{2a} + \frac{2ab}{2a}$$

$$+ \frac{(-b^3 + a^3 + 3ab^2 - 3a^2 b)}{3a^2} + \frac{b^3 + a^2 b - 2ab^2}{2a^2}$$

$$+ \frac{(-b^3 - a^3 - 3ab^2 - 3a^2 b)}{3a^2} + \frac{b^3 + a^2 b + 2ab^2}{2a^2}$$

$$- \frac{b^2}{2a} + \frac{b^3}{3a^2} - \frac{b^3}{2a^2} = b$$

Variance = V(X) = $E[X - E(X)]^2$

$$= E(X - b)^2 = \int_{-\infty}^{\infty} (x - b)^2 f(x)\,dx$$

$$= \int_{b-a}^{b} (x - b)^2 . \frac{1}{a}\left(\frac{1 + x - b}{a}\right) dx + \int_{b}^{b+a} (x - b)^2 \frac{1}{a}\left[1 - \frac{(a - b)}{a}\right] dx$$

$$= \frac{1}{a} \int_{b-a}^{b} \left[(x - b)^2 + \frac{(x - b)^3}{a}\right] dx + \frac{1}{a} \int_{b}^{b+a} \left[(x - b)^2 - \frac{(x - b)^3}{a}\right] dx$$

$$= \frac{1}{a}\left[\frac{(x - b)^3}{3} + \frac{(x - b)^4}{4a}\right]_{b-a}^{b} + \frac{1}{a}\left[\frac{(x - b)^3}{3} - \frac{(x - b)^4}{4a}\right]_{b}^{b+a}$$

$$= \frac{1}{a} . \frac{a^3}{3} - \frac{a^4}{4a^2} + \frac{a^3}{3a} - \frac{a^4}{4a^2}$$

$$= \frac{2a^3}{3a} - \frac{a^4}{2a^2} = \frac{2a^2}{3} - \frac{a^2}{2} = \frac{a^2}{6}$$

**Example: -** Find mean and Variance for the following distribution

$$f(x) = \begin{cases} \frac{(x - 1)^3}{4}, & 1 \le x \le 3 \\ 0 \quad , & \text{otherwise} \end{cases}$$

**Solution: -** Check for $\int_{-\infty}^{\infty} f(x)\,dx = 1$

As mean = E(X) = $\int_{-\infty}^{\infty} x f(x)\,dx = \int_{1}^{3} \frac{(x - 1)^3}{4} . x\,dx$

$$= \frac{1}{4}\int_1^3 (x^4 - x - 3x^3 + 3x^2)\,dx = \frac{13}{5}$$

Variance = V(X) = E[X – E(X)$^2$] = $E\left(X - \frac{13}{6}\right)^2$

$$= \int_{-\infty}^{\infty} \left(x - \frac{13}{5}\right)^2 f(x)\,dx$$

$$= \frac{1}{4}\int_1^3 \left(x - \frac{13}{5}\right)^2 \frac{(x-1)^3}{4}\,dx$$

$$= \frac{1}{4}\left\{\left[\left(x - \frac{13}{5}\right)^2 \frac{(x-1)^4}{4}\right]_1^3 - \frac{1}{2}\int_1^2 \left(x - \frac{13}{5}\right)(x-1)^4\,dx\right\}$$

$$= \frac{1}{4}\left\{4 \times \frac{4}{25} - \frac{1}{2}\left[\left(x - \frac{13}{5}\right)\frac{(x-1)^5}{5}\right]^3 + \frac{1}{2} \times \frac{1}{5}\left[\frac{(x-1)^6}{6}\right]_1^3\right\}$$

$$= \frac{4}{25} - \frac{4}{5} \times \frac{2}{5} + \frac{4}{15}$$

$$= \frac{4}{25} - \frac{8}{25} + \frac{4}{15} = -\frac{4}{25} + \frac{4}{15} = \frac{8}{75}$$

**Theorem: -** If X is a r.v., then V(aX + b) = $a^2$ V(X), where a and b are constants.

**Proof: -** Let Y = aX + b, then

E(Y) = aE(X) + b

Then V(aX + b) = V(Y) = E[Y – E(Y)]$^2$

= E[aX + b – aE(X) – b]$^2$

= $a^2E(X^2) + a^2[E(X)]^2 - 2a^2[E(X)]^2$

= $a^2E(X^2) - a^2[E(X)]^2$

= $a^2[E(X^2) - [E(X)]^2] = a^2$ V(X)

**Cor**(i) If b = 0, then V(ax) = $a^2$ V(X)

(ii) If a = 0, then V(b) = 0

(iii) If a = 1, then V(X + b) = V(X).

**Theorem: -** Prove that $V(X \pm Y) = V(X) \pm V(Y) \pm 2\,Cov(X, Y)$ and $V(X \pm Y) = V(X) + V(Y)$ provided X and Y are independent r.v.

**Proof: -** $V(X + Y) = E[(X + Y) - E(X + Y)]^2$

$$= E[(X + Y) - E(X) - E(Y)]^2$$

$$= E[\{X - E(X)\} + \{Y - E(Y)\}]^2$$

$$= E[X - E(X)]^2 + [Y - E(Y)]^2 + 2E[(X - E(X)][Y - E(Y)]$$

$$= E[X - E(X)]^2 + E[Y - E(Y)]^2 + 2E[\{X - E(X)\}\{Y - E(Y)\}]$$

$$= V(X) + V(Y) + 2\,cov(X, Y)$$

Similarly $V(X - Y) = V(X) + V(Y) - 2cov(X, Y)$

$$V(X \pm Y) = V(X) + V(Y) \pm 2cov(X, Y)$$

If X and Y are independent, then $cov(X, Y) = 0$

$\therefore \quad V(X \pm Y) = V(X) + V(Y) \pm 2.0$

$$= V(X) + V(Y).$$

**Covariance: -** If X and Y are two r.v., then cov between then is defined as

$$Cov(X, Y) = E[[X - E(X)[Y - E(Y)]]$$

$$= E[XY - XE(Y) - YE(X) + E(X)\,E(Y)]$$

$$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y)$$

We can also express it as

$$Cov(X, Y) = E[[X - E(X)][Y - E(Y)]]$$

$$= \sum_i \sum_j (x_i - \overline{X})(y_j - \overline{Y}) f(x_i, y_j) \text{ for discrete case}$$

$$Cov(X, Y) = E[XY - \overline{X}Y - X\overline{Y} + \overline{X}\,\overline{Y}]$$

$$= E(XY) - \overline{X}E(Y) - \overline{Y}E(X) + \overline{X}\,\overline{Y}$$

$$= E(XY) - \overline{X}\,\overline{Y} - \overline{Y}\,\overline{X} + \overline{X}\,\overline{Y}$$

$$= E(XY) - \overline{X}\,\overline{Y}$$

If X and Y are independent r.v., then

$$E(XY) = E(X)\,E(Y)$$

$$Cov(X, Y) = E(X)\,E(Y) - \overline{X}\,\overline{Y} = \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} = 0$$

⇒ Cov(X, Y) = 0 if X and Y are independent.

**Remark**

i) $Cov(ax, by) = E[(aX - E(ax))(bY - E(bY))]$

$= E[(aX - aE(X))(bY - BE(Y))]$

$= E[a[X - E(X)] . b[Y - E(Y)]]$

$= ab[E(X - E(X))(Y - E(Y))]$

$= ab\, Cov(X, Y)$

⇒ Covariance is not independent of change of scale.

ii) $Cov(X + a, Y + b) = E[\{(X + a) - E(X - a)\}\ \{Y + b - E(Y + b)\}]$

$= E[\{(X + a) - E(X) - a\}\ \{Y + b - E(Y) - b\}$

$= E[\{X - E(X)\}\ \{Y - E(Y)\}]$

$= Cov(X, Y)$

Thus Cov(X, Y) is independent of change of origin but is not independent of change of scale.

**MEAN DERIVATION FOR CONTINUOUS CASE**

$$E[|X - a|] = \int_{-\infty}^{\infty} |x - a| f(x)dx$$

$$\text{Variance} = \sigma_x^2 = E[X - E(X)]^2 = \int_{-\infty}^{\infty} (x - \overline{X})^2 f(x)dx$$

**Absolute moment**

Let X be r. v. with p.d.f f(x), then its $r^{th}$ absolute moment about any point a is given by

$$E[|X - a|^r] = \int_{-\infty}^{\infty} |x - a|^r f(x)\, dx$$

For Variance i.e. $\mu_2 = E[X - E(x)]^2$

$= E[X^2 - 2XE(X) + E^2(X)]$

$= E(X^2) - 2E(X)E(X) + E^2(X)$ $[\Theta\ E(X) = \mu_2]$

$\mu_2 = E(X^2) - 2[E(X)]^2 + E^2(X)$

$= E(X^2) - [E(X)]^2$

$\mu_2 = \mu_2' - \mu_1'^2$

Effect of change of origin & scale on moment

$$U = \frac{X - a}{h}$$

Now $E[U] = \overline{U} = E\left(\frac{X-a}{h}\right) = \frac{1}{h}[E(X) - E(a)]$

$$\overline{U} = \frac{1}{h}(\overline{X} - a) \Rightarrow \overline{X} = a + h\overline{U}$$

Then $\mu_r = E[X - \overline{X}]^r = E[a + hU - a - h\overline{U}]^r$

$= E[h^r(U - \overline{U})^r] = h^r E[U - \overline{U}]^r$

Where $E[U - \overline{U}]^r = r^{th}$ moment of U about mean

**MOMENT GENERATING FUNCTION (M.G.F)**

M.G.F of a r.v. X with p.d.f. f(x) is given by

$$M_X(t) = E[e^{tx}]$$

where t is real number.

Then $M_X(t)$ is known as Moment generating function about origin.

Now, M.G.F about Mean is given by

$$M_{X-\mu}(t) = E[e^{t(X-\mu)}]$$

If X is a discrete r.v.: -

MGF about origin is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} f(x) \qquad \dots (1)$$

or $M_X(t) = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \dots\dots + \frac{t^rX^r}{r!} + \dots\right]$

$$= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \dots\dots + \frac{t^r}{r!}R(X^r) + \dots$$

$$M_X(t) = 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \dots\dots \frac{t^r}{r!}\mu'_r + \dots$$

where $\mu'_r$ = coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, is the $r^{th}$ moment about origin.

The moments can also be obtained from the M.G.F through following relations:

$$\mu'_r = \frac{d^r}{dt^r} M_X(t) \Big/_{t=0}$$

$$\mu'_1 = \frac{d}{dt} M_X(t) \Big/_{t=0}$$

$$\mu'_2 = \frac{d^2}{dt^2} M_X(t) \Big/_{t=0} \quad \text{etc.}$$

when X is continuous r.v., then MGF about origin is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

& MGF about mean is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(x-\mu)} f(x)\, dx$$

and MGF about any point 'a' is

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx$$

As $\quad M_X(t) = \sum_x e^{tx} f(x) = \sum_x f(x)\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + ...... + \frac{t^rx^r}{r!}\right]$

$\therefore \quad \frac{d}{dt} M_X(t) = \sum_{x=0}^{\infty} f(x)\left[x + \frac{2tx^2}{2!} + \frac{3t^2x^3}{3!} + ...... + \frac{rt^{r-1}x^{r-1}}{r} + .....\right]$

$\therefore \quad \frac{d}{dt} M_X(t) = \sum_{x=0}^{\infty} xf(x) = \sum_{x=1}^{\infty} xf(x) = \mu'_1 = 1^{st}$ moment about origin.

**Example: -** Let x be a r.v. with p.d.f

$$f(x) = \begin{cases} x & , \quad 0 \le x \le 1 \\ 2 - x & , \quad 1 \le x \le 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find MGF for this distribution. Also determine mean & variance.

**Solution: -** First we shall prove that given distribution is probability distribution or not i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ or not

Now $\int_{-\infty}^{\infty} f(x)\,dx = \left( \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{\infty} \right) f(x)\,dx$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x) + 0$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} - 0 + 4 - 2 - 2 + \frac{1}{2}$$

$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow$ Given distribution is a probability distribution

Mean = E(X) = $\int_{0}^{1} xf(x)dx + \int_{1}^{2} xf(x)dx$

$$= \int_{0}^{1} x.x\,dx + \int_{1}^{2} x(2-x)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 - \frac{1}{3} = \frac{2}{3} + 3 - \frac{8}{3} = 1$$

$\Rightarrow \quad E(X^2) = \int_{0}^{1} x^2 f(x)\,dx + \int_{1}^{2} x^2 f(x)dx$

$$= \int_{0}^{1} x^2.xdx + \int_{1}^{2} x^2(2-x)\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \frac{2}{3}(8) - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{1}{2} - 4 + \frac{14}{3} = \frac{3 - 24 + 28}{6} = \frac{7}{6}$$

$\therefore$ Variance = $\frac{7}{6} - 1^2 = \frac{1}{6}$

Now $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

$$= \int_0^1 e^{tx}\, x dx + \int_1^2 (2 - x)e^{tx} dx$$

$$= \left[x\frac{e^{tx}}{t}\right]_0^1 - \int_0^1 (1)\frac{e^{tx}}{t}dx + \left[(2-x)\frac{e^{tx}}{t}\right]_1^2 - \int_1^2 (-1)\frac{e^{tx}}{t}dx$$

$$= (1)\frac{e^t}{t} - \frac{1}{t^2}\left[e^{tx}\right]_0^1 + \left[0 - 1.\frac{e^t}{t}\right] + \frac{1}{t^2}\left[e^{tx}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{1}{t^2}\left[e^t - 1\right] - \frac{e^t}{t} + \frac{1}{t^2}\left[e^{2t} - e^t\right]$$

$$= -\frac{1}{t^2}e^t + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} = \frac{1}{t^2}[e^{2t} - 2e^t + 1]$$

$$M_X(t) = \left(\frac{e^t - 1}{t}\right)^2$$

Now $\mu_1' = \frac{d}{dt}M_X(t)\Big/_{t=0}$ $\qquad \left[\Theta\ \mu_r' = \frac{d^r}{dt^r}M_X(t)\Big/_{t=0}\right.$

$$\mu_2' = \frac{d}{dt}M_X(t)\Big/_{t=0}$$

then variance, $\mu_2 = \mu_2' - \mu_1'^2$

also $M_X(t) = \frac{1}{t^2}\left[\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+.....\right)-2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+....\right)+1\right]$

$$= \frac{1}{t^2}\left[(4-2)\frac{t^2}{2!}+(8-2)\frac{t^3}{3!}+(16-2)\frac{t^4}{4!}+......\right]$$

$$= \frac{1}{t^2}\left[\frac{2t^2}{2}+\frac{6t^3}{6}+\frac{7t^4}{12}+......\right]$$

$$M_X(t) = \left[1+\frac{t}{1!}+\left(\frac{7}{6}\right)\frac{t^2}{2!}+......\right]$$

$\mu_1' =$ coefficient of $\frac{t}{1!} \rightarrow$ Mean

$\mu_2' =$ coefficient of $\frac{t^2}{2!} = \frac{7}{6}$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$

**Property: - (1)** $M_{CX}(t) = M_X(Ct)$

**Proof: -** By definition,

$M_{cX}(t) = E[e^{tcX}] = E[e^{(ct)X}]$

$M_{cX}(t) = M_X(ct)$

**(2)** MGF of the sum of a number of independent r. v. is equal to the product of heir respective MGF's i.e.

$$M_{X_1+X_2+.....+x_n}(t) = M_{X_1}(t).M_{X_2}(t).......M_{X_n}(t)$$

**Proof: -** By definition,

$$M_{X_1+X_2+.....+x_n}(t) = E\left[e^{t(X_1+X_2+.....+X_n)}\right]$$

$$= E\left[e^{tX_1}.e^{tX_2}........e^{tX_n}\right]$$

If $X_1, X_2,....., X_n$ are independent r.v., then function $e^{tx_1}, e^{tx_2}, .....e^{tx_n}$ are also independent.

$\therefore$ $M_{X_1+X_2+.....+x_n}(t) = E\left[\left(e^{tX_1}\right)E\left(e^{tX_2}\right)........E\left(e^{tX_n}\right)\right]$

$$= M_{X_1}(t).M_{X_2}(t)......M_{X_n}(t)$$

**(3)** Effect of change of origin and scale on M.G.F

Let us transform X to the new variable U by changing both the origin and scale in X as follows: -

**Proof: -** If X is a r.v., then

$$U = \frac{X-a}{h} \quad \Rightarrow \quad X = a + hU \quad \text{where a \& h are constants}$$

Now $M_X(t) = E[e^{tX}] = E[e^{t(a+hU)}]$

$= E[e^{at}, e^{thU}]$

$= e^{at} E[e^{(th)U}]$

where $M_X(t)$ is the m.g.f. of X about origin.

$\Rightarrow \quad M_X(t) = e^{at} M_U(th)$

$\Rightarrow \quad M_U(th) = e^{-at} M_X(t)$

Put $t\,h = t' \quad \Rightarrow \quad t = \frac{t'}{h}$

then $M_U(t') = e^{\frac{-at'}{h}} M_X\left(\frac{t'}{h}\right)$ = M.G.F of U (about origin)

**Standard normal variate**

If X is a r.v., then the variable Z defined by

$Z = \frac{X-\mu}{\sigma}$ is called standard normal variate

where $\mu$, $\sigma$ are Mean & standard derivation, respectively

Also $E(Z) = 0$

Variance $(Z) = 1$

Now $E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu)$

$E(Z) = \frac{1}{\sigma}[E(X)-\mu] = \frac{1}{\sigma}[\mu-\mu] = 0$

and $V(Z) = E\left(\frac{X-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\left[E(X-\mu)^2\right]$

$\Rightarrow \quad V(Z) = \frac{1}{\sigma^2}.\sigma^2 = 1 \qquad [\because \sigma^2 = E(X-\mu)^2]$

If $X \sim N(\mu, \sigma^2)$,then $Z \sim N(0, 1)$

The m.g.f of standard variate Z is $M_Z(t) = e^{-\mu t/\sigma} M_X(\frac{t}{\sigma})$

**Example: -** Let X is a discrete r.v.

$P(X = r) = pq^{r-1}$, r = 1, 2,…..

where p is the probability of success in one trial and q is the probability of failure in one trial. Then find MGF, Mean & variance.

