M.SC. MATHEMATICS

MAL-524
ORDINARY DIFFERENTIAL EQUATIONS – II

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CONTENTS

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Chapter-1

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS -I

Objectives

This chapter will be devoted to explaining the main concepts of the systems of linear differential equations. Some theorems concerning the fundamental matrix of such systems will be proved. Relations between Wronskian and linear independence/dependence of solutions of such systems will be developed.

Introduction

We have already studied single differential equation of different types and obtained the existence and uniqueness of solution of initial value problem of first order equations which are not necessarily linear. But in some practical situations we have to deal with more than one differential equation with many variables or depending upon a single variable. Such system of equations arise quite naturally in the analysis of certain physical situations. There is a very important class of differential equations known as linear differential equations, for which a general and elaborate theory is available. Apart from their theoretical importance, these equations are of great significance in physics and engineering in the problem of oscillation and electric circuits among others. This chapter extends the theory to a system of linear equations which give rise to the study of matrix differential equation, which will include both homogeneous and non-homogeneous type.

Types of Linear Systems

The general linear system of two first-order differential equations in two unknown functions x and y is of the form

$$a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)x + a_4(t)y = F_1(t),$$

$$b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)x + b_4(t)y = F_2(t) \tag{1}$$

A solution of system (1) is an ordered pair of real functions (f, g) such that x = f(t),   y = g(t) simultaneously satisfy both equations of the system (1) on some real interval $a \le t \le b$.

The general linear system of three first-order differential equations in three unknown functions x, y and z is of the form

$$a_1(t)\frac{dx}{dt}+a_2(t)\frac{dy}{dt}+a_3(t)\frac{dz}{dt}+a_4(t)x+a_5(t)y+a_6(t)z=F_1(t),$$

$$b_1(t)\frac{dx}{dt}+b_2(t)\frac{dy}{dt}+b_3(t)\frac{dz}{dt}+b_4(t)x+b_5(t)y+b_6(t)z=F_2(t), \tag{2}$$

$$c_1(t)\frac{dx}{dt}+c_2(t)\frac{dy}{dt}+c_3(t)\frac{dz}{dt}+c_4(t)x+c_5(t)y+c_6(t)z=F_3(t).$$

A solution of system (2) is an ordered pair of real functions (f, g, h) such that x = f(t),   y = g(t), z = h(t) simultaneously satisfy all three equations of the system (2) on some real interval $a \le t \le b$.

We shall consider the standard type as a special case of linear system (1), which is of the form

$$\frac{dx}{dt}=a_{11}(t)x+a_{12}(t)y+F_1(t),$$

$$\frac{dy}{dt}=a_{21}(t)x+a_{22}(t)y+F_2(t). \tag{3}$$

This is the so-called normal form in the case of two linear differential equations in two unknown functions. We shall assume that the functions a_{11}, a_{12}, F_1, a_{21}, a_{22}, and F_2 in (3) are all continuous on a real interval $a \le t \le b$. If $F_1(t)$ and $F_2(t)$ are zero for all t, then the system (3) is called homogeneous; otherwise, the system is said to be non-homogeneous. An example of such a system with variable coefficients is

$$\frac{dx}{dt}=t^2x+(t+1)y+t^3,$$

$$\frac{dy}{dt}=te^tx+t^3y-e^t,$$

while one with constant coefficients is

$$\frac{dx}{dt} = 5x + 7y + t^2,$$

$$\frac{dy}{dt} = 2x - 3y + 2t.$$

The normal form in the case of a linear system of three differential equations in three unknown functions x, y, and z is

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + a_{13}(t)z + F_1(t),$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + a_{23}(t)z + F_2(t),$$

$$\frac{dz}{dt} = a_{31}(t)x + a_{32}(t)y + a_{33}(t)z + F_3(t)$$

An example of such a system with constant coefficients is

$$\frac{dx}{dt} = 3x + 2y + z + t,$$

$$\frac{dy}{dt} = 2x - 4y + 5z - t^2,$$

$$\frac{dz}{dt} = 4x + y - 3z + 2t + 1.$$

The normal form in the general case of a linear system of n differential equations in n unknown functions $x_1, x_2 \ldots\ldots, x_n$ is

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots\ldots\ldots + a_{1n}(t)x_n + F_1(t),$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots\ldots\ldots + a_{2n}(t)x_n + F_2(t),$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots\ldots\ldots + a_{nn}(t)x_n + F_n(t). \tag{4}$$

An important fundamental property of a normal linear system (4) is its relationship to a single nth – order linear differential equation in one unknown function. Consider the so-called normalized (the coefficient of the highest derivative is one) n-th order linear differential equation.

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots\ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = F(t) \tag{5}$$

in the one unknown function x. Let

$$x_1 = x, \quad x_2 = \frac{dx}{dt},$$

$$x_3 = \frac{d^2x}{dt^2}, \ldots\ldots, x_{n-1} = \frac{d^{n-2}x}{dt^{n-2}}, \quad x_n = \frac{d^{n-1}x}{dt^{n-1}}. \tag{6}$$

From (6) we have

$$\frac{dx}{dt} = \frac{dx_1}{dt}, \qquad \frac{d^2x}{dt^2} = \frac{dx_2}{dt}, \ldots\ldots, \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-1}}{dt}, \qquad \frac{d^n x}{dt^n} = \frac{dx_n}{dt}. \tag{7}$$

Then using both (6) and (7), the single n-th order equation (5) can be transformed into

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2, \\
&\frac{dx_2}{dt} = x_3, \\
&\quad\vdots \\
&\frac{dx_{n-1}}{dt} = x_n \\
&\frac{dx_n}{dt} = -a_n(t)x_1 - a_{n-1}(t)x_2 - \ldots - a_1(t)x_n + F(t),
\end{aligned} \tag{8}$$

which is a special case of the normal linear system (4) of n equations in n unknown functions.

Homogeneous Linear Systems

We shall now assume that $F_1(t)$ and $F_2(t)$ in the system (3) are both zero for all t and consider the basic theory of the resulting homogeneous linear system.

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y,$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y, \tag{9}$$

Theorem 1.1

Hypothesis Let

$$x = f_1(t), \qquad x = f_2(t)$$

and

$$y = g_1(t), \qquad y = g_2(t), \tag{10}$$

be two solutions of the homogeneous linear system (9). Let c_1 and c_2 be two arbitrary constants.

Conclusion Then

$$x = c_1 f_1(t) + c_2 f_2(t), \tag{11}$$

$$y = c_1 g_1(t) + c_2 g_2(t)$$

is also a solution of the system (9)

The solution (11) is called a linear combination of the solutions (10).

Definition

Let

$$x = f_1(t), \qquad x = f_2(t),$$

and

$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (9). These two solutions are linearly dependent on the interval $a \le t \le b$ if there exist constants c_1 and c_2, not both zero, such that

$$c_1 f_1(t) + c_2 f_2(t) = 0,$$

$$c_1 g_1(t) + c_2 g_2(t) = 0, \tag{12}$$

for all t such that $a \le t \le b$.

Definition

Let

$$x = f_1(t), \qquad x = f_2(t),$$

and

$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (9). These two solutions are linearly independent on the interval $a \leq t \leq b$ if they are not linearly dependent $a \leq t \leq b$. That is, the solutions $x = f_1(t)$, $y = g_1(t)$ and $x = f_2(t)$, $y = g_2(t)$ are linearly independent on $a \leq t \leq b$ if

$$c_1 f_1(t) + c_2 f_2(t) = 0,$$

$$c_1 g_1(t) + c_2 g_2(t) = 0, \tag{13}$$

for all t such that $a \leq t \leq b$

$$c_1 = c_2 = 0.$$

Definition

Let

$$x = f_1(t), \qquad x = f_2(t),$$

and

$$y = g_1(t), \qquad y = g_2(t),$$

be two solutions of the homogeneous linear system (9). The determinant

$$\begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} \tag{14}$$

is called the Wronskian of these two solutions. We denote it by W(t). We may now state the following useful criterion for the linear independence of two solutions of system (9).

Theorem 1.2

Two solutions

$$x = f_1(t), \qquad x = f_2(t),$$

and

$$y = g_1(t), \qquad y = g_2(t),$$

of the homogeneous linear system (9) are linearly independent on a interval $a \leq t \leq b$ if and only if their Wronskian determinant.

$$W(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} \tag{15}$$

is different from zero for all t such that $a \le t \le b$. Concerning the values of W(t), we also state the following results.

Theorem 1.3

Let W(t) be the Wronskian of two solutions of homogeneous linear system (9) on the interval $a \le t \le b$. Then either $W(t) = 0$ for all $t \in [a, b]$ or $W(t) = 0$ for no $t \in [a, b]$.

Example

Let us employ theorem 1.2 to verify the linear independence of the solutions

$$x = e^{5t}, \qquad x = e^{3t},$$

and

$$y = -3e^{5t}, \qquad y = -e^{3t},$$

of the system

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = 3x + 6y$$

We have

$$W(t) = \begin{vmatrix} e^{5t} & e^{3t} \\ -3e^{5t} & -e^{3t} \end{vmatrix} = 2e^{8t} \neq 0$$

on every closed interval $a \le t \le b$. Thus by Theorem 1.2 the two solutions are indeed linearly independent on $a \le t \le b$.

Before proceeding further, we state without proof the following two theorems from algebra.

Theorem A A system of n homogeneous linear algebraic equations in n unknowns has a nontrivial solution if and only if the determinant of coefficients of the system is equal to zero.

Theorem B A system of n linear non-homogeneous algebraic equations in n unknowns has a unique solution if and only if the determinant of coefficients of the system is unequal to zero.

Characteristic Values and Characteristic Vectors.

Let A be a given n x n square matrix of real numbers, and let S denote the set of all n x 1 column vectors of numbers. Now consider the equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{14}$$

in the unknown vector $\mathbf{x} \in S$, where λ is a number. Clearly the zero vector $\mathbf{0}$ is a solution of this equation for every number λ. We investigate the possibility of finding nonzero vectors $\mathbf{x} \in S$ which are solutions of (14) for some choice of the number λ.

Definitions

A **characteristic value** (or eigenvalue) of the matrix A is a number λ for which the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nonzero vector solution $\mathbf{x}$.

A **characteristic vector** (or eigenvector) of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some number λ.

The Matrix method for homogeneous linear systems with constant coefficients: n equations in n unknown functions:

We consider a homogeneous linear system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\end{aligned} \tag{15}$$

where the coefficients a_{ij}, ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$), are real constants.

We will express this system in vector-matrix notation. We introduce the n x n constant matrix of real numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{16}$$

and the vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{17}$$

Then by definition of the derivative of a vector,

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix};$$

and by multiplication of a matrix by a vector, we have

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{pmatrix}$$

Comparing the components of d**x**/dt with the left members of (15) and the components of A**x** with the right members of (15), we see that system (15) can be expressed as the homogeneous linear vector differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{18}$$

The real constant matrix A that appears in (18) and is defined by (16) is called the coefficient matrix of (18).

Definition

By a solution of the system (15), that is, of the vector differential equation (18), we mean an n x 1 column-vector function

$$\boldsymbol{\varphi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix},$$

whose components, $\phi_1,\ \phi_2,\ldots,\phi_n$ each have a continuous derivative on the real interval a ≤ t ≤ b, and which is such that

$$\frac{d\phi_1(t)}{dt} = a_{11}\phi_1(t) + a_{12}\phi_2(t) + \ldots + a_{1n}\phi_n(t),$$

$$\frac{d\phi_2(t)}{dt} = a_{21}\phi_1(t) + a_{22}\phi_2(t) + \ldots + a_{2n}\phi_n(t),$$

$$\vdots$$

$$\frac{d\phi_n(t)}{dt} = a_{n1}\phi_1(t) + a_{n2}\phi_2(t) + \ldots + a_{nn}\phi_n(t),$$

for all t such that a ≤ t ≤ b. In other words, the components $\phi_1,\ \phi_2,\ldots,\phi_n$ of $\boldsymbol{\varphi}$ are such that

$$\begin{aligned} x_1 &= \phi_1(t) \\ x_2 &= \phi_2(t) \\ &\vdots \\ x_n &= \phi_n(t) \end{aligned}$$

simultaneously satisfy all n equations of the system (15) identically on a ≤ t ≤ b.

Theorem 1.4

Any linear combination of n solutions of the homogeneous linear system (15) is itself a solution of the system (15)

Theorem 1.5

There exist sets of n linearly independent solutions of the homogeneous linear system (15). Every solution of the system can be written as a linear combination of any n linearly independent solutions of (15).

Definition

Let

$$\boldsymbol{\varphi_1} = \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{pmatrix}, \boldsymbol{\varphi_2} = \begin{pmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{pmatrix}, \ldots, \boldsymbol{\varphi_n} = \begin{pmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{pmatrix}$$

be n linearly independent solutions of the homogeneous linear system (15). Let c_1, $c_2,\ldots,c_n$ be n arbitrary constants. Then the solution

$$\mathbf{x} = c_1\boldsymbol{\varphi}_\mathbf{1}(t) + c_2\boldsymbol{\varphi}_\mathbf{2}(t) + \ldots + c_n\boldsymbol{\varphi}_\mathbf{n}(t),$$

that is,

$$\begin{gathered} x_1 = c_1\phi_{11}(t) + c_2\phi_{12}(t) + \ldots + c_n\phi_{1n}(t), \\ x_2 = c_1\phi_{21}(t) + c_2\phi_{22}(t) + \ldots + c_n\phi_{2n}(t), \\ \vdots \\ x_n = c_1\phi_{n1}(t) + c_2\phi_{n2}(t) + \ldots + c_n\phi_{nn}(t), \end{gathered}$$

is called a general solution of the system (15)

Theorem 1.6

n solutions $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n}$ of the homogeneous linear system (15) are linearly independent on an interval $a \le t \le b$ if and only if

$$W(\boldsymbol{\varphi}_\mathbf{1},\boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n})(t) \neq 0$$

for all $t \in [a, b]$

Concerning the values of $W(\boldsymbol{\varphi}_\mathbf{1},\boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n})$, we also state the following result.

Theorem 1.7

Let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n}$ be n solutions of the homogeneous linear system (15) on an interval $a \le t \le b$. Then either $W(\boldsymbol{\varphi}_\mathbf{1},\boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n})(t) = 0$ for all $t \in [a, b]$ or $W(\boldsymbol{\varphi}_\mathbf{1},\boldsymbol{\varphi}_\mathbf{2},\ldots,\boldsymbol{\varphi}_\mathbf{n})(t) = 0$ for no $t \in [a, b]$.

Preliminary Definitions and Notations

If A is a matrix of complex numbers (a_{ij}) with n rows and n columns, define the norm, $|A|$, of A by

$$|A| = \sum_{i,j=1}^{n} |a_{ij}| \tag{1}$$

In case **x** is an n – dimensional vector, represented as a matrix of n rows and one column, then the vector magnitude coincides with the norm of **x** as defined by (1). The norm satisfies the following properties

(i) $|A+B| \leq |A| + |B|$

(ii) $|AB| \leq |A||B|$

(iii) $|Ax| \leq |A||x|$

where A and B are matrices, and **x** is an n-dimensional vector.

The distance between two matrices A and B is defined by $|A-B|$, and this distance satisfies the usual properties of a metric.

The zero matrix will be denoted by 0, and the unit matrix by E. These n-by-n matrices will be denoted by 0_n and E_n, respectively. Note that $|0_n| = 0$, and $|E_n| = n$, and not 1.

The complex conjugate matrix of A = (a_{ij}), denoted by $\overline{A}$, is defined by $\overline{A} = (\overline{a}_{ij})$, where $\overline{a}_{ij}$ is the complex conjugate of a_{ij}. The transposed matrix of A, denoted by A', is defined by A' = (a_{ji}). The conjugate transposed matrix of A is A* = $\overline{A}'$. Note that $|A^*| = |A'| = |\overline{A}| = |A|$. Also (AB)* = B* A*. The determinant of A is denoted by det A.

If det A = 0, then A is said to be singular. A nonsingular matrix A possesses an inverse (or reciprocal), A^{-1}, which satisfies

$$AA^{-1} = A^{-1} A = E$$

The polynomial in λ of degree n, det (λE-A), is called the characteristic polynomial of A, and its roots are the characteristic roots of A. If these roots are denoted by λ_i, i = 1, . . . , n, then clearly

$$\det\ (\lambda E - A) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

Two n-by-n complex matrices A and B are said to be similar if there exists a nonsingular n-by-n complex matrix P such that

$$B = P\ A\ P^{-1}$$

If A and B are similar, then they have the same characteristic polynomial,

for

$$
\begin{aligned}
\det\ (\lambda E - A) &= \det\ (P\,(\lambda E - A)P^{-1}) \\
&= \det\ P \det(\lambda E - A) \det P^{-1} \\
&= \det\ (\lambda E - A)
\end{aligned}
$$

In particular, the coefficients of the powers of λ in det $(\lambda E - A)$ are invariant under similarity transformations. Two of the most important invariants are det A and tr A, the determinant and trace of A, respectively.

If $\{A_m\}$ is a sequence of matrices, this sequence is said to be convergent if, given any $\in > 0$, there exists a positive integer $N_\in$ such that

$$\left|A_q - A_p\right| < \varepsilon \quad \text{whenever } p, q > N_\in$$

The sequence $\{A_m\}$ is said to have a limit matrix A if, given any $\in > 0$, there exists a positive integer $N_\in$, such that

$$\left|A_m - A\right| < \varepsilon \quad \text{whenever } m > N_\in$$

Clearly $\{A_m\}$ is convergent if and only if each of the component sequences is convergent, and this implies that $\{A_m\}$ is convergent if and only if there exists a limit matrix to which it tends.

The infinite series

$$\sum_{m=1}^{\infty} A_m$$

is said to be convergent if the sequence of partial sums is convergent, and the sum of the series is defined to be the limit matrix of the partial sums.

The following fundamental result concerning the canonical form of a matrix is assumed.

Theorem 1.8 Every complex n - by- n matrix A is similar to a matrix of the form

$$J = \begin{pmatrix} J_0 & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ . & . & . & \ddots & . \\ 0 & 0 & 0 & \cdots & J_s \end{pmatrix}$$

where J_0 is a diagonal matrix with diagonal $\lambda_1, \lambda_2, \ldots, \lambda_q$, and

$$J_i = \begin{pmatrix} \lambda_{q+i} & 1 & 0 & 0 & \Lambda & 0 & 0 \\ 0 & \lambda_{q+i} & 1 & 0 & \Lambda & 0 & 0 \\ . & . & . & . & \Lambda & . & . \\ 0 & 0 & 0 & 0 & \Lambda & \lambda_{q+i} & 1 \\ 0 & 0 & 0 & 0 & \Lambda & 0 & \lambda_{q+i} \end{pmatrix} \qquad (i = 1, \ldots, s)$$

The λ_j, $j = 1, \ldots, q + s$, are the characteristic roots of A, which need not all be distinct. If λ_j is a simple root, then it occurs in J_0, and therefore, if all the roots are distinct, A is similar to the diagonal matrix.

$$\mathrm{J} = \begin{pmatrix} \lambda_1 & 0 & 0 & \Lambda & 0 \\ 0 & \lambda_1 & 0 & \Lambda & 0 \\ . & . & . & \mathrm{K} & . \\ 0 & 0 & 0 & \mathrm{K} & \lambda_n \end{pmatrix}$$

From Theorem 1.8 it follows immediately that

$$\det \mathrm{A} = \prod \lambda_i, \qquad \operatorname{tr} \mathrm{A} = \sum \lambda_i$$

where the product and sum are taken over all roots, each root counted a number of times equal to its multiplicity. The J_i are of the form

$$J_i = \lambda_{q+i} E_{ri} + Z_i$$

where J_i has r_i rows and columns, and

$$Z_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \Lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & \Lambda & 0 & 0 \\ . & . & . & . & \mathrm{K} & . & . \\ 0 & 0 & 0 & 0 & \Lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \Lambda & 0 & 0 \end{pmatrix}$$

An equally valid form of J_i is $\lambda_{q+i} E_{ri} + \gamma Z_i$, where γ is any constant not zero. The matrix Z_i^2 has its diagonal of 1s moved one element to the right from that of Z_i and all other elements zero. From this it follows that $Z_i^{r_i}$ is a matrix which contains all zeros except for a single 1 in the first row and last column. Hence $Z_i^{r_i}$ is the zero matrix, and Z_i is nilpotent.

A particular series which is of great importance for the study of linear equations is the one defining the exponential of a matrix A, namely,

$$e^A = E + \sum_{m=1}^{\infty} \frac{A^m}{m!} \tag{2}$$

where A^m represents the m-th power of A. The series defining e^A is convergent for all A, since for any positive integers p, q,

$$\left| \sum_{m=p+1}^{p+q} \frac{A^m}{m!} \right| \leqq \sum_{m=p+1}^{p+q} \frac{|A|^m}{m!}$$

and the r.h.s. represents the Cauchy difference for the series $e^{|A|}$ which is convergent for all finite $|A|$. Also

$$\left| e^A \right| \le (n-1) + e^{|A|} \tag{3}$$

For matrices, it is not in general true that $e^{A+B} = e^A e^B$, but this relation is valid if A and B commute.

Every matrix A satisfies its characteristic equation det $(\lambda E - A) = 0$ and this remark is sometimes useful for the actual calculation of e^A. As a simple example, if

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then det $(\lambda E - A) = \lambda^2 = 0$, and therefore $A^2 = 0$, which implies $A^m = 0$, $m > 1$. Hence,

$$e^A = E + A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Using all these observations, we will prove the basic theorem:

If B is a nonsingular matrix, then it will be shown that there exists a matrix A (called a logarithm of B) such that $e^A = B$. Indeed, if B is in the canonical form J of Theorem 1.8, it is evident that A can be taken as

$$
A = \begin{pmatrix} A_0 & 0 & 0 & \Lambda & 0 \\ 0 & A_1 & 0 & \Lambda & 0 \\ . & . & . & K & . \\ 0 & 0 & 0 & \Lambda & A_s \end{pmatrix}
$$

provided that $e^{A_j} = J_j$, $j = 0, 1, \ldots, s$. It is also easily verified that a suitable A_0 is given by

$$
A_0 = \begin{pmatrix} \log\lambda_1 & 0 & \Lambda & 0 \\ 0 & \log\lambda_2 & \Lambda & 0 \\ . & . & K & . \\ 0 & 0 & \Lambda & \log\lambda_q \end{pmatrix}
$$

Clearly

$$
J_j = \lambda_{q+j} \left(E_{rj} + \frac{1}{\lambda_{q+j}} Z_j \right)
$$

where Z_j is the nilpotent matrix defined after Theorem 1.8. Since large powers of Z_j all vanish, the series

$$
\sum_{k=1}^{\infty} (-1)^{k+1} k^{-1} (\lambda_{q+j})^{-k} Z_j^k
$$

has only a finite number of terms, and is thus convergent. Define

$$
\log \left(E_{rj} + \frac{1}{\lambda_{q+j}} Z_j \right)
$$

to be this series, which is, of course, a polynomial in $\lambda_{q+j}^{-1} Z_j$. Thus

$F\left(\lambda_{q+j}^{-1} Z_j\right) = \exp\left[\log\left(E_{rj} + \lambda_{q+j}^{-1} Z_j\right)\right]$

is a polynomial in $\lambda_{q+j}^{-1} Z_j$. On the other hand, from

$1 + x = e^{\log(1+x)}$

$$
= 1 + \left(x - \frac{1}{2}x^2 + \Lambda \right) + \frac{1}{2!}\left(x - \frac{1}{2}x^2 + \Lambda \right)^2 + \Lambda, \qquad |x| < 1
$$

It follows that, when the right member is rearranged, the coefficients of x^k, $k \geq 2$, are all zero, while the coefficient of x is 1. This implies the same result for F, and proves that

$$\exp [\log (E_{rj} + \lambda_{q+j}^{-1} Z_j)] = E_{rj} + \lambda_{q+j}^{-1} Z_j$$

From this follows readily that a suitable A_j, j=1, . . .,s, is given by

$$A_j = (\log \lambda_{q+j}) E_{rj} + \log \left(E_{rj} + \frac{1}{\lambda_{q+j}} Z_j \right)$$

Using the fact that for any matrix M,

$$(PMP^{-1})^k = PM^k P^{-1} \qquad (k = 1, 2, \ldots)$$

one readily see that

$$Pe^M P^{-1} = e^{PMP^{-1}}$$

From this it follows that the result just sketched for a canonical matrix B is valid for any nonsingular matrix B. Indeed, if $J = e^A$ and $B = PJP^{-1}$, then $B = e^A$, where $\tilde{A} = PAP^{-1}$

Definition

If Φ is an n x n matrix of functions defined on a real t-interval I (the functions may be real or complex), then Φ is said to be continuous, differentiable or analytic on I, if every element of Φ is continuous, differentiable or analytic function of t on I.

$$\Phi = \begin{bmatrix} \phi_{11}(t) & \Lambda\Lambda & \phi_{1n}(t) \\ \Lambda\Lambda & \Lambda\Lambda & \Lambda\Lambda \\ \Lambda\Lambda & \Lambda\Lambda & \Lambda\Lambda \\ \phi_{n1}(t) & \Lambda\Lambda & \phi_{nn}(t) \end{bmatrix}, t \in I$$

If Φ is differentiable on I, then Φ' denotes the matrix of derivatives i.e.

$$\Phi' = [\phi'_{ij}(t)]$$

Note: If Φ and ψ are two differentiable n x n matrix functions, then

$$(\Phi\psi)' = \Phi'\psi + \Phi\psi'$$

and that $\Phi'\psi \neq \Phi\psi'$ in general.

Remark

If Φ is a non-singular matrix and $\Phi'(t)$ exist, then show that Φ^{-1} is differentiable at t and find $\left(\Phi^{-1}\right)'$.

Proof As $A^{-1} = \text{Adj } A / |A|$

$$\Phi^{-1} = \frac{\widetilde{\Phi}}{\det \Phi}, \qquad \text{where } \widetilde{\Phi} = \left(\widetilde{\phi}_{ij}\right) \quad (1)$$

and $\widetilde{\phi}_{ij}$ is the cofactor of ϕ_{ji}. Equation (1) shows that Φ^{-1} is differentiable at t as Φ is differentiable at t.

[Adj A = Transpose of matrix of co-factors]

Now we find $\left(\Phi^{-1}\right)'$.

We know that $\Phi\,\Phi^{-1} = E$

$$\Rightarrow \quad \left(\Phi\ \Phi^{-1}\right)' = E' = 0$$

$$\Rightarrow \quad \Phi\left(\Phi^{-1}\right)' = -\,\Phi'\ \Phi^{-1}$$

Pre-multiplying both sides by Φ^{-1}

$$\Rightarrow \quad \left(\Phi^{-1}\right)' = -\,\Phi^{-1}\ \Phi'\ \Phi^{-1}$$

where $\det \Phi \neq 0$

Theorem 1.9

The set of all solutions of the system

$$x'(t) = A(t)\,x(t), \qquad (1)$$

$$x(t_0) = x_0, \qquad t, t_0 \in I$$

forms an n-dimensional vector space over the field of complex numbers.