**Solution: -** $M_X(t) = E[e^{tX}]$

$$= \sum_{r=1}^{\infty} e^{tr} pq^{r-1}$$

$$\Rightarrow \quad M_X(t) = \frac{p}{q}\sum_{r=1}^{\infty} e^{tr} q^r = \frac{p}{q}\left(qe^t\right)\sum_{r=1}^{\infty}\left(qe^t\right)^{r-1}$$

$$= \frac{p}{q}.qe^t\left[1 + qe^t + (qe^t)^2 + ......\right]$$

$$\Rightarrow \quad M_X(t) = pe^t \frac{1}{(1-qe^t)} \qquad \text{[sum of G.P]}$$

Now $$\mu_1' = \frac{d}{dt}M_X(t)\Big/_{t=0} = \frac{(1-qe^t).pe^t - pe^t.(-qe^t)}{(1-qe^t)^2}\Big/_{t=0}$$

$$= \frac{pe^t - pqe^{2t} + PQE^{2T}}{(1-qe^t)^2}\Big/_{t=0}$$

$$= \frac{p}{(1-q)^2}$$

$$\Rightarrow \quad \mu_1' = \frac{p}{(1-q)^2} = \frac{p}{p^2} \qquad [\Theta 1 - q = p]$$

$$\Rightarrow \quad \mu_1' = \frac{1}{p} = \text{Mean}$$

and $$\mu_2' = \frac{d^2}{dt^2}M_X(t)\Big/_{t=0}$$

$$= \frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right)\Big/_{t=0}$$

$$= \left. \frac{(1-qe^t)pe^t - pe^t.2(1-qe^t)(-qe^t)}{(1-qe^t)^4} \right/_{t=0}$$

$$= \left. \frac{(1-qe^t)pe^t + 2pqe^{2t}}{(1-qe^t)^3} \right/_{t=0}$$

$$= \left. \frac{pe^t + pqe^{2t}}{(1-qe^t)^3} \right/_{t=0}$$

$$= \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$\therefore$ Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$\Rightarrow$ $\mu_2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$

**CUMULANT GENERATING FUNCTION AND CUMULANTS**

The logarithm of m.g.f about origin of a r.v. X is called second m'g.f or cumulant generating function (c.g.f.)

i.e. $K_X(t) = \log M_X(t) = \sum_r K_r \frac{t^r}{r!}$

and $K_r$ = coefficient of $\frac{t^r}{r!}$ in $K_X(t)$

This $K_r$ is known as $r^{th}$ cumulant

Now $K(t) = \log M_X(t) = \log[E(e^{tx})]$

$$= \log\left[ E\left( 1 + tX + \frac{t^2X^2}{2!} + \dots\dots + \frac{t^rX^r}{r!} + \dots\dots \right)\right]$$

$$= \log\left[ 1 + \left( \mu_1't + \mu_2'\frac{t^2}{2!} + \dots\dots + \mu_r'\frac{t^r}{r!} \right)\right]$$

$\Rightarrow$ $K_X(t) = \left[\left( \mu_1't + \mu_2'\frac{t^2}{2!} + \dots\mu_r'\frac{t^r}{r!} + \dots\dots \right) - \frac{1}{2}\left( \mu_1' + \mu_2'\frac{t^2}{2!} + \dots + \mu_r' + \frac{t^r}{r!} + \dots\dots \right)\right.$

$$+\frac{1}{3}\left(\mu_1' t+\mu_2' \frac{t^2}{2!}+\ldots.+\mu_r' \frac{t^r}{r!}+\ldots.\right)^3$$

$$\left.-\frac{1}{4}\left(\mu_1' t+\mu_2' \frac{t^2}{2!}+\ldots.+\mu_r' \frac{t^r}{r!}\right)^4+\ldots..\right] \qquad \ldots(1)$$

Now $K_X(t) = K_1 t + K_2 \frac{t^2}{2!}+K_3\frac{t^3}{3!}+K_4\frac{t^4}{4!}+\ldots\ldots$ …(2)

Equating coefficients of like terms in equation (1), (2) we get

$K_1$ = coefficient of $\frac{t}{1!}=\mu_1' = \text{Mean}$

$K_2 = \mu_2' - \mu_1' = \mu_2$ = (variance)

$K_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \mu_3$

$$\frac{K_4}{4!}=\frac{\mu_4^1}{4!}-\frac{1}{2}\left(\frac{\mu_2'^2}{4}+\frac{2\mu_1'\mu_3'}{3!}\right)+\frac{1}{3}\frac{3\mu_1'^2\mu_2'}{2}-\frac{\mu_1'^4}{4}$$

$\Rightarrow$ $K_4 = \mu_4' - 3\mu_2'^2 - 4\mu_1'\mu_3' + 12\mu_1'^2\mu_2' - 6\mu_1'^4$

$= \left(\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4\right) - 3\left(\mu_2'^2 - 2\mu_2'\mu_1'^2 + \mu_1'^4\right)$

$\Rightarrow$ $K_4 = \mu_4 - 3K_2^2 = \mu_4 - 3\mu_2^2$

Also $K_r = \frac{d^r}{dt^r}K_X(t)$

**Characteristic function**

Characteristic function of a variable X is defined as $\phi_X(t) = E[e^{itX}]$

If X is continuous r. v., then

$\phi_X(t) = \int e^{itx} f(x)\, dx$

If X is discrete r. v, then

$\phi_X(t) = \sum e^{itx} f(x)$

# LESSSON 6 THEORETICAL DISCRETE DISTRIBUTIONS

**BINOMIAL DISTRIBUTION**

Let n represent the number of trials in an event.

$x \to$ consecutive successes

$(n-x) \to$ failure

then $ppp\ldots..p\ qq\ldots..q = p^x q^{n-x}$

But x successes in n trials can occur in ${}^nC_x$ ways.

$\therefore$ Probability of x successes in any order in n trials = ${}^nC_x\, p^x\, q^{n-x}$, here n and p are called parameters.

Therefore, probability density function for a Binomial distribution is

$$P(X = x) = B(n, p; x)$$

$$= f(x) = {}^nC_x\, p^x\, q^{n-x},\ x = 0, 1, 2\ldots., n$$

Also $p + q = 1$ and $\sum f(x) = 1$

**Moments about origin of Binomial distribution**

The r[th] moment of the Binomial distance about origin is

$$\mu'_r = E[X^r] = \sum_{x=0}^{n} x^r f(x)$$

$$\Rightarrow \quad \mu'_r = \sum_{x=0}^{n} x^r\ {}^nC_x\, p^x\, q^{n-x}$$

when r = 1, 1[st] moment, $\mu'_1 = \sum_{x=0}^{n} x\ {}^nC_x\, p^x\, q^{n-x}$

$$\Rightarrow \quad \mu'_1 = \sum_{x=0}^{n} \frac{x\,\lfloor n}{\lfloor n-x\,\lfloor x}\ p^x\, q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{\lfloor n}{\lfloor n-x\ \ \lfloor x-1}\ p^x\, q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{\lfloor n-1}{\lfloor n-x\ \lfloor x-1}\ p^{x-1}\, q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{\lfloor n-1}{\lfloor (n-1)-(x-1)\ \ \lfloor x-1}\ \ p^{x-1}\, q^{(n-1)\text{-}(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} p^{x-1}\ q^{(n-1)-(x-1)}$$

$$= np\ (q + p)^{n-1} \qquad \left[\Theta \sum f(x) = (q+p)^n = 1\right]$$

$\Rightarrow \qquad \mu_1' = np = \text{Mean}$

Now Second moment about origin is $\mu_2' = E(X^2) = \sum x^2 f(x)$

$$\mu_2' = \sum_{x=0}^{n} \left[x(x-1) + x\right] {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\ {}^nC_x p^x q^{n-x} + \sum_{x=0}^{n} x\ \ {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\ \ {}^nC_x\ p^x\ q^{n-x} + \sum_{x=1}^{n} x\ \ {}^nC_x\ \ p^x\ \ q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{x(x-1)\ \underline{|n}}{x(x-1)\ \underline{|x-2}\ \underline{|n-x}} p^x\ q^{n-x} + np$$

$$\Rightarrow \qquad \mu_2' = \sum_{x=2}^{n} \frac{n(n-1)\underline{|n-2}}{\underline{|x-2}\ \ \underline{|n-x}}\ p^x\ q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{\underline{|n-2}}{\underline{|x-2}\ \underline{|(n-2)-(x-2)}}\ p^{x-2}\ q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\ p^{x-2}\ q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2.(1) + np$$

$\Rightarrow \qquad \mu_2' = n(n-1)p^2 + np$

Also variance, $\mu_2 = \mu_2' - \mu_1'^2$

$\Rightarrow \qquad \mu_2 = n(n-1)p^2 + np - n^2p^2$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$\Rightarrow \qquad \mu_2 = np(1 - p) = npq$

So, variance = npq

Also variance ≤ Mean

i.e. $npq \le np$ as $q \le 1$

and variance = Mean if q = 1

Now $\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3 \; f(x)$

As $x^3 = x(x-1)(x-2) + 3x\,(x-1) + x$

So $\mu_3' = \sum_{x=0}^{n} [x(x-1)(x-2) + 3x(x-1) + x] f(x)$

$$= \sum_{x=0}^{n} x(x-1)(x-2)\; {}^nC_x\, p^x\, q^{n-x} + 3\sum_{x=0}^{n} x(x-1)\; {}^nC_x\; p^x\; q^{n-x}$$

$$+ \sum_{x=0}^{n} x\; {}^nC_x\, p^x\; q^{n-x}$$

$$= \sum_{x=3}^{n} x(x-1)(x-2)\; {}^nC_x\; p^x\; q^{n-x} + 3\sum_{x=2}^{n} x(x-1)\; {}^nC_x\; p^x\; q^{n-x}$$

$$+ \sum_{x=1}^{n} x\; {}^nC_x\; p^x\; q^{n-x}$$

$$= \sum_{x=3}^{n} x(x-1)(x-2) \frac{\underline{|n}}{x(x-1)(x-2)\,\underline{|x-3}\;\underline{|n-x}}\; p^x\, q^{n-x}$$

$$+ 3n(n-1)p^2 + np$$

$$= n(n-1)(n-2) \sum_{x=3}^{n} \frac{\underline{|n-3}}{\underline{|x-3}\;\underline{|(n-3)-(x-3)}}\; p^x\; q^{n-x} + 3n(n-1)p^2 + np$$

$$\mu_3' = n(n-1)(n-2)p^3 \sum_{x=3}^{n} {}^{n-3}C_{x-3}\; p^{x-3}\; q^{(n-3)-(x-3)} + 3n(n-1)p^2 + np$$

$\Rightarrow \quad \mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

Also 3rd moment about mean is

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3[n(n-1)p^2 + np]\, np + 2n^3\, p^3$$

$$= (n^2-n)(n-2)p^3 + 3n^2p^2 - 3np^2 + np - 3n^3p^3 + 3n^2p^3 - 3n^2p^2 + 2n^3p^3$$

$$= n^3p^3 - n^2p^3 - 2n^2p^3 + 2np^3 - 3np^2 + np - n^3p^3 + 3n^2p^3$$

$$= np + 2np^3 - 3np^2$$

$$= np(1-3p + 2p^2) = np(p-1)(2p-1)$$

$$= np(-q)(2p-p-q) = npq(q-p)$$

$$\mu_4' = \sum_{x=0}^{n} x^4 \; {}^nC_x \; p^x \; q^{n-x}$$

Now $x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$

So $\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

and $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3\mu_1'^4$

$\Rightarrow$ $\mu_4 = npq[1 + 3(n-2)pq]$

**Moment generating function (M.G.F) about origin for a Binomial distribution**

$$M_X(t) = E[e^{tx}] = \sum e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \; {}^nC_x \, p^x \, q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x \; q^{n-x} (pe^t)^x \qquad [\because e^{tx} = (e^t)^x]$$

$$M_X(t) = (q + pe^t)^n$$

Now $\mu_r' = \left. \frac{d^r}{dt^r} M_X(t) \right/_{t=0}$

$\Rightarrow$ $\mu_1' = \left. \frac{d}{dt} M_X(t) \right/_{t=0}$

$$\mu_1' = \left. \frac{d}{dt}(q + pe^t)^n \right/_{t=0}$$

$$= \left. n(q + pe^t)^{n-1} pe^t \right/_{t=0}$$

$$= np\,(q + p)^{n-1} = np$$

Also $\mu_2' = \left. \frac{d^2}{dt^2} M_X(t) \right/_{t=0}$

$$= \left. \frac{d}{dt}\left[ n(q + pe^t)^{n-1} pe^t \right] \right/_{t=0}$$

$$= np\frac{d}{dt}\left[e^{t}(q+pe^{t})^{n-1}\right]\Big/_{t=0}$$

$$= np[e^{t}(q + pe^{t})^{n-1} + 1\,(n-1)(q + pe^{t})^{n-2}pe^{t}]\ t = 0$$

$$= np[\ 1(q + p)^{n-1} + 1(n-1)(q+p)^{n-2}p]$$

$$= np[1 + (n-1)p] = np + n(n-1)p^{2}$$

**Pearson's Coefficients**

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3},\ \beta_2 = \frac{\mu_4}{\mu_2^2},\ \alpha_1 = \sqrt{\beta_1},\ \alpha_2 = \beta_2 - 3$$

$$\beta_1 = \frac{(1-2p)^2}{npq}, \qquad \beta_2 = 3 + \frac{(1-6pq}{npq}$$

$$\alpha_1 = \frac{1-2p}{\sqrt{npq}}, \qquad \alpha_2 = \frac{1-6pq}{npq}$$

**Characteristic function**

$$\phi_X(t) = E[e^{itx}] = (q + pe^{it})^{n}$$

**Cumulants**

Cumulant generating function (c.g.f) of a Binomial distribution is

$$K_X(t) = \log M_X(t)$$

$$= \log (q + pe^{t})^{n} = n \log (q + pe^{t})$$

$$= n \log \left[ q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ......\right)\right]$$

$$= n \log \left[ (q+p) + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ........\right)\right]$$

$$= n \log \left[ 1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ........\right)\right]$$

using $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3}.......$, we get

$$K_X(t) = n\left[ p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ....\right) - \frac{p^2}{2}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ....\right)^2 + \frac{p^3}{3}\left(t + \frac{t^2}{2!} + ....\right)^3 ....\right]$$

The $r^{th}$ cumulant is

$K_r$ = coefficient of $\frac{t^r}{r!}$ in $K_X(t) = \left.\frac{d^r}{dt^r} K_X(t)\right/_{t=0}$

So $\quad K_1 = np = \mu_1'$

$K_2 = np - np^2$ = coefficient of $\frac{t^2}{2!}$

$\Rightarrow \quad K_2 = np(1-p) = npq = \mu_2 \qquad [\because p + q = 1]$

& $\quad \frac{K_3}{3!} = \frac{np}{3!} - \frac{np^2}{2}\frac{2}{2!} + \frac{np^3}{3}$

$\Rightarrow \quad K_3 = np - 3np^2 + 2p^3n = np(1-3p+2p^2)$

$= np(p-1)(2p-1)$

$= np(-q)[2p-p-q]$

$= npq(q-p) = \mu_3$

**Example: -** p = probability of getting a head = ½

q = probability of not getting a head = ½

The probability of getting x heads in a random throw of 10 coins is

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10}; x = 0, 1, 2, ......, 10$$

$\therefore \quad$ Probability of getting at least 7 heads is

$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$

$$= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\}$$

$$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024}$$

## Recurrence Relation

$$\mu_{r+1} = pq[nr\,\mu_{r-1} + \frac{d}{dp}\mu_r]$$

where $\mu_r$ is the $r^{th}$ moment about mean.

Now, $\mu_r = E[(X-np)^r] = \sum_{x=0}^{n}(x-np)^r\ {}^nC_x\, p^x\, q^{n-x}$

Differentiating w.r.t.p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n}\left[({}^nC_x\ p^x\ q^{n-x})r(x-np)^{r-1}(-n) + (x-np)^r\ {}^nC_x\ \frac{d}{dp}(p^x\, q^{n-x})\right]$$

$$= -nr\sum_{x=0}^{n}(x-np)^{r-1}\ {}^nC_x\, p^x\, q^{n-x}$$

$$+\sum_{x=0}^{n}(x-np)^r\ {}^nC_x\left[q^{n-x}xp^{x-1} + p^x(n-x)q^{n-x-1}(-1)\right]$$

$$\left[\Theta\ p = 1-q,\ q = 1-p\ \therefore \frac{dq}{dp} = -1\right]$$

$$= -nr\sum_{x=0}^{n}(x-np)^{r-1}\ f(x) + \sum_{x=0}^{n}(x-np)^r\ {}^nC_x\, p^x\, q^{n-x}\left[\frac{x}{p} - \frac{n-x}{q}\right]$$

$$= -nr\,\mu_{r-1} + \sum_{x=0}^{n}(x-np)^r\ {}^nC_x\, p^x\, q^{n-x}\,\frac{(x-np)}{pq}$$

$$\left[\therefore \frac{x}{p} - \frac{(n-x)}{q} = \frac{xq-np+xp}{pq} = \frac{x(p+q)-np}{pq} = \frac{x-np}{pq}\right]$$

$$\Rightarrow \quad \frac{d\mu_r}{dp} = -nr\mu_{r-1} + \sum_{x=0}^{n}\frac{(x-np)^{r+1}}{pq}\ {}^nC_x\, p^x\, q^{n-x}$$

$$= -\,nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

## M.G.F about its Mean

$M_{X-np}(t) = [qe^{-pt} + pe^{tq}]^n$

Now by definition $M_{X-np}(t) = \sum_{x=0}^{n} e^{(x-np)t} f(x)$

$\Rightarrow \quad M_{X-np}(t) = \sum_{x=0}^{n} e^{(x-np)t}\ {}^nC_x p^x q^{n-x}$

$$= \sum_{x=0}^{n} e^{xt}\ e^{-npt}\ {}^nC_x\, p^x\, q^{n-x}$$

$$= e^{-npt} \sum_{x=0}^{n} {}^nC_x\, q^{n-x} (pe^t)^x$$

$$= e^{-npt}(q + pe^t)^n$$

$$= (e^{-pt})^n (q + pe^t)^n$$

$$= [qe^{-pt} + pe^{t(1-p)}]^n$$

$\Rightarrow \quad M_{X-np}(t) = [qe^{-pt} + pe^{qt}]^n \qquad [\because p + q = 1]$

$$M_{X-np}(t) = \left[ q\left(1 - pt + \frac{p^2t^2}{2!} - \frac{p^3t^3}{3!} + \ldots\ldots\right) + p\left(1 + qt + \frac{q^2t^2}{2!} + \frac{q^3t^3}{3!} + \ldots\right)^n \right]$$

$$= \left[ (p+q) + pq(p+q)\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + \ldots \right]^n$$

As $\quad p + q = 1$, we have

$$M_{X-np}(t) = \left[ 1 + \left( pq\frac{t^2}{2!} + pq(q-p)\frac{t^3}{3!} + \ldots\ldots \right) \right]^n$$

$$= 1 + n\left[ pq\frac{t^2}{2!} + pq(q-p)\frac{t^3}{3!} + \ldots\ldots\ldots \right]$$

$$+ \frac{n(n-1)}{2!}\left[ pq\frac{t^2}{2!} + pq(q-p)\frac{t^3}{3!} + \ldots\ldots\ldots \right]^2 + \ldots\ldots$$

$$= 1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{3!} + \ldots\ldots\ldots$$

$\therefore \quad \mu_1 = 0$

$\mu_2$ = coefficient of $\frac{t^2}{2!}$ = npq

$\mu_3$ = coefficient of $\frac{t^3}{3!}$ = npq(q – p)

**Example: -** In 8 throw of a dice, 5 or 6 is considered as success. Find the mean number of successes and S.D.

**Solution: -** Here n = 8, p = $\frac{2}{6} = \frac{1}{3}$, q = $\frac{2}{3}$

$\therefore$ Mean = np = 8 × $\frac{1}{3} = \frac{8}{3}$

and SD = $\sqrt{npq} = \sqrt{\frac{8}{3} \times \frac{2}{3}} = \frac{4}{3}$

**Example: -** 6 dice are thrown 729 times. How many times due you expect at least 3 dice to show five or six?

**Solution: -** n = 6, N = 729

p = $\frac{2}{6} = \frac{1}{3}$, x =3, q = $\frac{2}{3}$

$\therefore$ $P(X\geq 3) = f(3) + f(4) + f(5) + f(6)$ …(1)

Now $f(x) = {}^nC_x\, p^x\, q^{n-x}$

$\therefore$ $(1) \Rightarrow P(X\geq 3) = \sum_{x=3}^{6} {}^nC_x\, p^x\, q^{n-x}$

$$= {}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^6 = \frac{233}{729}$$

No. of times, at least 3 dice will show

5 or 6 = $NP(X \geq 3) = 729 \times \frac{233}{729}$

= 233

# Fitting of Binomial distribution

By fitting of Binomial distribution we mean to find out the theoretical or expected frequencies for all values of X = 0, 1, 2…., n

The frequency F(x) for X = x is

$$F(x) = N\,f(x) = N\sum_{x=0}^{n} {}^{n}C_x\, p^x\, q^{n-x}$$

for x = 0

$$F(0) = Nq^n \qquad \ldots(1)$$

Also $\dfrac{f(x)}{f(x-1)} = \dfrac{{}^{n}C_x\, p^x\, q^{n-x}}{{}^{n}C_{x-1}\, p^{x-1} q^{n-x+1}}$

$$\frac{Nf(x)}{Nf(x-1)} = \frac{n-x+1}{x}.\frac{p}{q}$$

$\Rightarrow$ $\dfrac{F(x)}{F(x-1)} = \left(\dfrac{n-x+1}{x}\right)\dfrac{p}{q}$

$\Rightarrow$ $F(x) = \left(\dfrac{n-x+1}{x}\right).\dfrac{p}{q}F(x-1) \qquad \ldots(2)$

where x = 1, 2, …..,n

# POISSON DISTRIBUTION

Poisson distribution can be obtained from Binomial distribution under three conditions

(i) no. of trials is large i.e. $n \to \infty$

(ii) probability of success is very small i.e. $p \to 0$

(iii) the mean no. of successes is finite, i.e. np = m(say)

In case of Binomial distribution,

$$f(x) = {}^{n}C_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2, \ldots n$$