Proof

First we shall show that the set of all solutions forms a vector space and then establish that it is of dimension n.

Let x_1 and x_2 be the two solutions of (1)

Then $$x_1'(t) = A(t)x_1(t), \textit{ and } \quad x_2'(t) = A(t)x_2(t)$$

Now for any constants c_1 and c_2, we get

$$\frac{d}{dt}[c_1x_1 + c_2x_2] = c_1x_1' + c_2x_2' = c_1A(t)x_1 + c_2A(t)x_2$$

$$= A(t)\,[c_1x_1 + c_2x_2]$$

so that $$[c_1x_1 + c_1x_2]' = A(t)\,[c_1x_1 + c_2x_2]$$

which proves that if x_1 and x_2 are two solutions of (1), then $c_1x_1 + c_2x_2$ is also a solution of (1). This shows that the solutions form a vector space.

We note that each solution is an n-tuple. More precisely it is a column vector consisting of n components. We shall show that this vector space of solutions is of dimension n. For this we have to prove that the solution space contains n- linearly independent vectors which span the space.

Let $e_i = (0, 0, \ldots, 1, 0\ 0\ldots0)$ where 1 is in the i-th place. We know that $\{e_i, i = 1, 2, 3, \ldots,n\}$ is the standard basis for R^n. We shall construct a basis of solutions with the help of e_i's. By the existence theorem, given $t_0 \in I$, there exist solutions x_i, $i = 1, 2, 3\ldots$ such that

$$x_1(t_0) = e_1,\ x_2(t_0) = e_2,\ldots.,\ x_n(t_0) = e_n \tag{2}$$

We shall show that $x_1, x_2, \ldots, x_n$ is a linearly independent set which spans the space of solutions.

If $x_1, x_2, \ldots., x_n$ are not linearly independent in R_n, there must exist scalars $c_1, c_2,\ldots.c_n$, not all zero, such that

$$c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) = 0,\ t\in I \tag{3}$$

Since (3) is true for all $t\in I$, it is true in particular for $t = t_0$ so that we have

$$c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0) = 0 \tag{4}$$

Using (2) is (4), we get

$$c_1e_1 + c_2e_2 + \ldots + c_ne_n = 0 \tag{5}$$

which is a contradiction to the hypothesis that $e_1, e_2,\ldots e_n$ is a linearly independent set in R^n.

This proves that x_i's are linearly independent.

Let x be any solution of (1) on I such that $x(t_0) = x_0$. Since $x_0 \in R^n$, there exists unique scalars c_i, i = 1, 2,… n such that

$$x(t_0) = \sum_{i=1}^{n} c_i e_i$$

Since $x_i(t_0) = e_i$, we get $$x(t_0) = \sum_{i=1}^{n} c_i x_i(t_0)$$

Hence, the function $\sum_{i=1}^{n} c_i x_i$ is a solution on I which takes the value x_0 at t_0. By uniqueness of solutions, this must be equal to x on I so that we get

$$x = \sum_{i=1}^{n} c_i x_i$$

Therefore every solution x is a unique linear combination of the solution (x_i) which proves the theorem.

Definition

The set of all n linearly independent solutions of (1) is called a **fundamental set** of solutions of (1).

Example 1.1

Find the fundamental system of solutions of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{in } [0,1]$$

Since t and 2t are continuous in [0, 1], the matrix A(t) is a continuous matrix in [0, 1]

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ 2tx_2 \end{bmatrix}$$

so that we get

$$x_1' = tx_1 \text{ and } x_2' = 2tx_2$$

Solving these two equations, we gat $x_1 = c_1 e^{t^2/2}$, $x_2 = c_2 e^{t^2}$. Thus the vector space solutions are $x_1 = \begin{bmatrix} e^{t^2/2} \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ e^{t^2} \end{bmatrix}$. Since $x_1(t) \neq kx_2(t)$ for any t in [0, 1], the two vectors are linearly independent. Further the dimension of the vector space of solutions is 2.

Wronskian of Vector Functions

Consider the system

$$x'(t) = A(t)\,x(t),\ t \in I \tag{1}$$

(where A is a continuous matrix) having n linearly independent solutions on I. In any general situation we can take the solution as a vector function $\phi = (\phi_1, \phi_2, \ldots \phi_n)$ where we have

$$\phi_1(t) = \begin{bmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{bmatrix}, \phi_2(t) = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{bmatrix} \ldots \phi_n(t) = \begin{bmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{bmatrix} \tag{2}$$

in each of the vectors, the first subscript indicates the row and the second shows the vector solution. For example $\phi_{23}(t)$ shows the vector component in the second row of the vector $\phi_3(t)$. Since we are concerned with the linear dependence or otherwise of the vector solutions, we introduce the Wronskian of the vector valued functions ϕ_1, $\phi_2, \ldots$ ϕ_n as given in (2).

Definition

Then n x n determinant consisting of components of ϕ_1, $\phi_2, \ldots$ ϕ_n as

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

is called the Wronskian of n vector functions ϕ_1, $\phi_2, \ldots$ ϕ_n and is denoted by W(ϕ_1, $\phi_2, \ldots$ ϕ_n). Its value at any point $t_0 \in I$ is denoted by W(ϕ_1, $\phi_2, \ldots$ ϕ_n) (t_0). With the help of the Wronskian of $\phi_1, \phi_2, \ldots \phi_n$, we shall obtain the condition of the vector functions for linear dependence and independence and then derive the criterion for linear dependence or otherwise of the solutions of the homogeneous linear vector differential equation (1).

Theorem 1.10

The vector functions, $\phi_1, \phi_2, \ldots \phi_n$ are linearly dependent on I, then the Wronskian W($\phi_1, \phi_2, \ldots \phi_n$) (t) = 0 for every t $\in$ I.

Proof

If $\phi_1, \phi_2, \ldots \phi_n$ are linearly dependent on I, there exist n scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t) = 0 \tag{3}$$

for all t $\in$ I. Using the components of ϕ_i, i = 1, 2, 3… n and considering an arbitrary point $t_0 \in$ I, the above single vector equation is equivalent to the following n equations in n unknowns $c_1, c_2, \ldots c_n$ s.t.

$$c_1\phi_{11}(t_0) + c_2\phi_{12}(t_0) + \ldots + c_n\phi_{1n}(t_0) = 0$$

$$c_1\phi_{21}(t_0) + c_2\phi_{22}(t_0) + \ldots + c_n\phi_{2n}(t_0) = 0$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$c_1\phi_{n1}(t_0) + c_2\phi_{n2}(t_0) + \ldots + c_n\phi_{nn}(t_0) = 0$$

Since the scalars $c_1, c_2, \ldots, c_n$ are not all zero, the above homogeneous system has a non-trivial solution. Using theorem A, the determinant of the coefficients of above system of equations is zero. So we get

$$\begin{vmatrix} \phi_{11}(t_0) & \phi_{12}(t_0) & \mathrm{K} & \phi_{1n}(t_0) \\ \phi_{21}(t_0) & \phi_{22}(t_0) & \mathrm{K} & \phi_{2n}(t_0) \\ \mathrm{K} & \mathrm{K} & \mathrm{K} & \mathrm{K} \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \mathrm{K} & \phi_{nn}(t_0) \end{vmatrix} = 0$$

which gives W($\phi_1, \phi_2, \ldots \phi_n$)($t_0$) = 0. Since t_0 is an arbitrary point of I, we must have W($\phi_1, \phi_2, \ldots \phi_n$) (t) = 0 for all t $\in$ I.

Note In the above theorem, the vector functions are not the solutions of any linear homogeneous system. So the theorem is valid for any n-vector functions and in particular, the theorem is true for the solutions of homogeneous linear vector differential equations.

Theorem 1.11 If the vector functions $\phi_1, \phi_2,\ldots \phi_n$ are the solutions of the system x' = A(t) x on I, and the Wronskian $W(\phi_1, \phi_2,\ldots \phi_n)(t_0) = 0$ for some $t_0 \in I$, then $\phi_1, \phi_2,\ldots \phi_n$ are linearly dependent on I.

Proof Consider the system of equations

$$\begin{aligned}
&c_1\phi_{11}(t_0) + c_2\phi_{12}(t_0) + \ldots + c_n\phi_{1n}(t_0) = 0\\
&c_1\phi_{21}(t_0) + c_2\phi_{22}(t_0) + \ldots + c_n\phi_{2n}(t_0) = 0\\
&\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad\qquad (1)\\
&c_1\phi_{n1}(t_0) + c_2\phi_{n2}(t_0) + \ldots + c_n\phi_{nn}(t_0) = 0
\end{aligned}$$

in the n-unknown scalars $c_1, c_2,\ldots c_n$. In the above homogeneous system, the determinant of the coefficients is $W(\phi_1, \phi_2,\ldots \phi_n)(t_0)$, which is zero by hypothesis. So the system has non-trivial solution $c_1, c_2,\ldots c_n$, that is, there exist constants $c_1, c_2,\ldots c_n$, not all zero, satisfying the above equations, (using theorem A).

If $\phi_i = (\phi_{1i}, \phi_{2i} \ldots \phi_{ni})$, i = 1, 2,….n, then using the components, the above system can be written in an equivalent form as a vector equation.

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \ldots + c_n\phi_n(t_0) = 0 \qquad (2)$$

Now consider the vector functions ϕ defined as

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t)+\ldots+ c_n\phi_n(t),\ t \in I \qquad (3)$$

Since $\phi_1, \phi_2,\ldots \phi_n$ are the solutions of the system, their linear combination ϕ is also a solution of the system. Using (2), we have $\phi(t_0) = c_1\phi(t_0) + c_2\phi(t_0)\ldots+ c_n\phi_n(t_0) = 0$. Hence by using the result that x(t) = 0 is the only solution of the initial value problem x' = A(t)x, $x(t_0) = 0$, where t, $t_0 \in I$ and A(t) is a continuous matrix on I, $\phi(t) = 0$ for all $t \in I$. That is, $c_1\phi_1(t) + c_2\phi_2(t)+\ldots+ c_n\phi_n(t) = 0$ for all $t \in I$ where $c_1, c_2,\ldots,c_n$ are not all zero. Thus, using the definition, $\phi_1, \phi_2,\ldots \phi_n$ are linearly dependent.

Corollary If $\phi_1, \phi_2, \dots \phi_n$ are the solutions of the system $x' = A(t)x$, $t \in I$. Then either $W(\phi_1, \phi_2, \dots \phi_n)(t)$ is zero for all $t \in I$, or $W(\phi_1, \phi_2, \dots \phi_n)(t)$ is never zero on I.

Proof To prove this let us assume $W(\phi_1, \phi_2, \dots \phi_n)(t) = 0$ for some t or $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for any $t \in I$. If $W(\phi_1, \phi_2, \dots \phi_n)(t) = 0$ for some $t \in I$., then by theorem 1.11 $\phi_1, \phi_2, \dots \phi_n$ are linearly dependent on I so that $W(\phi_1, \phi_2, \dots \phi_n)(t) = 0$ for all $t \in I$ or $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for any $t \in I$.

Theorem 1.12 Let the vector functions $\phi_1, \phi_2, \dots \phi_n$ be n solutions of the homogeneous linear vector differential equation $x' = A(t)x$ on I. Then the n-solutions are linearly independent on I if and only if $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for all $t \in I$.

Proof By theorems 1.10 and 1.11, the solutions are linearly dependent on [a, b] if and only if $W(\phi_1, \phi_2, \dots \phi_n)(t) = 0$ for all $t \in I$. Therefore the solutions are linearly independent on I if and only if $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for some $t_0 \in [a, b]$. By the corollary of the previous theorem $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for some $t_0 \in I$ if and only if $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for all $t \in I$.

Theorem 1.13 There exists a fundamental set of solutions of the homogeneous linear vector differential equation.

$$x' = A(t)x, \quad t \in I. \tag{1}$$

Proof. Since we are concerned with solution vectors in R^n**,** let us consider the standard basis $\{e_i, I = 1, 2, 3 \dots n\}$ where $e_i = (0, 0, \dots, 1, 0, \dots 0)$. Here 1 is in the i-th place and zeros in all other places. Let $\phi_1, \phi_2, \dots \phi_n$ be n vector solutions which satisfy

$$\phi_1(t_0) = e_1,\ \phi_2(t_0) = e_2,\ \phi_n(t_0) = e_n \text{ for } t_0 \in I$$

Then

$$W(\phi_1, \phi_2, \dots \phi_n)(t_0) = W(e_1, e_2, \dots e_n) = \begin{vmatrix} 1 & 0 & 0 & \Lambda & 0 & 0 \\ 0 & 1 & 0 & \Lambda & 0 & 0 \\ M & M & M & M & M & M \\ 0 & 0 & 0 & \Lambda & 1 & 0 \\ 0 & 0 & 0 & \Lambda & 0 & 1 \end{vmatrix} \neq 0$$

Hence by corollary of Theorem 1.11, $W(\phi_1, \phi_2, \dots \phi_n)(t) \neq 0$ for any $t \in I$ and thus $\phi_1, \phi_2, \dots \phi_n$ are linearly independent on I.

Theorem 1.14 Let $\phi_1, \phi_2, \ldots \phi_n$ be a fundamental set of solutions of the homogeneous linear vector differential equation

$$x' = A(t)\, x \text{ on } I \tag{1}$$

and let ϕ be any arbitrary solution on the interval I. Then there exist unique scalars $c_1, c_2, \ldots c_n$ such that

$$\phi = c_1\phi_1 + c_2\phi_2 + \ldots + c_n\phi_n \text{ on } I$$

that is, ϕ can be expressed as a suitable linear combination of the fundamental set of solutions.

Proof Let us suppose $\phi(t_0) = u_0$, where $u_0 = (u_{10}, u_{20} \ldots u_{n0})$ is a constant vector. Now consider the linear non-homogeneous algebraic system

$$\begin{aligned}
&c_1\phi_{11}(t_0) + c_2\phi_{12}(t_0) + \ldots + c_n\phi_{1n}(t_0) = u_{10}\\
&c_1\phi_{21}(t_0) + c_2\phi_{22}(t_0) + \ldots + c_n\phi_{2n}(t_0) = u_{20}\\
&\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots\\
&c_1\phi_{n1}(t_0) + c_2\phi_{n2}(t_0) + \ldots + c_n\phi_{nn}(t_0) = u_{n0}
\end{aligned} \tag{2}$$

of n-equations in n-unknowns. Since $\phi_1, \phi_2, \ldots \phi_n$ are linearly independent solutions of (1), $W[\phi_1, \phi_2, \ldots \phi_n](t_0) \neq 0$. But $W[\phi_1, \phi_2, \ldots \phi_n](t_0)$ is the determinant of the coefficients of the system of equations (2). Since $W[\phi_1, \phi_2, \ldots \phi_n](t) \neq 0$, the system of equations (2) has a unique solution for $c_1, c_2, \ldots, c_n$, that is, there exist unique set of scalars $c_1, c_2, \ldots c_n$ such that

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + c_n\phi_n(t_0) = u_0 \tag{3}$$

Hence we have $\qquad \phi(t_0) = u_0 = \sum_{k=1}^{n} c_k\phi_k(t_0)$

Now consider the vector function $\psi(t) = \sum_{k=1}^{n} c_k\phi_k(t)$. Since any linear combination of solutions of (1) is also a solution of (1), we have

$$\psi(t_0) = \sum_{k=1}^{n} c_k\phi_k(t_0) \tag{4}$$

From (3) and (4), we have $\psi(t_0) = \phi(t_0)$.

Hence by the uniqueness of solutions ψ(t) = ϕ(t) for all t ∈ I. Thus, $\phi(t) = \sum_{k=1}^{n} c_k \phi_k(t)$ for all t ∈ I is solution of (1).

Fundamental Matrix and its Properties

We know that the solutions of the system x'(t) = A(t)x(t), t ∈ I (1)

form a vector space of dimension n. So it has n linearly independent solutions forming a basis. We can take the independent solutions as a vector $\phi = (\phi_1, \phi_2, \ldots \phi_n)$, where

$$\phi_1(t) = \begin{bmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{bmatrix} \quad \phi_2(t) = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{bmatrix} \ldots \phi_n(t) = \begin{bmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{bmatrix} \tag{2}$$

Definition 1 Let ψ be a matrix whose columns are the solutions (2) of the given system (1). This matrix ψ is called a solution matrix, since it satisfies the matrix equation x'(t) = A(t) x(t), t ∈ I

Definition 2 If the columns of the solution matrix are linearly independent, then the solution matrix is called a fundamental matrix.

According to Theorem 1.9, the n independent solutions of homogeneous system exist so that the fundamental matrix exists. We shall derive the differential equation satisfied by det ψ.

Definition

The trace of an n x n matrix A is given by the formula $A = \sum_{j=1}^{n} a_{jj}$. That is, the trace of A is the sum of its main diagonal elements.

Theorem 1.15 Abel-Liouville Formula

Let the functions

$$\boldsymbol{\varphi}_\mathbf{k} = \begin{pmatrix} \phi_{1k} \\ \phi_{2k} \\ \vdots \\ \phi_{nk} \end{pmatrix}, (k = 1, 2, \ldots, n$$

be n solutions of the homogeneous linear vector differential equation

$$\mathbf{x}' = A(t)\mathbf{x} \tag{1}$$

on the real interval [a, b]; let t_0 be any point of [a, b]; and let W denotes the Wronskain of ϕ_1,. ϕ_2,… ϕ_n. Then

$$W(t) = W(t_0)\exp\left[\int_{t_0}^{t} trA(s)ds\right] \tag{2}$$

for all $t \in [a, b]$

Proof We differentiate the Wronskian determinant

$$W = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

to obtain

$$W' = \begin{vmatrix} \phi'_{11} & \phi'_{12} & \cdots & \phi'_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi'_{21} & \phi'_{22} & \cdots & \phi'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots$$

$$+ \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi'_{i1} & \phi'_{i2} & \cdots & \phi'_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & \vdots & \vdots \\ \phi'_{n1} & \phi'_{n2} & \cdots & \phi'_{nn} \end{vmatrix} \tag{3}$$

where, primes denote derivatives with respect to t. Thus W' is the sum of n determinants, in each of which the elements of precisely one row are differentiated. Since ϕ_k, (k=1,2…,n), satisfies the vector differential equation (1), we have

$\phi_k' = A\phi_k$, (k =1, 2,…,n), and so $\phi_{ik}' = \sum_{j=1}^{n} a_{ij}\phi_{jk}$, (i = 1, 2,…n; j =1, 2,…n). Substitute these for the indicated derivatives in each of the n determinants in the preceding expression (3) for W'. Then the i-th determinant, (i = 1, 2,….n) in (3) becomes

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1}' & \phi_{i2}' & \cdots & \phi_{in}' \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{ij}\phi_{j1} & \sum_{j=1}^{n} a_{ij}\phi_{j2} & \cdots & \sum_{j=1}^{n} a_{ij}\phi_{jn} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

Writing out each of the indicated sums in the preceding determinant and using fundamental properties of determinants, we see that it breaks up into the following sum of n determinants:

$$a_{i1}\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + a_{i2}\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots$$

$$+ a_{ii}\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots + a_{in}\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

Each of these n determinants has two equal rows, except the ith one, and the coefficient of this exceptional one is a_{ii}. Since a determinant having two equal rows is zero, this leaves only the single exceptional determinant having the coefficient a_{ii}. Thus we have

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi'_{i1} & \phi'_{i2} & \cdots & \phi'_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = a_{ii} \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} \text{ for each i = 1, 2,…n}$$

Using this identity with i = 1, 2,….n, we replace each of the n determinants in (3) accordingly. Thus, (3) takes the form

$$W' = a_{11} \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + a_{22} \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots$$

$$+ a_{ii} \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \ldots + a_{nn} \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

That is, W' = $\left[\sum_{j=1}^{n} a_{jj}\right] W$, and so

$$W' = (\text{tr } A)\, W. \tag{4}$$

or, W satisfies the first order scalar homogeneous linear differential equation

$$\frac{dW(t)}{dt} = [trA(t)]W(t)$$

Integrating this, we at once obtain

$$W(t) = c \exp\left[\int_{t_0}^{t} trA(s)ds\right]$$

Letting $t = t_0$, we find that $c = W(t_0)$, and hence we obtain the required Abel-Liouville formula

$$W(t) = W(t_0) \exp\left[\int_{t_0}^{t} trA(s)ds\right] \qquad (5)$$

Note From Abel-Liouville Formula, we can conclude that if det $\Phi(t) \neq 0$ for some $t \in I$, Then det $\Phi(t) \neq 0 \quad \forall\, t \in I$

Theorem 1.16 A solution matrix Φ of the matrix differential equation

$$x' = A(t)x, \ t \in I \qquad (1)$$

is a fundamental matrix if and only if det $\Phi(t) \neq 0$ for any $t \in I$.

Proof

First we shall prove the necessary part of the theorem. Assuming that the given solution matrix Φ is a fundamental matrix of (1), we shall prove that det $\Phi(t) \neq 0$ for any $t \in I$. Let the column vectors of Φ be $\phi_j = 1, 2, \ldots n$. Let ψ be any solution of (1). Since (ϕ_j) forms a basis of solutions, ψ can be expanded as linear combination of ϕ_j' s, that is, there exist unique non-zero constants $c_1, c_2, \ldots c_n$ such that

$$\psi = \sum_{i=1}^{n} c_i \phi_i = c_1\phi_1 + c_2\phi_2 + \ldots + c_n\phi_n$$

which we can write in the matrix form as

$$\psi = [\phi_1, \phi_2 \ldots \phi_n]\begin{bmatrix} c_1 \\ c_2 \\ :: \\ c_n \end{bmatrix} = \Phi C \qquad (2)$$

where C is the unique column vector. Equation (2) gives n-linear equations of the form

$$\psi_i = \sum_{j=1}^{n} \phi_{ij} c_j, \qquad i = 1,2,\ldots n$$

in the unique constants $c_1, c_2, \ldots c_n$. Since the above non-homogeneous system has a unique solution $c_1, c_2, \ldots c_n$ for a fix $\tau \in I$, we get det $\Phi(\tau) \neq 0$ for a fixed $\tau \in I$. (Using Theorem B).

From Abel-Liouville formula

$$\det \Phi(t) = \det \Phi(\tau) \exp\left[\int_{\tau}^{t} trA(s)ds \right]$$

Using det $\Phi(\tau) \neq 0$ in the above integral equation, $\Phi(t) \neq 0$ for any t $\in$ I.

To prove the converse, let $\Phi(\tau)$ be any solution matrix of the system such that det $\Phi(t) \neq 0$ for any t $\in$ I. Now we shall prove that $\Phi(t)$ is a fundamental matrix. Since $\det \Phi(t) \neq 0$ for any t $\in$ I, its Wronskian $W(\phi_1, \phi_2, ...\phi_3)(t) \neq 0$. Hence by theorem 1.12, ϕ_1, ϕ_2….., ϕ_n are linearly independent. In other words, the column vectors of the solution matrix Φ(t) are linearly independent. Hence Φ is a fundamental matrix of (1).

Corollary

Two different homogeneous systems cannot have the same fundamental matrix.

If Φ(t) is the fundamental matrix of the given homogeneous linear system (1), we have

$$\Phi'(t) = A(t)\Phi(t) \tag{1}$$

and $\det \Phi(t) \neq 0$ for any t $\in$ I by the theorem so that is inverse, namely, $\Phi^{-1}(t)$ exists for every t $\in$ I. Post multiplying both sides of (1) by $\Phi^{-1}(t)$, we get A(t) = $\Phi'(t)$ $\Phi^{-1}(t)$. Hence Φ determines A uniquely for the system. Therefore, it cannot be a fundamental matrix for another homogeneous system.

Note The above theorem is true only for the solution matrices as there are matrices Φ with linearly independent columns but with det Φ = 0. For example, let $\Phi(t) = \begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix}$. The column vectors are linearly independent for

$$c_1 \begin{pmatrix} t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 0 \end{pmatrix} = 0$$

implies $c_1 t + c_2 t^2 = 0$ which in turn implies $c_1 = 0$, and $c_2 = 0$. Further det Φ(t) = 0. Thus we have a Φ(t) with linearly independent columns for which det Φ(t) = 0 because $\phi_1 = \begin{pmatrix} t \\ 0 \end{pmatrix} and\ \phi_2 = \begin{pmatrix} t^2 \\ 0 \end{pmatrix}$ are not the solutions of any linear homogeneous system.

Theorem 1.17

(i) If Φ is a fundamental matrix of the system $x' = Ax$ and C is constant non-singular matrix, then ΦC is a fundamental matrix.

(ii) Every fundamental matrix of the system is of this type for some non-singular matrix.

Proof

(i) Since Φ is a fundamental matrix, $\det \Phi \neq 0$ by Theorem 1.16 so that it is non-singular. By hypothesis, C is a non-singular matrix. Since the product of two non-singular matrices is non-singular, ΦC is non-singular so that $\det \Phi C \neq 0$. Hence by Theorem 1.16 ΦC is a fundamental matrix.

(ii) Let Φ_1 and Φ_2 be two fundamental matrices. Assume that

$$\Phi_2 = \Phi_1 \psi \tag{1}$$

and we will show that ψ is a constant non-singular matrix. Now using matrix differentiation, we get from (1)

$$\Phi_2' = \Phi_1' \psi + \Phi_1 \psi' \tag{2}$$

Since Φ_1 and Φ_2 are solutions of $x' = Ax$, we get

$$\Phi_2' = A\Phi_2, \quad \Phi_1' = A\Phi_1 \tag{3}$$

With the help of (3) and (1) in the given equation, we have

$$\Phi_2' = A\Phi_2 = \Phi_1' \psi + \Phi_1 \psi' = A\Phi_1 \psi + \Phi_1 \psi' = A\Phi_2 + \Phi_1 \psi'$$

Hence we get

$$A\Phi_2 = A\Phi_2 + \Phi_1 \psi' \text{ which gives } \Phi_1 \psi' = 0 \tag{4}$$

Since Φ_1 is non-singular (4) implies $\psi' = 0$ or $\psi = C$, where C is a constant matrix. Thus we have $\Phi_2 = \Phi_1 C$. Hence we get $C = \Phi_1^{-1}\Phi_2$. Since both Φ_1 and Φ_2 are non-singular, C is a constant non-singular matrix.

Hence, the proof of the theorem

Note

If Φ is a fundamental matrix of the system, x' = Ax and C is any constant non-singular matrix, then CΦ is not in general a fundamental matrix.

Proof

If possible, let CΦ be a fundamental matrix of the system x' = Ax. Then it should satisfy the equation (C Φ)' = ACΦ, that is, CΦ' = ACΦ.

Since C is non-singular, premultiplying by C^{-1}, we get Φ' = C^{-1} ACΦ which shows that Φ is a fundamental matrix of x' = C^{-1} ACx, but two different homogeneous systems cannot have the same fundamental matrix. Hence a contradiction.

Example 1.2 Find the fundamental matrix of the system

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} -t & t & 0 \\ 0 & -t & 0 \\ 0 & 0 & t \end{bmatrix}$$

Now the system x' = Ax is equivalent to the set of differential equations

$$x_1'(t) = -\text{tx}_1 + \text{tx}_2,\ x_2'(t) = -\text{tx}_2,\ x_3'(t) = \text{tx}_3$$

Solving these three equations, we obtain the solutions as

$$x_1 = \frac{t^2}{2} e^{-t^2/2},\ x_2 = e^{-t^2/2},\ x_3 = e^{t^2/2}$$

Hence the solution matrix is

$$\varphi(t) = \begin{bmatrix} \frac{t^2}{2} e^{-t^2/2} & 0 & 0 \\ 0 & e^{-t^2/2} & 0 \\ 0 & 0 & e^{t^2/2} \end{bmatrix}$$

Now let the column vectors of φ(t) be

$$\phi_1(t) = \begin{bmatrix} \frac{t^2}{2} e^{-t^2/2} \\ 0 \\ 0 \end{bmatrix}, \ \phi_2(t) = \begin{bmatrix} 0 \\ e^{-t^2/2} \\ 0 \end{bmatrix}, \ \phi_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{t^2/2} \end{bmatrix}$$

Since $W[\phi_1, \phi_2, \phi_3]$ (t) = $\frac{t^2}{2} e^{-t^2/2} \neq 0$, the vectors ϕ_1, ϕ_2 and ϕ_3 are linearly independent. Hence the solution matrix φ is a fundamental matrix.

Example 1.3 If $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 4 & -2 \end{bmatrix}$, find the determinant of the fundamental matrix φ satisfying φ(0)= E.

We know from Abel Liouville formula

$$\det \varphi(t) = \det \varphi(\tau) \exp \int_\tau^t trA \, \mathrm{ds} \qquad (1)$$

Now let us choose $\tau = 0$. Hence det φ(0) = det E = 1. For the given matrix A,

$$trA = 1 + 1 - 2 = 0.$$

Using the above values in (1), we get

$$\det \varphi(t) = 1 \exp \int_\tau^t 0 \, \mathrm{dt} = 1 \exp(0) = 1$$

Hence det φ(t) = 1.

Example 1.4 Check whether the matrix $\varphi(t) = \begin{bmatrix} e^t & 1 & 0 \\ 1 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a fundamental matrix for the system x' = Ax, t $\in$ I.

We know that φ (t) is a fundamental matrix if and only if det φ(t) ≠ 0.

For the given matrix det φ(t) = 1 $[e^t e^{-t} - 1] = 0$ by expanding φ(t) along the last row. Since det φ(t) = 0, φ(t) cannot be a fundamental matrix of given system.

Definition Adjoint system

If Φ is a fundamental matrix of LH system

$$x' = A(t)x \tag{1}$$

Then (as proved earlier)

$$\left[\Phi^{-1}\right]' = -\Phi^{-1}\Phi'\Phi^{-1}$$

$$\therefore \left[\Phi^{-1}\right]' = -\Phi^{-1}A\ \Phi\,\Phi^{-1}$$

[as Φ satisfies (1), $\therefore \Phi' = A\Phi$]

$$\Rightarrow \qquad \therefore \left[\Phi^{-1}\right]' = -\Phi^{-1}A$$

Taking conjugate transpose on both sides

$$\left(\left(\Phi^{-1}\right)'\right)^* = -\left(\Phi^{-1}A\right)^*$$

$$= \quad -A^*\left(\Phi^{-1}\right)^*$$

$$= \quad -A^*\Phi^{*-1} \qquad [(AB)^* = B^*A^*]$$

$$\Rightarrow \left(\Phi^{*-1}\right)' = \ -A^*\left(\Phi^{*-1}\right) \tag{2}$$

Also det $\left(\Phi^*\right)^{-1}$ is $\neq 0$ as $\det\Phi \neq 0$.

Therefore $\left(\Phi^{-1}\right)^* = \Phi^{*-1}$ is a fundamental matrix for the system

$$x' = -A^*(t)x \tag{3}$$

This system (3) is called the adjoint to LH system (1).

Theorem 1.18 If Φ is a fundamental matrix for LH system

$$x' = A(t)\,x \tag{1}$$

then ψ is a fundamental matrix for its adjoint $x' = -A^*(t)x$, iff $\psi^*\Phi = C$, where C is a constant non-singular matrix.

Proof. Given Φ is a fundamental matrix of (1)

$\therefore$ Φ^{*-1} is a fundamental matrix of adjoint system

$$x' = -A^*(t)x \quad (2)$$

Let ψ be a fundamental matrix of (2). Thus ψ and Φ^{*-1} are fundamental matrices of the same system (2). Therefore, by theorem 1.17

$$\psi = (\Phi^*)^{-1} D \quad (3)$$

for some constant non-singular matrix D.

Now $\quad \psi^* = D^*\left((\Phi^*)^{-1}\right)^*$

$$= D^*\left((\Phi^{-1})^*\right)^* = D^*\Phi^{-1}$$

$\Rightarrow \quad \psi^* \Phi = D^*\Phi^{-1}\Phi = D^*E = D^*$

and let D* = C

Then

$\psi^*\Phi = C$, where C is a constant non-singular matrix. [as D is non-singular $\Rightarrow$ D* is non-singular]

Conversely Suppose that

$$\psi^* \Phi = C \quad (4)$$

where C is a constant non-singular matrix.

Claim: If Φ is a fundamental matrix of (1) and satisfies condition (4), then ψ is a fundamental matrix of (3)

Now given Φ is a fundamental matrix, so det $\Phi \neq 0 \Rightarrow \Phi^{-1}$ exists. So from equation (4)

$\Rightarrow \quad \psi^* = C\,\Phi^{-1}$

Apply conjugate Transpose operation

$$(\psi^*)^* = (C\Phi^{-1})^*$$

$$\psi = (\Phi^{-1})^* C^* = (\Phi^*)^{-1} C^*$$

Since $\left(\Phi^*\right)^{-1}$ is a fundamental matrix of adjoint system (2). Then by Theorem 1.17 $\left(\Phi^*\right)^{-1} C^*$ is also a fundamental matrix of ajdoint system (2). Consequently ψ is also a fundamental matrix of adjoint system (2). Hence the proof.

Theorem 1.19

Hypothesis. Let f be a nontrivial solution of the nth-order homogeneous linear differential equation

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0. \tag{1}$$

Conclusion The transformation

$$x = f(t)v \tag{2}$$

reduces Equation (1) to an (n-1)th-order homogeneous linear differential equation in the dependent variable *w*, where *w* = *dv/dt*.

Proof. Let *x* = *f(t)v*. Then

$$\frac{dx}{dt} = f(t)\frac{dv}{dt} + f'(t)v,$$

$$\frac{d^2x}{dt^2} = f(t)\frac{d^2v}{dt^2} + 2f'(t)\frac{dv}{dt} + f''(t)v,$$

$$\vdots$$

$$\frac{d^n x}{dt^n} = f(t)\frac{d^n v}{dt^n} + nf'(t)\frac{d^{n-1}v}{dt^{n-1}} + \frac{n(n-1)}{2!}f''(t)\frac{d^{n-2}v}{dt^{n-2}} + \ldots + f^{(n)}(t)v.$$

Substituting these expressions into the differential equation (1), we have

$$a_0(t)\left[f(t)\frac{d^n v}{dt^n} + nf'(t)\frac{d^{n-1}v}{dt^{n-1}} + \ldots + f^{(n)}(t)v\right]$$

$$+ a_1(t)\left[f(t)\frac{d^{n-1}v}{dt^{n-1}} + (n-1)f'(t)\frac{d^{n-2}v}{dt^{n-2}} + \ldots + f^{(n-1)}(t)v\right]$$

$$+ \ldots + a_{n-1}(t)\left[f(t)\frac{dv}{dt} + f'(t)v\right] + a_n(t)f(t)v = 0$$

or

$$a_0(t)f(t)\frac{d^n v}{dt^n}+\left[na_0(t)f'(t)+a_1(t)f(t)\right]\frac{d^{n-1}v}{dt^{n-1}}+\Lambda$$

$$+\left[na_0(t)f^{(n-1)}(t)+\Lambda\ +a_{n-1}(t)f(t)\right]\frac{dv}{dt}$$

$$+\left[a_0(t)f^{(n)}(t)+a_1(t)f^{(n-1)}(t)+\Lambda\ +a_{n-1}(t)f'(t)+a_n(t)f(t)\right]v=0. \tag{3}$$

Now since f is a solution of Equation (1), the coefficient of v is zero. Then, letting $w=dv/dt$, Equation (3) reduces to the (n-1)th-order equation in w,

$$A_0(t)\frac{d^{n-1}w}{dt^{n-1}}+A_1(t)\frac{d^{n-2}w}{dt^{n-2}}+\Lambda\ +A_{n-1}(t)w=0, \tag{4}$$

where

$A_0(t)=a_0(t)f(t),$

$A_1(t)=na_0(t)f'(t)+a_1(t)f(t),\ldots\ldots,$

$A_{n-1}(t)=na_0(t)f^{(n-1)}(t)+\ldots\ldots+a_{n-1}(t)f(t).$

Now suppose that $w_1, w_2,\ldots w_{n-1}$ is a known fundamental set of equation (4). Then $v_1, v_2,\ldots v_{n-1}$ defined by

$$v_1(t)=\int w_1(t)dt, \qquad v_2(t)=\int w_2(t)dt,\ldots\ldots\ldots\ldots v_{n-1}(t)=\int w_{n-1}(t)dt$$

is a set of (n-1) solutions of equation (3). Also, the function v_n such that $v_n(t) = 1$ for all t is a solution of Equation (3). These n solutions $v_1,v_2\ldots, v_n$ of Equation (3) are linearly independent. Then, using (2) we obtain n solutions f_i, where $f_i(t) = f(t)v_i(t)$ (i =1,2..,n) of the original nth - order equation. The n solutions f_i, so defined are also linearly independent and thus constitute a fundamental set of equation (1).

One may extend Theorem 1.12 to show that if m (where m < n) linearly independent solutions of equation (1) are known, then equation (1) may be reduced to a homogeneous linear equation of order (n-m).

Reduction of the order of a homogeneous system

If m ($0 < m < n$) linearly independent solutions of (LH) are known, it is possible to reduce the order of (LH) by m, and hence a linear system of order n-m only need be solved.

Suppose $\varphi_1,\ldots, \varphi_m$ are m linearly independent vectors which are solutions of (LH) on an interval I. Let φ_i have components φ_{ij} ($i = 1, \ldots,n$). Then the rank of the n-by-m matrix with elements φ_{ij} ($i =1, \ldots,n; j = 1,\ldots,m$) at every $t \in I$ is m, because of the linear independence of its columns. This means that for each $t \in I$ there is an m-by-m determinant in this matrix which does not vanish there. Take any $t_0 \in I$ and assume for the moment that the determinant of the matrix Φ_m whose elements are φ_{ij} ($i = 1, \ldots,m$; $j = 1. . .,m$) is not zero at t_0. Then, by the continuity of det Φ_m in its elements φ_{ij} and the continuity of the functions φ_{ij} near t_0, one has that det $\Phi(t) \neq 0$ for t in some interval $\widetilde{I}$ containing t_0. Let $\widetilde{I}$ be any such interval; the reduction process will be outlined for $\widetilde{I}$. (The idea behind the process is a modification of the variation of constants).

Let the matrix U have the vectors $\varphi_1,..., \varphi_m$ for its first m columns and the vectors $e_{m+1}, \ldots, e_n$ for its lat n –m columns, where e_j is the column vector with all elements 0 except for the j-th which is 1. Clearly U is non-singular on $\widetilde{I}$. The substitution.

$$x = Uy \tag{1}$$

is made in (LH). [Note that $x = \varphi_j$ ($j = 1, \ldots, m$) in (1) corresponds to $y = e_j$ ($j = 1,\ldots, m$). Thus the substitution (1) may be expected to yield a system in y which will have e_j, $j =1, \ldots, m$, as solutions]. Using (1) in (LH) we get

$$U' y + U y' = AUy$$

Writing this out gives

$$\sum_{j=1}^{m}\varphi'_{ij}\, y_j + \sum_{j=1}^{m}\varphi_{ij} y'_j = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{ik}\varphi_{kj} y_j + \sum_{k=m+1}^{n} a_{ik} y_k \qquad (i = 1, \ldots, m)$$

$$\sum_{j=1}^{m}\varphi'_{ij}\, y_j + y'_i + \sum_{j=1}^{m}\varphi_{ij} y'_j = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{ik}\varphi_{kj} y_j + \sum_{k=m+1}^{n} a_{ik} y_k \qquad (i = m+1,\ldots,n)$$

Expressing the fact that the vectors φ_j with components φ_{ij} are solutions of (LH),

$$\varphi'_{ij} = \sum_{k=1}^{n} a_{ik}\varphi_{kj} \qquad (i = 1, \ldots, n, j = 1, \ldots, m)$$

we get

$$\sum_{j=1}^{m} \varphi_{ij} y'_j = \sum_{k=m+1}^{n} a_{ik} y_k \qquad (i = 1, \ldots, m)$$

$$y'_i + \sum_{j=1}^{m} \varphi_{ij} y'_j = \sum_{k=m+1}^{n} a_{ik} y_k \qquad (i = m + 1, \ldots, n) \tag{2}$$

Since det $\Phi_m \neq 0$ on $\widetilde{I}$, the first set of equations in (2) may be solved for $y'_i \ (j = 1,\ldots,m)$ in terms of φ_{ij}, a_{ik} and y_k (k = m + 1,…,n), and these values of $y'_i \ (j = 1,\ldots,n)$ may then be put into the second set of formulas of (2). This gives a set of first-order equations satisfied by the y_i (i = m + 1, …,n) of the type

$$y_i^{'} + \sum_{k=m+1}^{n} b_{ik} y_k \qquad (i = m + 1, \ldots, n) \tag{3}$$

that is, a linear system of order n – m.

Now suppose $\overline{\psi}_{m+1},\ldots\overline{\psi}_n[\overline{\psi}_j$ having components ψ_{ij} (i, j = m+1,....,n)] is a fundamental set on $\widetilde{I}$ for the system (3). Let $\overline{\Psi}_{n-m}$ denote the matrix with elements $\psi_{ij}(i, j = m+1,\ldots,n)]$. Clearly $\det \overline{\Psi}_{n-m}(t) \neq 0$ on $\widetilde{I}$. For each j = m + 1,…, n, let ψ_{ij} (i = 1, …,m) be solved for by quadratures (that is, by integration) from the relations

$$\sum_{j=1}^{m} \varphi_{ij}\psi'_{jp} = \sum_{k=m+1}^{n} a_{ik}\psi_{kp} \tag{4}$$

$$i = 1, \ldots, m \qquad p = m + 1, \ldots, n$$

Let ψ_p (p = m+1, . . ., n) denote the vectors having components ψ_{ip} (i = 1, . . ., n), and let

$$\psi_p = e_p \qquad (p = 1, \ldots, m)$$

Since ψ_p, p = 1,…, n, satisfy (3) and the first set of equations of (2), they must also satisfy the second set of equations of (2), and therefore ψ_p, p = 1,…, n, are solutions of (2). Thus, if now Ψ is the matrix with columns ψ_p, p = 1,…, n, and if

$$\Phi = U\Psi$$

then Φ is a matrix solution of (LH) on $\widetilde{I}$. U is nonsingular. Since $\det\Psi = \det\overline{\Psi}_{n-m}$ on $\widetilde{I}$, it follows that Φ is nonsingular on $\widetilde{I}$ and hence a fundamental solution of (LH) on $\widetilde{I}$.

The above procedure is summarized in the following theorem.

Theorem 1.20 Let $\varphi_1,\ldots, \varphi_m$ $(m < n)$ be m known linearly independent solutions of (LH) with φ_j $(j = 1, \ldots, m)$ having components φ_{ij} $(i = 1, \ldots,n)$. Assume the determinant of the matrix with elements φ_{ij} $(i, j = 1,\ldots,m)$ is not zero on some subinterval $\widetilde{I}$ of I. Then the construction of a set of n linearly independent solutions of (LH) on $\widetilde{I}$ can be reduced to the solution of a linear system (3) of order n – m, plus quadratures (4), using the substitution (1).

The restriction that the matrix Φ_m should be nonsingular on an interval will now be removed. It is clear that the n-by-m matrix with elements φ_{ij} $(i = 1, \ldots, n;\ j =1, \ldots, m)$, has rank m because of the independence of the solutions φ_j, $j = 1,\ldots m$. Thus, at any $t = t_0$, there is a non-singular m-by-m matrix obtained by taking m rows, $i_1, \ldots, i_m$, of the n-by-m matrix. By continuity, this matrix is nonsingular over some interval $\widetilde{I}$.

Summary

The students are made familiar with some preliminary definitions and fundamental results of linear homogeneous system. Relation between fundamental matrix and Wronskian of solution functions have been developed. Lastly procedure for reduction of order of a homogeneous linear system has been explained in detail.

Keywords Linear systems, fundamental matrix, Wornskian, variation of constant, reduction of order.

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Chapter-2

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS -II

Objectives

The main objectives of this chapter include finding out solution of non-homogeneous linear systems, linear systems with constant coefficients, systems with periodic coefficients along with the study of linear differential equation of order n.

Non-homogeneous Linear Equations: Variation of Parameters

Consider the second order differential equation

$$L(t)y = a_0(t)y'' + a_1(t)y' + a_2(t)y = b(t) \tag{1}$$

Important Fact

If you know the general solution of the associated homogeneous problem L(t) y = 0 then you can always reduce the construction of the general solution of (1) to the problem of finding two primitives (antiderivatives). The method for doing this is called variation of parameters.

We shall first illustrate the method of variation of parameters on second order equations in the normal form

$$L(t)y = y'' + p(t)y' + q(t)y = g(t). \tag{2}$$

You can put the general equation (1) into normal form by simply dividing by $a_0(t)$.

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the homogeneous problem L(t)y = 0 associated with (2). The general solution of the homogeneous problem is then given by

$$y = Y_H(t) = c_1Y_1(t) + c_2Y_2(t) \tag{3}$$

The idea of the method of variation of parameters is to seek solutions of (2) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t). \quad (4)$$

In other words you simply replace the arbitrary constants c_1 and c_2 in (3) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the title of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (2) is satisfied, and the other of which is chosen to simplify the resulting system.

Let us see how this is done. Differentiating (4) yields

$$y' = u_1(t)\, Y_1'(t) + u_2(t)\, Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t) \quad (5)$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0 \quad (6)$$

whereby (5) simplifies to

$$y' = u_1(t)\, Y_1'(t) + u_2(t)\, Y_2'(t). \quad (7)$$

Differentiating (7) then yields

$$y'' = u_1(t)\, Y_1''(t) + u_2(t)\, Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t). \quad (8)$$

Now substituting (4), (7), and (8) into (2) and using the fact that $Y_1(t)$ and $Y_2(t)$ are solutions of $L(t)y = 0$, we find that

$$\begin{aligned}
g(t) &= L(t)y \\
&= y'' + p(t)y' + q(t)y \\
&= u_1(t)\, Y_1''(t) + u_2(t)\, Y_2''(t) + u_1'(t)\, Y_1'(t) + u_2'(t)\, Y_2'(t) \\
&\quad + p(t)u_1(t)\, Y_1'(t) + p(t)u_2(t)\, Y_2'(t) \\
&\quad + q(t)u_1(t)Y_1(t) + q(t)u_2(t)Y_2(t) \\
&= u_1(t)\,[Y_1''(t) + p(t)\, Y_1'(t) + q(t)\, Y_1(t)] \qquad (9) \\
&\quad + u_2(t)\,[Y_2''(t) + p(t)\, Y_2'(t) + q(t)\, Y_2(t)] \\
&\quad + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) \\
&= u_1(t)[L(t)Y_1(t)] + u_2(t)[L(t)Y_2(t)]
\end{aligned}$$

$$+u_1'(t)Y_1'(t)+u_2'(t)Y_2'(t)$$

$$=u_1'(t)Y_1'(t)+u_2'(t)Y_2'(t).$$

Here we have used the fact that $L(t)Y_1(t) = 0$ and $L(t)Y_2(t)= 0$ to see that many terms in the expression for $L(t)y$ cancel. The resulting system that governs $u_1(t)$ and $u_2(t)$ is thereby given by (6) and (9):

$$u_1'(t)Y_1(t)+u_2'(t)Y_2(t)=0 \tag{10}$$

$$u_1'(t)Y_1'(t)+u_2'(t)Y_2'(t)=g(t).$$

This is a linear system of two algebraic equations for $u_1'(t)$ and $u_2'(t)$. Because

$$Y_1(t)Y_2'(t)-Y_2(t)Y_1'(t)=W(Y_1,Y_2)(t)\neq 0,$$

One can always solve this system to find

$$u_1'(t)=\frac{Y_2(t)g(t)}{W(Y_1,Y_2)(t)},\quad u_2'(t)=\frac{Y_1(t)g(t)}{W(Y_1,Y_2)(t)}.$$

Letting $U_1(t)$ and $U_2(t)$ be any primitives of the respective right-hand sides above, one see that

$$u_1(t) = c_1 + U_1(t),\ u_2(t) = c_2 + U_2(t),$$

whereby (4) yields the general solution

$$y = c_1Y_1(t) + U_1(t)Y_1(t) + c_2Y_2(t) + U_2(t)Y_2(t)$$

Notice that this decomposes as $y = Y_H(t) + Y_p(t)$ where

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t), \quad Y_P(t) = U_1(t)Y_1(t) + U_2(t)Y_2(t).$$

The best way to apply this method in practice is not to memorize one of the various formulas for the final solution given in the book, but rather to construct the linear system (10), which can then be rather easily solved for $u_t'(t)$ and $u_2'(t)$. Given $Y_1(t)$ and $Y_2(t)$ a fundamental set of solutions to the associated homogeneous problem, you proceed as follows.

1) Write the equation in the normal form

$$y'' + p(t)y' + q(t)y = g(t)$$

2) Write the form of the solution you seek:

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)$$

3) Write the algebraic linear system for $u_t^{'}(t)$ and $u_2^{'}(t)$

$$u_t^{'}(t)Y_1(t) + u_2^{'}(t)Y_2(t) = 0,$$

$$u_t^{'}(t)Y_1^{'}(t) + u_2^{'}(t)Y_2^{'}(t) = g(t)$$

The form of the left hand sides of this system mimics the form of the solution we seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_t^{'}(t)$ and $u_2^{'}(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1^{'}(t)$ *and* $Y_2^{'}(t)$.

4) Solve the algebraic system for $u_t^{'}(t)$ and $u_2^{'}(t)$. This is always very easy to do, especially if you start with the first equation.

5) Integrate to find $u_1(t)$ and $u_2(t)$. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. Remember to include the constants of integration, c_1 and c_2.

6) Substitute the result into the form of the solution you wrote down in step 2. If the problem is an initial-value problem, you must determine c_1 and c_2 from the initial conditions.

Example 2.1 Find the general solution of

$$y'' + y = sec(t)$$

Before presenting the solution, notice that while this equation has constant coefficients the driving is not of the form that would allow you to use the method of undetermined coefficients. You should be able to recognize this right away and thereby see that the only method you can use to solve this problem is variation of parameters.

The equation is in normal form. Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1 \cos(t) + c_2 \sin(t)$$

Hence, we will seek a solution of the form

$$y = u_1(t)\cos(t) + u_2(t)\sin(t),$$

where
$$u_t'(t)\cos(t) + u_2'(t)\sin(t) = 0,$$
$$-u_t'(t)\sin(t) + u_2'(t)\cos(t) = \sec(t).$$

Solving this system by any means you choose, yields

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}, \qquad u_2'(t) = 1$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \ln(|\cos(t)|), \quad u_2(t) = c_2 + t.$$

Therefore the general solution is

$$y = c_1\cos(t) + c_2\sin(t) + \ln(|\cos(t)|)\cos(t) + t\sin(t).$$

Non-Homogeneous Linear Systems

Now we shall consider non-homogeneous linear systems of the type

$$x' = A(t)x + B(t),\ t \in I \tag{1}$$

where A is an n x n matrix of continuous functions on I and B is the continuous vector on I which is not identically zero. As in the case of non-homogeneous linear equations, we shall first explain how the solution of (1) is closely related to the solution of the corresponding homogeneous system

$$x' = A(t)x,\ t \in I \tag{2}$$

More precisely, we shall show that any solution of (1) is the sum of a particular solution of (1) and the solution of (2) given by the solution vectors. Since we can find the solutions of the homogeneous system, the problem is to find a particular solution of (1). Once we know the fundamental matrix of the homogeneous system, we can find a solution of the non-homogeneous system by the method of variation of parameters. First we shall formulate the theorem giving the general solution of (1).

Theorem 2.1 Let ϕ_0 be any solution of the non-homogeneous system (1) and ϕ_1, ϕ_2, ... ,ϕ_n be a basis of solutions of the corresponding homogeneous system (2) and let

$c_1, c_2, \ldots\ldots.., c_n$ be constants. Then

(i) the vector function

$$\phi_0 + \sum_{k=1}^{n} c_k \phi_k \tag{3}$$

is also a solution of the non-homogeneous system for every choice of $c_1, c_2, \ldots\ldots c_n$.

(ii) an arbitrary solution of the non-homogeneous system (1) is of the form (3) for a suitable choice of $c_1, c_2, \ldots\ldots c_n$.

Proof (i) Let $\psi(t) = \phi_0 + \sum_{k=1}^{n} c_k \phi_k$.

We shall show that $\psi(t)$ is a solution of (1). Since ϕ_0 and $\sum_{k=1}^{n} c_k \phi_k$ are solutions of (1) and (2) respectively, we have

$$\phi_0' = A(t)\phi_0 + B(t) \tag{4}$$

$$\left(\sum c_k \phi_k\right)' = A(t)\sum_{k=1}^{n} c_k \phi_k \tag{5}$$

Now
$$[\psi(t)]' = \phi_0' + \sum_{k=1}^{n} c_k \phi_k' \tag{6}$$

Using (4) and (5) in (6), we get

$$[\psi(t)]' = A(t)\phi_0 + B(t) + A(t)\sum_{k=1}^{n} c_k \phi_k$$

$$= A(t)\left[\phi_0 + \sum_{k=1}^{n} c_k \phi_k\right] + B(t)$$

Therefore $[\psi(t)]' = A(t)\ \psi(t) + B(t)$ which shows that $\psi(t)$ is a solution of (1) for every choice of $c_1, c_2, \ldots\ldots c_n$.