Taking limit case of f(x) in Binomial distribution, we get f(x) for Poisson distribution,

$$f(x) = \frac{m^x e^{-m}}{x!}, \qquad x = 0, 1, 2, 3, \ldots..$$

Now $n \to \infty$, $p \to 0$, $np = m$

i.e. $p = \dfrac{m}{n}$, $q = 1 - \dfrac{m}{n}$ as $n \to \infty$

Probability of x successes in Binomial distance is

$$f(x) = {}^{n}C_x\, p^x\, q^{n-x}$$

Probability of x successes in Poisson distance is

$$f(x) = \lim_{n\to\infty} {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$= \lim_{n\to\infty} {}^{n}C_{x} \left(\frac{m}{n}\right)^{x} \left(1-\frac{m}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{m}{n}\right)^{x} \left(1-\frac{m}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}}\; \frac{n(n-1)(n-2)....(n-x-1)\underline{\underline{|n-x}}}{\underline{|n-x}\,(n^{x})} \left(1-\frac{m^{n-x}}{n}\right)$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}}\; \frac{n(n-1)(n-2)....(n-x+1)}{n^{x}} \left(1-\frac{m}{n}\right)^{n} \left(1-\frac{m}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right) ..... \left(\frac{n-x-1}{n}\right) \left(1-\frac{m}{n}\right)^{n} \left(1-\frac{m}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}} 1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) ..... \left(1-\frac{x-1}{n}\right)\left(1-\frac{m}{n}\right)^{n} \left(1-\frac{m}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}} \left(1-\frac{m}{n}\right)^{n} \qquad [\Theta \text{Rest of the terms tends to unity as } n\to\infty]$$

$$= \lim_{n\to\infty} \frac{m^{x}}{\underline{|x}} \left(1-\frac{m}{n}\right)^{n} \qquad (1)$$

Now $$\lim_{n\to\infty} \left(1-\frac{m}{n}\right)^{n} = \lim_{n\to\infty} \left(1-n.\frac{m}{n}+\frac{n(n-1)}{2!}\frac{m^{2}}{n^{2}}-\frac{n(n-1)(n-2)}{3!}\frac{m^{3}}{n^{3}}+....\right)$$

$$= \lim_{n\to\infty} \left(1-m+\frac{m^{2}}{2!}-\frac{m^{3}}{3!}+.....\right)$$

$$= e^{-m}$$

$\therefore$ from (1)

$$f(x) = \frac{m^{x}}{\underline{|x}} e^{-m}, \; x = 0, 1, 2,.....$$

## Moments of Poisson distribution

$\mu'_{r} = E[X^{r}] = r^{th}$ moment about origin

$\therefore \qquad \mu_1' = E[X] = \sum_{x=0}^{\infty} x\, f(x)$

$$= \sum_{x=0}^{\infty} x\, e^{-m} \frac{m^x}{\underline{|x}} = \sum_{x=0}^{\infty} \frac{x e^{-m} m^x}{x\underline{|x-1}}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-m} m^x}{\underline{|x-1}}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{\underline{|x-1}}$$

$$= m e^{-m} \left[ 1 + \frac{m}{\underline{|1}} + \frac{m^2}{\underline{|2}} + \ldots.. \right]$$

$\Rightarrow \qquad \mu_1' = m e^{-m} e^{m} = m$

$\Rightarrow \qquad \mu_1' = m = \text{Mean}$

Now $\qquad \mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 e^{-m} \frac{m^x}{\underline{|x}}$

$$= \sum_{x=0}^{\infty} \left[ \frac{x(x-1) + x}{\underline{|x}} \right] e^{-m} m^x$$

$$= e^{-m} \left[ \sum_{x=0}^{\infty} \frac{x(x-1)}{\underline{|x}} m^x + \sum_{x=0}^{\infty} \frac{x\, m^x}{\underline{|x}} \right]$$

$$= e^{-m} \sum_{x=2}^{\infty} \frac{m^x}{\underline{|x-2}} + e^{-m} \sum_{x=1}^{\infty} \frac{m^x}{\underline{|x-1}}$$

$$= m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{\underline{|x-2}} + m$$

$$= m^2 e^{-m} . e^{m} + m$$

$\Rightarrow \qquad \mu_2' = m^2 + m$

$\therefore \qquad$ Variance, $\mu_2 = \mu_2' - \mu_1'$

$\Rightarrow \qquad \mu_2 = m^2 + m - m^2 = m \Rightarrow \mu_2 = m$

So $\qquad$ mean = Variance

## Moment generating function about origin

M.G.F about origin is by definition

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-m} m^x}{\lfloor x} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{\lfloor x}$$

$$= e^{-m}\left[1 + me^t + \frac{(me^t)^2}{\lfloor 2} + .....\right]$$

$$= e^{-m} . e^{me^t}$$

$$\Rightarrow \quad M_X(t) = e^{m(e^t - 1)} \qquad ...(1)$$

## Moment generating function about mean

$$M_{X-m}(t) = \sum_{x=0}^{\infty} e^{t(x-m)} f(x)$$

$$= \sum_{x=0}^{\infty} e^{t(x-m)} \frac{e^{-m} m^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} e^{tx} e^{-mt} e^{-m} \frac{m^x}{\lfloor x}$$

$$= e^{-(1+t)m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{\lfloor x}$$

$$= e^{-m(1+t)} e^{me^t} = e^{m(e^t - t - 1)} \qquad ...(2)$$

$$= e^{m\left(1+t+\frac{t^2}{\lfloor 2}+\frac{t^3}{\lfloor 3}+.....\right)-t-1} = e^{m\left(\frac{t^2}{\lfloor 2}+\frac{t^3}{\lfloor 3}+.....\right)}$$

Now $\mu_r' = \left.\frac{d^r}{dt^r} M_X(t)\right/_{t=0}$

$$\therefore \quad \mu_1' = \frac{d}{dt} M_X(t) \Big/_{t=0}$$

$$= \frac{d}{dt} e^{m(e^t-1)} \Big/_{t=0}$$

$$= e^{m(e^t-1)} me^t \Big/_{t=0}$$

$$\mu_1' = m$$

$$\text{and} \quad \mu_2' = \frac{d^2}{dt^2} M_X(t) \Big/_{t=0}$$

$$= \frac{d}{dt}\left[me^t\, e^{m(e^t-1)}\right] \Big/_{t=0}$$

$$= m\left[e^t\, e^{m(e^t-1)} + e^t\, e^{m(e^t-1)} me^t\right] \Big/_{t=0}$$

$$= m[1.e^0 + 1.\ 1.\ m.\ 1]$$

$$= m(1 + m) = m^2 + m$$

**Characteristic function**

$$\phi_X(t) = e^{m(e^{it}-1)} = E\left[e^{itx}\right]$$

# Cumulant generating function about origin

$$K_X(t) = \log M_X(t) = \log\, e^{m(e^t-1)}$$

$$= m(e^t - 1)$$

$$\Rightarrow \quad K_X(t) = m\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + ......\right]$$

The $r^{th}$ cumulant is

$$K_r = \text{coefficient of } \frac{t^r}{\lfloor r} \text{ in } K_X(t)$$

$\therefore$ $K_1 = m, K_2 = m, K_3 = m, \ldots\ldots$

i.e. $K_r = m$ , for r =1, 2, 3,…..

All cumulants are equal for Poisson distribution

## Recurrence Relation for Moments

$$\mu_{r+1} = m[r\mu_{r-1} + \frac{d\mu_r}{dm}]$$

Now $r^{th}$ central moment by definition is

$$\mu_r = E[(X - m)^r]$$

$$= \sum_{x=0}^{\infty}(x-m)^r\ f(x) = \sum_{x=0}^{\infty}(x-m)^r\ \frac{e^{-m}m^x}{\underline{|x}}$$

$$\therefore \quad \frac{d\mu_r}{dm} = \sum_{x=0}^{\infty}\left[r(x=m)^{r-1}(-1)e^{-m}\frac{m^x}{\underline{|x}}\right] + \sum_{x=0}^{\infty}\left[(x-m)^r e^{-m}(-1)\frac{m^x}{\underline{|x}}\right]$$

$$+\sum_{x=0}^{\infty}(x-m)^r e^{-m} x\frac{m^{x-1}}{\underline{|x}}$$

$$= -r\sum_{x=0}^{\infty}(x-m)^{r-1}\ \frac{e^{-m}m^x}{\underline{|x}} + \sum_{x=0}^{\infty}(x-m)^r\ \frac{e^{-m}m^x}{\underline{|x}}\left[\frac{x}{m}-1\right]$$

$$= -r\ \sum_{x=0}^{\infty}\ (x-m)^{r-1}\frac{e^{-m}m^x}{\underline{|x}} + \sum_{x=0}^{\infty}(x-m)^r\ \frac{e^{-m}m^x}{\underline{|x}}\left(\frac{x-m}{m}\right)$$

$$= -r\ \mu_{r-1} + \frac{1}{m}\sum_{x=0}^{\infty}(x-m)^{r+1}\frac{e^{-m}m^x}{\underline{|x}}$$

$$\Rightarrow \quad \frac{d\mu_r}{dm} = -r\mu_{r-1} + \frac{1}{m}\mu_{r+1}$$

$$\Rightarrow \quad \frac{d\mu_r}{dm} + r\mu_{r-1} = \frac{1}{m}\mu_{r+1}$$

$$\Rightarrow \quad \mu_{r+1} = m[r\mu_{r-1} + \frac{d\mu_r}{dm}]$$

## Recurrence relation or Fitting of Poisson distribution

$$f(x) = \frac{e^{-m}m^x}{\underline{|x}}$$

$$f(x+1) = \frac{e^{-m}m^{x+1}}{\underline{|x+1}}$$

$$\therefore \quad \frac{f(x+1)}{f(x)} = \frac{m}{x+1}$$

$\Rightarrow \quad f(x + 1) = \dfrac{mf(x)}{x+1}$ …(1)

Now $\quad Nf(x) = F(x)$

$\therefore \quad F(0) = Nf(0) = Ne^{-m}$

$\therefore$ (1) $\Rightarrow Nf(x + 1) = \left(\dfrac{m}{x+1}\right), f(x)N$

$\Rightarrow \quad F(x + 1) = \dfrac{m}{x+1}F(x)\ , x = 0,1,2,.....$ …(2)

Put x = 0 in (2), w get

$$F(1) = m\,F(0) = m\,Ne^{-m}$$

$$F(2) = \frac{m}{2}F(1) = \frac{m^2}{2}Ne^{-m}$$

$$F(3) = \frac{m}{3}F(2) = \frac{m^3}{6}Ne^{-m},.....$$

**Additive Property of Poisson variate**

If $X_1$, $X_2$ are two independent Poisson variates with parameters $m_1$, $m_2$.Then their sum $X_1 + X_2$ is also a Poisson varite with Parameter $(m_1 + m_2)$

i.e. $\quad X_1 + X_2 \sim P(m_1 + m_2;x)$

**Proof: -** If X is a Poisson variable, then

$$f(x) = \frac{e^{-m}m^x}{\underline{|x}}, x = 0, 1, 2….$$

& $\quad M_{x_1}(t) = e^{m_1(e^t-1)}$

$$M_{X_2}(t) = e^{m_2(e^t-1)}$$

"The MGF of sum of two independent r.v. is equal to the product of their MGF."

By using this result, we have

$$M_{X_1+X_2}(t) = M_{X_1}(t)\,.\,M_{X_2}(t)$$

$$= e^{m_1(e^t-1)}\,.\,e^{m_2(e^t-1)}$$

$$= e^{(e^t-1)(m_1+m_2)}$$

This is the MGF of Poisson variate $X_1 + X_2$ with parameter $(m_1 + m_2)$.

**Example: -** Show that in a Poisson distribution with unit mean, deviation about mean is $\frac{2}{e}$ times the standard deviation.

**Solution: -** Here mean = 1

Variance = SD = 1 [$\because$ In Poisson distance mean = variance]

$\therefore$ M.D = Mean Deviation = E[|X−1|]

$$= \sum_{x-0}^{\infty} \frac{|x-1|e^{-1}}{\lfloor x}$$

$$= e^{-1}\sum_{x=0}^{\infty} \frac{|x-1|}{\lfloor x}$$

$$= \frac{1}{e}\left[1+\frac{1}{\lfloor 2}+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 4}+.....\right]$$

Now general term of the series

$$\frac{1}{\lfloor 2}+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 4}+.....\text{is}$$

General term = $\frac{n}{\lfloor n+1} = \frac{n+1-1}{\lfloor n+1}$

$$= \frac{1}{\lfloor n} - \frac{1}{\lfloor n+1}$$

Putting n = 1, 2, 3….

$$\text{M.D} = \frac{1}{e}\left[1+\left(1-\frac{1}{\lfloor 2}\right)+\left(\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}\right)+\left(\frac{1}{\lfloor 3}-\frac{1}{\lfloor 4}\right)+.....\right]$$

$$= \frac{1}{e}(1 + 1) = \frac{2}{e}\times 1 = \frac{2}{e}\times \text{S.D.}$$

# M.G.F about mean

M.G.F about mean for Binomial distribution is

$$M_{X-np}(t) = E[e^{t(X-np)}]$$

$$= [qe^{-pt} + pe^{(1-p)t}]^n = [qe^{-pt} + pe^{qt}]^n$$

$$= e^{-npt}[q + pe^t]^n$$

$\therefore$ $\log M_{X-np}(t) = -npt + n \log\left[q\left(1+\frac{pe^t}{q}\right)\right]$

$$= -npt + n \log q + n \log\left(1+\frac{pe^t}{q}\right)$$

Put np = m, we get

$$\text{Log } M_{X-np}(t) = -mt + n \log\left(1-\frac{m}{n}\right) + n \log\left[1+\frac{me^t}{n\left(1-\frac{m}{n}\right)}\right]$$

$$= -mt + n\left[-\frac{m}{n}-\frac{m^2}{2n^2}-\frac{m^3}{3n^3}....\right] + n\left[\frac{me^t}{n\left(1-\frac{m}{n}\right)} - \frac{1}{2}\left\{\frac{me^t}{n\left(1-\frac{m}{n}\right)}\right\}^2\right.$$

$$\left. + \frac{1}{3}\left\{\frac{me^t}{n\left(1-\frac{m}{n}\right)}\right\}^3\right]$$

$$= -mt - m + \frac{me^t}{\left(1-\frac{m}{n}\right)} - \frac{1}{2n}\left\{\frac{me^t}{1-\frac{m}{n}}\right\}^2 - \frac{m^2}{2n} + 0\left(\frac{1}{n^2}\right)$$

$$= -mt - m + \frac{me^t}{1-\frac{m}{n}} + 0\left(\frac{1}{n}\right)$$

As $n\to\infty$, we have

$$\lim_{n\to\infty} \log M_{X-np}(t) = -mt - m + me^t$$

$\Rightarrow$ $$\lim_{n\to\infty} M_{X-np}(t) = e^{me^t - mt - m}$$

$$= e^{m(e^t - t - 1)}$$

= M.G.F about mean of Poisson distance

**Example: -** If X is a Poisson variate s.t. P(X = 1) = P(X = 2), find P(X = 4).

**Solution: -** Probability of x successes in Poisson distribution is

$$P(X = x) = f(x) = \frac{e^{-m} m^x}{\lfloor x}$$

$$\therefore \quad P(X = 1) = \frac{e^{-m} m}{\lfloor 1} = e^{-m}\, m \qquad \ldots(1)$$

$$P(X = 2) = \frac{e^{-m} m^2}{\lfloor 2} = \frac{m^2}{2} e^{-m} \qquad \ldots(2)$$

Since P(X = 1) = P(X = 2)

$\therefore$ from (1) & (2), we get

$$m\, e^{-m} = \frac{m^2}{2} e^{-m} \Rightarrow m = 2$$

Therefore for X = 4

$$P(X = 4) = \frac{e^{-m} m^4}{\lfloor 4} = \frac{e^{-2}(2)^4}{24}$$

$$= \frac{16}{24} e^{-2} = 0.09$$

Let $e^{-2} = a$

$\Rightarrow$ $\log_{10} a = -2 \log_{10} e$

$\Rightarrow$ $\log_{10} a = -2(0.4343) = -0.8686$

$= \bar{1}.1314$

$\Rightarrow$ a = antilog ($\bar{1}$.1314) = 0.1353

**Example: -**Six coins are tossed, 6400 times, using Poisson distribution, what is the probability of getting six heads x times?

**Solution: -** When a coin is tossed, probability of getting head is = $\frac{1}{2}$

When 6 coins are tossed, probability of getting 6 heads $= \left(\frac{1}{2}\right)^2 = \frac{1}{64}$ = p(say)

$$\text{Mean} = np = 6400 \times \frac{1}{64} = 100 \text{ (Mean)}$$

$\Rightarrow \qquad m = 100$

Probability of getting six heads x times

$$= \frac{e^{-m} m^x}{\lfloor x} \text{ with } m = 100$$

$$= \frac{e^{-100}(100)^x}{\lfloor x}$$

**Example: -** After correcting 50 pages of the proof of a book, the proof reader finds that there are on the average 2 errors of 5 pages. How many pages would one expect to find with 0, 1, 2, 3 & 4 errors in 1000 pages of first print of the book?

**Solution**: - Now Mean = $\frac{2}{5} = 0.4 = M$ (say)

X→ Poisson distance,

$$f(x) = P(X = x) = \frac{e^{-m} m^x}{\lfloor x}$$

$$= \frac{e^{-0.4}(0.4)^x}{\lfloor x}$$

Expected number of pages with 1000 pages is

$$1000\ f(x) = \frac{1000 e^{-0.4}(0.4)^x}{\lfloor x}$$

Now we want to find f(0), f(1), f(2), f(3), f(4) by using Recurrence Relation,

$$f(x + 1) = \frac{m}{x+1} f(x$$

| X = x | f(x) | Expected number of pages = 1000 f(x) |
|---|---|---|
| 0 | $f(0) = e^{-0.4} = 0.6703$ | $670.3 \approx 670$ |
| 1 | $f(1) = \frac{0.4}{0+1} f(0) = 0.4 \times 0.6703$ = 0.26812 | $268.12 \approx 268$ |
| 2 | $f(2) = \frac{0.4}{1+1} f(1) = 0.053624$ | $53.62 \approx 54$ |

| | | |
|---|---|---|
| 3 | $f(3) = \frac{0.4}{2+1} f(2) = 0.0071298$ | $7.1298 \approx 7$ |
| 4 | $f(4) = \frac{0.4}{3+1} f(3) = 0.0007129$ | $\approx 1$ |

**Example: -** In a certain factory making blades, there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distance to calculate the approximate. Number of packets containing no defective, one defective & 2 defective blades respectively in a consignment of 10000 packets.

$(e^{-0.02} = 0.9802)$

**Solution: -** Let p→probability of blade being defective = $\frac{1}{500}$

n = 10

$\therefore$ Mean = m = np = $\frac{1}{50} = 0.02$

Number of packets containing x defective blades,

$$F(x) = \frac{Ne^{-m}m^x}{\underline{|x}}$$

$$= \frac{10000 \times e^{-0.02}(0.02)^x}{\underline{|x}}$$

$$= \frac{10000 \times 0.9802(0.02)^x}{\underline{|x}}$$

$$F(x) = \frac{9802(0.02)^x}{\underline{|x}}$$

| X | F(x) |
|---|---|
| 0 | $F(0) = \frac{9802(0.02)^0}{\underline{|0}} = 9802$ |
| 1 | $F(1) = \frac{9802(0.02)^1}{\underline{|1}} = 196$ |

2

$$F(2) = \frac{9802(0.02)^2}{\lfloor 2}$$

**Mode of Poisson distribution**

Let the most probable values of r.v. X be r i.e. r is mode.

Then P(X = r) should be greater than P(X = r + 1) and P(X = r– 1)

i.e

$$\frac{e^{-m}m^{r-1}}{\lfloor r-1} < \frac{e^{-m}m^{r}}{\lfloor r} > \frac{e^{-m}m^{r+1}}{\lfloor r+1}$$

$$\Rightarrow \quad \frac{m^{r-1}}{\lfloor r-1} < \frac{m.m^{r-1}}{r\lfloor r-1} > \frac{m^{r-1}m^2}{(r+1)r\lfloor r-1}$$

$$\Rightarrow \quad 1 < \frac{m}{r} > \frac{m^2}{r(r+1)}$$

$\Rightarrow \quad r(r + 1) < m(r + 1) > m^2$

$\Rightarrow \quad r < m \quad \& (r + 1) > m$

$\Rightarrow \quad r < m \quad \& r > m - 1$

$\Rightarrow \quad m - 1 < r < m$

Thus if m is an **integer,** there shall be two modes m and (m–1).

If m is not an integer, then mode is the integral value between (m–1) & m.

**Example: -** A Poisson distance a double mode at X = 3 and X = 4, what is the probability that X will have one or the other of these two values.