(ii) Let ϕ be any arbitrary solution of (1). Then

$$\phi'(t) = A(t)\phi(t) + B(t) \tag{7}$$

We shall show that $\phi - \phi_0$ is a solution of the corresponding homogeneous equation

(2).

Now $$\frac{d}{dt}[\phi-\phi_0]=\frac{d\phi}{dt}-\frac{d\phi_0}{dt} \qquad (8)$$

Using (4) and (7) in (8), we get

$$\frac{d}{dt}[\phi - \phi_0] = A(t)\phi(t) + B(t) - [A(t)\,\phi_0 + B(t)]$$

$$= A(t)[\phi(t) - \phi_0]$$

which proves that $\phi - \phi_0$ satisfies the corresponding homogeneous equation (2). Since $\phi_1, \phi_2, \ldots \phi_n$ are the basis of solution vectors, there exist constants $c_1, c_2, \ldots\ldots c_n$ such that

$$\phi-\phi_0=\sum_{k=1}^{n}c_k\phi_k \qquad or \qquad \phi=\phi_0+\sum_{k=1}^{n}c_k\phi_k$$

for a suitable choice of $c_1, c_2, \ldots\ldots c_n$ which completes the proof of the theorem.

The following theorem helps us to find a particular solution of the non-homogeneous system by the method of variation of parameters, once we know the fundamental matrix of the given corresponding homogeneous system.

Theorem 2.2 If $\varphi(t)$ is the fundamental matrix of the homogeneous system

$$x'(t) = A\,(t)\,x(t),\ t \in 1 \qquad (1)$$

then ψ defined by

$$\psi(t)=\varphi(t)\int_{t_0}^{t}\varphi^{-1}(s)B(s)ds,\ t\in I \qquad (2)$$

is a solution of the initial value problem of the non-homogeneous system

$$x'(t) = A(t)x(t) + B(t),\ x(t_0) = 0 \qquad (3)$$

Proof The method of proof is to assume a differentiable vector function u(t) so that

$$\psi(t)=\varphi(t)u(t), t\in I,\ \psi(t_0)=0 \qquad (4)$$

is a solution of the non-homogeneous equation (3).

Since $\psi(t)$ is a solution of (3), we have

$$\psi'(t) = A(t)\psi(t) + B(t)$$

Substituting for $\psi(t)$ from (4), we get

$$\psi'(t) = A(t)\varphi(t)u(t) + B(t) \tag{5}$$

Since $\varphi(t)$ is a fundamental matrix of (1),

$$\varphi'(t) = A(t)\varphi(t) \tag{6}$$

Now differentiating (4), we get for any $t \in I$,

$$\psi'(t) = \varphi'(t)u(t) + \varphi(t)u'(t) \tag{7}$$

Using (6) in (7), we get

$$\psi'(t) = A(t)\varphi(t)u(t) + \varphi(t)u'(t) \tag{8}$$

Equating the expressions for $\psi(t)$ from (5) and (8)

$$A(t)\varphi(t)u(t) + \varphi(t)u'(t) = A(t)\varphi(t)u(t) + B(t)$$

which gives

$$\varphi(t)u'(t) = B(t) \tag{9}$$

Since $\varphi(t)$ is a fundamental matrix, $\det \varphi(t) \neq 0$, so that it is non-singular. Hence we get on premultiplying (9) by $\varphi^{-1}(t)$,

$$u'(t) = \varphi^{-1}(t)B(t) \tag{10}$$

Integrating equation (10), we obtain

$$u(t) = \int_{t_0}^{t} \varphi^{-1}(s)B(s)ds, \qquad t_0, t \in I \tag{11}$$

Substituting the above value of u(t) in (4), we get

$$\psi(t) = \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)B(s)ds, \ t \in I, \tag{12}$$

We shall show that (12) is indeed a solution of (3). We do this by direct verification

$$\psi'(t) = \varphi'(t)\int_{t_0}^{t} \varphi^{-1}(s)B(s)ds + \varphi(t)\varphi^{-1}(t)B(t)$$

$$= A\,\varphi(t)u(t) + B(t) = A\psi(t) + B(t), \qquad \text{(using (11))}$$

which proves that $\psi'(t) = A\psi(t) + B$, showing that (12) is a solution of (3).

Note If x_h is the vector solution of the corresponding homogeneous equation $x' = A(t)x$, $x(t_0) = 0$, then Theorem 2.1 gives the general solution of (3) as

$$\psi(t) = x_h(t) + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)B(s)ds, \quad t \in I \tag{13}$$

These formulas (12) and (13) are called methods of variation of parameters for non-homogeneous linear systems.

The following example illustrates the above theorem.

Example 2.2 Obtain the solution $\psi(t)$ of the initial value problem

$$x' = Ax + B(t),\ \psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

where $$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ B(t) = \begin{bmatrix} \sin at \\ \cos bt \end{bmatrix} \tag{2}$$

To solve the above non-homogeneous matrix equation, first we need the fundamental matrix of $x'(t) = Ax(t)$ with the given data.

The given equation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is equivalent to

$$x_1' = x_1 \text{ and } x_2' = 2x_2 \tag{3}$$

Solving equation (3), we get

$$x_1(t) = e^t \text{ and } x_2(t) = e^{2t}$$

Hence the solution vectors are $\phi_1(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\phi_2(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$.

Since $W[\phi_1, \phi_2](t) \neq 0$ for any t, ϕ_1 and ϕ_2 are linearly independent so that $\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$ is a solution matrix φ. Since det $\varphi(t) \neq 0$, the above solution matrix is a fundamental matrix by theorem 1.9.

To determine ψ(t), we need the inverse of φ(t). If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the inverse of φ(t), then we have

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} ae^t & be^t \\ ce^{2t} & de^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding matrix entries on both sides, we get $ae^t = 1$, $be^t = 0$, $ce^{2t} = 0$, $de^{2t} = 1$ so that we get $a = e^{-t}$, $b = 0$, $c = 0$ and $d = e^{-2t}$. Hence the inverse matrix is $\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$.

If ψ(t) is the solution of (2), from Theorem 2.2, we get

$$\psi(t) = \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)B(s)ds, \quad t, t_0 \in I \tag{4}$$

Substituting φ(t), $\varphi^{-1}(t)$, B(t) in (4), we get

$$\psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} \sin at \\ \cos bt \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{-t}\sin at \\ e^{-2t}\cos bt \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \dfrac{e^{-t}}{1+a^2}(-\sin at - a\cos at) + c_1 \\ \dfrac{e^{-2t}}{4+b^2}(-2\cos bt + b\sin bt) + c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+a^2}(-\sin at - a\cos at) + c_1 e^t \\ \frac{1}{4+b^2}(-2\cos bt + b\sin bt) + c_2 e^{2t} \end{bmatrix}$$

Using the initial conditions at t = 0,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-a}{1+a^2} + c_1 \\ \frac{-2}{4+b^2} + c_2 \end{bmatrix}$$

Equating the corresponding elements, we get

$$c_1 = \frac{a}{1+a^2}, c_2 = 1 + \frac{2}{4+b^2} = \frac{6+b^2}{1+a^2}$$

Hence the solution matrix $\psi(t)$ is

$$\psi(t) = \begin{bmatrix} \frac{1}{1+a^2}(-\sin at - a\cos at) + \frac{a}{1+a^2}e^t \\ \frac{1}{4+b^2}(-2\cos bt + b\sin bt) + \frac{(6+b^2)}{(4+b^2)}e^{2t} \end{bmatrix}$$

which gives the solution $\psi(t)$ as

$$\begin{bmatrix} (1+a^2)^{-1}(ae^t - (\sin at + a\cos at] \\ (4+b^2)^{-1}[(6+b^2)e^{2t} + (b\sin bt - 2\cos bt)] \end{bmatrix}$$

Theorem 2.3 Prove that the solution

$$\phi(t) = \Phi(t)\int_\tau^t \Phi^{-1}(s)b(s)ds, \qquad t \in I \tag{1}$$

of NH system x' = A(t)x + b can be written as

$$\phi(t) = \Psi^{*^{-1}}(t)\int_\tau^t \Psi^*(s)b(s)ds \tag{2}$$

where Ψ is a fundamental matrix of the adjoint system

$$x' = -A^*(t)x \tag{3}$$

Proof We know that if Φ is a fundamental matrix of x' = A(t)x; Then Φ^{*-1} is a fundamental matrix of x' = -A^*(t)x (Adjoint system).

$\therefore$ Ψ and Φ^{*-1} are fundamental matrices of same system (3). Therefore, by theorem 1.10

$$\Psi = \left(\Phi^*\right)^{-1} C, \quad \text{for some constant non-singular matrix C.}$$

$$\Rightarrow \qquad \Psi^* = C^* \Phi^{-1}$$

$$\Rightarrow \qquad \Phi^{-1} = \left(C^*\right)^{-1} \Psi^* \tag{4}$$

$$\Rightarrow \qquad \left(\Phi^{-1}\right)^{-1} = \left(\Psi^*\right)^{-1} C^*$$

$$\Rightarrow \qquad \Phi = \left(\Psi^*\right)^{-1} C^* \tag{5}$$

Put these values of Φ and Φ^{-1} from (4) and (5) in (1), we get

$$\phi(t) = \Psi^{*-1}(t) C^* \int_\tau^t \left(C^*\right)^{-1} \Psi^*(s) b(s) ds$$

$$\phi(t) = \Psi^{*-1}(t) \int_\tau^t \Psi^*(s) b(s) ds \qquad \text{as } C^* C^{*-1} = \text{Identity}$$

Hence the proof.

Theorem 2.4 ϕ(t) can be written as

$$\phi(t) = \Phi(t) \int_\tau^t \Psi^*(s) b(s) ds$$

provided $\Psi^*\Phi = E$.

Proof Proof follows from Theorem 2.3. The result of theorem 2.3 is

$$\phi(t) = \Psi^{*-1}(t) \int_\tau^t \Psi^*(s) b(s) ds \tag{1}$$

Now we find out $\Psi^{*-1}(t)$.

From given relation

$$\Psi^{*}\Phi = E$$

$$\Rightarrow \qquad \Psi^{*} = E\,\Phi^{-1}$$

$$\Rightarrow \qquad \Psi^{*^{-1}}(t) = \Phi\, E^{-1} = \Phi\, E = \Phi$$

Use this in equation (1)

$$\phi(t) = \Phi(t)\int_{\tau}^{t} \Psi^{*}(s)b(s)ds\,.$$

Linear Systems with Constant Coefficients

In this section we shall obtain the solution of the linear homogeneous system

$$x' = Ax,\ t \in I \tag{1}$$

where A is a constant matrix and I is an interval of R.

Theorem 2.5 (i) The general solution of (1) is $x(t) = e^{tA}c$ where c is an arbitrary constant column matrix.

(ii) If (1) satisfies the initial condition $x(t_0) = x_0$, $t_0 \in I$, its solution is $x(t) = e^{(t-t_0)A}x_0$.

(iii) The fundamental matrix $\varphi(t)$ of the system is $\varphi(t) = e^{tA}$.

Proof (i) Let x(t) be a solution of (1). Then we have

$$x'(t) = Ax(t) \quad \text{or} \quad x'(t) - Ax(t) = 0,\ t \in I \tag{2}$$

Let us define a vector $u(t) = e^{-tA}\,x(t)$, $t \in I$. Then differentiating

$$u'(t) = e^{-tA}\,(-A)x(t) + e^{-tA}\,x'(t) \tag{3}$$

so that

$$u'(t) = e^{-tA}\,[\,-Ax(t) + x'(t)]$$

Using (2) in (3), we get $u'(t) = e^{-tA}[0] = 0$. Hence $u(t) = c$ is a constant vector for $t \in I$.

Since $u(t) = c$, we get $c = e^{-tA}\,x(t)$.

Premultiplying both sides by e^{tA}, we get

$$x(t) = e^{tA}\,c \tag{4}$$

(ii) When t = t_0, $x(t_0) = e^{t_0 A}c$ so that we get c = $e^{-t_0 A}x(t_0)$. Using this value of c in (4)

$$x(t) = e^{tA} . e^{-t_0 A} x(t_0) = e^{(t-t_0)A} x_0$$

(iii) If $\varphi(t)$ is a fundamental matrix, it satisfies differential equation (1) and det $\varphi(t) \neq 0$. We shall show that e^{tA} has these twin properties so that e^{tA} is a fundamental matrix of the system.

Let us take $\varphi(t) = e^{tA}$. Then we have

$$\varphi'(t) = \lim_{\Delta t \to 0} \frac{e^{(t+\Delta t)A} - e^{tA}}{\Delta t} = \lim_{\Delta t \to 0} Ae^{tA}\left(\frac{e^{\Delta t A} - 1}{A\,\Delta t}\right)$$

Since $$\lim_{\Delta t \to 0}\left(\frac{e^{\Delta t A} - 1}{A\,\Delta t}\right) = 1, \varphi'(t) = Ae^{tA} = A\varphi(t)$$

Thus e^{tA} satisfies the given differential equation. Further we note that $\varphi(0)$ = E and from Theorem 1.15

$$\det \varphi(t) = \det \varphi(0) \exp [(\mathrm{tr}A)\,(t - t_0)] \neq 0$$

Hence det $\varphi(t) \neq 0$ so that $\varphi(t)$ is a fundamental matrix of the system.

Corollary The solution of the system x' = Ax, x(0) = E, t $\in$I is x(t) = e^{tA}.

Proof From (ii), the solution is x(t) = e^{tA} x(0) = e^{tA} E = e^{tA}

Note 1 The fundamental matrix of the system (1) can be determined by considering the linearly independent sets in R^n. We know that $e_1, e_2, \ldots e_n$ is a basis in R^n. Let ϕ_1, ϕ_2, … ϕ_n be the solutions corresponding to the initial value problem $x(t_0) = e_1$, $x(t_0) = e_2, \ldots, x(t_0) = e_n$. Then if $t_0 = 0$, by (ii) of Theorem 2.1 $\phi_1(t) = e^{tA}e_1$, $\phi_2(t) = e^{tA}e_2 \ldots \phi_n(t)\, e^{tA}e_n$.

Hence $$\varphi(t) = e^{tA}[e_1, e_2, \ldots e_n] = e^{tA}\begin{bmatrix} 1 & 0 & 0 & \Lambda & 0 \\ 0 & 1 & 0 & \Lambda & 0 \\ \Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\ 0 & 0 & \Lambda & & 1 \end{bmatrix} = e^{tA}E$$

Thus $\varphi(t) = e^{tA}E$ is a fundamental matrix of the system.

Note 2 We cannot solve the matrix differential equation x'(t) = A(t) x(t) as the first order linear differential equation by finding an integrating factor.

That is x(t) = exp $\left(\int_{t_0}^{t} A(s)ds\right)$ can not be a solution of the equation for

$$x'(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right)A(t) = x(t)A(t)$$

Hence x(t) = exp$\left(\int_{t_0}^{t} A(s)ds\right)$ is a solution if and only if exp$\left(\int_{t_0}^{t} A(s)ds\right)$and A(t) commute. They commute if A is a constant matrix or a diagonal matrix. Hence the above method of solutions is valid only for a constant or a diagonal matrix.

Example 2.3 Find the fundamental matrix of x' = Ax, where

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

The fundamental matrix is e^{tA}. So we shall find e^{tA} for the given matrix A. For this we shall find A^2, A^3, A^n, … so that

$$e^{tA} = 1 + \frac{tA}{1!} + \frac{t^2}{2!}A^2 + ... + \frac{t^n}{n!}A^n + ...$$

Now $$A^2 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1^3 & 0 \\ 0 & a_2^3 \end{bmatrix}$$

Proceeding similarly, we get $A^n = \begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix}$

Hence $$e^{tA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\frac{t}{1!} + \frac{t^2}{2}\begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix}$$

$$+ \frac{t^3}{3!}\begin{bmatrix} a_1^3 & 0 \\ 0 & a_2^3 \end{bmatrix} + ... + \frac{t^n}{n!}\begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix} + ...$$

$$= \begin{bmatrix} 1+\frac{ta_1}{1!}+\frac{t^2a_1^2}{2!}+\ldots+\frac{t^na_1^n}{n!}+\ldots & 0 \\ 0 & 1+\frac{ta_2}{1!}+\frac{t^2a_2^2}{2!}+\ldots+\frac{t^na_2^n}{n!}+\ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{ta_1} & 0 \\ 0 & e^{ta_2} \end{bmatrix}$$

Example 2.4 Find the solution and the fundamental matrix of

$$x' = Ax \text{ where } A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation of A is det $(A - \lambda E) = 0$.

Now $\quad\quad \det(A - \lambda E) = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$

which gives $\quad\quad (5 -\lambda)\ (2 -\lambda) - 4 = 0$

Hence the characteristic equation is

$$\lambda^2 - 7\lambda + 6 = 0$$

that is $\quad\quad (\lambda^2 - 7\lambda + 6) = (\lambda- 6)\ (\lambda - 1) = 0$

Thus the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 1$.

Now we shall find the eigenvectors corresponding to the distinct eigenvalues.

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the eigenvector corresponding to the eigvenvalue $\lambda_1 = 6$, then

$$(A-6E)\ x = 0.$$

Hence $\quad\quad (A - 6E)x = \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

which is $\quad\quad \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus we have $\quad\quad \begin{bmatrix} -x_1 & +4x_2 \\ x_1 & -4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which gives $-x_1 + 4x_2 = 0$ and $x_1 - 4x_2 = 0$

we can choose $x_1 = 4$ and $x_2 = 1$ so that the eigenvector corresponding to $\lambda_1=6$ is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Let us find the eigenvector corresponding to $\lambda_2 = 1$.

Now $$(A - E)\, x = \begin{bmatrix} -1+5 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 & 4x_2 \\ x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $x_1 + x_2 = 0$. So we can choose $x_1 = 1$ and $x_2 = -1$

Thus the eigenvector corresponding to $\lambda_2 = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

These eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent. The solution vectors are $e^{6t}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Using these, the fundamental matrix $\varphi(t)$ is given by

$$\varphi(t) = \begin{bmatrix} 4e^{6t} & e^{t} \\ e^{6t} & -e^{t} \end{bmatrix}$$

The solution of the matrix equation is

$$x(t) = \alpha_1 e^{6t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \alpha_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Linear Systems with Periodic Coefficients

If $x(t)$ is the solution of the linear system

$$x' = Ax \tag{1}$$

Then the solution $x(t)$ is said to be periodic with period ω if $x(t + \omega) = x(t)$. The question arises, under what conditions (1) admits a periodic solution. If it admits a periodic solution, it is of interest to note the nature of A. Besides answering this question about the periodic solution and the nature of A, we will also investigate the case of the solutions when the matrix A is of minimal period, that is, $A(t + \omega) = A(t)$, $\omega \neq 0$ and $- \infty < t < \infty$.

Theorem 2.6 The system (1) admits a non zero periodic solution of period ω if and only if $E- e^{A\omega}$ is singular where E is the identity matrix.

Proof According to Theorem 2.5, the solution x(t) of (1) is given by

$$x(t) = e^{At}c \tag{1}$$

where c is an arbitrary non-zero vector. x(t) is periodic with period ω if and only if

$$x(t) = x(t + \omega) = e^{A(t+\omega)}c \tag{2}$$

Using (1) in (2), we get $\quad e^{At}c - e^{A(t+\omega)}c = 0$

that is $\quad e^{At}[E - e^{A\omega}]\, c = 0$

Since $e^{At} \neq 0$, (1) has a solution if and only if $[E - e^{A\omega}]\, c = 0$. Since c is non-zero constant vector, the system (1) has a non-zero periodic solution of period ω if and only if $\det [E- e^{A\omega}] = 0$. This implies $E - e^{A\omega}$ is singular.

The next theorem characterizes the non-zero periodic solution of the non-homogeneous equation

$$x'(t) = A(t)x + f(t) \tag{1}$$

where f is a continuous vector valued function on $(-\infty, \infty)$.

Theorem 2.7 Let f(t) be a periodic function with period ω. Then the solution of (1) is periodic of period ω if and only if $x(0) = x(\omega)$.

Proof To prove the necessity of the condition, let x(t) be a periodic solution of (1) with non-zero period ω. Then $x(t + \omega) = x(t)$. Taking $t = 0$, $x(\omega) = x(0)$ which proves the necessity of the condition.

To prove the sufficiency of the condition, let x(t) be a solution of (1) satisfying the condition $x(0) = x(\omega)$. On this assumption, we shall prove that the solution is periodic.

Now $\quad x'(t + \omega) = Ax(t + \omega) + f(t + \omega) \quad (2)$

We make the substitution

$$u(t) = x\,(t + \omega) \tag{3}$$

so that we have
$$u'(t) = x'\,(t + \omega) \tag{4}$$

Using (4) and (3) in (2), we get

$$u'(t) = Au(t) + f(t), \text{ since } f(t + \omega) = f(t)$$

Hence $u(t)$ is a solution of (1) and also $u(0) = x(\omega) = x(0)$ by hypothesis. Since the solution of (1) is unique it cannot have two different solutions $u(t)$ and $x(t)$. Thus we get $x(t) = u(t) = x(t + \omega)$ which gives $x(t) = x(t + \omega)$, showing that the solution $x(t)$ is periodic with period ω.

In our previous study, we obtained the solution of a non-homogeneous equation with the help of the corresponding homogeneous equation. In connection with the solution of the periodic equation, we have the following theorem.

Theorem 2.8 Let $f(t)$ be a continuous periodic function of period ω on $(-\infty, \infty)$. A necessary and sufficient condition for the system.

$$x' = Ax + f(t) \tag{1}$$

to have a unique periodic solution of period ω is that the corresponding homogeneous system.

$$x' = Ax \tag{2}$$

has no non-zero periodic solution of period ω.

Proof We know by Theorem 2.2, a general solution of (1) is

$$x(t) = x_h + \varphi(t) \int_0^t \varphi^{-1}(s) f(s) ds \tag{3}$$

where φ is the fundamental matrix of the system.

Further

$$x_h(t) = e^{At}c,\ \varphi(t) = e^{At},\ \varphi^{-1}(s) = e^{-As} \tag{4}$$

Substituting (4) in (3), we get

$$x(t) = e^{At}c + \int_0^t e^{At}e^{-As} f(s)ds = e^{At}c + \int_0^t e^{A(t-s)} f(s)ds \qquad (5)$$

where c is a constant vector.

Further from (5), we get

$$x(0) = c \qquad (6)$$

According to the previous theorem, x(t) is non-zero periodic solution of (1) if and only if

$$x(0) = x(\omega) \qquad (7)$$

From (6) and (7), we get

$$x(0) = x(\omega) = c \qquad (8)$$

Now $$x(\omega) = e^{A\omega}c + \int_0^{\omega} e^{A(\omega-s)} f(s)ds$$

Using (8) in the above, we have

$$c = e^{A\omega}c + \int_0^{\omega} e^{A(\omega-s)} f(s)ds$$

Hence $$(E - e^{A\omega})c = \int_0^{\omega} e^{A(\omega-s)} f(s)ds \qquad (9)$$

Hence there is a unique, periodic solution for (1) if and only if (9) has a unique solution for c for any periodic function f. It has unique solution for c if and only if $(E-e^{A\omega})$ is non-singular which implies and is implied by Theorem 2.6, the system (2) has no non-zero periodic solution of period ω. This completes the proof of the theorem.

After the study of the system x' = Ax with x(t) as a periodic function of period ω, we shall discuss the solution of the same system when it is of minimal period in the sense that $A(t + \omega) = A(t)$, $\omega \neq 0$, $t \in (-\infty, \infty)$. We shall consider in this section a system x' = A(t) x, where A is of minimal period. The next question is, if φ(t) is a fundamental matrix what is about $\varphi(t + \omega)$? The following theorem states the $\varphi(t + \omega)$ is also a fundamental matrix.

Representation Theorem

Theorem 2.9 Let $\varphi(t)$ denotes the fundamental matrix of the system

$$x' = Ax \tag{1}$$

where A is of minimal period ω. Then $\varphi(t + \omega)$ for $t \in (-\infty, \infty)$ is also a fundamental matrix of (1), and corresponding to each such φ, there exists (i) a periodic non-singular matrix P such that $P(t + w) = P(t)$ and (ii) a constant matrix B such that $\varphi(t) = P(t)e^{tB}$.

Proof Since $\varphi(t)$ is a fundamental matrix of (1), it is a solution matrix of (1) so that $\varphi'(t) = A(t)\, \varphi(t)$.

Now $$\varphi'(t + \omega) = A(t + \omega)\, \varphi\, (t + \omega) \tag{2}$$

Since A is of minimal period

$$A(t + \omega) = A(t) \tag{3}$$

Using (3) in (2), we get $\varphi'\, (t + \omega) = A(t)\, \varphi\, (t + \omega)$

Further $\det \varphi(t + \omega) \neq 0$, for if $\det \varphi(t + \omega) = 0$ implies $\det \varphi(t) = 0$ for $\omega = 0$ contradicting that φ is a fundamental matrix. Hence $\varphi\, (t + \omega)$ is a fundamental matrix.

Since $\varphi(t)$ and $\varphi(t + \omega)$ are solution matrices of (1), there exists a non-singular matrix C such that $\varphi(t + \omega) = \varphi(t)\, C$ by theorem 1.10. Since C is a constant non-singular matrix, there exist a matrix B s.t.

$$C = e^{\omega B} \text{ (by theorem 1.1)}$$

Hence we can take $$\varphi(t + \omega) = \varphi(t)\, e^{\omega b} \tag{4}$$

Let us define $$P(t) = \varphi(t)e^{-tB}$$

We shall now show that P is a periodic function with period ω and non-singular.

Now $$P(t + \omega) = \varphi(\, t + \omega)e^{-(t + \omega)B} \tag{5}$$

Using (4) in (5) we get $P(t + \omega) = \varphi(t)e^{\omega B}.\ e^{-(t+\omega)B} = P(t)$. Hence $P(t + \omega) = P(t)$.

Thus P(t) is periodic with period ω.

φ(t) is a fundamental matrix, so det φ(t) ≠ 0.

Hence $\qquad \det P(t) = \det \varphi(t)e^{-\omega B} = \det\varphi(t).\ \det e^{-\omega B} \neq 0$

so that P(t) is a non-singular matrix.

From the definition $\varphi(t) = P(t)e^{\omega B}$, where P is periodic and non-singular.

Basic Theory of the nth-order Homogeneous Linear Differential Equation

A. Fundamental Results

In this section, we shall be concerned with the single nth order homogeneous linear scalar differential equation

$$a_0(t)\ \frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0, \tag{1}$$

where $a_0, a_1,\ldots,a_{n-1}, a_n$ are continuous on an interval $a \leq t \leq b$ and $a_0(t) \neq 0$ on $a \leq t \leq b$. Let L_n denote the formal nth – order linear differential operator defined by

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{d}{dt} + a_n(t) \tag{2}$$

Then differential equation (1) may be written

$$L_n x = 0 \tag{3}$$

If we divide through by $a_0(t)$ on $a \leq t \leq b$, we obtain the equation

$$\frac{d^n x}{dt^n} + \frac{a_1(t)d^{n-1}x}{a_0(t)dt^{n-1}} + \ldots + \frac{a_{n-1}(t)dx}{a_0(t)dt} + \frac{a_n(t)x}{a_0(t)} = 0 \tag{4}$$

This equation, in which the coefficient of $\frac{d^n x}{dt^n}$ is 1, is said to be normalized.

Theorem 2.10

1. Consider the differential equation

$$a_0(t)\ \frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0,$$

where a_0, $a_1 \ldots$, a_{n-1}, and a_n are continuous on the interval $a \le t \le b$, and $a_0(t) \ne 0$ on $a \le t \le b$.

2. Let t_0 be a point of the interval $a \le t \le b$, and let c_0, c_1, … c_{n-1} be a set of n real constants.

Conclusion The exists a unique solution ϕ of (1) such that

$\phi(t_0) = c_0$, $\phi'(t_0) = c_1, \ldots$, $\phi^{(n-1)}(t_0) = c_{n-1}$, and this solution is defined over the entire interval $a \le t \le b$.

Corollary

Hypothesis The function ϕ is a solution of the homogeneous equation (1) such that

$$\phi(t_0) = 0,\ \phi'(t_0) = 0 \ldots,\ \phi^{(n-1)}(t_0) = 0 \tag{6}$$

where t_0 is a point of the interval $a \le t \le b$ on which the coefficients a_0, $a_1, \ldots$, a_n are all continuous and $a_0(t) \ne 0$.

Conclusion $\phi(t) = 0$ for all t such that $a \le t \le b$.

Proof First note that ϕ such that $\phi(t) = 0$ for all $t \in [a, b]$ is indeed a solution of the differential equation (1) which satisfies the initial conditions (6). But by Theorem 2.10 the initial value problem composed of Equation (1) and conditions (6) has a unique solution on $a \le t \le b$. Hence the stated conclusion follows.

We have already studied that a single nth-order differential equation is closely related to a certain system of n first order differential equations. We now investigate this relationship more carefully in the case of the nth order homogeneous linear scalar differential equation.

$$a_0(t)\,\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0$$

Let

$$x_1 = x\,, \qquad x_2 = \frac{dx}{dt}$$

$$x_3 = \frac{d^2x}{dt^2}, \qquad \ldots$$

$$x_{n-1} = \frac{d^{n-2}x}{dt^{n-2}}, \qquad x_n = \frac{d^{n-1}x}{dt^{n-1}} \tag{7}$$

Differentiating (7), we have

$$\frac{dx}{dt} = \frac{dx_1}{dt}, \; \frac{d^2x}{dt^2} = \frac{dx_2}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-1}}{dt}, \; \frac{d^n x}{dt^n} = \frac{dx_n}{dt}. \tag{8}$$

The first (n -1) equations of (8) and the last (n -1) equations of (7) at once give

$$\frac{dx_1}{dt} = x_2, \; \frac{dx_2}{dt} = x_3 \ldots, \; \frac{dx_{n-1}}{dt} = x_n \tag{9}$$

Now assuming $a_0(t) \neq 0$ on $a \leq t \leq b$, (1) is equivalent to

$$\frac{d^n x}{dt^n} = -\frac{a_n(t)}{a_0(t)}\, x - \frac{a_{n-1}(t)}{a_0(t)} \frac{dx}{dt} - \ldots - \frac{a_1(t)}{a_0(t)} \frac{d^{n-1}x}{dt^{n-1}}$$

Using both (7) and (8) this becomes

$$\frac{dx_n}{dt} = -\frac{a_n(t)}{a_0(t)}\, x_1 - \frac{a_{n-1}(t)}{a_0(t)} x_2 - \ldots - \frac{a_1(t)}{a_0(t)} x_n \tag{10}$$

Combining (9) and (10) we have

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ &\vdots \\ \frac{dx_{n-1}}{dt} &= x_n \\ \frac{dx_n}{dt} &= -\frac{a_n(t)}{a_0(t)} x_1 - \frac{a_{n-1}(t)}{a_0(t)} x_2 - \ldots - \frac{a_1(t)}{a_0(t)} x_n. \end{aligned} \tag{11}$$

This is a special homogeneous linear system. In vector notation, it is the homogeneous linear vector differential equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} \tag{12}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}$$

Now suppose f satisfies the nth-order homogeneous linear differential equation (1). Then

$$a_0(t)f^{(n)}(t) + a_1(t)f^{(n-1)}(t) + \ldots + a_{n-1}(t)\, f'(t) + a_n(t)f(t) = 0, \tag{13}$$

for $t \in [a, b]$. Consider the vector $\boldsymbol{\varphi}$ defined by

$$\boldsymbol{\varphi}(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \vdots \\ \phi_{n-1}(t) \\ \phi_n(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ f'(t) \\ f''(t) \\ \vdots \\ f^{(n-2)}(t) \\ f^{(n-1)}(t) \end{pmatrix} \tag{14}$$

From (13) and (14) we see at once that

$$\begin{aligned} \phi_1'(t) &= \phi_2(t), \\ \phi_2'(t) &= \phi_3(t), \\ &\vdots \\ \phi_{n-1}'(t) &= \phi_n(t), \end{aligned} \tag{15}$$

$$\phi_n'(t) = -\frac{a_n(t)}{a_0(t)}\phi_1(t) - \frac{a_{n-1}(t)}{a_0(t)}\phi_2(t) - \ldots - \frac{a_1(t)}{a_0(t)}\phi_n(t)$$

Comparing this with (11), we see that the vector $\boldsymbol{\varphi}$ defined by (14) satisfies the system (11).

Conversely, suppose

$$\boldsymbol{\varphi}(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{pmatrix}$$

satisfies system (11) on [a, b]. Then (15) holds for all t $\in$ [a, b]. The first (n -1) equations of (15) give

$$
\begin{aligned}
&\phi_2(t) = \phi_1'(t) \\
&\phi_3(t) = \phi_2'(t) = \phi_1''(t) \qquad (16) \\
&\qquad\vdots \\
&\phi_n(t) = \phi_{n-1}'(t) = \phi_{n-2}''(t) = \cdots = \phi_1^{(n-1)}(t),
\end{aligned}
$$

and so $\phi_n'(t) = \phi_1^{[n]}(t).$ These last equation of (15) then becomes

$$\phi_1^{[n]}(t) = -\frac{a_n(t)}{a_0(t)}\phi_1(t) - \frac{a_{n-1}(t)}{a_0(t)}\phi_1'(t) - \ldots - \frac{a_1(t)}{a_0(t)}\phi_1^{[n-1]}(t)$$

or $$a_0(t)\phi_1^{[n]}(t) + a_1(t)\phi_1^{[n-1]}(t) + \cdots + a_{n-1}(t)\phi_1'(t) + a_n(t)\phi_1(t) = 0$$

Thus ϕ_1 is a solution f of the nth order homogeneous linear differential equation (1) and (16) shows that, in fact,

$$\boldsymbol{\varphi}(t) = \begin{pmatrix} f(t) \\ f'(t) \\ f''(t) \\ \vdots \\ f^{(n-1)}(t) \end{pmatrix}$$

We have thus obtained the following fundamental result.

Theorem 2.11

Consider the n-th order homogeneous linear differential equation

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + \ldots + a_n(t)x = 0,$$

and the corresponding homogeneous linear system (11). If f is a solution of (1) on [a, b], then

$$\varphi = \begin{pmatrix} f \\ f' \\ f'' \\ \vdots \\ f^{(n-1)} \end{pmatrix} \tag{17}$$

is a solution of (11) on [a, b]. Conversely, if

$$\varphi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

is a solution of (11) on [a, b], then its first component ϕ_1 is a solution of (1) on [a, b] and φ is, in fact, of the form (17)

Definition

Let $f_1, f_2, \ldots, f_n$ be n real functions, each of which has an (n-1)th derivative on $a \le x \le b$. The determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

where the primes denote derivatives, is called Wronskian of the n functions $f_1, f_2, \ldots, f_n$. We denote it by $W(f_1, f_2, \ldots, f_n)$ and denote its value at t by $W(f_1, f_2, \ldots, f_n)(t)$.

Let $f_1, f_2, \ldots, f_n$ be n solutions of the n-th order homogeneous linear differential equation (1) on $a \le t \le b$, and let

$$\boldsymbol{\varphi_1} = \begin{pmatrix} f_1 \\ f_1' \\ \vdots \\ f_1^{(n-1)} \end{pmatrix}, \quad \boldsymbol{\varphi_2} = \begin{pmatrix} f_2 \\ f_2' \\ \vdots \\ f_2^{(n-1)} \end{pmatrix}, \ldots, \quad \boldsymbol{\varphi_n} = \begin{pmatrix} f_n \\ f_n' \\ \vdots \\ f_n^{(n-1)} \end{pmatrix}$$

be the corresponding solutions of homogeneous linear system (11) on $a \le t \le b$. By definition, the Wronskian of the n solutions $f_1, f_2, \ldots f_n$ of (1) is

$$\begin{vmatrix} f_1 & f_2 & \cdots & fn \\ f_1' & f_2' & \cdots & f_n' \\ \mathrm{M} & \mathrm{M} & & \mathrm{M} \\ f_1^{(n-1)} & f_2^{(n-1)} & \Lambda & f_n^{(n-1)} \end{vmatrix}$$

Now note that by definition of the Wronskian of n solutions of (1) this is also the Wronskian of the n solution $\varphi_1, \varphi_2, \ldots, \varphi_n$ of (1). That is,

$$W(f_1, f_2, \ldots, f_n)(t) = W(\varphi_1, \varphi_2, \ldots, \varphi_n)(t)$$

for all $t \in [a, b]$. Now we know that, either $W(\varphi_1, \varphi_2, \ldots, \varphi_n)(t) = 0$ for all $t \in [a, b]$ or $W(\varphi_1, \varphi_2, \ldots, \varphi_n)(t) = 0$ for no $t \in [a, b]$. Thus either $W(f_1, f_2, \ldots, f_n)(t) = 0$ for all $t \in [a, b]$ or $W(f_1, f_2, \ldots, f_n)(t) = 0$ for no $t \in [a, b]$.

Able's Identity or Abel Liouville formula

Theorem 2.11 Let $x_1, x_2, \ldots. x_n$ be n linearly independent solutions of

$$L(x) = a_0(t)\, x^n + a_1(t)\, x^{(n-1)} + \ldots + a_n(t)\, x = 0$$

where $a_0(t) \neq 0$ for any $t \in I$ and $a_0, a_1, a_2, \ldots a_n$ are continuous functions of I and let $t_0 \in I$. Then

$$W(x_1, x_2, \ldots. x_n)(t) = W(x_1, x_2, \ldots. x_n)(t_0) \exp\left[-\int_{t_0}^{t} \frac{a_1(s)}{a_0(s)} ds\right] \qquad (1)$$

Proof We shall first prove the formula for n = 2 and then give a proof for general n.

$$\text{Now} \qquad W(x_1, x_2) = x_1 x_2' - x_1' x_2$$

$$W'(x_1, x_2) = x_1 x_2'' - x_1'' x_2$$

Since x_1 and x_2 are solutions of

$$a_0(t)x'' + a_1(t)\, x' + a_2(t)\, x = 0 \qquad (2)$$

whe shall get the values of x_1'' *and* x_2'' from (2)

$$a_0 x_1'' = -\mathrm{a}_1 x_1' - a_2 x_1$$

$$a_0 x_2'' = -\mathrm{a}_1 x_2' - a_2 x_2$$

Using these values, we get

$$W'(x_1, x_2) = \frac{x_1}{a_0}(-a_1 x_2^{'} - a_2 x_2) - \frac{x_2}{a_0}(-a_1 x_1^{'} - a_2 x_1)$$

Thus a_0 $W'(x_1, x_2) = -a_1(x_1 x_2^{'} - x_2 x_1^{'}) = -a_1 \; W(x_1, x_2)$. Thus the Wronskian $W(x_1, x_2)$ satisfies a linear differential equation of the first order

$$W'(x_1, x_2) = -\frac{a_1}{a_0} W(x_1, x_2)$$

or we have $$W'(x_1, x_2) + \frac{a_1}{a_0} W(x_1, x_2) = 0$$

Hence the integrating factor is exp $\left[\int_{t_0}^{t} \frac{a_1(s)}{a_0(s)} ds \right]$. Therefore its solution is

$$W(x_1, x_2)(t) = c \; \exp\left[-\int_{t_0}^{t} \frac{a_1(s)}{a_0(s)} ds \right]$$

where c is a constant. Taking $t = t_0$, we find

$$W(x_1, x_2)(t_0) = c$$

so that the solution for n = 2 is given by

$$W(x_1, x_2)(t) = W(x_1, x_2)(t_0) \; \exp\left[-\int_{t_0}^{t} \frac{a_1(s)}{a_0(s)} ds \right]$$

Now, we shall prove the case for any general n, let us denote $W(x_1, x_2, ... x_n)$ by W. Since W is a determinant of order n, its derivative W' is the sum of n determinants $V_1, V_2, \ldots V_n$ so that

$$W' = V_1 + V_2 + \ldots + V_n$$

where each determinant V_k is obtained by differentiating only one row, keeping the other (n-1) rows as they are. Hence we have

$$W'=\begin{vmatrix} x_1' & x_2' & \dots & x_n' \\ x_1' & x_2' & \dots & x_n' \\ x_1'' & x_2'' & \dots & x_n'' \\ \dots & \dots & \dots & \dots \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1'' & x_2'' & \dots & x_n'' \\ x_1'' & x_2'' & \dots & x_n'' \\ \dots & \dots & \dots & \dots \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{vmatrix}$$

$$+\ldots+\begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1' & x_2' & \dots & x_n' \\ \dots & \dots & \dots & .. \\ x_1^{(n)} & x_2^{(n)} & \dots & x_n^{(n)} \end{vmatrix} \tag{3}$$

In the first (n -1) determinants, two rows are identical in (3) and therefore they are all zero. Hence we are left with the last determinant only. Hence the last determinant in (3) is

$$W'=\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_1' & x_2' & x_3' & \dots & x_n' \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{(n)} & x_2^{(n)} & \dots & \dots & x_n^{(n)} \end{vmatrix} \tag{4}$$

Since $x_1, x_2,\ldots, x_n$ are the solutions of $L(x) = 0$,

$$a_0 x_1^{(n)} = -a_1 x_1^{(n-1)} - \ldots - a_n x_1$$

$$a_0 x_2^{(n)} = -a_1 x_2^{(n-1)} - \ldots - a_n x_2$$

$$a_0 x_3^{(n)} = -a_1 x_3^{(n-1)} - \ldots - a_n x_3$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_0 x_n^{(n)} = -a_1 x_n^{(n-1)} - \ldots - a_n x_n \tag{5}$$

Using the set of equations (5), we can eliminate $x_1^{(n)}, x_2^{(n)},\ldots, x_n^{(n)}$ in the determinant (4)

$$W'=\frac{1}{a_0}\begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1' & x_2' & \dots & x_n' \\ x_2'' & x_2'' & \dots & x_n'' \\ \dots & \dots & \dots & \dots \\ x_1^{(n-2)} & x_2^{(n-2)} & \dots & x_n^{(n-2)} \\ (-a_1 x_1^{(n-1)} - \ldots - a_n x_1) & (-a_1 x_2^{(n-1)} - \ldots - a_n x_2) & \dots & (-a_1 x_n^{(n-1)} - \ldots - a_n x_n) \end{vmatrix} \tag{6}$$

Now in the determinant (6), let us multiply the first row by a_n, second row by a_{n-1} and so on up to (n-1)-th row by a_2 and add these to the last row. Then using (5) in the resulting determinant, we get

$$W' = \frac{1}{a_0}\begin{vmatrix} x_1 & x_2 & \dots & x_3 \\ x_1' & x_2' & \dots & x_n' \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -a_1 x_1^{(n-1)} & -a_1 x_2^{(n-1)} & \dots & -a_1 x_n^{(n-1)} \end{vmatrix} = -\frac{a_1}{a_0}\begin{vmatrix} x_1 & x_2 & \dots & x_3 \\ x_1' & x_2' & \dots & x_n' \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{vmatrix}$$

Thus we are led to a differential equation

$$W' = -\frac{a_1}{a_0}W \tag{7}$$

Hence the integrating factor of (7) is exp $\left[\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}ds\right]$. So the solution of (7) is

W(t) = c exp $-\left[\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}ds\right]$. Now taking $t = t_0$, we get $c = W(t_0)$ so that the solution of

(7) is $W(t) = W(t_0)$ exp $-\left[\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}ds\right]$ where $W(t_0)$ represents the value of the Wronskian at $t = t_0$. This completes the proof the theorem.

Corollary 1 If a_0 and a_1 are constants, then from the theorem we get

$$W(t) = W(t_0)\exp\left[-\frac{a_1}{a_0}(t - t_0)\right]$$

Corollary 2 Then n solutions $x_1, x_2, \dots, x_n$ of $L(x) = 0$ on an interval I are linearly independent on I if and only if $W(x_1, x_2, \dots, x_n)(t_0) \neq 0$ for any point $t_0 \in I$.

The proof follows from the fact the $W(x_1, x_2, \dots, x_n)(t) \neq 0$ implies $W(x_1, x_2, \dots, x_n)(t_0) \neq 0$ by the theorem.

Note Since the calculations of the Wronskian at any pint $t \in I$ is difficult, the theorem and the Corollary 1 are useful to find it in terms of the Wronskian at $t = 0$ or $t = 1$

which may be easier to calculate. We shall illustrate this by considering a homogeneous equation of order 3.

Example 2.5 Compute the Wronskian of the three independent solutions of $x^3 - x'' - x' + x = 0$ in [0, 1].

The roots of the characteristic equation are -1, 1, 1. Hence the three solutions are $x_1(t) = e^{-t}$, $x_2(t) = e^t$ and $x_3(t) = te^t$. We shall be finding $W(x_1, x_2, x_3)$ on [0, 1]. Now

$$W(x_1, x_2, x_3)(t) = \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & e^t + te^t \\ e^{-t} & e^t & 2e^t + te^t \end{vmatrix}$$

Let us find $W(x_1, x_2, x_3)(0)$, that is, the Wronskian at t = 0.

$$W(x_1, x_2, x_3)(0) = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Expanding the determinant along the first row, we find $W(x_1, x_2, x_3)(0) = 4$.

By the theorem

$$W(x_1, x_2, x_3)(t) = W(x_1, x_2, x_3)(0) \exp\left[-\int_0^t (-1)ds\right]$$

which gives $W(x_1, x_2, x_3)(t) = 4e^t$.

This proves that the solutions are linearly independent.

Theorem 2.12 (i) The operator L is linear operator on the space of n-times differentiable functions.

(ii) If $x_1, x_2, \ldots, x_n$ are solutions of $L(x) = 0$ and $c_1, c_2 \ldots, c_n$ are arbitrary constants, then $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is a solution of $L(x) = 0$

Proof To prove L is linear, let x_1 and x_2 be any two solutions of $L(x) = 0$. For any two, scalars c_1 and c_2 we shall show that $L(c_1x_1 + c_2x_2) = c_1L(x_1) + c_2L(x_2)$. For this let us consider

$$L[c_1x_1 + c_2x_2](t)$$

$$=a_0(t)\,[c_1x_1^{(n)}(t) + c_2x_2^{(n)}(t)] + a_1(t)[c_1x_1^{(n-1)}(t) + c_2x_2^{(n-1)}(t)]$$

$$+\ldots+ a_n(t)[c_1x_1(t) + c_2x_2(t)]$$

$$= c_1[a_0(t)x_1^{(n)}(t) + a_1(t)x_1^{(n-1)}(t) +\ldots+ a_n(t)x_1(t)]$$

$$+ c_2[a_0(t)x_2^{(n)}(t) + a_1(t)x_2^{(n-1)}(t) +\ldots+ a_n(t)x_2(t)]$$

$$= c_1L(x_1)(t) + c_2L(x_2)(t) = [c_1L(x_1) + c_2L(x_2)](t)$$

Thus we have

$$L[c_1x_1+ c_2x_2](t) = [c_1L(x_1) + c_2L(x_2)]t \text{ for all } t \in I$$

Hence $\qquad L(c_1x_1+ c_2x_2) = c_1L(x_1) + c_2L(x_2)$

which proves that L is a linear operator on the space of n-times differentiable functions on I.

(ii) The given differential equation is

$$L(x) = a_0(t)x^{(n)} + a_1(t)x^{(n-1)}+\ldots+ a_n(t)x = 0 \tag{1}$$

where $a_0(t) \neq 0$ for any $t \in I$.

Since $x_1, x_2, \ldots,x_n$ are solutions of (1)

$$L(x_1)(t) = 0,\ L(x_2)(t) = 0,\ldots,\ L(x_n)(t) = 0 \text{ for all } t \in I \tag{2}$$

Let us takes $x = c_1x_1+ c_2x_2 +\ldots+c_nx_n$ where $c_1, c_2,\ldots,c_n$ are arbitrary constants.

Since L is linear, wet get

$$L(x) = c_1L(x_1) + c_2L(x_2) + \ldots+ c_nL(x_n)$$

which gives

$$L(x)\,(t) = c_1L(x_1)(t)+ c_2L(x_2)(t) +\ldots+ c_nL(x_n)(t) \tag{3}$$

for all $t \in I$. Using (2) in (3) we get $L(x)(t) = 0$ for all $t \in I$. In other words,

$x = c_1x_1+ c_2x_2 +\ldots+c_nx_n$ is a solution of (1).

Note For non-linear equations, the above theorem is not necessarily true as shown by the following example.

Example 2.6 Consider the differential equation

$$x'' = - x'^2 \tag{1}$$

First note that the given differential equation is not linear because of the appearance of x'^2, it cannot be written in the linear form.

Let us take $x' = y$ so that (1) becomes

$$\frac{dy}{dt} = -y^2 \tag{2}$$

Integrating (2), we get $\frac{1}{y} = t + a$ so that $\frac{dt}{dx} = t + a.$ Hence the solution is $x(t)=\log(t+a) + b$ where a and b are arbitrary constants. Thus, we can take the two solutions as $x_1(t) = \log(t+a) + b$, $x_2(t) = \log (t+a)$.

Let us check whether $x = c_1x_1 + c_2x_2$ can be a solution of (1) for arbitrary constants c_1 and c_2. Choose $c_1 = 4$, $c_2 = -2$. Then we have

$$x(t) = 4 [\log (t + a) + b] - 2\log (t + a) \tag{3}$$

Now from (3), $x''(t) = - \frac{2}{(t+a)^2}$ *and* $x'(t) = \frac{2}{(t+a)}$

Hence $x''(t) \neq - [x'(t)]^2$ which shows that this x(t) does not satisfy equation (1). This proves that the Theorem 2.12 is not necessarily true for non-linear equations.

Theorem 2.13 $x_1, x_2, x_3,\ldots x_n$ are linearly independent solutions of the homogeneous linear differential equation,

$$L(x) = a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = 0 \tag{1}$$

where $a_0(t) \neq 0$ for any $t \in I$ and $a_0, a_1, a_2,\ldots a_n$ are continuous functions on I if and only if the Wronskian

$$W(x_1, x_2,\ldots,x_n)(t) \neq 0 \tag{2}$$

for every $t \in I$.

Proof First, we prove the necessity of the condition.

Let us suppose that the solutions $x_1, x_2, \ldots,x_n$ are linearly independent solutions of (1) and prove that

$$\mathbf{W}(x_1, x_2, \ldots, x_n)(t) \neq 0 \text{ for every } t \in I.$$

If this condition is not satisfied, let us suppose on the contrary that there exists a $t_0 \in I$ such that the Wronskian

$$\mathbf{W}(x_1, x_2, \ldots, x_n)(t_0) = 0 \tag{3}$$

Hence, (3) implies that the determinant of the coefficients of the following set of homogeneous equations in $c_1, c_2, \ldots, c_n$ is zero:

$$c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0) = 0$$

$$c_1x_1'(t_0) + c_2x_2'(t_0) + \ldots + c_nx_n'(t_0) = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$c_1x_1^{(n-1)}(t_0) + c_2x_2^{(n-1)}(t_0) + \ldots + c_nx_n^{(n-1)}(t_0) = 0 \tag{4}$$

Hence using hypothesis (3), the above system of n linear equations has a non trivial solution for the n-unknowns $c_1, c_2, \ldots c_n$ which means that not all the constants $c_1, c_2, \ldots c_n$ are zero. Let us take one such solution to be $c_1, c_2, \ldots c_n$ itself.

Having determined the non-zero coefficients $c_1, c_2, \ldots c_n$ from the set of n-equations (4), let us define the function x as follows:

$$x = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{5}$$

Since $x_1, x_2, \ldots x_n$ are the solutions of (1), using (ii) of Theorem 2.12, it follows that x is a solution of (1), that is $L(x)\, t = 0$ for every $t \in I$.

Now using (5) in the set of equations (4), we get

$$x(t_0) = 0,\ x'(t_0) = 0,\ x''(t_0) = 0, \ldots, x^{(n-1)}(t_0) = 0 \tag{6}$$

The above (6) is nothing but the initial conditions of $L(x) = 0$ at $t = t_0 \in I$. Hence $L(x) = 0$ and (6) together give the initial value problem of (1). Since the solution of the initial value problem is unique, we get $x(t) = 0$ for all $t \in I$. This implies

$$c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) = 0 \text{ for all } t \in I. \tag{7}$$

But (7) contradicts the fact that $x_1, x_2, \ldots x_n$ are linearly independent on I. This contradiction proves that our assumption $W(x_1, x_2, \ldots x_n)(t) = 0$ for some $t_0 \in I$ is wrong. Hence

$$W(x_1, x_2, \ldots x_n)(t) \neq 0 \text{ for all } t \in I.$$

To prove the converse, let us assume that $W(x_1, x_2, \ldots x_n)(t) \neq 0$ for any $t \in I$ and show that $x_1, x_2, \ldots x_n$ are linearly independent. If they are not linearly independent, let $c_1, c_2, \ldots c_n$ be non zero constants such that

$$c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) = 0 \text{ for all } t \in I. \tag{8}$$

Differentiating (8) successively upto order $(n - 1)$, we get

$$c_1x_1'(t) + c_2x_2'(t) + \ldots + c_nx_n'(t) = 0$$

$$c_1x_1''(t) + c_2x_2''(t) + \ldots + c_nx_n''(t) = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$c_1x_1^{(n-1)}(t) + c_2x_2^{(n-1)}(t) + \ldots + c_nx_n^{(n-1)}(t) = 0 \tag{9}$$

for all $t \in I$.