**Solution: -** $P(X = x) = \frac{e^{-m}m^x}{\lfloor x}$

$$\left.\begin{aligned} \therefore \quad P(X=3) &= \frac{e^{-m}m^3}{\lfloor 3} \\ \& \quad P(X=4) &= \frac{e^{-m}m^4}{\lfloor 4} \end{aligned}\right\} \quad \ldots(1)$$

Since the Poisson distance has a double mode at X = 3 and X = 4, so

$P(X = 3) = P(X = 4)$

$$\Rightarrow \quad \frac{e^{-m}m^3}{\lfloor 3} = \frac{e^{-m}m^4}{\lfloor 4}$$

$\Rightarrow \qquad m = \dfrac{\lfloor\underline{4}}{\lfloor\underline{3}} = 4$

$\therefore \qquad$ from (1), $P(X = 3) = \dfrac{e^{-4}4^3}{\lfloor\underline{3}} = \dfrac{32}{3}e^{-4}$

$\& \ P(X = 4) = \dfrac{e^{-4}4^4}{\lfloor\underline{4}} = \dfrac{e^{-4}.4.4^3}{4\lfloor\underline{3}}$

$= \dfrac{e^{-4}.4^3}{\lfloor\underline{3}} = \dfrac{32}{3}e^{-4}$

Now P[(X = 3) Y (X = 4)] = P(X =3) + P(X = 4)

$= \dfrac{64}{3}e^{-4}$

$= \dfrac{64}{3}(0.0183) = 0.3904$

**Example: -** If X is a Poisson variate with mean m, show that $\dfrac{X-m}{\sqrt{m}}$ is a variable with mean **zero** and variance **unity.**

**Solution: -** We know that

Mean = E(X) = m

Let $Z = \dfrac{X-m}{\sqrt{m}}$

then $\quad E(Z) = E\left(\dfrac{X-m}{\sqrt{m}}\right) = \dfrac{1}{\sqrt{m}}E(X-m)$

$= \dfrac{1}{\sqrt{m}}\left[E(X) - E(m)\right]$

$= \dfrac{1}{\sqrt{m}}\left[m - m\right] = 0$

and variance $= E\left(\dfrac{X-m}{\sqrt{m}}\right)^2$

$= \dfrac{1}{m}E(X-m)^2 = \dfrac{1}{m}\mu_2$

$$= \frac{1}{m}.m = 1$$

Also find M.G.F of this variable and show that it approaches $e^{\frac{1}{2}t^2}$ as m→∞

$$\text{Let } Y = \frac{X - m}{\sqrt{m}}$$

then $$M_Y(t) = E[e^{t^4}] = E\left[e^{t\frac{(X-m)}{\sqrt{m}}}\right]$$

$$= e^{-\frac{mt}{\sqrt{m}}} E\left(e^{\frac{tx}{\sqrt{m}}}\right)$$

$$= e^{-\sqrt{m}t} \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{\lfloor x} e^{\frac{tx}{\sqrt{m}}}$$

$$= e^{-\sqrt{m}t} e^{-m} \sum_{x=0}^{\infty} \frac{(me^{t/\sqrt{m}})^x}{\lfloor x}$$

$$= e^{-m-\sqrt{m}t}\left[1 + \frac{me^{t/\sqrt{m}}}{\lfloor 1} + \frac{(me^{t/\sqrt{m}})^2}{\lfloor 2} + .....\right]$$

$$= e^{-m-\sqrt{m}t} e^{me^{t/\sqrt{m}}}$$

$$= e^{-m-\sqrt{m}t+m\left(1 + \frac{t}{\sqrt{m}} + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3\, m\sqrt{m}} + ....\right)}$$

$$M_Y(t) = e^{\frac{t^2}{2} + \frac{t^3}{\lfloor 3\, \sqrt{m}} + .....}$$

As m→∞, $\lim_{m\to\infty} M_4(t) = e^{t^2/2}$ which is MGF of standard normal variate (Normal Distribution)?

# GEOMETRIC DISTRIBUTION

Suppose we have a series of independent trials or repetitions and on each trial or repetition, the probability of success 'p' remains the same. Then the probability preceding the first success is given by

$$pq^x$$

where X→Number of failures preceding the 1st success in series of independent trials.

and p = P(success in a single trial)

eg., in tossing a coin, the possible outcomes of experiment are

S, FS, FFS, ……FF,…..FS,……..

X = 0, 1, 2,……, x

and corresponding probability are

$p, qp, q^2p, \ldots\ldots, q^xp\ldots$

∴ Probability distribution of X,

$$f(x) = P(X = x) = \begin{cases} pq^x & , \quad x = 0,1,2,\ldots\ldots\infty \\ 0 & , \quad \text{otherwise} \end{cases}$$

Also $\sum_x f(x) = 1$

Now $\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} pq^x$

$$= p + pq + pq^2 + pq^3 + \ldots..$$

$$= p(1 + q + q^2 + \ldots\ldots)$$

$$= p.\frac{1}{1-q} \qquad \left[\text{sum of inf inite G.P.S} + \frac{a}{1-r}\right]$$

$$= p.\frac{1}{p} = 1$$

**Moments of Geometric distribution**

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} xf(x) = \sum_{x=1}^{\infty} xpq^x$$

$$= pq\left(\sum_{x=1}^{\infty} xq^{x-1}\right)$$

$$= pq(1 + 2q + 3q^2 + \ldots\ldots)$$

$$= pq(1 - q)^{-2}$$

$$\Rightarrow \quad \mu_1' = pq.\frac{1}{p^2} = \frac{q}{p} \qquad \ldots(1)$$

$$\Rightarrow \quad \mu_1' = \text{Mean} = \frac{q}{p}$$

$$\mu_2' = \sum_{x=0}^{\infty} x^2 f(x) = \sum_{x=0}^{\infty} [x(x-1) + x] pq^x$$

$$= \sum_{x=2}^{\infty} x(x-1)pq^x + \sum_{x=1}^{\infty} x\,pq^x$$

$$= 2pq^2 \sum_{x=2}^{\infty} \frac{x(x-1)}{2.1} q^{x-2} + \frac{q}{p} \qquad \text{[from (1)]}$$

$$= 2pq^2(1-q)^{-3} + \frac{q}{p} = \frac{2q^2}{p^2} + \frac{q}{p}$$

which is 2nd moment about origin.

$$\therefore \quad \text{variance, } \mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p}$$

$$= \frac{q}{p^2}(q + p) = \frac{q}{p^2}$$

# MGF about origin

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\, pq^x$$

$$= p\sum_{x=0}^{\infty} (qe^t)^x = p[1 - qe^t]^{-1}$$

$$= \frac{p}{1 - qe^t}$$

# Moments from MGF

$$\mu_1' = \frac{d}{dt} M_X(t) \Big/_{t=0} = \frac{p(-1)(-qe^t)}{(1 - qe^t)^2} \Big/_{t=0}$$

$$= \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) \Big/_{t=0} = \frac{d}{dt}\left[\frac{pqe^t}{(1-qe^t)}\right]\Big/_{t=0}$$

$$= pq\left[\frac{(1-qe^t)^2 e^t - e^t.2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]\Big/_{t=0}$$

$$= pq\left[\frac{(1-qe^t)e^t + 2qe^{2t}}{(1-qe^t)^3}\right]\Big/_{t=0}$$

$$= pq\left[\frac{(1-q)+2q}{(1-q)^3}\right]$$

$$= pq\left(\frac{p+2q}{p^3}\right) = \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\mu_3' = \frac{d^3}{dt^3} M_X(t) \Big/_{t=0}$$

$$= pq \frac{d}{dt}\left[e^t(1-qe^t)^{-2} + 2qe^{2t}(1-qe^t)^{-3}\right]\Big/_{t=0}$$

$$= pq\ [e^t(1-qe^t)^{-2} + e^t(-2)(1-qe^t)^{-3}(-qe^t) + 2qe^{2t}.\ 2(1-qe^t)^{-3} + 2qe^{2t}(-3)$$
$$(1-qe^t)^{-4}(-qe^t)]$$

$$= pq[(1-q)^{-2} + 2q(1-q)^{-3} + 4q(1-q)^{-3} + 6q^2e(1-q)^{-4}]$$

$$\Rightarrow \qquad \mu_3' = \left[\frac{1}{p^2} + \frac{6q}{p^3} + \frac{6q^2}{p^4}\right] pq$$

$$\mu_3' = \frac{q}{p} + \frac{6q^2}{p^2} + \frac{6q^3}{p^3}$$

$$= \frac{q}{p} + \frac{6q^2}{p^3}(p+q)$$

$$= \frac{q}{p} + \frac{6q^2}{p^3}.1 = \frac{q}{p} + \frac{6q^2}{p^3}$$

$$\mu'_4 = \frac{d^4}{dt^4} M_X(t) \Big/_{t=0}$$

$$= pq \frac{d}{dt}\left[e^t (1-qe^t)^{-2} + 2qe^{2t}(1-qe^t)^{-3} + 4qe^{2t}(1-qe^t)^{-3} + 6q^2e^{3t}(1-qe^t)^{-4}\right] \Big/_{t=0}$$

$$= pq \left[e^t (1-qe^t)^{-2} + e^t qe^t .2(1-qe^t)^{-3} + 4qe^{2t}(1-qe^t)^{-3} + 6q^2e^{3t}(1-qe^t)^{-4} + 8qe^{2t}(1-qe^t)^{-3} + 12q^2e^{3t}(1-qe^t)^{-4} + 18q^2e^{3t}(1-qe^t)^{-4} + 24q^2e^{3t}(1-qe^t)^{-5}(qe^t)\right] \Big/_{t=0}$$

$$= pq \,[(1-q)^{-2} + 2q\,(1-q)^{-3} + 4q\,(1-q)^{-3} + 6q^2\,(1-q)^{-4} + 8q\,(1-q)^{-3} + 12q^2\,(1-q)^{-4} + 18q^2\,(1-q)^{-4} + 24q^3\,(1-q)^{-5}]$$

$$= pq \left[\frac{1}{p^2} + \frac{14q}{p^3} + \frac{36q^2}{p^4} + \frac{24q^3}{p^5}\right]$$

$$\mu'_4 = \frac{q}{p} + \frac{14q^2}{p^2} + \frac{36q^3}{p^3} + \frac{24q^4}{p^4}$$

# LESSON 7 THEORETICAL CONTINUOUS DISTRIBUTIONS

## Uniform or rectangular distribution

A **continuous** r.v. X is said to have continuous uniform (or rectangular) distribution over an interval [a, b] if its p.d.f. is given by

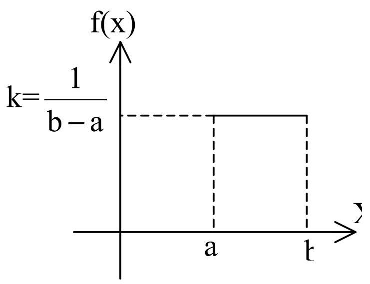
$$f(x) = \begin{cases} K & , \quad a \le x \le b \\ 0 & , \quad \text{otherwise} \end{cases}$$

also $\int_a^b f(x)\ dx = 1$

$\Rightarrow \quad \int_a^b K\,dx = 1 \quad \Rightarrow [Kx]_a^b = 1$

$\Rightarrow \quad K(b-a) = 1 \quad \Rightarrow K = \frac{1}{b-a}$

$$\therefore \quad f(x) = \begin{cases} \frac{1}{b-a} & , \quad a \le x \le b \\ 0 & , \quad \text{otherwise} \end{cases}$$




### Moments

$$\mu_r' = E[X^r] = \int_a^b x^r\ f(x)\ dx$$

$$\therefore \quad \mu_r' = \int_a^b x^r \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{(b-a)} \left(\frac{x^{r+1}}{r+1}\right)_a^b$$

$$= \frac{1}{(b-a)(r+1)}\left[b^{r+1} - a^{r+1}\right]$$

$$\therefore \qquad \mu_1' = \frac{1}{(b-a)}\left(\frac{b^2 - a^2}{2}\right) = \frac{b+a}{2}$$

$$\Rightarrow \qquad \mu_1' = \text{Mean} = \frac{b+a}{2}$$

$$\mu_2' = \frac{1}{(b-a).3}(b^3 - a^3) = \frac{(b-a)(b^2 + a^2 + ab)}{3(b-a)}$$

$$= \frac{1}{3}(b^2 + a^2 + ab)$$

Variance, $\mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{1}{3}(b^2 + a^2 + ab) - \frac{1}{4}\ (b^2 + a^2 + 2ab)$$

$$= \frac{b^2}{12} + \frac{a^2}{12} - \frac{2ab}{12}$$

$$= \frac{1}{12}(b^2 + a^2 + 2ab) = \frac{(a-b)^2}{12}$$

$$\Rightarrow \qquad \mu_2 = \frac{1}{12}(b-a)^2$$

**MOMENT GENERATING FUNCTION**

$$M_X(t) = \int_a^b e^{tx} . \frac{1}{(b-a)}\, dx$$

$$\Rightarrow \qquad M_X(t) = \frac{1}{(b-a)}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{t(b-a)}\left[e^{bt} - e^{at}\right]$$

**Characteristic function**

$$\phi_X(t) = \int_a^b e^{itx}\ f(x)\, dx = \frac{e^{ibt} - e^{iat}}{it(b-a)}$$

## Mean deviation

$$E[|X-Mean|] = \int_a^b \left| x - \frac{(a+b)}{2} \right| . \frac{1}{(b-a)} dx$$

$$= \frac{1}{(b-a)} \int_a^b \left| x - \left( \frac{a+b}{2} \right) \right| dx \qquad \ldots(1)$$

Put $t = x - \frac{a+b}{2} \quad \Rightarrow \quad dt = dx$

when $x = b, \; t = a - \frac{(a+b)}{2} = \frac{a-b}{2}$

when $x = b, \; t = b - \frac{(a+b)}{2} = \frac{b-a}{2}$

$$\therefore (1) \Rightarrow E[|X - Mean|] = \frac{1}{(b-a)} \int_{-\left(\frac{b-a}{2}\right)}^{\frac{b-a}{2}} |t| dt$$

$$= \frac{2}{(b-a)} \int_0^{b-a/2} t\, dt = \frac{2}{(b-a)} \left[ \frac{t^2}{2} \right]_0^{\frac{b-a}{2}}$$

$$= \frac{2}{2(b-a)} . \frac{(b-a)^2}{4} = \frac{b-a}{4}$$

**Example: -** Show that for the rectangular distance if $f(x) = \frac{1}{2a}$, $-a < x < a$,

The MGF about origin is $\frac{\sinh at}{at}$.

Also show that moments of even order are given by $\mu_{2n} = \frac{a^{2n}}{2n+1}$

**Solution: -** By definition,

$$M_X(t) = \int_{-a}^{a} e^{tx} . \frac{1}{2a} dx = \frac{1}{2at} \left[ e^{tx} \right]_{-a}^{a}$$

$$= \frac{1}{2at} (e^{at} - e^{-at}) = \frac{1}{2at} . 2 \sinh at$$

$\Rightarrow \quad M_X(t) = \dfrac{\sinh at}{at}$

$$= \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + ......\right]$$

$$= 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + ....... \qquad ...(1)$$

Since there are no terms with odd powers of t in $M_X(t)$, all moments of odd order about origin are zero i.e.

$\mu'_{2n+1}$ (about origin) = 0

In particular, $\mu'_1 = 0$ i.e. Mean = 0

$\Rightarrow \quad \mu'_r$ (about origin) = $\mu_r$ (about Mean)

$$\left[\because \ \mu'_r = E[X^r], \qquad \mu_r = E[(X-\mu)^r] \right.$$

$$\left. \text{if } \mu = 0 = \text{Mean, then} \qquad \mu_r = E[X^r] = \mu'_r \right]$$

Therefore $\mu_{2n+1} = 0$

The moments of even order are given by

$$\mu'_{2n} = \mu_{2n} = \text{Coefficient of } \frac{t^{2n}}{\lfloor 2n} \text{ in } M_X(t)$$

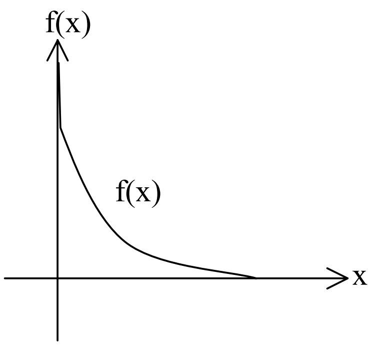
$$\Rightarrow \quad \mu_{2n} = \frac{a^{2n}}{2n+1} \qquad \text{[using (1)]}$$

# THE EXPONENTIAL DISTRIBUTION

A **continuous** r.v. X assuming **non-negative** values is said to have an exponential distribution with parameter **θ > 0** if its p.d.f. is given by

$$f(x) = \begin{cases} \theta e^{-\theta x} & , \quad x \ge 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Also $\displaystyle\int_0^\infty f(x)\,dx = 1$

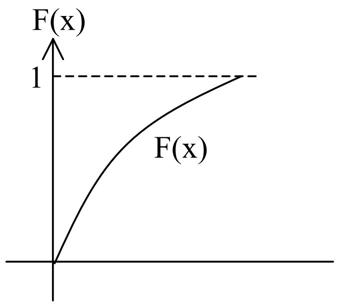


The Cumulative Distribution function

F(x) is given by,

$$F(x) = \int_0^x f(x)\,dx = \theta\int_0^x e^{-\theta x}\,dx$$

$$= \begin{cases} 1-e^{-\theta x}, & x \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$




**MOMENT GENERATING FUNCTION**

$$M_X(t) = E[e^{tx}] = \theta\int_0^\infty e^{tx}e^{-\theta x}\,dx$$

$$= \theta\int_0^\infty e^{(\theta-t)x}\,dx$$

$$= \theta\left[\frac{e^{-(\theta-t)x}}{-(\theta-t)}\right]_0^\infty$$

$$= \frac{\theta}{\theta-t}, \quad \theta > t$$

$$\Rightarrow \qquad M_X(t) = \frac{\theta}{\theta-t} = \frac{\theta}{\theta\left(1-\frac{t}{\theta}\right)} = \frac{1}{1-\frac{t}{\theta}}$$

$$= \left(1-\frac{t}{\theta}\right)^{-1} = 1+\frac{t}{\theta}+\frac{t^2}{\theta}+\frac{t^3}{\theta}+\ldots\ldots \qquad \ldots(1)$$

$$= \sum_{r=0}^{\infty}\left(\frac{t}{\theta}\right)^r$$

$$(1) \Rightarrow \quad M_X(t) = 1 + \frac{t}{\theta} + \frac{\lfloor\underline{2}}{\theta^2}\frac{t^2}{\lfloor\underline{2}} + \frac{\lfloor\underline{3}}{\theta^3}\frac{t^3}{\lfloor\underline{3}} + .......$$

$$\therefore \qquad \mu_r' = E[X^r] = \text{coefficient of } \frac{t^r}{\lfloor\underline{r}} \text{ in } M_X(t)$$

$$= \frac{\lfloor\underline{r}}{\theta^2}, \qquad r = 1, 2, 3, ........$$

$$\therefore \qquad \text{Mean} = \mu_1' = \frac{1}{\theta}$$

$$\mu_2' = \frac{2}{\theta^2}$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

Also by definition,

$$\text{Mean} = \int_0^\infty x\, f(x)\, dx = \int_0^\infty x\, \theta\, e^{-\theta x}\, dx$$

$$= \theta\left[\left(x\frac{e^{-\theta x}}{-\theta}\right)_0^\infty - \int_0^\infty (1)\frac{e^{-\theta x}}{-\theta}\, dx\right]$$

$$= \theta\left[0 + (-1)\frac{e^{-\theta x}}{\theta^2}\right]_0^\infty$$

$$= -\frac{1}{\theta}\left[e^{-\theta x}\right]_0^\infty = \frac{-1}{\theta}[0-1] = \frac{1}{\theta}$$

$$E(X^2) = \mu_2' = \int_0^\infty x^2\, f(x)\, dx = \int_0^\infty x^2 . \theta e^{-\theta x}\, dx$$

$$= \theta\left[\left(x^2 . \frac{e^{-\theta x}}{-\theta}\right)_0^\infty - \int_0^\infty 2x . \frac{e^{-\theta x}}{-\theta}\, dx\right]$$

$$= \theta\left[0 - 0 + \frac{2}{\theta}\int_0^\infty x e^{-\theta x}\, dx\right]$$

$$= \frac{2}{\theta}\left[\theta\int_0^\infty x e^{-\theta x}\, dx\right] = \frac{2}{\theta} . \frac{1}{\theta} = \frac{2}{\theta^2}$$

Variance = E[X–E(x)]$^2$

$$= E\left[X - \frac{1}{\theta}\right]^2 = E(X^2) - \frac{1}{\theta^2}$$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

## Mean Deviation

$$E|X-E(X)| = E\left[\left|X - \frac{1}{\theta}\right|\right]$$

$$= \int_0^\infty \left|x - \frac{1}{\theta}\right| \theta e^{-\theta x}\ dx$$

$$= \frac{\theta}{\theta}\int_0^\infty |x\theta - 1| e^{-\theta x}\ dx = \int_0^\infty |x\theta - 1|\ e^{-\theta x}\ dx$$

Put $\theta x = y \Rightarrow \theta dx = dy$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$\therefore$ $$E[|X-E(X)|] = \int_0^\infty |y-1| e^{-y} \frac{dy}{\theta}$$

$$= \frac{1}{\theta}\int_0^\infty |y-1| e^{-y}\ dy \qquad \ldots(1)$$

when $|y-1| = -(y-1)$ for $y-1 < 0$

$\Rightarrow$ $|y-1| = 1-y,$ $y < 1$

and $|y-1| = y-1$ for $y-1 > 0$

$= y-1$ for $y > 1$

$\therefore (1)$ $$E[|X-E(X)|] = \frac{1}{\theta}\left[\int_0^1 (1-y)e^{-y}\ dy + \int_1^\infty (y-1)^{-y}\ dy\right]$$

$$= \frac{1}{\theta}\left[\left((1-y)\frac{e^{-y}}{-1}\right)_0^1 - \int_0^1 (-1)\frac{e^{-y}}{-1} dy + \left((y-1)\frac{e^{-y}}{-1}\right)_1^\infty\right.$$

$$- \int_1^{\infty} 1.\frac{e^{-y}}{-1} dy \Bigg]$$

$$= \frac{1}{\theta}\left[-0+1-\left(\frac{e^{-y}}{-1}\right)_0^1 + 0 - 0 + \left(\frac{e^{-y}}{-1}\right)_1^{\infty}\right]$$

$$= \frac{1}{\theta}\left[1+(e^{-1}-1)-(0-e^{-1})\right]$$

$$= \frac{1}{\theta}[e^{-1}+e^{-1}] = \frac{2}{\theta}e^{-1}$$

## Cumulants

The c.g.f. is

$$K_X(t) = \log M_X(t) = \log\left(1-\frac{t}{\theta}\right)^{-1} = -\log\left(1-\frac{t}{\theta}\right)$$

$$= \left[\frac{t}{\theta}+\frac{1}{2}\left(\frac{t}{\theta}\right)^2+\frac{1}{3}\left(\frac{t}{\theta}\right)^3+.......\right]$$

$$= \sum_{r=1}^{\infty}\left(\frac{t}{\theta}\right)^r\frac{1}{r} = \sum_{r=1}^{\infty}\frac{t^r}{\lfloor\underline{r}}\left(\frac{\lfloor\underline{r-1}}{\theta^r}\right)$$

$\Rightarrow$ $K_r$ = coefficient of $\frac{t^r}{\lfloor\underline{r}}$ in $K_X(t)$

$$= \frac{\lfloor\underline{r-1}}{\theta^r} \quad , r = 1, 2, 3, .......$$

$\therefore$ $K_1 = \frac{1}{\theta}$ , $K_2 = \frac{1}{\theta^2}$

$K_3 = \frac{2}{\theta^3}$ , $K_4 = \frac{6}{\theta^4}$

**Example: -** Let X have the probability density function $f(x) = \theta e^{-\theta x}$ for $x > 0$ and zero, otherwise. Prove that if the +ve part of x-axis is divided into intervals of equal length h starting at the origin, then the probabilities that X will lie in successive intervals from a G.P. with common ratio $e^{-\theta h}$.

**Solution: -** $P(0 \le X \le h) = \int_0^h f(x)\,dx = \int_0^h \theta e^{-\theta x}\,dx$

$$= \theta\left[\frac{e^{-\theta x}}{-\theta}\right]_0^h = -1\left[e^{-\theta h} - 1\right]$$

$$= 1 - e^{\theta h}$$

$$P(h \le X \le 2h) = \int_h^{2h} \theta e^{-\theta h}\,dx$$

$$= -\frac{\theta}{\theta}\left[e^{-\theta x}\right]_h^{2h} = -\left[e^{-2\theta h} - e^{-\theta h}\right]$$

$$= e^{-\theta h}\,(1 - e^{-\theta h})$$

$$P(2h \le X \le 3h) = -\left[e^{-\theta x}\right]_{2h}^{3h}$$

$$= e^{-2h\theta} - e^{-3h\theta} = e^{-2\theta h}(1 - e^{-\theta h})$$

and so on.

Thus the probabilities that X will lie in successive intervals from a G.P. with common ratio $e^{-\theta h}$.

**Example: -** The r.v. X has the exponential distribution with density function given by

$$f(x) = \begin{cases} 2e^{-2x} & , \quad x > 0 \\ 0 & , \quad x \le 0 \end{cases}$$

(i) what is the probability that X is not smaller than two?

(ii) Show that $\frac{\text{Mean}}{\text{S.D}} = 1$

(iii) Find its M.G.F

**Solution: -(i)** Probability that X is not smaller than two

$$P(X \ge 2) = \int_2^{\infty} f(x)\,dx = 2\int_2^{\infty} e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_2^{\infty} = -1\left[0 - e^{-4}\right] = \frac{1}{e^4}$$

**(ii)** Mean = $\mu_1' = E(X) = \int_0^{\infty} x f(x)\,dx$

$$= 2\int_0^\infty x\, e^{-2x}\ dx$$

$$= 2\left[\left(x\frac{e^{-2x}}{-2}\right)_0^\infty + \int (1)\frac{e^{-2x}}{2}\ dx\right]$$

$$= 2\left[0 - 0 - \frac{1}{4}\left(e^{-2x}\right)_0^\infty\right]$$

$$= 2\left[-\frac{1}{4}(0-1)\right] = 2.\frac{1}{4} = \frac{1}{2}$$

$$\mu_2' = E(X^2) = \int_0^\infty x^2 f(x)\ dx \qquad = 2\int_0^\infty x^2 e^{-2x}\ dx$$

$$\mu_2' = 2\left[\left(x^2\,\frac{e^{-2x}}{-2}\right)_0^\infty + \int_0^\infty 2x.\frac{e^{-2x}}{+2}\ dx\right] = 2\left[0 - 0 + \frac{1}{2}.2\int_0^\infty xe^{-2x}\ dx\right] = 2\left[\frac{1}{2}.\frac{1}{2}\right] = \frac{1}{2}$$

Variance, $\mu_2 = \mu_2' - \mu_1' = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore$ S.D. $= \sqrt{\mu_2} = \frac{1}{2}$

$\therefore$ $\frac{\text{Mean}}{\text{S.D}} = \frac{1/2}{1/2} = 1$

**(iii)** M.G.F., $M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx}\ f(x)\ dx$

$$= \int_0^\infty e^{tx}(2)e^{-2x}\ dx \qquad = 2\int_0^\infty e^{(t-2)x}\ dx = 2\left[\frac{e^{(t-2)x}}{(t-2)}\right]_0^\infty$$

$$= \frac{2}{(t-2)}\left[e^{-(2-t)x}\right]_0^\infty = \frac{2}{t-2}[0-1] \quad = \frac{-2}{t-2} = \frac{2}{2-t},\quad 2 > t$$

# NORMAL DISTRIBUTION

A r.v. X is said to have Normal distribution with parameters $\mu$ and $\sigma^2$, called Mean & variance respectively if its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad , \ -\infty < x < \infty; \sigma > 0; -\infty < \mu < \infty$$

If $\quad Z = \dfrac{X-\mu}{\sigma} \Rightarrow z = \dfrac{x-\mu}{\sigma}$

Here $\quad$ Mean = E(z) = 0

and $\quad$ variance = V(Z) = 1 = S.D.

then $\quad f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad , \ -\infty < z < \infty$

then f(z) is known as Standard Normal distribution.

Now $\quad \displaystyle\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx$

Put $z = \dfrac{x-\mu}{\sigma} \quad \Rightarrow \quad \sigma\, dz = dx$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} .\sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \, dz$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\, dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} \, dz \qquad \text{[ΘIntegrand is even function of z]}$$

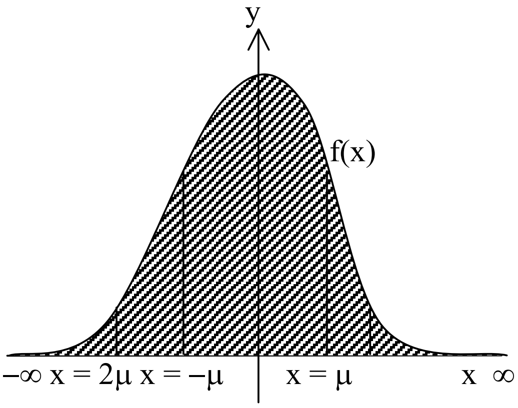
Put $\quad \dfrac{1}{2}z^2 = t$

$\Rightarrow \quad z\, dz = dt \Rightarrow dz = \dfrac{dt}{z} = \dfrac{dr}{\sqrt{2t}}$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\, dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \frac{dt}{\sqrt{2\pi}} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} e^{-1/2} \, dt$$

$$= \frac{1}{\sqrt{\pi}}.\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \qquad \ldots(1)$$

$\Rightarrow$ it is a probability density function.




This is called *Normal distribution* or Normal probability curve or *Gaussian distribution.*

This curve is symmetrical w.r.t. y-axis. The area under the curve gives the probability density function, $f(x) = P(0 \le X \le x)$ where x is constant.

Mean, $\mu_1' = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{\infty} x.\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

Put $z = \frac{x-\mu}{\sigma}$ ...(2)

$\Rightarrow$ $x = \mu + \sigma z \Rightarrow dx = \sigma\, dz$

$\therefore$ $\mu_1' = \int_{-\infty}^{\infty} (\mu + \sigma z).\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, \sigma\, dz$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2}\, dz$$

$\Rightarrow$ $\mu_1' = \mu(1) + 0$ [from(1)] [$\Theta$ integrand $z e^{-\frac{1}{2}z^2}$ is an odd function of z]

$\Rightarrow$ $\mu_1' = \mu$ (Mean)

Variance, $\mu_2 = E[(X-\mu)^2]$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 \frac{e^{-\frac{1}{2}z^2}}{\sigma} . \sigma\, dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

Put $\frac{z^2}{2} = t \Rightarrow dz = \frac{dt}{\sqrt{2t}}$

$$\mu_2 = \frac{\sqrt{2}}{\pi}\sigma^2 \int_0^{\infty} 2t\, e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2}{\sqrt{\pi}}\sigma^2 \int_0^{\infty} e^{-t}\, t^{1/2}\, dt$$

$$= \frac{2}{\sqrt{\pi}}\sigma^2 \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}}\sigma^2 . \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \mu_2 = \frac{\sigma^2}{\sqrt{\pi}}.\sqrt{\pi} = \sigma^2 \quad \Rightarrow \quad \mu_2 = \sigma^2$$

## Normal distribution as a limiting case of Binomial distribution

It can be derived under these conditions: -

(i) n, the number of trials is infinitely large i.e. $n \to \infty$

(ii) neither p nor q is very small.

p.d.f. of Binomial distribution is

$$f(x) = {}^nC_x\, p^x\, q^{n-x}$$

$$= \frac{\lfloor n}{\lfloor x \lfloor n-x} p^x q^{n-x}, \ x = 0, 1, 2, \ldots, n \qquad \ldots(1)$$

$$Z = \frac{X - E(X)}{\sqrt{var(x)}} \quad \text{i.e.} \quad z = \frac{x - np}{\sqrt{npq}} \qquad \ldots(2)$$

where $x = 0, 1, 2, \ldots, n$

when $x = 0 \Rightarrow z = -\sqrt{\frac{np}{q}}$;   As $n \to \infty, z \to -\infty$

when $x = n \Rightarrow z = \frac{n - np}{\sqrt{npq}} = \frac{nq}{\sqrt{npq}} = \sqrt{\frac{nq}{p}}$

and  As $n \to \infty$,  $z \to \infty$

Thus in the limiting case as $n \to \infty$, z takes values $-\infty$ to $\infty$.

Using Stirling's approximation formula,

$$\lfloor r = \sqrt{2\pi}\, e^{-r}\, r^{r+\frac{1}{2}}$$

From (1) as $n \to \infty$, we get

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}\, p^x\, q^{n-x}}{\sqrt{2\pi}\, e^{-x}\, x^{x+\frac{1}{2}} \sqrt{2\pi}\; e^{-(n-x)}\, (n-x)^{n-x+\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{n^{n+\frac{1}{2}}\, p^x\, q^{n-x}}{x^{x+\frac{1}{2}}\, (n-x)^{n-x+\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{n^{n+\frac{1}{2}}\, p^x\, q^{n-x}}{x^{x+\frac{1}{2}}\, (n-x)^{n-x+\frac{1}{2}}} . \frac{\sqrt{npq}}{\sqrt{npq}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{n^{n+1}\, p^{x+\frac{1}{2}}\, q^{n-x+\frac{1}{2}}}{\sqrt{npq}\, x^{x+\frac{1}{2}}\, (n-x)^{n-x+\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{n^{n+1}}{\sqrt{npq}} . \frac{1}{x^{x+\frac{1}{2}}\, (n-x)^{n-x+\frac{1}{2}}} \frac{(n.q)^{n-x+\frac{1}{2}}}{n^{n-x+\frac{1}{2}}} . \frac{(np)^{x+\frac{1}{2}}}{n^{x+\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{\sqrt{npq}} \left(\frac{np}{x}\right)^{x+\frac{1}{2}} \left(\frac{nq}{n-x}\right)^{n-x+\frac{1}{2}} \qquad \ldots(3)$$

from (2), $x = np + z\sqrt{npq}$

$\therefore \qquad \frac{n}{np} = 1 + z\frac{\sqrt{npq}}{np} = 1 + z\sqrt{\frac{q}{np}}$ …(4)

and $\quad n - x = n - np - z\sqrt{npq}$

$$= n(1-p) - z\sqrt{npq} = nq - z\sqrt{npq}$$

$\therefore \qquad \frac{n-x}{nq} = 1 - z\sqrt{\frac{p}{nq}}$ …(5)

Hence probability difference of the distribution is

$$\lim_{n\to\infty} f(x)\,dx = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\sqrt{npq}}.\frac{1}{N}\,dx \qquad \text{…(6)}$$

where $N = \left(\frac{x}{np}\right)^{x+\frac{1}{2}}\left(\frac{n-x}{nq}\right)^{n-x+\frac{1}{2}}$

using (4) & (5), we get

$$N = \left(1+z\sqrt{\frac{q}{np}}\right)^{\left(np+z\sqrt{npq}+\frac{1}{2}\right)}\left(1-z\sqrt{\frac{p}{nq}}\right)^{\left(nq-z\sqrt{npq}+\frac{1}{2}\right)}$$

$$\therefore \log N = \left(np+z\sqrt{npq}+\frac{1}{2}\right)\log\left(1+z\sqrt{\frac{q}{np}}\right)+\left(nq-z\sqrt{npq}+\frac{1}{2}\right)\log\left(1-z\sqrt{\frac{p}{nq}}\right)$$

$$= \left(np+z\sqrt{npq}+\frac{1}{2}\right)\left[z\sqrt{\frac{q}{np}}-\frac{1}{2}z^2\frac{q}{np}+\frac{1}{3}z^3\frac{q^{3/2}}{(np)^{3/2}}.......\right]$$

$$+\left(nq-z\sqrt{npq}+\frac{1}{2}\right)\left[-z\sqrt{\frac{p}{nq}}-\frac{z^2}{2}\frac{p}{nq}-\frac{z^3}{3}\frac{p^{3/2}}{(nq)^{3/2}}.......\right]$$

$$= z\sqrt{npq}-\frac{z^2}{z}q+\frac{z^3}{3}\frac{q^{3/2}}{\sqrt{nq}}+......+z^2q-\frac{z^3}{2}\frac{q^{3/2}}{\sqrt{np}}+\frac{z}{2}\sqrt{\frac{q}{np}}-\frac{z^2}{4}\frac{q}{np}$$

$$+\frac{z^3}{6}\frac{q^{3/2}}{(np)^{3/2}}+\left[-z\sqrt{npq}-\frac{z^2}{2}p-\frac{z^3}{3}\frac{p^{3/2}}{\sqrt{nq}}+z^2p+\frac{z^3}{2}\frac{p^{3/2}}{\sqrt{nq}}-\frac{3}{2}\sqrt{\frac{p}{nq}}\right]$$

$$-\frac{z^2}{4}\frac{p}{nq}-\frac{z^3}{6}\frac{p^{3/2}}{(nq)^{3/2}}+........$$

Collecting the terms in decreasing power of n, we get

$$\log N = \frac{z^2}{2}(p+q) + \frac{1}{\sqrt{n}}\left[\frac{z^3}{\underline{|3}}\left(\frac{-q^{3/2}}{\sqrt{p}} + \frac{p^{3/2}}{\sqrt{q}}\right) + \frac{z}{2}\left(\sqrt{\frac{q}{p}} - \sqrt{\frac{p}{q}}\right)\right] + \ldots\ldots$$

$$= \frac{z^2}{2} + \frac{1}{\sqrt{n}}\left[\left(\frac{z^3}{\underline{|3}}\frac{p^2-q^2}{\sqrt{pq}}\right) + \frac{3}{2}\left(\frac{q-p}{\sqrt{pq}}\right)\right] + \text{[terms of higher powers } 1/\sqrt{n}\text{]}$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} \log N = z^{2/2}$$

or $\lim_{n\to\infty} \log N = z^{2/2} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{N} = e^{-z^{2/2}}$

Put this value in (6), we get the probability of the distribution of Z is given by

$$dP = \lim_{n\to\infty} f(x)\,dx = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\sqrt{npq}} . e^{-z/2}\sqrt{npq}\,dz \qquad [\Theta\ dz = \frac{dx}{\sqrt{npq}}]$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

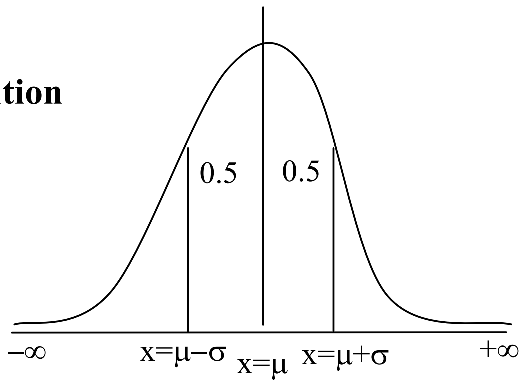
Hence probability function of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

The probability function of normal distribution with variable X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

## Some characteristics of Normal distribution




$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $-\infty < x < \infty$

and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ where $z = \frac{x-\mu}{\sigma}$

(1) The curve is Bell shaped & symmetrical about the line $x = \mu$

(2) Mean, Median & Mode coincides.

(3) Max. probability occurs at the point $x = \mu$ and is given by

$$[f(x)|_{max.} = \frac{1}{\sigma\sqrt{2\pi}}$$

(4) $\beta_1 = 0$, $\beta_2 = 3$

(5) All odd order moments are zero i.e. $\mu_{2n+1} = 0$

(6) Linear combination of independent normal variate is also a normal variate.

(7) Points of inflexion of the curve are given by $x = \mu \pm \sigma$

(8) Mean Derivation is $\frac{4}{5}\sigma$ (appox.)

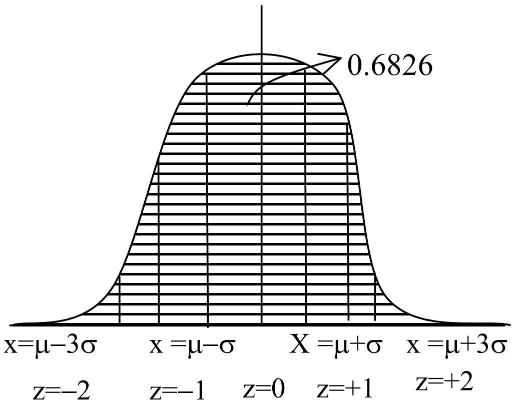
$$= \sqrt{\frac{2}{\pi}}\sigma$$

(9) Area property

$P(\mu - \sigma < \times < \mu + \sigma) = 0.6826$

$P(\mu - 2\sigma < \times < \mu + 2\sigma) = 0.9544$

$P(\mu - 3\sigma < \times \mu + 3\sigma) = 0.9973$




when $x = \mu - \sigma$ , $z = -1$

and $x = \mu + \sigma$ , $z = +1$

$\therefore$ $P(-1 < Z < +1) = 0.6826$

$P(-2 < Z < +2) = 0.9544$

$P(-3 < Z < +3) = 0.9973$

also $P(-z < Z < 0) = P(0 < Z < z)$ due to symmetry.