For a fixed t in I, (9) is a set on n homogeneous linear algebraic equations satisfied by $c_1, c_2, \ldots c_n$. Since the Wronskian of $x_1, x_2, \ldots x_n$ is not zero by hypothesis, the determinant of the coefficients of the above equations in the n unknowns $c_1, c_2, \ldots c_n$ is not zero. Hence there is only one trivial solution to the set of equations (9) namely $c_1 = c_2 = c_3 = \ldots c_n = 0$ contradicting that $c_1, c_2, \ldots, c_n$ are non-zero. This contradiction proves that $x_1, x_2, \ldots x_n$ are linearly independent. This completes the proof of the theorem.

Theorem 2.14 Let $x_1, x_2, \ldots x_n$ be n linearly independent solutions of $L(x) = 0$. Then any solution of the equation

$$L(x) = 0 \tag{1}$$

is of the form $x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ where $c_1, c_2, \ldots c_n$ are constants.

Proof Let x be any solution of (1). Let t_0 be a point of I such that $x(t_0) = a_1$, $x'(t_0) = a_2,\ldots,x^{(n-1)}(t_0)= a_n$ so that $L(x) = 0$ and these initial conditions form an initial value problem for $L(x) = 0$ having the solution $x(t)$.

We shall show that there exist unique constants $c_1, c_2,\ldots,c_n$ such that

$$y = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{2}$$

is solution of $L(x)$ with the initial condition

$$y(t_0) = a_1,\ y'(t_0) = a_2,\ldots,\ y^{(n-1)}(t_0) = a_n$$

Hence by theorem on the uniqueness of solution of the initial value problem, $x(t) = y(t)$ for $t \in I$ so that $x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is any solution of $L(x) = 0$.

Hence the proof is complete if we establish the existence of the unique constants $c_1, c_2,\ldots c_n$ satisfying (2). Let us write down the initial conditions for y as a solution of (1)

$$c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0) = a_1$$

$$c_1x_1'(t_0) + c_2x_2'(t_0) + \ldots + c_nx_n'(t_0) = a_2$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$c_1x_1^{(n-1)}(t_0) + c_2x_2^{(n-1)}(t_0) + \ldots + c_nx_n^{(n-1)}(t_0) = a_n \tag{3}$$

We shall show that the constants $c_1, c_2,\ldots c_n$ exist uniquely by hypothesis. We note that the determinant formed by the coefficients of $c_1, c_2,\ldots c_n$ in the above system of non-homogeneous linear equations is the Wronskian of the functions $x_1, x_2,\ldots x_{n-1}, x_n$ at the point t_0. Since $x_1, x_2,\ldots x_{n-1}, x_n$ are linearly independent on I, the Wronskian at $t = t_0$ is not zero. So the system of non-homogeneous equations (3) has unique solution $c_1, c_2,\ldots c_n$ which completes the proof of the theorem.

Theorem 2.15

Hypotheses

1. Let $f_1, f_2, \ldots, f_n$ be a set of n functions each of which has a continuous n-th derivative on $a \le t \le b$.

2. Suppose $W(f_1, f_2, \ldots, f_n)(t) \neq 0$ for all t on $a \le t \le b$.

Conclusion There exists a unique normalized (coefficient of $\frac{d^n x}{dt^n}$ is unity) homogeneous linear differential equation of order n (with continuous coefficients) which has $f_1, f_2, \ldots, f_n$ as a fundamental set on $a \le t \le b$. This equation is

$$\frac{W[f_1(t), f_2(t).....f_n(t), x]}{W[f_1(t), f_2(t).....f_n(t)]} = 0. \tag{1}$$

Proof The differential equation (1) is actually

$$\frac{\begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) & x \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) & x' \\ \vdots & \vdots & & \vdots & \vdots \\ f_1^{(n)}(t) & f_2^{(n)}(t) & \cdots & f_n^{(n)}(t) & x^{(n)} \end{vmatrix}}{\begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}} = 0. \tag{2}$$

The expansion of the numerator $W(x, f_1, f_2, \ldots, f_n)$ in equation (1) by the last column shows that (1) is a differential equation of order n, with the coefficient of $x^{(n)}$ as one. Thus, we get an equation of the form

$$\frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dx}{dt} + p_n(t)x = 0,$$

and so is a normalized homogeneous linear differential equation of order n. Also, by Hypotheses 1 and 2 the coefficients $p_i(i = 1, 2, \ldots n)$ are continuous on $a \le t \le b$. If any one of $f_1(t), f_2(t), \ldots, f_n(t)$ is substituted for x in equation (2), the resulting determinant in the numerator will have two identical columns. Thus each of the functions $f_1, f_2, .. f_n$ is a solutions of Equation (1) on $a \le t \le b$; and by Theorem 2.13 we see from Hypothesis 2 that these solution are linearly independent on $a \le t \le b$. Thus equation (1) is indeed an equation of the required type having $f_1, f_2, .. f_n$ as a fundamental set.

We now show that equation (1) is the only normalized n-th order homogeneous linear differential equation with continuous coefficients which has this property. Suppose there are two such equations

$$\frac{d^n x}{dt^n} + q_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \Lambda\ + q_{n-1}(t)\frac{dx}{dt} + q_n(t)x = 0,$$

$$\frac{d^n x}{dt^n} + r_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \Lambda\ + r_{n-1}(t)\frac{dx}{dt} + r_n(t)x = 0. \tag{3}$$

Then the equation

$$[q_1(t) - r_1(t)]\frac{d^{n-1}x}{dt^{n-1}} + \Lambda\ + [q_n(t) - r_n(t)]x = 0 \tag{4}$$

is a homogeneous linear differential equation of order at most (n-1), and the coefficients in equation (4) are continuous on $a \le t \le b$. Further, since $f_1, f_2,..f_n$ satisfy both the equations (3) these n functions are all solutions of Equation (4) on $a \le t \le b$.

We shall show that $q_1(t) - r_1(t) = 0$ for all t on $a \le t \le b$. To do so, let t_0 be a point of the interval $a \le t \le b$ and suppose that $q_1(t_0) - r_1(t_0) \neq 0$. Then there exist a subinterval I, $\alpha \le t \le \beta$, of $a \le t \le b$ containing t_0 such that $q_1(t) - r_1(t) \neq 0$ on I. Since the n solutions $f_1, f_2,..f_n$ of equation (4) are linearly independent on $a \le t \le b$, they are also linearly independent on I. Thus on I, equation (4) of order at most (n-1), has a set of n linearly independent solutions. But this is a contradiction. Thus there exists no $t_0 \in [a, b]$ such that $q_1(t_0) - r_1(t_0) \neq 0$. In other words, $q_1(t) - r_1(t) = 0$ for all t on $a \le t \le b$.

Similarly one can show that $q_k(t) - r_k(t) = 0$, $k = 2, 3\ldots,n$ for all t on $a \le t \le b$. Thus equation (3) are identical on $a \le t \le b$ and the uniqueness is proved

Example 2.7

Consider the function f_1 and f_2 defined, respectively by $f_1(t) = t$ and $f_2(t) = te^t$. We note that

$$W(f_1, f_2)(t) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2 e^t \neq 0 \quad \text{for } t \neq 0.$$

Thus by theorem 2.15 on every closed interval a ≤ t ≤ b not including t = 0 there exists a unique normalized second order homogeneous linear differential equation with continuous coefficients which has f_1 and f_2 as a fundamental set. Theorem 2.15 states that this equation is

$$\frac{W[t, te^t, x]}{W[t, te^t]} = 0.$$

Writing out the two Wronskians involved, this becomes

$$\frac{\begin{vmatrix} t & te^t & x \\ 1 & te^t + e^t & x' \\ 0 & te^t + 2e^t & x'' \end{vmatrix}}{\begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix}} = 0.$$

Since

$$\begin{vmatrix} t & t & x \\ 1 & t+1 & x' \\ 0 & t+2 & x'' \end{vmatrix} = t^2 \frac{d^2x}{dt^2} - t(t+2)\frac{dx}{dt} + (t+2)x \qquad and \qquad \begin{vmatrix} t & t \\ 1 & t+1 \end{vmatrix} = t^2$$

we see that this differential equation is

$$\frac{d^2x}{dt^2} - \left(\frac{t+2}{t}\right)\frac{dx}{dt} + \left(\frac{t+2}{t^2}\right)x = 0.$$

Adjoint Equations

Let L_n be the nth order linear differential operator given by

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \Lambda + a_n(t)$$

and the corresponding nth order linear differential equation is

$L_n(x) = 0.$

Then the linear differential operator given by

$$L_n^+ = \overline{L}_n \equiv (-1)^n \frac{d^n}{dt^n}[\overline{a}_0(t)] + (-1)^{n-1}\frac{d^{n-1}}{dt^{n-1}}[\overline{a}_1(t)] + ... + (-1)^1 \frac{d}{dt}[\overline{a}_{n-1}(t)] + \overline{a}_n(t)$$

is called the Adjoint operator of the operator L_n and the corresponding adjoint differential equation is

$\overline{L}_n(x) = 0$ i.e.

$$(-1)^n \frac{d^n}{dt^n}[\overline{a}_0(t)x] + (-1)^{n-1}\frac{d^{n-1}}{dt^{n-1}}[\overline{a}_1(t)x] + \ldots\ldots\ldots - \frac{d}{dt}[\overline{a}_{n-1}(t)x] + \overline{a}_n(t)x = 0 \qquad (1)$$

Equation (1) may be written as

$$(-1)^n[\overline{a}_0(t)x]^{(n)} + (-1)^{n-1}[\overline{a}_1(t)x]^{(n-1)} + \ldots\ldots\ldots + \overline{a}_n(t)x = 0$$

The coefficients $a_k(t) \in C^{n-k}$ i.e. a_k have continuous (n-k)th derivatives on I.

Definition

If the adjoint of a differential equation is the equation itself, then the equation is called self-adjoint differential equation.

Definition

If the adjoint of a differential operator is the operator itself, then the operator is called self-adjoint.

We have already studied that the linear system corresponding to linear differential equation

$$L_n x = 0 \qquad (1)$$

is, $\hat{x}' = A(t) + \hat{x}$ (2)

The adjoint system to the LS (2) is

$$\hat{x}' = -A^*(t) + \hat{x} \qquad (3)$$

where

$$-A^*(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \dfrac{\overline{a}_n}{\overline{a}_0} \\ -1 & 0 & \cdots & 0 & \dfrac{\overline{a}_{n-1}}{\overline{a}_0} \\ 0 & -1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \dfrac{\overline{a}_1}{\overline{a}_0} \end{bmatrix} \qquad (4)$$

Consider the special case with $a_0 = 1$.

Then equation (3) becomes

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \bar{a}_n \\ -1 & 0 & \cdots & 0 & \bar{a}_{n-1} \\ 0 & -1 & \cdots & 0 & \bar{a}_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \bar{a}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{5}$$

Theorem 2.14 (Lagrange Identity)

Let the coefficients $a_k(t)$ in the differential operator

$$L_n = a_0(t)\frac{d^n}{dt} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_n(t)$$

have continuous (n-k)th derivatives i.e. $a_k \in C^{n-k}$ on I (k = 0, 1, …,n). If u and v be any two (complex) continuous functions having nth order derivative on the interval I, then

$$\bar{v}L_n u - u\overline{L_n^+ v} = \frac{d}{dt}[P(u,v)],$$

where $P(u,v) = \sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{j-1}u^{(k-j)}(\bar{v}a_{n-k})^{(j-1)}\right].$

This P(u,v) is called bilinear concomitant associated with the operator L_n.

Proof Consider the expression

$U^{(k-1)}V - U^{(k-2)}V' + \ldots\ldots + (-1)^{k-2} U'V^{(k-2)} + (-1)^{k-1} UV^{(k-1)}$, k = 0,1,….,n.

Differentiating

$$\frac{d}{dt}\left[U^{(k-1)}V - U^{(k-2)}V' + \ldots\ldots + (-1)^{k-2} U'V^{(k-2)} + (-1)^{k-1} UV^{(k-1)}\right]$$

$= U^{(k)}V + U^{(k-1)}V' - U^{(k-1)}V' - U^{(k-2)}V'' + \ldots\ldots + (-1)^{k-2} U''V^{(k-2)} + (-1)^{k-2} U'V^{(k-1)}$

$+ (-1)^{k-1} U'V^{(k-1)} + (-1)^{k-1} UV^{(k)}$

Thus

$= VU^{(k)} + (-1)^{k-1} UV^{(k)}$

$$VU^{(k)} = (-1)^k UV^{(k)} + \frac{d}{dt}\left[U^{(k-1)}V - U^{(k-2)}V' + \dots\dots + (-1)^{k-2} U'V^{(k-2)} + (-1)^{k-1} UV^{(k-1)}\right] \tag{1}$$

k = 0,1,…..,n

Now

$$\bar{v}L_n u = \bar{v}\left[a_0 \frac{d^n u}{dt^n} + a_1 \frac{d^{n-1}u}{dt^{n-1}} + \Lambda + a_{n-1}\frac{du}{dt} + a_n u\right]$$

$$= \bar{v}a_0 \frac{d^n u}{dt^n} + \bar{v}a_1 \frac{d^{n-1}u}{dt^{n-1}} + \Lambda + \bar{v}a_{n-1}\frac{du}{dt} + \bar{v}a_n u \tag{2}$$

Now, we shall obtain all the terms in r.h.s. of equation (2). First we put U = u, V=$\bar{v}a_0$ and k = n in equation (1),

$$\bar{v}a_0 \frac{d^n u}{dt^n} = (-1)^n u \frac{d^n(\bar{v}a_0)}{dt^n} + \frac{d}{dt}\left[\bar{v}a_0 \frac{d^{n-1}u}{dt^{n-1}} - (\bar{v}a_0)' \frac{d^{n-2}u}{dt^{n-2}} + \Lambda\right.$$

$$\left. + (-1)^{n-2}\frac{du}{dt}(\bar{v}a_0)^{(n-2)} + (-1)^{n-1}u(\bar{v}a_0)^{(n-1)}\right] \tag{3}$$

Now put

U = u, V = $\bar{v}a_1$ and k = n-1 in equation (1)

$$\bar{v}a_1 \frac{d^{n-1}u}{dt^{n-1}} = (-1)^{n-1} u \frac{d^{n-1}(\bar{v}a_1)}{dt^{n-1}} + \frac{d}{dt}\left[\bar{v}a_1 \frac{d^{n-2}u}{dt^{n-2}} - (\bar{v}a_1)' \frac{d^{n-3}u}{dt^{n-3}} + \Lambda\right.$$

$$\left. + (-1)^{n-3}\frac{du}{dt}(\bar{v}a_1)^{(n-3)} + (-1)^{n-2}u(\bar{v}a_1)^{(n-2)}\right] \tag{4}$$

and so on.

Let U = u, V = $\bar{v}a_{n-1}$ and k = 1 in equation (1)

$$\bar{v}a_{n-1}\frac{du}{dt} = (-1)^1 u \frac{d(\bar{v}a_{n-1})}{dt} + \frac{d}{dt}\left[\bar{v}a_{n-1}u\right] \tag{5}$$

Finally put U = u, V = $\bar{v}a_n$ and k = 0, we get

$$\bar{v}a_n u = u\bar{v}a_n \qquad (6)$$

Using (3) – (6) in equation (2), we get

$$\bar{v}L_n u = u\left[(-1)^n \frac{d^n}{dt^n}(\bar{v}a_0) + (-1)^{n-1}\frac{d^{n-1}}{dt^{n-1}}(\bar{v}a_1) + \Lambda + (-1)^1 \frac{d}{dt}(\bar{v}a_{n-1}) + \bar{v}a_n\right] +$$

$$\frac{d}{dt}\left[\sum_{j=1}^{n}(-1)^{j-1}(u)^{(n-j)}(\bar{v}a_0)^{(j-1)}\right] + \frac{d}{dt}\left[\sum_{j=1}^{n-1}(-1)^{j-1}(u)^{(n-1-j)}(\bar{v}a_1)^{(j-1)}\right] +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$\frac{d}{dt}\left[\sum_{j=1}^{1}(-1)^{j-1}(u)^{(1-j)}(\bar{v}a_{n-1})^{(j-1)}\right]$$

Thus one obtains

$$\bar{v}L_n u - u\overline{L_n^+ v} = \frac{d}{dt}\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{j-1}(u)^{(k-j)}(\bar{v}a_{n-k})^{(j-1)}\right]$$

$$= \frac{d}{dt}[P(u,v)] \qquad (7)$$

where

$$\overline{L_n^+ v} = (-1)^n[a_0(t)\bar{v}]^{(n)} + (-1)^{n-1}[a_1(t)\bar{v}]^{(n-1)} + \ldots. + (-1)^1[a_{n-1}(t)\bar{v}]^{(1)} + a_n(t)\bar{v}$$

where, we have used the definition of adjoint operator as

$$L_n^+ v = (-1)^n[\bar{a}_0(t)v]^{(n)} + (-1)^{n-1}[\bar{a}_1(t)v]^{(n-1)} + \ldots. + (-1)^1[\bar{a}_{n-1}(t)v]^{(1)} + \bar{a}_n(t)v,$$

proving the result.

The Lagrange identity (7) holds for all continuous differentiable functions u(t) and v(t) defined over some solution domain I = {t| a ≤ t ≤ b}.

The functions u and v must be differentiable in order that $L_n u$ and $\overline{L_n^+ v}$ exist. This is the only restriction we place upon these functions.

Theorem 2.15 (Green's Formula)

If the a_k in L_n and u, v are the same as in theorem 2.14, then for any t_1, t_2, ∈ I,

$$\int_{t_1}^{t_2} \left(\bar{v} L_n u - u \overline{L_n^+ v} \right) dt = P(u,v)(t_2) - P(u,v)(t_1),$$

where P(u,v)(t) is the value at t of P(u,v).

Proof The integral of Lagrange identity (7) produces

$$\int_{t_1}^{t_2} \left(\bar{v} L_n u - u \overline{L_n^+ v} \right) dt = \left[P(u,v) \right]_{t_1}^{t_2}$$

$$= P(u,v)\big|_{t=t_2} - P(u,v)\big|_{t=t_1} = P(u,v)(t_2) - P(u,v)(t_1),$$

which is the Green's formula.

The nth-order Non-homogeneous Linear Equation

In this section we consider briefly the nth-order non homogeneous linear scalar differential equation

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1} x}{dt^{n-1}} + \Lambda + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = b(t) \tag{1}$$

Using the operator notation already introduced, we may write this as

$$L_n x = b(t), \tag{2}$$

Where, as before,

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \Lambda + a_{n-1}(t)\frac{d}{dt} + a_n(t).$$

We now prove the basic theorem dealing with equation (2).

Theorem 2.16

Hypothesis

1. Let v be any solution of the nonhomogeneous equation (2)

2. Let u be any solution of the corresponding homogeneous equation

$$L_n x = 0. \tag{3}$$

Conclusion Then u + v is also a solution of the non homogeneous equation (2).

Proof We have

$$L_n[u(t)+v(t)] = a_0(t)\frac{d^n}{dt}[u(t)+v(t)] + a_1(t)\frac{d^{n-1}}{dt^{n-1}}[u(t)+v(t)] + \ldots\ldots\ldots\ldots\ldots.$$

$$+ a_{n-1}(t)\frac{d}{dt}[u(t)+v(t)] + a_n(t)[u(t)+v(t)]$$

$$= a_0(t)\frac{d^n}{dt}u(t) + a_1(t)\frac{d^{n-1}}{dt^{n-1}}u(t) + \Lambda\ + a_{n-1}(t)\frac{d}{dt}u(t) + a_n(t)u(t)$$

$$+ a_0(t)\frac{d^n}{dt}v(t) + a_1(t)\frac{d^{n-1}}{dt^{n-1}}v(t) + \Lambda\ + a_{n-1}(t)\frac{d}{dt}v(t) + a_n(t)v(t)$$

$= L_n[u(t)] + L_n[v(t)]$

Now by Hypothesis 1, $L_n[v(t)] = F(t)$; and by Hypothesis 2, $L_n[u(t)] = 0$.

Thus $L_n[u(t) + v(t)] = F(t)$; that is, u+v is a solution of Equation (2).

In particular, if $f_1, f_2, \ldots\ldots f_n$ is a fundamental set of the homogeneous equation (3) and v is any particular solution of the non homogeneous equation (2) then

$$c_1f_1 + c_2f_2 + \ldots\ldots + c_nf_n + v$$

is also a solution of the non homogeneous equation (2). If a fundamental set $f_1, f_2, \ldots\ldots f_n$ of the corresponding homogeneous equation (3) is known, then a particular solution of the non homogeneous equation (2) can always be found by the method of variation of parameters.

Theorem 2.17

If $\{\phi_1, \ldots.\phi_n\}$ is a fundamental set of homogeneous linear differential equation

$L_nx = x^{(n)} + a_1x^{(n-1)} + \ldots\ldots. + a_nx = 0$

$a_k \in C$ on I, where C is set of complex functions. Then (prove that) the solution ψ of the non homogeneous equation

$L_nx = b(t)$ $\quad$ ($b \in C$ on I)

Satisfying $\hat{\psi}(\tau) = \hat{\xi}$ $\quad$ $(\tau \in I, |\hat{\xi}| < \infty)$

is given by

$$\psi(t) = \psi_h(t) + \sum_{k=1}^{n} \phi_k(t) \int_{\tau}^{t} \left\{ \frac{W_k(\phi_1, \ldots\ldots, \phi_n)(s)}{W(\phi_1, \ldots\ldots, \phi_n)(s)} \right\} b(s)ds$$

where ψ_h is the solution of $L_n x = 0$ for which $\hat{\psi}_h(\tau) = \hat{\xi}$ and $W_k(\phi_1, \ldots\ldots, \phi_n)$ is the determinant obtained from $W(\phi_1, \ldots\ldots, \phi_n)$ by replacing the kth column by (0, 0, ...,0, 1)

Proof We know that if φ is the fundamental matrix for the LS

$$x' = A(t)x \tag{1}$$

then

$$\psi(t) = \varphi(t) \int_{\tau}^{t} \varphi^{-1}(s)\, b(s)ds \tag{2}$$

is the solution of the non homogeneous system.

$$x' = A(t)x + b(t) \tag{3}$$

satisfying $\psi(\tau) = 0$. (by theorem 2.2)

Thus, we can say that $\hat{\psi}$ is the vector solution of the system

$$\hat{x}' = A(t)\hat{x} + \hat{b}(t) \tag{4}$$

such that $\hat{\psi}(\tau) = 0$

then ψ_1, the first component of $\hat{\psi}$ is given by [or general component, just named as ψ_1]

$$\psi_1(t) = \int_{\tau}^{t} \gamma_{1n}(t,s)\, b(s)ds \tag{5}$$

where $\gamma_{1n}(t,s)$ denotes the (1,n)th element of $\varphi(t)\ \varphi^{-1}(s)$ i.e. $\gamma_{1n}(t,s)$ is the element in the first row and nth column of the matrix $\varphi(t)\ \varphi^{-1}(s)$ [i.e. element obtained by multiplying the first row of $\varphi(t)$ with the nth column of $\varphi^{-1}(s)$]

Now, we know that (I, j)th element of $\phi(t)$ is $\phi_j^{(i-1)}$ i.e.

Then det $\varphi(t) = W(\phi_1, \ldots\ldots, \phi_n)(t)$.

Now, the element in the ith row and nth column of $\varphi^{-1}(s)$ is given by

$$= \frac{\overline{\phi}_{in}}{W(\phi_1, \ldots\ldots, \phi_n)(s)}, \tag{6}$$

As $A^{-1} = \dfrac{Adj\ A}{|A|}$, similarly

$$\varphi^{-1} = \frac{Adj\ \varphi}{\det \varphi = W(\phi_1, \ldots\ldots, \phi_n)}$$

where $\overline{\phi}_{in}$ is the cofactor of (n,i)th element $\phi_i^{(n-1)}$ in φ.

$\therefore$

$$\gamma_{1n}(t,s) = \phi_1(t)\frac{\overline{\phi}_{1n}}{W(\phi_1, \ldots\ldots, \phi_n)(s)} + \phi_2(t)\frac{\overline{\phi}_{2n}}{W(\phi_1, \ldots\ldots, \phi_n)(s)} + \cdots$$

$$+ \phi_n(t)\frac{\overline{\phi}_{nn}}{W(\phi_1, \ldots\ldots, \phi_n)(s)} \tag{7}$$

Thus from equation (7), we get

$$W(\phi_1, \ldots\ldots, \phi_n)(s)\gamma_{1n}(t,s) = \sum_{k=1}^{n} \phi_k(t)\,\overline{\phi}_{kn}(s)$$

where

$$\overline{\phi}_{kn} = \begin{vmatrix} \phi_1 & \cdots & \phi_{k-1} & 0 & \phi_{k+1} & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_{k-1}' & 0 & \phi_{k+1}' & \cdots & \phi_n' \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_{k-1}^{(n-1)} & 1 & \phi_{k+1}^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}$$

= $W_k(\phi_1, \ldots\ldots, \phi_n)$, as defined in the statement of the theorem.

[Explanation: As $\overline{\phi}_{1n}$ = cofactor of (n, 1)th element in φ.

$$= (-1)^{1+n} \begin{vmatrix} \phi_1 & \cdots & \cdots & \phi_n \\ \phi_2' & \cdots & \cdots & \phi_n' \\ \vdots & & & \\ \phi_2^{(n-2)} & \cdots & \cdots & \phi_n^{(n-2)} \end{vmatrix} \tag{a}$$

which is the same as the determinant

$$= \begin{vmatrix} 0 & \phi_2 & \cdots & \phi_n \\ 0 & \phi_2^{'} & \cdots & \phi_n^{'} \\ \vdots & & \cdots & \\ 0 & \phi_2^{(n-2)} & & \phi_n^{(n-2)} \\ 1 & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} \qquad \text{(b)}$$

expanded by column 1, i.e. $= (-1)^{1+n} \begin{vmatrix} \phi_2 & \cdots & \cdots & \phi_n \\ \phi_2^{'} & \cdots & \cdots & \phi_n^{'} \\ \vdots & & & \\ \phi_2^{(n-2)} & \cdots & \cdots & \phi_n^{(n-2)} \end{vmatrix}$ (c)

which is same as equation (a).

$$= W_1 (\phi_1, \ldots, \phi_n)$$

Thus we have written cofactor of $\phi_1^{(n-1)}$ as equation (b) instead of equation no. (a). Similarly $\overline{\phi}_{2n}$ = cofactor of (n, 2)th element $\phi_2^{(n-1)}$

$$= (-1)^{n+2} \begin{vmatrix} \phi_1 & \phi_3 & \cdots & \phi_n \\ \phi_1^{'} & \phi_3^{'} & \cdots & \phi_n^{'} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_3^{(n-2)} & \cdots & \phi_n^{(n-2)} \end{vmatrix} \qquad \text{(d)}$$

or

$$= \begin{vmatrix} \phi_1 & 0 & \phi_3 & \ldots & \phi_n \\ \phi_1^{'} & 0 & \phi_3^{'} & \cdots & \phi_n^{'} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & 0 & \phi_3^{(n-2)} & \ldots & \phi_n^{(n-2)} \\ \phi_1^{(n-1)} & 1 & \phi_3^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{vmatrix}$$

Now expanding this by 2^{nd} column, we get equation (d).