**Theorem: -** Prove that Mean, Median and Mode coincide.

**Proof: - Mode**

To calculate Mode, we have

$f'(x) = 0 \qquad , f''(x) < 0$

Now $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ …(1)

$\therefore$ $\log f(x) = -\log \sigma\ \sqrt{2\pi} - \dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2$

Differentiating w.r.t. x, we get

$$\frac{1}{f(x)} f'(x) = 0 - \frac{2}{2\sigma^2}(x-\mu) = \frac{-1}{\sigma^2}(x-\mu)$$

$\Rightarrow$ $f'(x) = -\dfrac{f(x)}{\sigma^2}(x-\mu)$ …(2)

Now $f'(x) = 0 \Rightarrow x - \mu = 0 \Rightarrow x = \mu$

Also $f''(x) = \dfrac{-1}{\sigma^2}\left[1.f(x) + (x-\mu)f'(x)\right]$

$= \dfrac{-1}{\sigma^2}\left[f(x) - (x-\mu)\dfrac{f(x)(x-\mu)}{\sigma^2}\right]$ from(2)

$= \dfrac{f(x)}{\sigma^2}\left[1 - \dfrac{(x-\mu)^2}{\sigma^2}\right]$

Put x = $\mu$, we get

$f''(x) = -\dfrac{1}{\sigma^2}\left[(f(x)\right]_{x=\mu} = \dfrac{-1}{\sigma^2}.\dfrac{1}{\sigma\sqrt{2\pi}} < 0$ [By using (1)]

Hence x = $\mu$ is the mode of Normal distribution.

## Median

Median is the middle value of variate. Let M be the Median distribution, then

$$\int_{-\infty}^{\infty} f(x)\, dx + \int_{M}^{\infty} f(x)\, dx = 1$$

Also $\displaystyle\int_{-\infty}^{M} f(x)\, dx = \int_{-\infty}^{M} f(x)\, dx$

So we have $\displaystyle\int_{-\infty}^{M} f(x)\, dx = \frac{1}{2}$

$$\Rightarrow \quad \int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

or $\quad I_1 + I_2 = \frac{1}{2}$

Now $\quad I_1 = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$

Put $\quad z = \frac{x-\mu}{\sigma}$

$$\therefore \quad I_1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z2/2} dz$$

$$\Rightarrow \quad I_1 = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz$$

Put $\quad z^2/2 = t \qquad \Rightarrow z\,dz = dt \quad \Rightarrow dz = \frac{dt}{\sqrt{2t}}$

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \frac{1}{\sqrt{2}} t^{-1/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}} . \sqrt{\pi} = \frac{1}{2}$$

$$\Rightarrow \quad I_2 = 0 \Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$\Rightarrow \quad M = \mu$

Hence Mean = Madian = Mode

## Moment generating function about origin

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx}\ f(x)\ dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}\ dx$$

Put $z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow$ $x = \mu + \sigma z$

$dx = \sigma\ dz$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)}\ e^{-\frac{1}{2}}.\sigma\ dz$$

$$M_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2t\sigma z)}\ dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2+t^2\sigma^2-2t\sigma z)+\frac{1}{2}t^2\sigma^2}\ dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\ e^{\frac{1}{2}t^2\sigma^2}\ dz$$

$$= \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\ dz$$

Put $z - t\sigma = \mu$ $\Rightarrow dz = d\mu$

$\therefore$ $$M_X(t) = \frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\mu^2}\ d\mu$$

$$= e^{\mu t+\frac{1}{2}t^2\sigma^2}(1) \qquad \left[\Theta\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\ dz = 1\right]$$

$$= e^{\mu t+\frac{1}{2}t^2\sigma^2} \qquad \ldots(1)$$

## Moment generating function about mean

$$M_{X-\mu}(t) = E[e^{t(X-\mu)}] = E[e^{tX}.e^{-\mu t}]$$

$$= e^{-\mu t}\, E[e^{tX}] = e^{-\mu t}\, M_X(t)$$

$$= e^{-\mu t}\, e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\frac{1}{2}\sigma^2 t^2}$$

Also by defination

$$M_{X-\mu}(t) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{t(x-\mu)}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= e^{-\mu t} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{tx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= e^{-\mu t}\,.\,e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\frac{1}{2}\sigma^2 t^2} \qquad \text{[from(1)]}$$

## Moment generating function about any point 'a'

$$M_{X-a}(t) = e^{t(\mu - a) + \frac{1}{2}\sigma^2 t^2}$$

By defination

$$M_{X-a}(t) = E[e^{t(X-a)}]$$

$$M_{X-a}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(x-a)}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= e^{-at}\,.\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx}\,.\, e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= e^{-at}.e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{t(\mu - a) + \frac{1}{2}\sigma^2 t^2}$$

## Cumulants

$$K_X(t) = \log M_X(t) = \log\left( e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right)$$

$$= \mu t + \frac{1}{2}\sigma^2 t^2$$

The 1st cumulants, $K_1$ = coefficient of t = $\mu_1'$ (Mean) = $\mu$

The $2^{nd}$ cumulants, $K_2 = \sigma^2$ = variance = $\mu_2$

$K_r = 0$ for r = 3, 4,…….

Thus $\mu_3 = K_3 = 0$ and $\mu_4 = K_4 + 3K_2^2 = 3\sigma^4$ $[\Theta\ K_4 = \mu_4 - K_2^2]$

Now $\beta_1 = \dfrac{\mu_4}{\mu_2} = \dfrac{3\sigma^2}{\sigma^4} = 3$ $(\beta_1 = 0, \beta_2 = 3)$

## Moments from Moment generating function

M.G.F about mean = $e^{\frac{1}{2}\sigma^2 t^2}$

$$= 1 + \frac{1}{2}t^2\sigma^2 + \frac{\left(\frac{t^2\sigma^2}{2}\right)^2}{\underline{|2}} + \frac{\left(\frac{1}{2}t^2\sigma^2\right)^3}{\underline{|3}} + ..... + \frac{\left(\frac{t^2\sigma^2}{2}\right)^n}{\underline{|n}} + ..... \quad (1)$$

from here,

(1), $\mu_r$ = coefficient of $\dfrac{t^r}{\underline{|r}}$ = rth moment about mean

Since there is no term with odd powers of t in (1), there fore all moments of odd order about Mean vanish.

i.e. $\mu_{2n+1} = 0$, n = 1, 2, ……

and $\mu_{2n}$ = coefficient of $\dfrac{t^{2n}}{\underline{|2n}}$ in (1) = $\dfrac{\sigma^{2n}\,\underline{|2n}}{2^n\,\underline{|n}}$

$$\left[\Theta\ \frac{t^{2n}\sigma^{2n}\cdot\underline{|2n}}{2^n\,\underline{|n}\,\underline{|2n}} = \left(\frac{\sigma^{2n}\,\underline{|2n}}{2^n\,\underline{|n}}\right)\frac{t^{2n}}{\underline{|2n}}\right]$$

$$\therefore \quad \mu_{2n} = \frac{\sigma^{2n}}{2^n\,\underline{|n}}\left[2n(2n-1)(2n-n).....4.3.2.1\right]$$

$$= \frac{\sigma^{2n}}{2^n\,\underline{|n}}[1.3.5\ldots\ldots(2n-1)]\,[2.4.6.(2n-2)2n]$$

$$= \frac{\sigma^{2n}}{2^n\,\underline{|n}}[1.3.5\ldots\ldots(2n-1)\,2^n\,[1.2\ldots..n]$$

$$= \frac{\sigma^{2n}}{2^n\,\underline{|n}}[1.3.5\ldots\ldots(2n-1)\,2^n\,\underline{|n}$$

$\Rightarrow \quad \mu_{2n} = \sigma^{2n}[\,1.3.5\ldots\ldots(2n-1)]$

$\therefore \quad \mu_2 = 1.\sigma^2 = \sigma^2$

$\mu_3 = 0, \qquad \mu_4 = 1.\,3\sigma^4$

$\mu_5 = 0, \qquad \mu_6 = 1.3.5\sigma^6$

$\beta_1 = 0, \qquad \beta_2 = 3$

Also $\mu_1'$ = moment about origin

$$= E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

So $\quad \mu_{2n+1}$ = odd moments about mean

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\,dx$$

and $\quad \mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\,dx$ = even order moments about mean

## Characteristic Function

$$\phi_X(t) = E[e^{itx}] = e^{i\mu t - \frac{1}{2}t^2\sigma^2}$$

## Mean deviation

$$\text{M.D} = \int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

Put $\quad z = \dfrac{x-\mu}{\sigma} \qquad \Rightarrow dz = \dfrac{1}{\sigma}\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| \frac{e^{-\frac{1}{2}z^2}}{\sigma}.\sigma\,dz$$

$\therefore \quad \text{M.D} = \dfrac{2}{\sqrt{2\pi}} \displaystyle\int_{0}^{\infty} \sigma z\, e^{-\frac{1}{2}z^2}\,dz \qquad$ [$\Theta$ integrand is eveb function of z]

Put $\frac{1}{2}z^2 = t$ $\Rightarrow \frac{1}{2}.2z\,dz = dt$

$\Rightarrow$ $z\,dz = dt$

$\therefore$ $$\text{M.D.} = \frac{2}{\sqrt{2\pi}}\sigma \int_0^{\infty} e^{-t}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\sigma\left[\frac{e^{-t}}{-1}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\sigma.[0-1] = \sqrt{\frac{2}{\pi}}\sigma$$

$$= \frac{4}{5}\sigma = 0.8\,\sigma \quad \Rightarrow \text{M.D} = \sqrt{\frac{2}{\pi}}\,\sigma$$

## Points of Inflexion of a Normal Curve

At the points of inflexion,

$$f''(x) = 0, \qquad f'''(x) \neq 0$$

Now $$f''(x) = -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

Put $f''(x) = 0$, $1 - \frac{(x-\mu)^2}{\sigma^2} = 0$

$\Rightarrow$ $(x-\mu)^2 = \sigma^2$ $\Rightarrow$ $x-\mu = \pm\sigma$

$\Rightarrow$ $x = \mu \pm \sigma$

Now $$f'''(x) = -\frac{f'(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] - \frac{f(x)}{\sigma^2}\left[\frac{-2(x-\mu)}{\sigma^2}\right]$$

$$= -\frac{f'(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] + \frac{2f(x)(x-\mu)}{\sigma^4}$$

At $x = \mu \pm \sigma$,

$$f'''(x) = \frac{-f'(x)}{\sigma^2}\left[1 - \frac{(\mu+\sigma-\mu)^2}{\sigma^2}\right] + \frac{2f(x)}{\sigma^4}(\mu+\sigma-\mu)$$

$\Rightarrow$ $$f'''(x) = \frac{-f'(x)}{\sigma^2}[1-1] + \frac{2f(x)}{\sigma^3}.\sigma$$

$$= 0 + \frac{2f(x)}{\sigma^3} = \frac{2f(\mu+\sigma)}{\sigma^3} \neq 0$$

At $x = \mu-\sigma$,

$$f'''(x) = -\frac{f'(x)}{\sigma^2}\left[1 - \frac{(-\sigma)^2}{\sigma^2}\right] + \frac{2f(x)}{\sigma^4}(-\sigma)$$

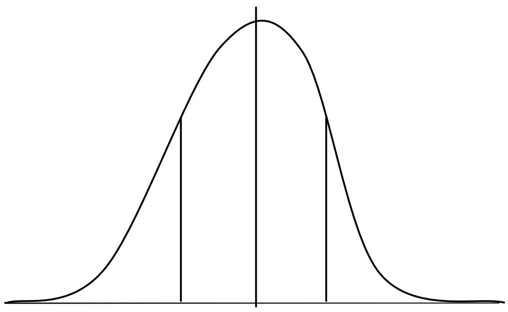
$$= -\frac{f'(x)}{\sigma^2}(1-1) - \frac{2f(x)}{\sigma^3} = 0 - \frac{2f(x)}{\sigma^3}$$

$$\Rightarrow \qquad f'''(x) = \frac{-2f(x)}{\sigma^3} = \frac{-2f(\mu-\sigma)}{\sigma^3} \neq 0$$

$$f'''(x) = \frac{(\pm\sigma)}{\sigma^4} f(\mu \pm \sigma)\left[3 - \frac{(\pm\sigma)^2}{\sigma^2}\right]$$

$$= \frac{\pm 2f(\mu \pm \sigma)}{\sigma^3} \neq 0$$

The S.D. is the distance of point of inflexion from the axis of symmetry.




**Example: -** $\mu_1'$ (about 10) = 40

$\mu_4'$ (about 50) = 48

Find Mean and S.D.

**Solution: -** By defination 1st moment about 10,

$$\mu_1' = \int_{-\infty}^{\infty} (x-10) f(x)\, dx = 40$$

$$\Rightarrow \qquad = \int_{-\infty}^{\infty} x f(x)\, dx - 10 \int_{-\infty}^{\infty} f(x)\, dx = 40$$

$$\Rightarrow \qquad = \int_{-\infty}^{\infty} x f(x) dx = 40 + 10 = 50 \qquad \left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$$

$$\therefore \qquad \text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx = 50$$

As Mean is 50.

$\mu'_4 = 48$(4th moment about Mean)

$\Rightarrow \quad 3\sigma^4 = 48 \qquad \Rightarrow \sigma^4 = 16 = 2^4$

$\Rightarrow \quad \sigma = \text{S.D.} = 2$

**Theorem: -** If $X_1$ and $X_2$ are **independent** normal variates, then $X_1 + X_2$ is also a normal variate.

**Proof: -** $X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

$$M_{X_1}(t) = E\left[e^{tX_1}\right] = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}$$

$$M_{X_2}(t) = E\left[e^{tX_2}\right] = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

As $X_1$and $X_2$ are independent,

$$\therefore \quad M_{X_1+X_2}(t) = M_{X_1}(t).M_{X_2}(t)$$

$$= e^{(\mu_1+\mu_2)t + \frac{1}{2}t^2(\sigma_1^2+\sigma_2^2)}$$

which is the M.G.F of Normal distance with Mean $\mu_1+\mu_2$ and variance $\sigma_1^2 + \sigma_2^2$

i.e. $\quad X_1 + X_2 \sim N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2)$

**Theorem: -** A linear combination of independent normal variates is also a normal variate. i.e.

if $\quad Y = a_1X_1 + a_1X_2 + \ldots\ldots + a_n + X_n$, then Y is also a normal variate.

**Proof: -** $X_i$ are independent normal variates.

$$\therefore \quad M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}t^2\sigma_i^2}$$

$$\text{and} \quad M_{X_i}(a_i t) = e^{\mu_i a_i t + \frac{1}{2}t^2\sigma_i^2 a_i^2} \qquad \ldots(1)$$

Now M.G.F of $Y = \sum_{i=1}^{n} a_i X_i$ is

$$M_{\sum a_i X_i}(t) = M_{a_1X_1+a_2X_2+\ldots\ldots+a_nX_n}(t)$$

$$= M_{a_1X_1}(t)M_{a_2X_2}(t)\ldots\ldots.M_{a_nX_n}(t) \qquad [\Theta X_i\text{'s are independent}]$$

$$= M_{X_1}(a_1t).M_{X_2}(a_2t)\ldots\ldots\ldots.M_{X_n}(a_nt) \qquad [\Theta M_{CX}(t) = M_X(Ct)]$$

$$= e^{\mu_1 a_1 t + \frac{1}{2}\sigma_1^2 a_1^2 t^2}.e^{\mu_2 a_2 t + \frac{1}{2}\sigma_2^2 a_2^2 t^2} \ldots\ldots\ldots. \qquad [\text{from}(1)]$$

$$= e^{t\left(\sum a_i\mu_i\right)+\frac{1}{2}t^2\left(\sum a_i^2\sigma_i^2\right)}$$

which is M.G.F of Normal distribution with **Mean =** $\sum a_i\mu_i$ and **Variance** = $\sum a_i^2\sigma_i^2$

**Remarks**

(1) If we take $a_1 = a_2 = 1$, $a_3 = a_4 = \ldots\ldots 0$,

then $X_1 + X_2 \sim N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2)$

if we take $a_1 = 1$, $a_2 = -1$, $a_3 = a_4 = \ldots\ldots.. 0$,

then $X_1 - X_2 \sim N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2)$

Thus we see that sum as well as the difference of two independent normal variates is also a normal variate.

**Note: -** If $X_i \sim N(\mu, \sigma^2)$, $X_i$ are identically distributed independent Normal variates, then their mean

$$\overline{X}_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Taking $a_1 = a_2 = \ldots\ldots.. = a_n = \frac{1}{n}$ in the result

$$Y = \sum a_i X_i \sim N\left(\sum a_i\mu_i, \sum a_i^2\sigma_i^2\right) \qquad \ldots(1)$$

then $\frac{1}{n}(X_1 + X_2 + \ldots\ldots + X_n) \sim N\left\{\frac{1}{n}\sum_{i=1}^{n}\mu_i,\ \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2\right\}$

$$\Rightarrow \quad \overline{X}_i \sim N\left\{\frac{1}{n}.n\mu,\ \frac{1}{n^2}n\sigma^2\right\}$$

[$\because$ As each $X_i$ has mean = $\mu$, variance = $\sigma^2$]

$$\Rightarrow \quad \overline{X}_i \sim N\left(\mu,\ \frac{1}{n}\sigma^2\right)$$

**Theorem: -** If $X \sim B(n, p)$, then prove that MGF of $\frac{X - np}{\sqrt{npq}}$ tends to $e^{t^2/2}$ as $n \to \infty$.

**Proof: -** Let $U = \frac{X - np}{\sqrt{npq}}$

$$f(x) = {}^nC_x\, p^x\, q^{n-x}$$

MGF of Binomial distribution is

$M_X(t) = (q + pe^t)^n$

$\therefore \quad M_U(t) = e^{-at/h} M_X\left(\frac{t}{h}\right)$ where $U = \frac{X-a}{h}$

$$M_U(t) = e^{-\frac{npt}{\sqrt{npq}}}\left[q + pe^{\frac{t}{\sqrt{npq}}}\right]^n$$

$$\therefore \quad \log M_U(t) = \frac{-npt}{\sqrt{npq}} + n \log \left[q + pe^{t/\sqrt{npq}}\right]$$

$$= \frac{-npt}{\sqrt{npq}} + n \log \left[1 + p\left\{\frac{t}{\sqrt{npq}} + \frac{t^2}{\lfloor 2\, npq} + \frac{t^3}{\lfloor 3\,(npq)^{3/2}} + \ldots\ldots\right\}\right]$$

$$= \frac{t^2}{2q}(1-p) + 0(n^{-1/2})$$

$$= \frac{t^2}{2} + 0\left(\frac{1}{\sqrt{n}}\right)$$

As $\quad n \to \infty, \qquad \log M_U(t) = t^2/2$

$\Rightarrow \qquad M_U(t) = e^{t^2/2}$

## Area property

If $X \sim N(\mu, \sigma^2)$

The probability that r.v. X will lie between $X = \mu$ and $X = x_1$ is given by

$$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\quad Z = \frac{X-\mu}{\sigma}$, then $z = \frac{x-\mu}{\sigma}$

If $\quad X = \mu$, then $Z = 0$

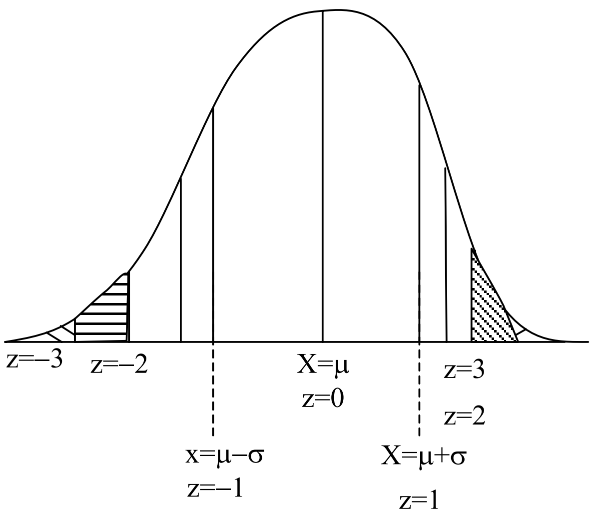
& when $X = x_1$, $Z = z_1$(say) where $z_1 = \frac{x_1 - \mu}{\sigma}$

$\therefore \quad P(\mu < X < x_1) = P(0 < Z < z_1)$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{1}{2}z^2} \, dz = \int_0^{z_1} \varphi(z) \, dz$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the probability function.