Thus

$\overline{\phi}_{kn}$ = cofactor element in nth row and kth, column i.e. (n, k)th element

$$= \begin{vmatrix} \phi_1 & \Lambda & \phi_{k-1} & 0 & \phi_{k+1} & \Lambda & \phi_n \\ \phi_1^{'} & & \phi_{k-1}^{'} & 0 & \phi_{k+1}^{'} & \Lambda & \phi_n^{'} \\ M & M & M & M & M & \Lambda & M \\ \\ \phi_1^{(n-1)} & M & \phi_{k-1}^{(n-1)} & 1 & \phi_{k+1}^{(n-1)} & \Lambda & \phi_1^{(n-1)} \end{vmatrix} \quad]$$

$\Rightarrow$

$W(\phi_1,\ldots,\phi_n)(s)\,\gamma_{1n}(t, s)$

$$= \sum_{k=1}^{n} \phi_k(t) W_k(\phi_{1,K,}\phi_n)(s) b(s)$$

Using this value of $\gamma_{1n}(t, s)$ in equation (5), we get

$$\psi_1(t) = \int_{\tau}^{t} \sum_{k=1}^{n} \phi_k(t) \frac{W_k(\phi_1,\Lambda\ \phi_n)(s)}{W(\phi_1,\Lambda\ \phi_n)(s)} b(s)ds$$

$$= \sum_{k=1}^{n} \phi_k(t) \int_{\tau}^{t} \frac{W_k(\phi_1,\Lambda\ \phi_n)(s)}{W(\phi_1,\Lambda\ \phi_n)(s)} b(s)ds$$

Then $\psi_1(t)$ is the solution of $L_n x = b(t)$

satisfying $\hat{\psi}_1(\tau) = 0$. (8)

we know that,

when $\psi_h(t)$ is the solution of $L_n x = 0$

s.t. $\hat{\psi}_h(\tau) = \hat{\xi}$

Then

$\psi(t) = \psi_1(t) + \psi_h(t)$ is a solution of the non-homogeneous equation $L_n x = b(t)$.

$$\hat{\psi}(\tau) = \hat{\psi}_1(\tau) + \hat{\psi}_h(\tau) = 0 + \hat{\xi} = \hat{\xi}$$

$$\therefore\ \psi(t) = \psi_h(t) + \sum_{k=1}^{n} \phi_k(t) \int_{\tau}^{t} \frac{W_k(\phi_1,\Lambda\ \phi_n)(s)}{W(\phi_1,\Lambda\ \phi_n(s)} b(s)ds \quad (9)$$

is a solution of

$$L_n x = b(t)$$

satisfying $\hat{\psi}(\tau) = \hat{\xi}$

Thus the variation of constant formula takes this special form (9) for nonhomogeneous linear differential equation of order n.

Summary

Method of variation of parameters is the tool, which makes it possible to find the solution of non-homogeneous system. Also fundamental matrix is found for linear system with constant coefficients and representation theorem is proved for linear system with periodic coefficients. Abel's Liouville formula and Lagrange Identity for an nth order linear differential equation are presented at the end of chapter.

Keywords Non-homogeneous, periodic, constant coefficients, adjoint equation, Lagrange identity.

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Chapter-3

NONLINEAR DIFFERENTIAL EQUATIONS - I

Objectives

Non-linear phenomena are of fundamental importance in various fields of science and engineering. In this chapter, attention is devoted to study the basic concepts of linear and non-linear autonomous systems. Particular emphasis is placed to study the nature of critical points of linear and non linear autonomous systems.

Introduction

The mathematical formulation of numerous physical problems results in differential equations which are actually nonlinear. In many cases it is possible to replace such as a nonlinear equation by a related linear equation which approximates the actual nonlinear equation. However, such a “linearization” is not always feasible; and when it is not, the original nonlinear equation itself must be considered. In this chapter we shall give a brief introduction to certain methods of approximation to study nonlinear equation.

Phase Plane, Paths and Critical Points

A. Basic concepts and Definitions

In this chapter we shall be concerned with second order nonlinear differential equations of the form

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right). \tag{1}$$

A specific example of such an equation is van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0, \tag{2}$$

where μ is a positive constant. Equation (2) may be put in the form (1) where

$$F\left(x, \frac{dx}{dt}\right) = -\mu(x^2 - 1)\frac{dx}{dt} - x.$$

Let us suppose that the differential equation (1) describes a certain dynamical system having one degree of freedom. The state of this system at time t is determined by the values of x (position) and dx/dt (velocity). The plane of the variables x and dx/dt is called a phase plane.

If we let y = dx/dt, we can replace the second-order equation (1) by the equivalent system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = F(x, y). \tag{3}$$

We can determine information about equation (1) from a study of the system (3). In particular we shall be interested in the configurations formed by the curves which the solutions of (3) define. We shall regard t as a parameter so that these curves will appear in the xy plane. Since y = dx/dt, this xy plane is simply the x, dx/dt phase plane .

More generally, we shall consider systems of the form

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y), \tag{4}$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t appears only in the differentials dt of the left members and not explicitly in the functions P and Q on the right, is called an autonomous system. We shall now proceed to study the configurations formed in the xy phase plane by the curves which are defined by the solutions of (4).

From existence and uniqueness theorem, it follows that, given any number t_0 and any pair (x_0, y_0) of real number, there exists a unique solution

$$x = f(t),$$

$$y = g(t), \tag{5}$$

of the system (4) such that

$$f(t_0) = x_0$$

$$g(t_0) = y_0.$$

If f and g are not both constant functions, then (5) defines a curve in the xy plane which we shall call a path (or orbit or trajectory) of the system (4).

If the ordered pair of functions defined by (5) is a solution of (4) and t_1 is any real number, then the ordered pair of functions defined by

$$x = f(t - t_1),$$

$$y = g(t - t_1), \qquad (6)$$

is also a solution of (4). Assuming that f and g in (5) are not both constant functions and that $t_1 \neq 0$, the solutions defined by (5) and (6) are two different solutions of (4). These two different solutions are simply different parameterizations of the same path. We thus observe that the terms solution and path are not synonymous. On the one hand, a solution of (4) is an ordered pair of functions (f, g) such that $x = f(t)$, $y = g(t)$ simultaneously satisfy the two equations of the system (4) identically; on the other hand, a path of (4) is a curve in the xy phase plane which may be defined parametrically by more than one solution of (4).

Eliminating t between the two equations of the system (4), we obtain the equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \qquad (7)$$

This equation gives the slope of the tangent to the path of (4) passing through the point (x, y), provided the functions P and Q are not both zero at this point. The one parameter family of solutions of (7) thus provides the one-parameter family of paths of (4). However, the description (7) does not indicate the directions associated with these paths.

At point (x_0, y_0) at which both P and Q are zero, the slope of the tangent to the path, as defined by (7), is indeterminate. Such points are singled out in the following definition.

Definition

Given the autonomous system

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y) \tag{4}$$

a point (x_0, y_0) at which both

$$P(x_0, y_0) = 0 \text{ and } Q(x_0, y_0) = 0$$

is called a critical point (equilibrium point or singular point) of (4).

Example 3.1

Consider the linear autonomous system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -x. \tag{8}$$

We find the general solution of this system as

$$x = c_1 \sin t - c_2 \cos t,$$

$$y = c_1 \cos t + c_2 \sin t,$$

where c_1 and c_2 are arbitrary constants. The solution satisfying the conditions $x(0) = 0$, $y(0) = 1$ is found to be

$$x = \sin t,$$

$$y = \cos t. \tag{9}$$

This solution defines a path C_1 in the xy plane. The solution satisfying the condition $x(0) = -1$, $y(0) = 0$ is

$$x = \sin (t - \pi/2),$$

$$y = \cos (t - \pi/2), \tag{10}$$

The solution (10) is different from the solution (9), but (10) also defines the same path C_1. That is, the ordered pairs of functions defined by (9) and (10) are two different solutions of (8) which are different parameterization of the same path C_1. Eliminating t from either (9) or (10) we obtain the equation $x^2 + y^2 = 1$ of the path C_1 in the xy phase plane. Thus the path C_1 is the circle with centre at (0, 0) and radius 1. From either (9) or (10) we can see that the direction associated with C_1 is the clockwise direction.

Eliminating t between the equations of the system (8) we obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{11}$$

which gives the slope of the tangent to the path of (8) passing through the point (x, y), provided $(x, y) \neq (0, 0)$.

The one –parameter family of solutions

$$x^2 + y^2 = c^2$$

of equation (11) gives the one-parameter family of paths in the xy phase plane. Several of these are shown in figure 3.1. The path C_1 referred to above is, of course, that for which c = 1.

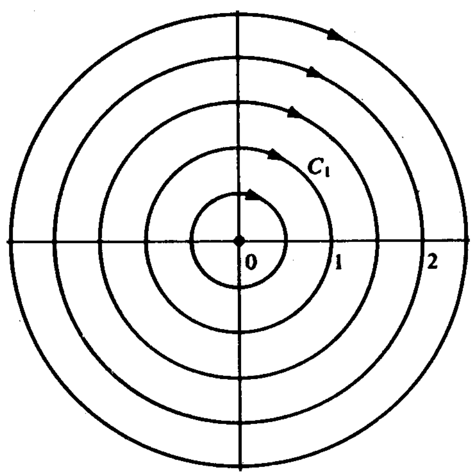


Figure 3.1

Looking back at the system (8), we see that P(x, y) = y and Q(x, y) = -x. Therefore the only critical point of the system is the origin (0, 0). Given any real number t_0, the solution x = f(t), y = g(t) such that $f(t_0) = g(t_0) = 0$ is simply

$$x = 0,$$

$$y = 0, \quad \text{for all } t$$

We now introduce certain basic concepts dealing with critical points and paths

Definition

A critical point (x_0, y_0) of the system (4) is called **isolated** if there exists a circle

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of (4) within this circle.

In what follows we shall assume that every critical point is isolated.

Note For convenience, we shall take the critical point (x_0, y_0) to be the origin (0, 0). There is no loss of generality in doing this, for if $(x_0, y_0) \neq (0, 0)$, then the translation of coordinates $\xi = x - x_0$, $\eta = y - y_0$ transforms (x_0, y_0) into the origin in the $\xi\eta$ plane.

Definition

Let C be a path of the system (4), and let $x = f(t)$, $y = g(t)$ be a solution of (4) which represents C parametrically. Let (0, 0) be a critical point of (4). We shall say that the path C approaches the critical point (0, 0) as $t \to +\infty$ if

$$\lim_{t\to+\infty} f(t) = 0, \qquad \lim_{t\to+\infty} g(t) = 0. \tag{12}$$

Thus when we say that a path C defined parametrically by $x = f(t)$, $y = g(t)$ approaches the critical point (0, 0) at $t \to +\infty$, we understand the following: a point R tracing out C according to the equations $x = f(t)$, $y = g(t)$ will approach the point (0, 0) at $t \to +\infty$. This approach of a path C to the critical point (0, 0) is independent of the solution actually used to represent C. That is, if C approaches (0, 0) as $t \to +\infty$, then (12) is true for all solutions $x = f(t)$, $y = g(t)$ representing C.

In like manner, a path C_1 approaches the critical point (0, 0) as $t \to -\infty$ if

$$\lim_{t\to-\infty} f_1(t) = 0, \qquad \lim_{t\to-\infty} g_1(t) = 0.$$

where $x = f_1(t)$, $y = g_1(t)$ is a solution defining the path C_1.

Definition

Let C be a path of the system (4) which approaches the critical point (0, 0) of (4) as t $\to +\infty$, and let x = f(t), y = g(t) be a solution of (4) which represents C parametrically. We say that C enters the critical point (0, 0) as t $\to +\infty$ if

$$\lim_{t \to +\infty} \frac{g(t)}{f(t)} \tag{13}$$

exists or if the quotient in (13) becomes either positively or negatively infinite as t $\to + \infty$.

We observe that the quotient g(t)/f(t) in (13) represents the slope of the line joining critical point (0, 0) and a point R with coordinates [f(t), g(t)] on C. Thus when we say that a path C enters the critical point (0, 0) as t $\to + \infty$ we mean that the line joining (0, 0) and a point R tracing out C approaches a define “limiting” direction as t $\to + \infty$.

B. Types of Critical points

We shall now discuss certain types of critical points .

1 Center

(1) The critical point (0, 0) of Figure 3.2 is called a center. Such a point is surrounded by an infinite family of closed paths, members of which are arbitrarily close to (0, 0) but is not approached by any path either as t $\to + \infty$ or as t $\to -\infty$.

Definition

The isolated critical point (0, 0) of (4) is called a **center** if there exists a neighbourhood of (0, 0) which contains a countably infinite number of closed paths P_n(n = 1, 2, 3….,), each of which contains (0, 0) in its interior, and which are such that the diameters of the paths approach 0 as n $\to \infty$ [but (0, 0) is not approached by any path either as t $\to + \infty$ or t $\to - \infty$]

1. We define a neighbourhood of (0, 0) to be the set of all points (x, y) lying within some fixed (positive) distance d of (0, 0).

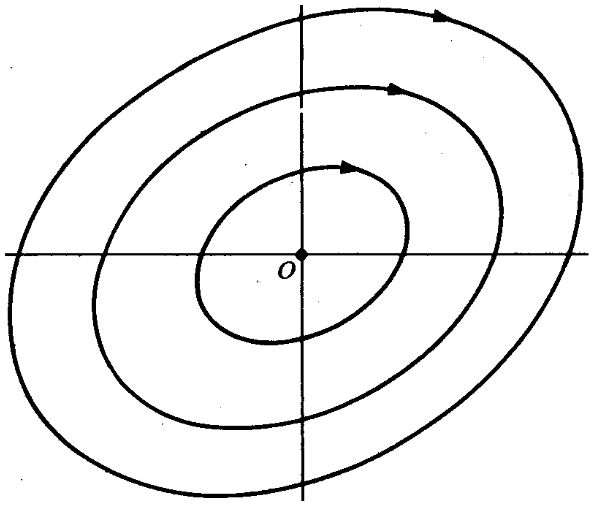


Figure 3.2

2. An infinite set is called countable if it can be put into one-to-one correspondence with the set of all positive integers. An example of a countable set is the set of all rational numbers.

3. An infinite set is uncountable if it is not countable. An example of an uncountable set is the set of all real numbers.

2. Saddle Point

The critical point (0, 0) of Figure 3.3 is called a saddle point. Such a point may be characterized as follows:

1. It is approached and entered by two half line paths (AO and BO) $t \to +\infty$, these two paths forming the geometric curve AB.

2. It is approached and entered by two half line paths (CO and DO) as $t \to -\infty$, these two paths forming the geometric curve CD.

3. Between the four half-line paths described in (1) and (2) there are four domains R_1, R_2, R_3, R_4, each containing an infinite family of semi-hyperbolic like paths which do not approach O as $t \to +\infty$ or as $t \to -\infty$, but which

become asymptotic to one or another of the four half line paths as t → + ∞ and as t → - ∞.

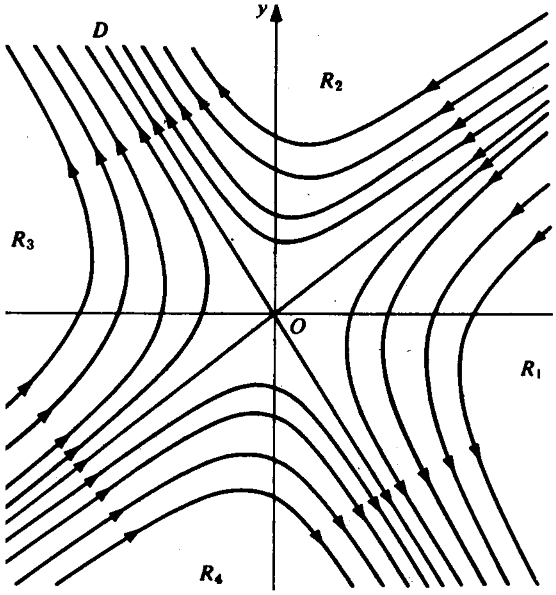


Figure 3.3

Definition

The isolate critical point (0, 0) of (4) is called a **saddle** point if there exists a neighbourhood of (0, 0) in which the following two conditions hold:

1. There exist two paths which approach and enter (0, 0) from a pair of opposite directions as t → + ∞ and there exist two paths which approach and enter (0, 0) from a different pair of opposite directions as t → - ∞.

2. In each of the four domains between any two of the four directions in (1) there are infinitely many paths which are arbitrarily close to (0, 0) but which do not approach (0, 0) either as t → + ∞ or as t → - ∞.

3. Spiral Point

The critical point (0, 0) of figure 3.4 is called a spiral point (or focal point). Such a point is approached in a spiral-like manner by an infinite family of paths as t → + ∞ (or as t → - ∞). Observe that while the paths approach O, they do not enter it. That is a point R tracing such a path C approaches O as t → + ∞ (or as t → - ∞), but the line OR does not tend to a definite direction, since the path constantly winds about O.

Definition

The isolated critical point (0, 0) of (4) is called a **spiral** point (or **focal** point) if there exists a neighbourhood of (0, 0) such that every path P in this neighbourhood has the following properties:

1. P is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0;

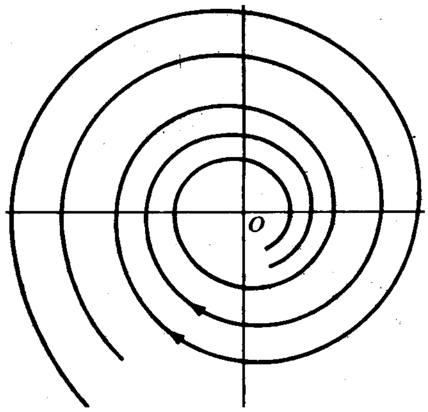


Figure 3.4

2. P approaches (0, 0) as $t \to +\infty$ (or as $t \to -\infty$); and

3. P approaches (0, 0) in a spiral like manner, winding around (0, 0) an infinite number of times as $t \to +\infty$ (or as $t \to -\infty$).

4. Node

The critical point (0, 0) of Figure 3.5 is called a node. Such a point is not only approached but also entered by an infinite family of paths as $t \to +\infty$ (or as $t \to -\infty$). That is a point R tracing such a path not only approaches O but does so in such a way that the line OR tends to a definite direction as $t \to +\infty$ (or as $t \to -\infty$). For the node shown in Figure 3.5 there are four rectilinear paths, AO, BO, CO, and DO. All other paths are like "semiparabolas." As each of these semiparabolic-like paths approaches O, its slope approaches that of the line AB.

Definition

The isolated critical point (0, 0) of (4) is called a **node** if there exists a neighbourhood of (0, 0) such that every path P in this neighbourhood has the following properties:

1. P is defined for all $t > t_0$ [or for all $t < t_0$] for some number t_0

2. P approaches (0, 0) as $t \to +\infty$ (or as $t \to -\infty$); and

3. P enters (0, 0) as $t \to +\infty$ [or as $t \to -\infty$]

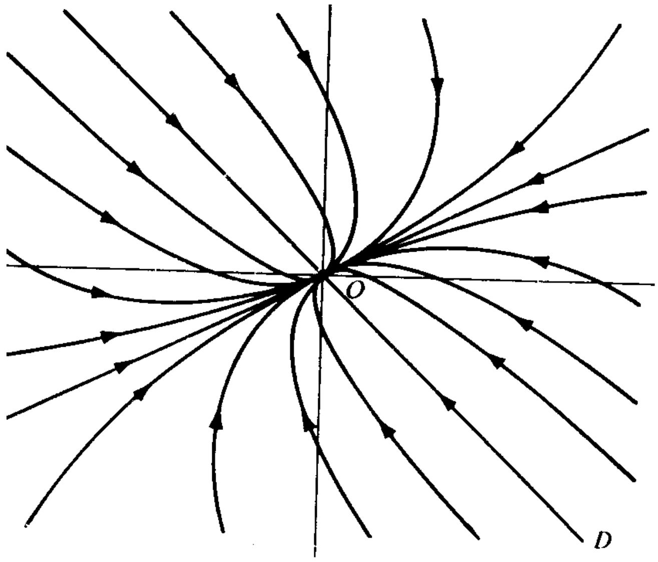


Figure 3.5

C. Stability

We assume that (0, 0) is an isolated critical point of the system

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y) \tag{4}$$

and proceed to introduce the concepts of stability and asymptotic stability for this critical point.

Definition

Assume that (0, 0) is an isolated critical point of the system (4). Let C be path of (4); let x = f(t), y = g(t) be a solution of (4) defining C parametrically. Let

$$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2} \tag{14}$$

denotes the distance between the critical point (0, 0) and the point R: [f,(t), g(t)] on C. The critical point (0, 0) is called **stable** if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that the following is true: Every path C for which

$$D(t_0) < \delta \text{ for some value } t_0 \tag{15}$$

is defined for all $t \geq t_0$ and is such that

$$D(t) < \varepsilon \quad \text{for } t_0 \leq t < \infty. \tag{16}$$

Let us analyze this definition, making use of Figure 3.6. The critical point (0, 0) is said to be stable if, corresponding to every positive number ε, we can find another positive number δ which dose "something" for us. Now what is this "something"? To answer this we must understand what the inequalities in (15) and (16) mean. According to (14), the inequality $D(t_0) < \delta$ for some value t_0 in (15) means that the distance between the critical point (0, 0) and the point R on the path C must be less than δ at $t = t_0$. This means that at $t = t_0$, R lies within the circle K_1 of radius δ about (0, 0) (Figure 3.6). Likewise the inequality $D(t) < \varepsilon$ for $t_0 \leq t < \infty$ in (16) means that the distance between (0, 0) and R is less than ε for all $t \geq t_0$ and hence that for $t \geq t_0$, R lies within the circle K_2 of radius ε about (0, 0). Now we can understand the "something" which the number δ does for us. If (0, 0) is stable, then every path C which is inside the circle K_1 of radius δ at $t = t_0$ will remain inside the circle K_2 of radius $t \geq t_0$. Roughly speaking, if every path C stays as close to (0, 0) as we want it to (that is, within distance ε) after it once gets close enough (that is within distance δ), then (0, 0) is stable.

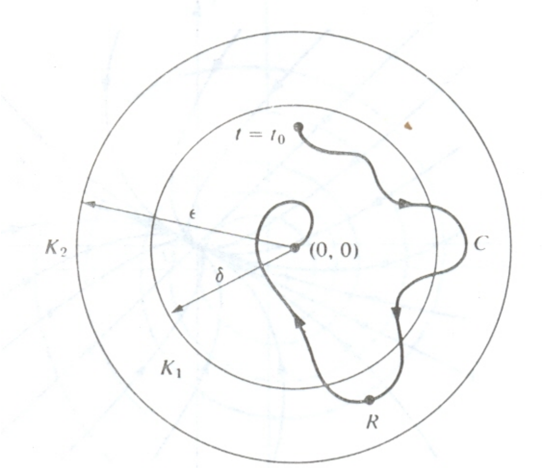


Figure 3.6

Definition

Assume that (0, 0) is an isolated critical point of the system (4). Let C be a path of (4); and let x = f(t), y = g(t) be a solution of (4) representing C parametrically. Let

$$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2} \tag{14}$$

denotes the distance between the critical point (0, 0) and the point R: [f(t), g(t)] on C. The critical point (0, 0) is called asymptotically stable if (1) it is stable and (2) there exists a number $\delta_0 > 0$ such that if

$$D(t_0) < \delta_0 \tag{17}$$

for some value t_0, then

$$\lim_{t \to +\infty} f(t) = 0, \quad \lim_{t \to +\infty} g(t) = 0, \tag{18}$$

To analyze this definition, note that condition (1) requires that (0, 0) must be stable. That is, every path C will stay as close to (0, 0) as we desire after it once gets sufficiently close. But asymptotic stability is a stronger condition than mere stability. For, in addition to stability, the condition (2) requires that every path that gets sufficiently close to (0, 0) [see 17] ultimately approaches (0, 0) as $t \to +\infty$ [see 18]. Note that the path C of figure 3.6 has this property.

Definition

A critical point is called **unstable** if it is not stable.

As illustration of stable critical points, we point out that the center in Figure 3.2, the spiral point in Figure 3.4, and the node in Figure 3.5 are all stable. Of these three, the spiral point and the node are asymptotically stable. If the directions of the paths in Figures 3.4 and 3.5 have been reversed, the spiral point and the node of these respective figures would have both been unstable. The saddle point of Figure 3.3 is unstable

Exercise 3.1

For the autonomous system in Exercise 1 (a) find the real critical points of the system, (b) obtain the differential equation which gives the slope of the tangent to the paths of the system, and (c) solve this differential equation to obtain the one-parameter family of paths.

1. $$\frac{dx}{dt} = x - y^2,$$

$$\frac{dy}{dt} = x^2 - y.$$

2. Consider the linear autonomous system

$$\frac{dx}{dt} = x,$$

$$\frac{dy}{dt} = x + y.$$

(a) Find the solution of this system which satisfies the conditions x(0) = 1, y(0) = 3.

(b) Find the solution of this system which satisfies the conditions x(4) = e, y(4) = 4e.

(c) Show that the two different solutions found in (a) and (b) both represent the same path.

(d) Find the differential equation which gives the slope of the tangent to the paths, and solve it to obtain the one-parameter family of paths.

Answers to Exercise

1. (a) (0, 0), (1, 1); (b) $\frac{dy}{dx} = \frac{x^2 - y}{x - y^2}$; (c) $x^3 - 3xy + y^3 = c$

2. (a) $x = e^t$, $y = 3e^t + te^t$; (b) $x = e^{t-3}$, $y = te^{t-3}$;

(c) The equation of the common path is y = x(ln |x| + 3);

(d) The family of paths is y = x(ln |x| + c).

Critical Points and Paths of Linear Systems

A. Basic-Theorems

In this chapter we want to study nonlinear differential equations and the corresponding nonlinear autonomous systems of the form (4). We shall be interested in classifying the critical points of such nonlinear systems. We shall see that under appropriate circumstances we may replace a given nonlinear system by a related linear system and then employ this linear system to determine the nature of the critical point of the given system. Thus in this section we shall first investigate the critical points of a linear autonomous system.