The definite integral $\int_0^{z_1} \varphi(z) \, dz$ is known as normal probability Integral & gives the area under Normal curve between ordinates at $Z = 0$& $Z = z_1$. These are tabulated for different values of $z_1$ at interval of 0.01




$P(\mu - \sigma < X < \mu + \sigma) = P(-1 < Z < 1)$

$= \int_{\mu-\sigma}^{\mu+\sigma} f(x) \, dx = \int_{-1}^{1} \varphi(z) \, dz = 2\int_0^1 \varphi(z) \, dz$ [Θ curve is symmetrical]

$= 2(0.3413)$

$= 0.6826$

and $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2)$

$$= \int_{-2}^{2} \varphi(z) \, dz = 2\int_0^2 \varphi(z) \, dz$$

$$= 2(0.4772) = 0.9544$$

and $(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3)$

$$= 2(0.4987) = 0.9974$$

and outside range$\mu \pm 3\sigma$

$P(|X-\mu| > 3\sigma) = P(|Z| > 3)$

$= 1-P(-3 < Z < 3)$

$= 1-0.9974 = 0.0026$

**Example: -** If $X \sim N(12, 16)$

Find **(i)** $P(X \geq 20)$ **(ii)** $P(X \leq 20)$ **(iii)** $P(0 \leq X \leq 12)$

**Solution: - (i)** Here $\mu = 12, \sigma = 4$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-12}{4}$$

$\therefore$ $P(X \geq 20) = P(Z \geq 2)$

$= 0.5 - P(0 \leq Z < 2)$

$= 0.5 - 0.4772 = 0.0228$

**(ii)** $P(X \leq 20) = 1- P(X \geq 20)$

$= 1-0.0228 = 0.9772$

**(iii)** $P(0 \leq X \leq 12) = P(-3 \leq Z \leq 0) = P(0 \leq Z \leq 3)$ (due to symmmtrical)

$\therefore$ $P( 0 \leq X \leq 12) = 0.4987$

**(iv)** Find x\` when P(X > x\`) = 0.24

**Solution: -** when X = x\`, then $Z = \frac{x'-12}{4} = z_1$ (say) …(1)

$\therefore$ $P(Z > z_1) = 0.24$

$P(0 < Z < z_1) = 0.5 - 0.24 = 0.26$

From table, $z_1 = 0.71$

$\therefore(1)\Rightarrow$ $z_1 = \frac{x'-12}{4} = 0.71$

$\Rightarrow$ $x' = 14.84$

**Example: -** $X \sim N(\overset{\mu}{30}, \overset{\sigma^2}{25})$

Find (i) $P(26 \leq X \leq 40)$ (ii) $X \geq 45$ (iii) $|X-30| > 5$

**Solution: -** Here $\mu = 30, \sigma = 5$

(i) when X = 26, $Z = \frac{X-\mu}{\sigma} = \frac{26-30}{5} = -0.8$

& when X = 40, $Z = \frac{40-30}{5} = 2.0$

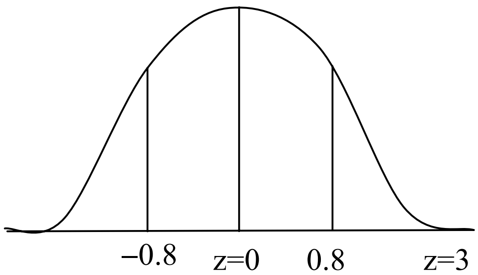
$\therefore$ $P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$

$= P(-0.8 \le Z < 0.8) + P(0 \le Z \le 2)$

$= P(0 \le Z \le 0.8) + P(0 \le Z < 2)$

$= 0.2881 + 0.4772$

$P(26 \le X \le 40) = 0.7653$




(ii) $P(X \ge 45) = P(Z \ge 3) = 0.5 - P(0 \le Z \le 3)$

$= 0.5 - 0.4987 = 0.0013$

(iii) $P(|X-30|>5)$

Put $|X-30| = |Y|$ $\therefore |Y| \le 5$

$\Rightarrow$ $-5 \le Y \le 5$

$\therefore$ $P(|X-30| \le 5) = P(25 \le X \le 35)$

$= P(-1 \le Z \le 1) = 2P(0 \le Z \le 1)$

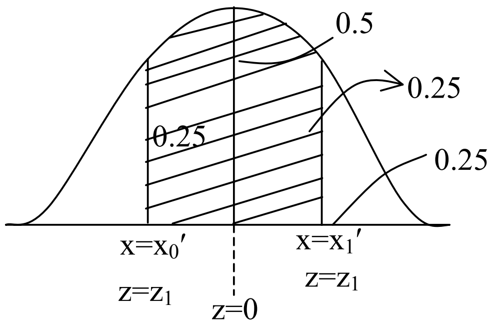
Then $P(|X-30| > 5) = 1 - P(|X-30| \le 5)$

$= 1 - 0.6826 = 0.3174$

**Example: -** $X \sim N(12, 16)$

Find $x_0^1$ and $x_1^1$ when $P(x_0^1 \le X \le x_1^1) = 0.50$ and $P(X > x_1^1) = 0.25$

**Solution: -**



when $X = x_1'$, $Z = \frac{x_1' - 12}{4} = z_1$ (say)

when $X = x_0'$, $Z = \frac{x_0' - 12}{4} = -z_1$ (from figure)

$P(X > X_1') = 0.25 \quad \Rightarrow P(Z > z_1) = 0.25$

and $[P(Z > z_1) = 0.5 - P(0 \le Z \le z_1) = 0.25]X$

$\therefore \quad P(0 < Z < z_1) = 0.5 - P(Z > z_1) = 0.25$

from table, $z_1 = 0.67$

Hence $\frac{x_1' - 12}{4} = 0.67$

$\Rightarrow \quad x_1' = 0.67 \times 4 + 12 = 14.68$

and $\frac{x_0' - 12}{4} = -0.67 \Rightarrow x_0' = 9.32$

## LESSON 8 MULTIPLE AND PARTIAL CORRELATION

INTRODUCTION.

When the value of the one variable is associated with or influenced by other variable, e.g., the age of husband and wife, the height of father and son, the supply and demand of a commodity and so on, Karl Pearson's coefficient of correlation can be used as a measure of linear relationship between them. But sometimes there is correlation between many variables and the value of one variable may be influenced by many others, e.g., the yield of crop per acre say ($X_1$) depends upon quality of seed ($X_2$), fertility of soil ($X_3$), fertilizer used ($X_4$), irrigation facilities ($X_5$) etc. Whenever we are interested in studying the joint effect of a group of variable upon a variable not included in that group , our study is that of **multiple correlation and multiple regression.** The correlation and regression between only two variates after eliminating the linear effect of other variates in them is called the **partial correlation and partial regression**.

**Yule's Notation:**

Let us consider a distribution involving three random variables $X_1$, $X_2$ and $X_3$. Then the equation of the plane of regression of $X_1$ on $X_2$ and $X_3$ is

$$X_1 = a + b_{12.3} X_2 + b_{13.2} X_3 \qquad (1)$$

Without loss of generality, we can assume the variables $X_1$, $X_2$, and $X_3$ have been measured from their respective means, so that

$$E(X_1) = E(X_2) = E(X_3) = 0$$

Hence on taking expectation of both sides in (1), we get $a = 0$

Thus the plane of regression of $X_1$, $X_2$, and $X_3$ becomes

$$X_1 = b_{12.3} X_2 + b_{13.2} X_3 \qquad (2)$$

The coefficient $b_{12.3}$ and $b_{13.2}$ are known as the **partial regression coefficients** of $X_1$ on $X_2$ and of $X_1$ on $X_3$ , respectively. The quantity

$e_{1.23} = b_{12.3} X_2 + b_{13.2} X_3$

is called the estimate of $X_1$ as given by the plane of regression (1) and the quantity.

$X_{1.23} = X_1 - b_{12.3} X_2 - b_{13.2} X_3$

is called the **error of estimate or residual**.

In the general case of n variable $X_1, X_2, \ldots X_n$ the equation of the plane of regression of $X_1$ on $X_2, X_3, \ldots X_n$ becomes

$X_1 = b_{12.34\ldots n} X_2 + b_{13.24\ldots n} X_3 + \ldots b_{1n.23\ldots(n-1)} X_n$

The error of estimate or residual is given by

$X_{1.23\ldots n} = X_1 - b_{12.34\ldots n} X_2 + b_{13.24\ldots n} X_3 + \ldots b_{1n.23\ldots(n-1)} X_n$.

The notation used here are due to Yule. The subscripts before the dot (.) are known as **primary subscripts** and those after the dot are called **secondary subscripts**. The order of regression coefficient is determined by the number of secondary subscripts, e.g.,

$b_{12.3}, b_{12.34}, \ldots, b_{12.34\ldots n}$

are the regression coefficients of order 1.2,……(n-2) respectively. Thus in general, a regression coefficient with p-secondary subscripts will be called a regression co-efficient of order ‘p’. It may be noted that the order in which the secondary subscripts are written is immaterial but the order of the primary subscripts is important, e.g., in $b_{12.34\ldots n}$, $X_2$ is independent while $X_1$ is dependent variable but in $b_{21.34\ldots n}$, $X_1$ is independent while $X_2$ is dependent variable. Thus of the two primary subscripts, former refer to dependent variable and the latter to independent variable.

The order of the residual is also determined by the number of secondary subscripts in it, e.g., $X_{1.23}$, $X_{1.234}$,…….$X_{1.23.....n}$, are the residual of order 2,3,……. (n-1) respectively.

Remaks: In the following sequences we shall assume that the variable under consideration has been measured from their respective means.

**Plane of regression:**

The equation of the plane of regression of $X_1$ on $X_2$ and $X_3$ is

$$X_1 = b_{12.3} X_2 + b_{13.2} X_3 \qquad (3)$$

The constants b's in (3) are determined by the principle of leastsquares, i.e., by minimizing the sum of the squares of the residual, viz.,

$S = \sum X_{1.23}^2 = \sum (X_1 - b_{12.3} X_2 - b_{13.2} X_3)^2$,

the summation being extended to the given values (N in number) of the variables.

The normal equation of estimating $b_{12.3}$ and $b_{13.2}$ are

$$\left.\begin{aligned} \frac{\partial S}{\partial b_{12.3}} &= 0 = -2\sum X_2(X_1 - b_{12.3}X_2 - b_{13.2}X_3) \\ \frac{\partial S}{\partial b_{13.2}} &= 0 = -2\sum X_3(X_1 - b_{12.3}X_2 - b_{13.2}X_3) \end{aligned}\right\} \qquad (4)$$

i.e., $\sum X_2 X_{1.23} = 0$ and $\sum X_3 X_{1.23} = 0$ (5)

$$\Rightarrow \left.\begin{aligned} &\sum X_1X_2 - b_{12.3}\sum X_2^2 - b_{13.2}\sum X_2X_3 \\ &\sum X_1X_3 - b_{12.3}\sum X_2X_3 - b_{13.2}\sum X_3^2 \end{aligned}\right\} \qquad (6)$$

Since $X_i$'s are measured from their respective means, we have

$$\left.\begin{aligned}&\sigma_i^2 = \frac{1}{N}\sum X_i^2, \quad \mathrm{Cov}(X_i, X_j) = \frac{1}{N}\sum X_i X_j \\ &r_{ij} = \frac{\mathrm{Cov}(X_i, X_j)}{\sigma_i \sigma_j} = \frac{\sum X_i X_j}{N\sigma_i \sigma_j}\end{aligned}\right\} \tag{7}$$

Hence from (6), we get

$$r_{12}\,\sigma_1\,\sigma_2 - b_{12.3}\,\sigma_2^2 - b_{13.2}r_{23}\,\sigma_2\,\sigma_3 = 0$$

$$r_{13}\,\sigma_1\,\sigma_3 - b_{12.3}\,r_{23}\,\sigma_2\,\sigma_3 - b_{13.2}\,\sigma_3^2 = 0 \tag{8}$$

solving equations (10.30d) for $b_{12.3}$ and $b_{13.2,}$ we get

$$b_{12.3} = \frac{\begin{vmatrix} r_{12}\sigma_1 & r_{23}\sigma_3 \\ r_{13}\sigma_1 & \sigma_3 \end{vmatrix}}{\begin{vmatrix} \sigma_2 & r_{23}\sigma_3 \\ r_{23}\sigma_2 & \sigma_3 \end{vmatrix}} = \frac{\sigma_1}{\sigma_2}\frac{\begin{vmatrix} r_{12} & r_{23} \\ r_{13} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}} \tag{9}$$

Similarly, we obtain

$$b_{13.2} = \frac{\sigma_1}{\sigma_3}\frac{\begin{vmatrix} 1 & r_{12} \\ r_{23} & r_{13} \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}} \tag{10}$$

If we write

$$\omega = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix} \tag{11}$$

And $\omega_{ij}$ is the cofactor of the element in the ith row and jth column of $\omega$, we have from (9) and (10)

$$b_{12.3} = -\frac{\sigma_1}{\sigma_2}\frac{\omega_{12}}{\omega_{11}} \text{ and } b_{13.2} = -\frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}} \tag{12}$$

Substituting thes values in (3), we get the required equation of the plane of regression of $X_1$ on $X_2$ and $X_3$

$$X_1 = -\frac{\sigma_1}{\sigma_2}.\frac{\omega_{12}}{\omega_{11}}.X_2 - \frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}}.X_3 = 0 \qquad (13)$$

$$\Rightarrow \quad \frac{X_1}{\sigma_1}.\omega_{11} + \frac{X_2}{\sigma_2}.\omega_{12} + \frac{X_3}{\sigma_3}.\omega_{13} = 0$$

**or** Eliminating the coefficient $b_{12.3}$ and $b_{13.2}$ in (3) and (8), the required equation of the plane of regression of $X_1$ on $X_2$ and $X_3$ becomes

$$\begin{vmatrix} X_1 & X_2 & X_3 \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 & r_{23}\sigma_2\sigma_3 \\ r_{13}\sigma_1\sigma_3 & r_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{vmatrix} = 0$$

Dividing $C_1$, $C_2$ and $C_3$ by $\sigma_1$, $\sigma_2$ and $\sigma_3$ respectively and also $R_2$ and $R_3$ by $\sigma_2$ and $\sigma_3$ respectively, we get

$$\begin{vmatrix} \frac{X_1}{\sigma_1} & \frac{X_2}{\sigma_2} & \frac{X_3}{\sigma_3} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{X_1}{\sigma_1}.\omega_{11} + \frac{X_2}{\sigma_2}.\omega_{12} + \frac{X_3}{\sigma_3}.\omega_{13} = 0$$

Where $\omega_{ij}$ is defined in (11)

**Generalization:**

In general, the equation of the plane of regression of $X_1$ on $X_2$, $X_3$…… $X_n$ is

$X_1 = b_{12.34......n} X_2 + b_{13.24......n} X_3 + ........ b_{1n.23........(n-1)} X_n$ (14)

The sum of the squares of residuals is given by

$S = \sum X^2_{1.23......n}$

$= \sum( X_1 - b_{12.34......n} X_2 + b_{13.24......n} X_3 -........b_{1n.23........(n-1)}X_n$

Using the principle of least squares, the normal equations for estimating the (n-1), b's are

$$\left.\begin{aligned}
\frac{\partial S}{\partial b_{12.34.......n}} &= 0 = -2\sum X_2 (X_1 - b_{12.34..........n} X_2 - b_{13.24..........n} X_3 - ..........b_{1n.23..........(n-1)} X_n) \\
\frac{\partial S}{\partial b_{13.24.........n}} &= 0 = -2\sum X_3 (X_1 - b_{12.34.........n} X_2 - b_{13.24..........n} X_3 - ...........b_{1n.23..........(n-1)} X_n) \\
&. \\
&. \\
&. \\
&. \\
\frac{\partial S}{\partial b_{1n.23.........n}} &= 0 = -2\sum X_n (X_1 - b_{12.34.........n} X_2 - b_{13.24..........n} X_3 - ...........b_{1n.23..........(n-1)} X_n)
\end{aligned}\right\}$$

(15)

i.e., $\sum X_i X_{1.23.....n} = 0, \quad (i = 2,3,.......n)$ (16)

which on simplification after using (7)

$r_{12}\, \sigma_1\, \sigma_2 = b_{12.34.......n}\, \sigma_2^2 + b_{13.24.........n} r_{23}\, \sigma_2\, \sigma_3 + ......b_{1n.23.........(n-)} r_{2n}\, \sigma_2\, \sigma_n = 0$

$r_{13}\, \sigma_1\, \sigma_3 = b_{12.34.......n}\, r_{23}\, \sigma_2\, \sigma_3 + b_{13.24........n}\, \sigma_3^2 + .......b_{1n.23.........(n-)} r_{3n}\, \sigma_3\, \sigma_n = 0$

. . .

. . .

. . .

$r_{1n}\, \sigma_1\, \sigma_n = b_{12.34.......n}\, r_{2n}\, \sigma_2\, \sigma_n + b_{13.24........n}\, r_{3n}\, \sigma_3\, \sigma_n + ......b_{1n.23.........(n-)}\, \sigma_n^2 = 0$

(15)

Hence, the elimination of b's between (14) and (15) yields

$$\begin{vmatrix} X_1 & X_2 & X_3 & \ldots\ldots\ldots\ldots & X_n \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 & r_{23}\sigma_2\sigma_3 & \ldots\ldots\ldots\ldots & r_{23}\sigma_2\sigma_3 \\ r_{13}\sigma_1\sigma_3 & r_{23}\sigma_2\sigma_3 & \sigma_3^2 & \ldots\ldots\ldots\ldots & r_{3n}\sigma_3\sigma_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{1n}\sigma_1\sigma_n & r_{2n}\sigma_2\sigma_n & r_{3n}\sigma_3\sigma_n & \cdots\ldots\ldots & \sigma_n^2 \end{vmatrix} = 0$$

Dividing $C_1$, $C_2$ …….. ,$C_n$ by $\sigma_1$, $\sigma_2$…….$\sigma_n$ respectively and also $R_2$, $R_3$…..$R_n$ by $\sigma_2$, $\sigma_3$…. $\sigma_n$ respectively, we get

$$\begin{vmatrix} \frac{X_1}{\sigma_1} & \frac{X_2}{\sigma_2} & \frac{X_3}{\sigma_3} & \ldots\ldots & \frac{X_n}{\sigma_n} \\ r_{12} & 1 & r_{32} & \ldots\ldots\ldots & r_{2n} \\ r_{13} & r_{23} & 1 & \ldots\ldots & r_{3n} \\ \vdots & \vdots & \vdots & \ldots\ldots & \vdots \\ r_{1n} & r_{2n} & r_{3n} & \ldots\ldots\ldots & 1 \end{vmatrix} = 0 \qquad (18)$$

If we write

$$\omega = \begin{vmatrix} 1 & r_{12} & r_{13} & \ldots\ldots & r_{1n} \\ r_{21} & 1 & r_{23} & \ldots\ldots & r_{23} \\ r_{31} & r_{32} & 1 & \ldots\ldots & r_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \ldots\ldots & 1 \end{vmatrix} \qquad (19)$$

And $\omega_{ij}$ is the cofactor of the element in the ith row and jth column of $\omega$, we get from (18)

$$\Rightarrow \quad \frac{X_1}{\sigma_1}.\omega_{11} + \frac{X_2}{\sigma_2}.\omega_{12} + \frac{X_3}{\sigma_3}.\omega_{13} + \ldots\ldots\ldots\ldots\ldots + .\frac{X_n}{\sigma_n}.\omega_{1n} = 0 \qquad (20)$$

As the required equation of plane of regression of $X_1$ on $X_2$, $X_3$…… $X_n$.

Equation (20) can be re-written as

$$X_1 = -\frac{\sigma_1}{\sigma_2}.\frac{\omega_{12}}{\omega_{11}}.X_2 - \frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}}.X_3 - \ldots\ldots\ldots\frac{\sigma_1}{\sigma_n}\frac{\omega_{1n}}{\omega_{11}}.X_n = 0 \qquad (21)$$

Comparing (21) with (14), we get

$$\left.\begin{aligned} b_{12.34} &= -\frac{\sigma_1}{\sigma_2}\frac{\omega_{12}}{\omega_{11}} \\ b_{13.24} &= -\frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}} \\ \vdots \quad & \quad \vdots \\ b_{1n.23\ldots\ldots(n-1)} &= -\frac{\sigma_1}{\sigma_n}\frac{\omega_{1n}}{\omega_{11}} \end{aligned}\right\} \qquad (22)$$

**Properties of residuals**

**Property 1**. The sum of the product of any residual of order zero with any other residual of higher order is zero, provided the subscript of the former occurs among the secondary subscripts of the latter.

The normal equation for estimating b's in trivarate and n-variate distributions are

$$\sum X_2 X_{1.23} = 0, \ \sum X_3 X_{1.23} = 0$$

And $\sum X_i X_{1.23\ldots..n} = 0$ ; I = 2,3,……n

respectively. Here $X_{i,}$ (i = 1,2,3…….,n) can be regarded as a residual of order zero.

**Property 2**. The sum of the product of any two residual in which all the secondary subscripts of the first occur among the secondary subscripts of the second is unaltered if we omit any or all the secondary subscripts of the first.

e.g. $\sum X_{1.2} X_{1.23} = \sum X_1 X_{1.23}$

also $\sum X_{1.23}^2 = \sum X_1 X_{1.23}$

so $\sum X_{1.23}^2 = \sum X_{1.2} X_{1.23} = \sum X_1 X_{1.23}$

**Property 3**. The sum of the product of two residuals is zero if all the subscripts (primary as well as secondary) of the one occur among the secondary subscripts of the other, e.g.,

$$\sum X_{1.2} X_{3.12} = 0$$

### Variance of the Residuals

Let us consider the plane of regression of $X_1$ on $X_2, X_3 \ldots\ldots X_n$ as.

$$X_1 = b_{12.34\ldots\ldots n} X_2 + b_{13.24\ldots\ldots n} X_3 + \ldots\ldots\ldots b_{1n.23\ldots\ldots\ldots(n-1)} X_n$$

Since all the $X_i$'s are measured from their respective means, we have

$$E(X_i) = 0;\ i = 1,2,3,\ldots\ldots n \quad \Rightarrow E(X_{1.23\ldots\ldots n}) = 0$$

Hence the variance of the residual is given by

$$\sigma^2_{1.23\ldots n} = \frac{1}{N}\sum[X_{1.23\ldots n} - E(X_{1.23\ldots n})]^2 = \frac{1}{N}\sum X^2_{1.23\ldots n}$$

$$\sigma^2_{1.23\ldots n} = \frac{1}{N}\sum X_{1.23\ldots n} X_{1.23\ldots n} = \frac{1}{N}\sum X_1 X_{1.23\ldots n}$$

$$= \frac{1}{N}\sum X_1 (X_1 - b_{12.34\ldots n} X_2 - b_{13.24\ldots n} X_3 - \ldots\ldots\ldots b_{1n.23\ldots(n-1)} X_{n\,2}$$

$$= \sigma_1^2 - b_{12.34\ldots n} r_{12} \sigma_1 \sigma_2 - b_{13.24\ldots n} r_{13} \sigma_1 \sigma_3 - \ldots\ldots\ldots\ldots\ldots - b_{1n.23\ldots\ldots\ldots(n-1)} r_{1n} \sigma_1 \sigma_n$$

$$\Rightarrow \sigma_1^2 - \sigma^2_{1.23\ldots\ldots n} = b_{12.34\ldots n} r_{12} \sigma_1 \sigma_2 - b_{13.24\ldots n} r_{13} \sigma_1 \sigma_3 - \ldots\ldots\ldots\ldots\ldots - b_{1n.23\ldots\ldots\ldots(n-1)} r_{1n} \sigma_1 \sigma_n$$

Eliminating the b's we get

$$\begin{vmatrix} \sigma_1^2 - \sigma_{1.23......n}^2 & r_{12}\sigma_1\sigma & ...... & r_{1n}\sigma_1\sigma_n \\ r_{12}\sigma_1\sigma & \sigma_2^2 & ...... & r_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \vdots & \vdots \\ r_{1n}\sigma_1\sigma_n & r_{2n}\sigma_2\sigma_n & ..... & \sigma_2^2 \end{vmatrix} = 0$$

Dividing $R_1, R_2.....R_n$ by $\sigma_1, \sigma_2.... \sigma_n$ respectively and also $C_1, C_2$ …….. ,$C_n$ by $\sigma_1, \sigma_2......\sigma_n$ respectively, we get

$$\begin{vmatrix} 1 - \dfrac{\sigma_{1.23......n}^2}{\sigma_1^2} - & r_{12} & ...... & r_{1n} \\ r_{12} & 1 & ...... & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{1n} & r_{2n} & ..... & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & r_{12} & ...... & r_{1n} \\ r_{12} & 1 & ...... & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{1n} & r_{2n} & ..... & 1 \end{vmatrix} - \begin{vmatrix} \dfrac{\sigma_{1.23......n}^2}{\sigma_1^2} & r_{12} & ...... & r_{1n} \\ 0 & 1 & ...... & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & r_{2n} & ..... & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \omega - \frac{\sigma_{1.23......n}^2}{\sigma_1^2}\omega_{11} = 0$$

$$\therefore \quad \sigma_{1.23......n}^2 = \sigma_1^2 - \frac{\omega}{\omega_{11}} \qquad (23)$$

Remarks: In a tri-variate distribution,

$$\therefore \quad \sigma_{1.23......n}^2 = \sigma_1^2 - \frac{\omega}{\omega_{11}} \qquad (24)$$

Where $\omega$ and $\omega_{11}$ are defined in (11)

**Coefficient of Multiple Correlation:**

In a tri-variate distribution in which each of the variables $X_1$, $X_2$ and $X_3$ has N observations, the multiple correlation coefficient of $X_1$ on $X_2$ and $X_3$, usually denoted by $R_{1.23}$, is the simple correlation coefficient between $X_1$ and the joint effect of $X_2$ and $X_3$ on $X_1$. In other words $R_{1.23}$ is the correlation coefficient between $X_1$ and its assumed value as given by the plane of regression of $X_1$ on $X_2$ and $X_3$ i.e.

$e_{1.23} = b_{12.3} X_2 + b_{13.2} X_3$

we have,

$X_{1.23} = X_1 - b_{12.3} X_2 - b_{13.2} X_3 = X_1 - e_{1.23}$

$\Rightarrow e_{1.23} = X_1 - X_{1.23}$

Since $X_i$'s are measured from their respective means, we have

$E(X_{1.23}) = 0$ and $E(e_{1.23}) = 0$

By definition

$$R_{1.23} = \frac{Cov(X_1, e_{1.23})}{\sqrt{V(X_1)V(e_{1.23})}} \quad (25)$$

$Cov(X_1, e_{1.23}) = E[\{X_1-E(X_1)\}\{e_{1.23}-E(e_{1.23})\}] = E(X_1, e_{1.23})$

$= \sigma_1^2 - \sigma_{1.23}^2$

Also

$V(e_{1.23}) = \sigma_1^2 - \sigma_{1.23}^2$

$$R_{1.23} = \frac{\sigma_1^2 - \sigma_{1.23}^2}{\sqrt{\sigma_1^2(\sigma_1^2 - \sigma_{1.23}^2)}}$$

$$1 - R^2_{1.23} = \frac{\sigma^2_{1.23}}{\sigma^2_1} \qquad (26)$$

Using (24), we get

$$1 - R^2_{1.23} = \frac{\omega}{\omega_{11}} \qquad (27)$$

Where, $\omega = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}$

and $\omega_{11} = \begin{vmatrix} 1 & r_{23} \\ r_{32} & 1 \end{vmatrix}$

Hence from (27), we get

$$R^2_{1.23} = 1 - \frac{\omega}{\omega_{11}} = \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23}}{1 - r^2_{23}}$$

This formula expresses the multiple correlation coefficient in terms of the total correlation coefficients between the pairs of variables

**Generalisation:**

$$R_{1.23........n} = \frac{\sigma^2_1 - \sigma^2_{1.23..........n}}{\sqrt{\sigma^2_1(\sigma^2_1 - \sigma^2_{1.23........n})}}$$

$$R^2_{1.23.......n} = 1 - \frac{\sigma^2_{1.23........n}}{\sigma^2_1}$$

$$R^2_{1.23......n} = 1 - \frac{\omega}{\omega_{11}} \qquad (28)$$

Where $\omega$ and $\omega_{11}$ are defined in (19)

## Coefficient of Partial Correlation:

The partial correlation coefficient between $X_1$ and $X_2$, usually denoted by $r_{12.3}$ is given by

$$r_{12.3} = \frac{Cov(X_{1.3}, X_{2.3})}{\sqrt{Var(X_{1.3})Var(X_{2.3})}}$$

We have

$$Cov(X_{1.3}, X_{2.3}) = \frac{1}{N}\sum X_{1.3}X_{2.3} = \frac{1}{N}\sum X_1 X_{2.3}$$

$$= \frac{1}{N}\sum X_1(X_2 - b_{23}X_3) = \frac{1}{N}\sum X_1X_2 - b_{23}\sum X_1X_3$$

$$= r_{12}\,\sigma_1\,\sigma_2 - (r_{23}\,\sigma_2\,r_{13}\,\sigma_1\,\sigma_3)/\,\sigma_3$$

$$= \sigma_1\,\sigma_2(r_{12} - r_{23}\,r_{13})$$

$$Var(X_{1.3}) = \frac{1}{N}\Sigma X_{1.3}^2 = \frac{1}{N}\Sigma X_{1.3}X_{2.3}$$

$$= \frac{1}{N}\Sigma X_1 X_{2.3} = \frac{1}{N}\Sigma X_1(X_1 - b_{13}X_3)$$

$$= \frac{1}{N}\Sigma X_1^2 - b_{13}.\frac{1}{N}\Sigma X_1X_3$$

$$= \sigma_1^2 - r_{13}\frac{\sigma_1}{\sigma_3}r_{13}\sigma_1\sigma_3$$

$$= \sigma_1^2(1 - r_{13}^2)$$

Similarly, we can get $Var(X_{2.3}) = \sigma_2^2(1 - r_{23}^2)$

Hence, $$r_{12.3} = \frac{\sigma_1\sigma_2(r_{12} - r_{13}r_{23})}{\sqrt{\sigma_1^2(1 - r_{13}^2)\sigma_2^2(1 - r_{23}^2)}} = \frac{(r_{12} - r_{13}r_{23})}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

**Example**: From the data relating to the yield of grain ($X_1$),fertilizer used ($X_2$) and whether condition ($X_3$) for 18 crops the following correlation coefficients were obtained:

$r_{12} = 0.77$ , $r_{13} = 0.72$, $r_{23} = 0.52$

Fond the partial correlation coefficient $r_{12.3}$ and multiple correlation coefficient $R_{1.23}$

Solution:

$$r_{1.23} = \frac{(r_{12} - r_{13}r_{23})}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}} = \frac{(0.77 - 0.72 \times 0.52)}{\sqrt{(1-(0.72)^2)(1-(0.52)^2)}} = 0.62$$

$$R_{1.23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1-r_{23}^2} = \frac{(0.77)^2 + (0.72)^2 - 2(0.77)(0.72)(0.52)}{1-(0.52)^2} = 0.7334$$

$R_{1.23} =$ +0.8564

## Central Limit Theorem

If $X_1, X_2, \ldots\ldots\ldots X_n$ are independent and identically distributed variables with

$E(X_i) = \mu_1$

$V(X_i) = \sigma_1^2$ , $i = 1,2,\ldots\ldots n$

Then the sum $S_n = X_1 + X_2 + \ldots\ldots\ldots + X_n$ is asymptotically normal with mean $\mu = n\,\mu_1$ and variance $\sigma^2 = n\,\sigma_1^2$

Hence we make the following assumptions

(i) The variable are independent and identically distributed

(ii) $E(X^2_i)$ exists for $i = 1,2,\ldots..$

Proof: Let $M_1(t)$ denote the M.G.F. of each of the derivation $(X_i-\mu_1)$ and M(t) denote the M.G.F. of the standard variate

$$Z = (S_n-\mu)/\sigma$$

Since $\mu_1' = E(X_i - \mu_1) = 0, \quad \mu_2' = E(X_i - \mu_1)^2 = \sigma_1^2$

We have

$$\begin{aligned} M_1(t) &= \left(1 + \mu_1' t + \mu_2' \frac{t^2}{2} + \mu_3' \frac{t^2}{6} + ........\right) \\ &= \left[1 + \frac{t^2}{2}\sigma_1^2 + O(t^3)\right] \end{aligned} \qquad (1)$$

Where $O(t^3)$ contains terms of order $t^3$ and higher powers of t

We have

$$Z = \frac{S_n - \mu}{\sigma} = \frac{(X_1 + X_2 ....... + X_n}{\sigma} = \sum_{i=1}^{n}\left(\frac{X_i - \mu_1}{\sigma}\right)$$

And since $X_i$'s are independent, we get

$$M_Z(t) = M_{\sum\limits_{i=1}^{n}\left(\frac{X_i-\mu_1}{\sigma}\right)}(t) = M_{\sum\limits_{i=1}^{n} X_i - \mu_1}(\sigma / t)$$

$$\begin{aligned} &= \prod_{i=1}^{n}\left\{M_{(X_i-\mu_1)}(t/\sigma)\right\} = \left[M_1(t/\sigma)\right]^n = \left[M_{(X_i-\mu_1)}(t/\sqrt{n}\sigma_1\right]^n \\ &= \left[1 + \frac{t^2}{2n} + O(n^{-n/2})\right]^n \end{aligned} \qquad \text{From (1)}$$

For every fixed 't', the term $O(n^{-3/2}) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, as $n \rightarrow \infty$, we get

$$\lim_{n\to\infty} it\, M_Z(t) = \lim_{n\to\infty}\left[1+\frac{t^2}{2n}+O(n^{-n/2})\right]^n = \exp\left[\frac{t^2}{2}\right] = e^{\frac{t^2}{2}},$$

which is the M.G.F. of standard normal variate

Hence by uniqueness theorem of M.G.F.'s, $Z = (S_n-\mu)/\sigma$ is asymptotically $N(0, 1)$, or $S_n = X_1 + X_2 + \ldots\ldots\ldots + X_n$ is asymptotically $N(\mu, \sigma^2)$, where $\mu = n\,\mu_1$ and $\sigma^2 = n\,\sigma_1^2$

## Weak Law of Large Number:

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sequence of random variables and $\mu_1, \mu_2, \ldots\ldots, \mu_n$

Be their respective expectation and let

$B_n = \text{Var}(X_1 + X_2 + \ldots\ldots\ldots + X_n) < \infty$

$$\text{Then } P\left\{\left|\frac{X_1+X_2+\ldots\ldots+X_n}{n} - \frac{\mu_1+\mu_2+\ldots\ldots\mu_n}{n}\right| \le \varepsilon\right\} \ge 1-\eta$$

For all $n > n_0$, where $\varepsilon$ and $\eta$ are arbitrary small positive numbers, provided

$$\lim_{n\to\infty}\frac{B_n}{n^2}\to 0$$

**Proof.** Using Chebychev's inequality, to the random variable

$(X_1 + X_2 + \ldots\ldots\ldots + X_n)/n$, we get for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1+X_2+\ldots\ldots+X_n}{n} - E\left(\frac{X_1+X_2+\ldots\ldots+X_n}{n}\right)\right| \le \varepsilon\right\} \ge 1-\frac{B_n}{n^2\varepsilon^2}$$

$$\left[\text{Since}\quad \text{Var}\left(\frac{X_1+X_2+\ldots\ldots+X_n}{n}\right)\right] = \frac{1}{n^2}\text{Var}(X_1+X_2+\ldots\ldots+X_n) = \frac{B_n}{n^2}$$

$$\Rightarrow P\left\{\left|\frac{X_1 + X_2 + ...... + X_n}{n} - \left(\frac{\mu_1 + \mu_2 + ...... + \mu_n}{n}\right)\right| < \varepsilon\right\} \geq 1 - \frac{B_n}{n^2\varepsilon^2}$$

Since ε is arbitrary, we assume $\frac{B_n}{n^2\varepsilon^2}\to\infty$, as n becomes indefinitely large. Thus, having chosen two arbitrary small positive numbers ε and η, number $n_0$ can be found so that the inequality

$$\frac{B_n}{n^2\varepsilon^2} < \eta, \text{ is 1}$$

Is true for $n > n_0$. Therefore, we shall have

$$P\left\{\left|\frac{X_1 + X_2 + ...... + X_n}{n} - \frac{\mu_1 + \mu_2 + ........\mu_n}{n}\right| \leq \varepsilon\right\} \geq 1 - \eta\,.$$

## Chi-Square Distribution

Definition. If $X \sim N(\mu, \sigma^2)$ , then $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

And $Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2$ , is the chi-square variate with 1 d. f

In general, if $X_{i,}$(i=1.,2,……n) are n independent normal variates with men $\mu_i$ and variance $\sigma^2_{i,}$ (i=1.,2,……n), then

$$\chi^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$$ , is the chi-square variate with n d. f

**Chi-square distribution:** if $X_{i,}$(i=1.,2,……n) are independent $N(\mu_i, \sigma^2_i)$, we want the distribution of

$$\chi^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 = \sum_{i=1}^{n} U_i^2$$

Where $U_i = \frac{X_i - \mu_i}{\sigma_i}$

**Probability density function**

$$dP(\chi^2) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma(n/2)}.\left[\exp(-\frac{1}{2}\chi^2)\right](\chi^2)^{(n/2)-1}d\chi^2 \qquad 0 \le \chi^2 < \infty$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)}.\left[\exp(-\frac{\chi^2}{2})\right](\chi^2)^{(n/2)-1}d\chi^2,$$

Which is the required probability distribution function of chi-square distribution with n degree of freedom

**Students t distribution**

Definition. Let $x_i$ ( i=1, 2, 3,…,n) be a random sample of size n from a normal population with mean $\mu$ and variance $\sigma^2$. Then student ‘s t is defined by the statistic

$$t = \frac{\bar{x} - \mu}{s}\sqrt{n}\ ,$$

where $\bar{x}$ is the sample mean and $s^2$ is an estimate of population variance of $\sigma^2$ . It follows Student t –distribution with $\nu$ = (n-1) degrees of freedom with probability density function

$$f(t) = \frac{1}{\sqrt{\nu}\ B(\frac{1}{2}, \frac{\nu}{2})\{1 + \frac{t^2}{\nu}\}^{\frac{\nu+1}{2}}} \qquad -\infty < t < \infty .$$

## Moments

The probability curve of t is symmetrical about t=0 and therefore,

Mode=mean =median=0

All odd order moments are zero i.e. $\mu_{2r+1}$ =0

Even order moments are given by

$$\mu_{2r} = \mu_{2r}' = \int_{-\infty}^{\infty} t^{2r} f(t)dt = \frac{2}{\sqrt{\nu}\ B(\frac{1}{2}, \frac{\nu}{2})} \int_{0}^{\infty} \frac{t^{2r}}{\{1 + \frac{t^2}{\nu}\}^{\frac{\nu+1}{2}}} dt$$

$$= \frac{\nu^r B(\frac{\nu}{2} - r, r + \frac{1}{2})}{B(\frac{1}{2}, \frac{\nu}{2})}$$

Put r=1, then $\mu_2 = \dfrac{n}{n-2}$ , n>2,

is known as variance.

## F –distribution.

If U and V are two independent $\chi^2$ variates with $\nu_1$ and $\nu_2$ degrees of freedom, respectively, then F –statistic is defined by

$$F=\frac{\left(\frac{U}{\nu_1}\right)}{\left(\frac{V}{\nu_2}\right)}, \qquad (\nu_1 > \nu_2), \qquad \Big| \text{ as } \begin{array}{l} U \sim \chi^2_{\nu_1} \\ V \sim \chi^2_{\nu_2} \end{array}$$

In other words, F is defined as the ratio of two independent chi-square variates divided by their respective degrees of freedom and it follows Snedecor's F-distribution with ($\nu_1$, $\nu_2$) degrees of freedom with density function given by

$$f(F) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{F^{\frac{\nu_1}{2}-1}}{\left\{1+\frac{\nu_1}{\nu_2}F\right\}^{\frac{\nu_1+\nu_2}{2}}}, \quad 0 < F < \infty .$$

## Moments.

The rth moment about origin is given by

$$\mu_r' = E(F^r) = \int_0^\infty F^r f(F) dF = \int_0^\infty \frac{F^r \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{F^{\frac{\nu_1}{2}-1}}{\left\{1+\frac{\nu_1}{\nu_2}F\right\}^{\frac{\nu_1+\nu_2}{2}}} dF$$

$$\mu_r' = \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} B\left(r+\frac{\nu_1}{2}, \frac{\nu_2}{2}-r\right)$$

Putting r=1, we get **mean** = $\mu_1' = \frac{\nu_2}{\nu_2 - 2}$

Further $\mu_2' = \frac{\nu_2^2(\nu_1+2)}{\nu_2(\nu_2-2)(\nu_2-4)}, \quad \nu_2 > 4$

Now variance is obtained from $\mu_2 = \mu_2' - \mu_1'^2 = \frac{\nu_2^2}{(\nu_2 - 2)}\left[\frac{\nu_1^2 - 4 - \nu_1\nu_2 + 4\nu_1}{\nu_1(\nu_2 - 4)(\nu_2 - 2)}\right]$