We consider the linear system

$$\frac{dx}{dt} = ax + by, \qquad\qquad \frac{dy}{dt} = cx + dy, \tag{19}$$

where a, b, c, and d are real constants. The origin (0, 0) is clearly a critical point of (19). We assume that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, \tag{20}$$

and hence (0, 0) is the only critical point of (19).

We can find solutions of (19) of the form

$$x = Ae^{\lambda t}, \qquad\qquad y = Be^{\lambda t} \tag{21}$$

If (21) is to be a solution of (19), then λ must satisfy the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0, \tag{22}$$

called the characteristic equation of (19). Note that by condition (20), zero cannot be a root of the equation (22) in the problem under discussion.

Let λ_1 and λ_2 be the roots of the characteristic equation (22). We shall prove that the nature of the critical point (0, 0) of the system (19) depends upon the nature of the roots λ_1 and λ_2 . For this we must consider the following five cases:

1. λ_1 and λ_2 are real, unequal, and of the same sign.

2. λ_1 and λ_2 are real, unequal, and of opposite sign.

3. λ_1 and λ_2 are real and equal.

4. λ_1 and λ_2 are conjugate complex but not pure imaginary.

5. λ_1 and λ_2 are pure imaginary.

Case 1

Theorem 3.1

Hypothesis The roots λ_1 and λ_2 of the characteristic equation (22) are real, unequal, and of the same sign.

Conclusion The critical point (0, 0) of the linear system (19) is a node.

Proof We first assume that λ_1 and λ_2 are both negative and take $\lambda_1 < \lambda_2 < 0$. Then the general solution of the system (19) may be written as

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$
$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}, \qquad (23)$$

where A_1, B_1, A_2 and B_2 are definite constants and $A_1B_2 \neq A_2B_1$, and where c_1 and c_2 are arbitrary constants. Choosing $c_2 = 0$ we obtain the solutions

$$x = c_1 A_1 e^{\lambda_1 t},$$
$$y = c_1 B_1 e^{\lambda_1 t}. \qquad (24)$$

Choosing $c_1 = 0$ we obtain the solutions

$$x = c_2 A_2 e^{\lambda_2 t},$$
$$y = c_2 B_2 e^{\lambda_2 t}. \qquad (25)$$

For any $c_1 > 0$, the solution (24) represents a path consisting of "half" of the line $B_1x = A_1y$ of slope B_1/A_1. For any $c_1 < 0$, they represent a path consisting of the "other half" of this line. Since $\lambda_1 < 0$, both of these half-line paths approach (0,0) as $t \to +\infty$. Also, since $y/x = B_1/A_1$, these two paths enter (0, 0) with slope B_1/A_1.

Similarly, for any $c_2 > 0$ the solutions (25) represent a path consisting of "half" of the line $B_2x = A_2y$; while for any $c_2 < 0$, the path consists of the "other half" of this line. These two half-line paths also approach (0, 0) as $t \to +\infty$ and enter it with slope B_2/A_2.

Thus the solutions (24) and (25) provide us with four half-line paths which all approach and enter (0, 0) as $t \to +\infty$.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (23) represents nonrectilinear paths. Again, since $\lambda_1 < \lambda_2 < 0$, all of these paths approach (0, 0) as $t \to +\infty$. Also, since

$$\frac{y}{x} = \frac{c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}}{c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}} = \frac{(c_1 B_1 / c_2) e^{(\lambda_1 - \lambda_2)t} + B_2}{(c_1 A_1 / c_2) e^{(\lambda_1 - \lambda_2)t} + A_2},$$

we have

$$\lim_{t \to \infty} \frac{y}{x} = \frac{B_2}{A_2}$$

and so all of these paths enter (0, 0) with limiting slope B_2/A_2.

Thus all the paths (both rectilinear and nonrectilinear) enter (0, 0) as $t \to +\infty$, and all except the two rectilinear ones defined by (24) enter with slope B_2/A_2. Thus, the critical point (0, 0) is a node. Clearly, it is asymptotically stable. A qualitative diagram of the paths appears in Figure 3.7.

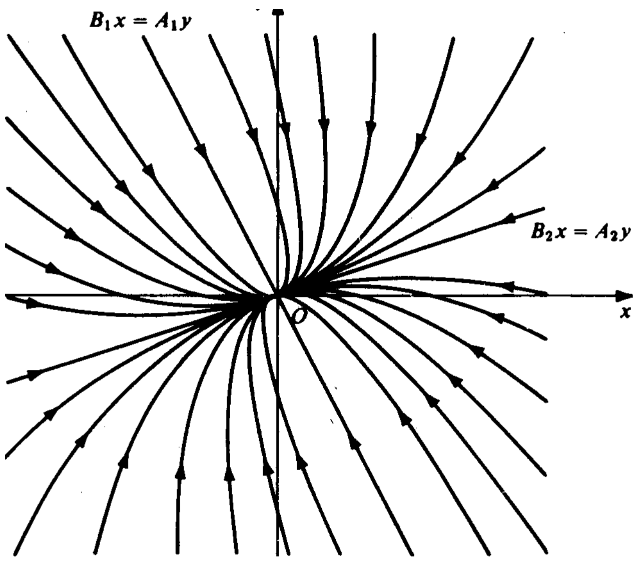


Figure 3.7

If now λ_1 and λ_2 are both positive and we take $\lambda_1 > \lambda_2 > 0$, we see that the general solution of (19) is still of the form (23) and particular solutions of the forms (24) and (25) still exist. The situation is the same as before, except all the paths approach and enter (0, 0) as $t \to -\infty$. The qualitative diagram of Figure 3.7 is unchanged, except that all the arrows now point in the opposite directions. The critical point (0, 0) is still a node, but in this case it is clear that it is unstable.

Theorem 3.2

Hypothesis The roots λ_1 and λ_2 of the characteristic equation (22) are real, unequal, and of opposite sign.

Conclusion The critical point (0, 0) of the linear system (19) is a saddle point.

Proof We assume that $\lambda_1 < 0 < \lambda_2$. The general solution of the system (19) may again be written in the form (23) and particular solutions of the forms (24) and (25) are again present.

For any $c_1 > 0$, the solutions (24) again represent a path consisting of "half" the line $B_1x = A_1y$; while for any $c_1 < 0$, they again represent a path consisting of the "other half" of this line. Also, since $\lambda_1 < 0$, both of these half-line paths still approach and enter (0, 0) as $t \to +\infty$.

Also, for any $c_2 > 0$, the solutions (25) represent a path consisting of "half" the line $B_2x = A_2y$; and for any $c_2 < 0$, the path which they represent consists of the "other half" of this line. But in this case, since $\lambda_2 > 0$, both of these half-line paths now approach and enter (0, 0) as $t \to -\infty$.

Once again, if $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (23) represents non-rectilinear paths. But here since $\lambda_1 < 0 < \lambda_2$, none of these paths can approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$. Further, none of them pass through (0, 0) for any t_0 such that $-\infty < t_0 < +\infty$. As $t \to +\infty$, we see from (23) that each of these non-rectilinear paths becomes asymptotic to one of the half-line paths defined by (25). As $t \to -\infty$, each of them becomes asymptotic to one of the paths defined by (24).

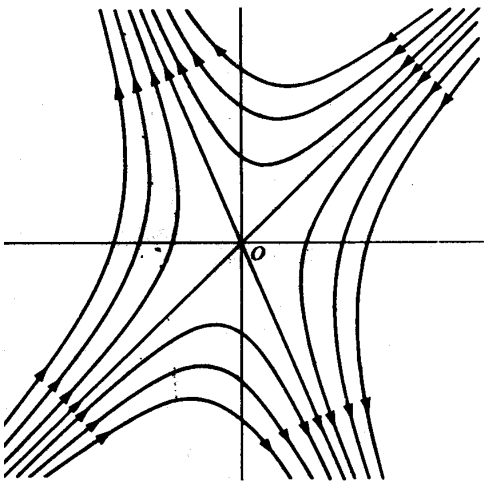


Figure 3.8

Thus there are two half-line paths which approach and enter (0, 0) as $t \to +\infty$ and two other half-line paths which approach and enter (0, 0) as $t \to -\infty$. All other paths are non-rectilinear paths which do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$, but which become asymptotic to one or another of the four half-line paths as $t \to +\infty$ and as $t \to -\infty$. Thus, the critical point (0, 0) is a saddle point. Clearly, it is unstable. A qualitative diagram of the paths appears in Figure 3.8.

Case 3

Theorem 3.3

Hypothesis The roots λ_1 and λ_2 of the characteristic equation (22) are real and equal.

Conclusion The critical point (0, 0) of the linear system (19) is a node.

Proof Let us first assume that $\lambda_1 = \lambda_2 = \lambda < 0$. We consider two subcases:

(a) $a = d \neq 0$, $b = c = 0$

(b) All other possibilities leading to a double root.

We consider first the special case (a). The characteristic equation (22) becomes

$$\lambda^2 - 2a\lambda + a^2 = 0$$

and hence $\lambda = a = d$. The system (19) now becomes

$$\frac{dx}{dt} = \lambda x,$$

$$\frac{dy}{dt} = \lambda y,$$

The general solution of this system is clearly

$$x = c_1 e^{\lambda t},$$

$$y = c_2 e^{\lambda t}, \tag{26}$$

where c_1 and c_2 are arbitrary constants. The paths defined by (26) for the various values of c_1 and c_2 are half-lines of all possible slopes. Since $\lambda < 0$, we see that each of these half-lines approaches and enters (0, 0) as $t \to +\infty$. That is, all the paths of the system enter (0, 0) as $t \to +\infty$. Thus, the critical point (0, 0) is a node. Clearly, it is asymptotically stable. A qualitative diagram of the paths appears in Figure 3.9.

If $\lambda > 0$, the situation is the same except that the paths enter (0, 0) as $t \to -\infty$, the node (0, 0) is unstable, and the arrows in figure 3.9 are all reversed.

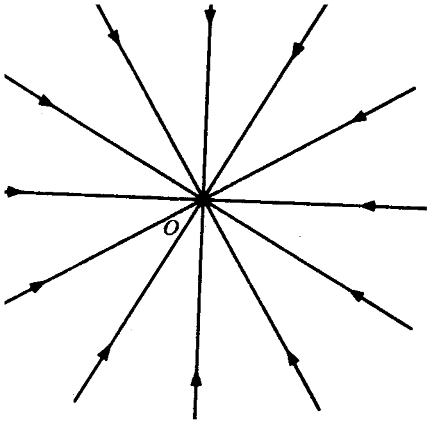


Figure 3.9

This type of node is sometimes called a star-shaped node.

Let us now consider case (b). Here the characteristic equation has the double root $\lambda < 0$, but we exclude the special circumstances of case (a). We know that the general solution of the system (19) may in this be written

$$x = c_1Ae^{\lambda t} + c_2(A_1t + A_2)e^{\lambda t},$$

$$y = c_1Be^{\lambda t} + c_2(B_1t + B_2)e^{\lambda t}, \tag{27}$$

where the A's and B's are definite constants, c_1 and c_2 are arbitrary constants, and $B_1/A_1 = B/A$. Choosing $c_2 = 0$ in (27) we obtain solutions

$$x = c_1Ae^{\lambda t},$$

$$y = c_1Be^{\lambda t}. \tag{28}$$

For any $c_1 > 0$, the solutions (28) represent a path consisting of “half" of the line $Bx = Ay$ of slope B/A; for any $c_1 < 0$, they represent a path which consists of the "other half" of this line. Since $\lambda < 0$, both of these half-line paths approach (0, 0) as $t \to +\infty$. Further, since $y/x = B/A$, both paths enter (0, 0) with slope B/A.

If $c_2 \neq 0$, the general solution (27) represents nonrectilinear paths. Since $\lambda < 0$, we see from (27) that

$$\lim_{t\to+\infty} x = 0, \qquad \lim_{t\to+\infty} y = 0.$$

Thus the nonrectilinear paths defined by (27) all approach (0, 0) as t $\to +\infty$. Also, since

$$\frac{y}{x} = \frac{c_1 B e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t}}{c_1 A e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t}} = \frac{(c_1 B / c_2) + B_2 + B_1 t}{(c_1 A / c_2) + A_2 + A_1 t}$$

we see that

$$\lim_{t\to+\infty} \frac{y}{x} = \frac{B_1}{A_1} = \frac{B}{A}$$

Thus all the nonrectilinear paths enter (0, 0) with limiting slope B/A.

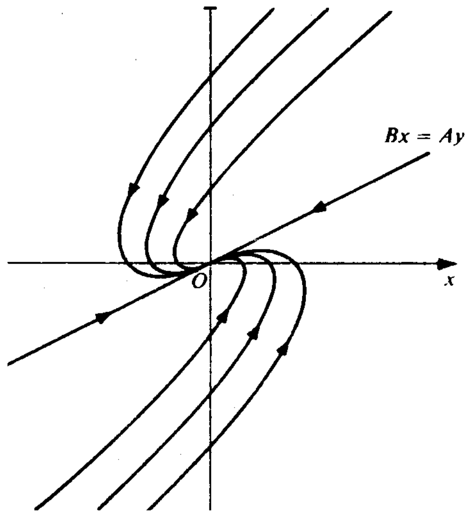


Figure 3.10

Thus all the paths (both rectilinear and nonrectilinear) enter (0, 0) as t $\to +\infty$ with slope B/A. Thus, the critical point (0, 0) is a node. Clearly, it is asymptotically stable. A qualitative diagram of the paths appears in Figure 3.10.

If $\lambda > 0$, the situation is again the same except that the paths enter (0, 0) as t $\to -\infty$, the node (0, 0) is unstable, and the arrows in Figure 3.10 are reversed.

Case 4

Theorem 3.4

Hypothesis The roots λ_1 and λ_2 of the characteristic equation (22) are conjugate complex with real part not zero (that is, not pure imaginary).

Conclusion The critical point (0, 0) of the linear system (19) is a spiral point.

Proof Since λ_1 and λ_2 are conjugate complex with real part not zero, we may write these roots as $\alpha \pm i\beta$, where α and β are both real and unequal to zero. Then the general solution of the system (19) may be written as

$$x = e^{\alpha t}[c_1(A_1 \cos \beta t - A_2 \sin \beta t) + c_2(A_2 \cos \beta t + A_1 \sin \beta t)]$$

$$y = e^{\alpha t}[c_1(B_1 \cos \beta t - B_2 \sin \beta t) + c_2(B_2 \cos \beta t + B_1 \sin \beta t)] \qquad (29)$$

Where A_1, A_2, B_1 and B_2 are definite real constants and c_1 and c_2 are arbitrary constants.

Let us first assume that $\alpha < 0$. Then from (29) we see that

$$\lim_{t \to +\infty} x = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} y = 0.$$

and hence all paths defined by (29) approach (0, 0) as $t \to +\infty$. We may rewrite (29) in the form

$$x = e^{\alpha t}[c_3 \cos \beta t + c_4 \sin \beta t),$$

$$y = e^{\alpha t}[c_5 \cos \beta t + c_6 \sin \beta t), \qquad (30)$$

where $c_3 = c_1A_1 + c_2A_2$, $c_4 = c_2A_1 - c_1A_2$, $c_5 = c_1B_1 + c_2B_2$ and $c_6 = c_2B_1 - c_1B_2$. Assuming c_1 and c_2 are real, the solutions (30) represent all paths in the real xy phase plane. We may now put these solutions in the form

$$x = K_1 e^{\alpha t} \cos(\beta t + \phi_1),$$

$$y = K_2 e^{\alpha t} \cos(\beta t + \phi_2), \qquad (31)$$

where $K_1 = \sqrt{c_3^2 + c_4^2}$, $K_2 = \sqrt{c_5^2 + c_6^2}$, and ϕ_1 and ϕ_2 are defined by the equations

$$\cos\phi_1 = \frac{c_3}{K_1}, \quad \cos\phi_2 = \frac{c_5}{K_2},$$

$$\sin\phi_1 = -\frac{c_4}{K_1}, \quad \sin\phi_2 = -\frac{c_6}{K_2}.$$

Let us consider

$$\frac{y}{x} = \frac{K_2 e^{\alpha t}\cos(\beta t + \phi_2)}{K_1 e^{\alpha t}\cos(\beta t + \phi_1)}. \qquad (32)$$

Letting K = K_2/K_1 and $\phi_3 = \phi_1 - \phi_2$, this becomes

$$\frac{y}{x} = \frac{K\cos(\beta t + \phi_1 - \phi_3)}{\cos(\beta t + \phi_1)}$$

$$= K\left[\frac{\cos(\beta t + \phi_1)\cos\phi_3 + \sin(\beta t + \phi_1)\sin\phi_3}{\cos(\beta t + \phi_1)}\right] \qquad (33)$$

$$= K[\cos\phi_3 + \sin\phi_3 \tan(\beta t + \phi_1)],$$

provided $\cos(\beta t + \phi_1) \neq 0$. As trigonometric functions involved in (32) and (33) are periodic, we conclude from these expressions that $\lim_{t\to+\infty} \frac{y}{x}$ does not exist and so the paths do not enter (0, 0). Instead, it follows from (32) and (33) that the paths approach (0, 0) in a spiral-like manner, winding around (0, 0) an infinite number of times as $t \to +\infty$. Thus, the critical point (0, 0) is a spiral point. Clearly, it is asymptotically stable. A qualitative diagram of the paths appears in Figure 3.11.

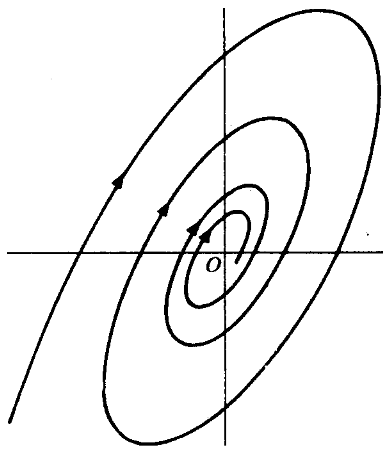


Figure 3.11

If $\alpha > 0$, the situation is the same except that the paths approach (0, 0) as $t \to -\infty$, the spiral point (0, 0) is unstable, and the arrows in Figure 3.11 are reversed.

Case 5

Theorem 3.5

Hypothesis The roots λ_1 and λ_2 of the characteristic equation (22) are pure imaginary.

Conclusion The critical point (0, 0) of the linear system (19) is a center.

Proof Since λ_1 and λ_2 are pure imaginary we may write them as $\alpha \pm i\beta$, where $\alpha = 0$ but β is real and unequal to zero. Then the general solution of the system (19) is of the form (29), where $\alpha = 0$. In the notation of (31) all real solutions may be written in the form

$$
\begin{aligned}
x &= K_1 \cos(\beta t + \phi_1), \\
y &= K_2 \cos(\beta t + \phi_2),
\end{aligned}
\tag{34}
$$

where K_1, K_2, ϕ_1 and ϕ_2 are defined as before. The solutions (34) define the paths in the real xy phase plane. Since the trigonometric functions in (34) oscillate indefinitely between ± 1 as $t \to +\infty$ and as $t \to -\infty$, the paths do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$. Rather it is clear from (34) that x and y are periodic functions of t and hence the paths are closed curves surrounding (0, 0), members of which are arbitrarily close to (0, 0). Indeed they are an infinite family of ellipses with center at (0, 0). Thus, the critical point (0, 0) is a center. Clearly, it is stable. However, since the paths do not approach (0, 0), the critical point is not asymptotically stable. A qualitative diagram of the paths appears in Figure 3.12.

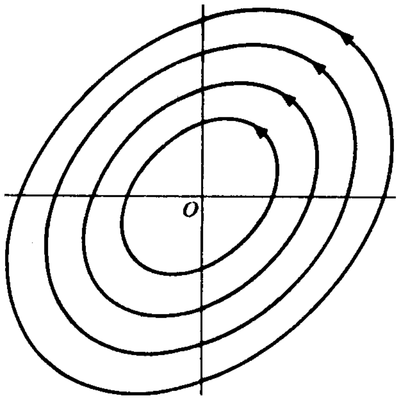


Figure 3.12

We summarize our results in Table 3.1. The stability results of column 3 of this table lead to Theorem 3.6.

Theorem 3.6

Consider the linear system

$$
\begin{aligned}
\frac{dx}{dt} &= ax + by, \\
\frac{dy}{dt} &= cx + dy,
\end{aligned}
\tag{19}
$$

where ad - bc $\neq$0, so that (0, 0) is the only critical point of the system.

1. If both roots of the characteristic equation (22) are real and negative or conjugate complex with negative real parts, then the critical point (0, 0) of (19) is asymptotically stable.

2. If the roots of (22) are pure imaginary, then the critical point (0, 0) of (19) is stable, but not asymptotically stable.

3. If either of the roots of (22) is real and positive or if the roots are conjugate complex with positive real parts, then the critical point (0, 0) of (19) is unstable.

Table 3.1

Nature of roots λ_1 and λ_2 of characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$	Nature of critical point (0, 0) of linear system $\frac{dx}{dt} = ax + by.$ $\frac{dy}{dt} = cx + dy.$	Stability of critical point (0, 0)
real, unequal, and of same sign	Node	asymptotically stable if roots are negative, unstable if roots are positive
real, unequal, and of opposite sign	Saddle point	Unstable
real and equal	node	asymptotically stable if roots are negative, unstable if roots are positive
conjugate complex but not pure imaginary	spiral point	asymptotically stable if real part of roots are negative, unstable if real part of roots are positive
pure imaginary	center	Stable, but not asymptotically stable.

B. Examples and Applications

Example 3.2

Determine the nature of the critical point (0, 0) of the system

$$\frac{dx}{dt} = 2x - 7y, \qquad\qquad (35)$$
$$\frac{dy}{dt} = 3x - 8y,$$

and determine whether or not the point is stable.

Solution The system (35) is of the form (19) where a = 2, b = - 7, c = 3, and d = - 8. The characteristic equation (22) is

$$\lambda^2 + 6\lambda + 5 = 0.$$

Hence the roots of the characteristic equation are λ_1 = -5, λ_2 = -1. Since the roots are real, unequal, and of the same sign (both negative), we conclude that the critical point (0, 0) of (35) is a node. Since the roots are real and negative the path is asymptotically stable.

Example 3.3

Determine the nature of critical point (0, 0) of the system

$$\frac{dx}{dt} = 2x - 4y, \qquad\qquad (36)$$
$$\frac{dy}{dt} = -2x + 6y,$$

and determine whether or not the point is stable.

Solution Here a = 2, b = 4, c = -2, and d = 6. The characteristic equation is

$$\lambda^2 - 8\lambda + 20 = 0.$$

and its roots are 4 ± 2i. Since these roots are conjugate complex but not pure imaginary, we conclude that the critical point (0, 0) of (36) is a spiral point. Since the real part of the conjugate complex roots is positive, the point is unstable.

Application to Dynamics.

The differential equation for the free vibrations of mass on a coil spring is

$$m\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}+kx=0 \tag{37}$$

where m > 0, $\alpha \geq 0$. and k > 0 are constants denoting the mass, damping coefficient, and spring constant, respectively.

The dynamical equation (37) is equivalent to the system

$$\begin{aligned}\frac{dx}{dt}&=y,\\ \frac{dy}{dt}&=-\frac{k}{m}x-\frac{\alpha}{m}y\end{aligned} \tag{38}$$

The solutions of the system (38) define the paths (phase trajectories) associated with the dynamical equation (37) in the xy phase plane. From (38) the differential equation of these paths is

$$\frac{dy}{dx}=-\frac{kx+\alpha y}{my}$$

We observe that (0, 0) is the only critical point of the system (38). The auxiliary equation of the differential equation (37) is

$$mr^2 + \alpha r + k = 0, \tag{39}$$

while the characteristic equation of the system (38) is

$$\lambda^2+\frac{\alpha}{m}\lambda+\frac{k}{m}=0 \tag{40}$$

The two equations (39) and (40) clearly have the same roots λ_1 and λ_1. Table 3.2, gives the form of the solution of the dynamical equation, the phase plane analysis of the critical point, and a brief interpretation.

Table 3.2

Damping factor α	Nature of roots of auxiliary and characteristic equation	Form of solution of dynamical equation $\omega = \sqrt{\frac{k}{m}}, \beta = \frac{\alpha}{2m}$	Nature of critical point (0, 0) in xy phase plane	Interpretation
$\alpha = 0$ (no damping)	pure imaginary	$x = c \cos(\omega t + \phi)$	(stable) center	Oscillatory motion. Displacement and velocity are periodic functions of time
$\alpha < 2\sqrt{km}$ under-damped)	conjugate complex with negative real parts	$x = ce^{-\beta t} \cos(\omega_1 t + \phi)$, where $\omega_1 = \sqrt{\omega^2 - \beta^2}$.	asymptotically stable spiral point	Damped oscillatory motion. Displacement and velocity $\to 0$ through smaller and smaller oscillations
$\alpha = 2\sqrt{km}$ (critically damped)	real, equal, and negative	$x = (c_1 + c_2 t)e^{-\beta t}$	asymptotically stable node	displacement and velocity $\to 0$ without oscillating
$\alpha > 2\sqrt{km}$ (over damped)	real, unequal, and negative	$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1 = -\beta + \sqrt{\beta^2 - \omega^2}$, $r_2 = -\beta - \sqrt{\beta^2 - \omega^2}$,	asymptotically stable node	displacement and velocity $\to 0$ without oscillating

Exercises 3.2

Determine the nature of the critical point (0, 0) of each of the linear autonomous systems in Exercises 1-4. Also, determine whether or not the critical point is stable.

1. $\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = 3x + y.$

2. $\frac{dx}{dt} = 3x + 4y, \quad \frac{dy}{dt} = 3x + 2y.$

3. $\frac{dx}{dt} = 2x - 4y, \quad \frac{dy}{dt} = 2x - 2y.$

4. $\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x + 5y.$

Answers to Exercise

1. Saddle point, unstable ;
2. Saddle point, unstable;
3. Center, stable;
4. Node, unstable

Critical Points and Paths of Nonlinear Systems

A. Basic Theorems on Nonlinear Systems

We now consider the nonlinear real autonomous system

$$\frac{dx}{dt} = P(x, y),$$

$$\frac{dy}{dt} = Q(x, y). \tag{4}$$

We assume that the system (4) has an isolated critical point which we shall choose to be the origin (0, 0). We now assume further that the functions P and Q in the right members of (4) are such the P(x, y) and Q(x, y) can be written in the form

$$P(x, y) = ax + by + P_1(x, y),$$

$$Q(x, y) = cx + dy + Q_1(x, y), \tag{41}$$

where

1. a, b, c, and d, are real constanst, and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

and

2. P_1 and Q_1 have continuous first partial derivatives for all (x, y), and are such that

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0 \tag{42}$$

Thus the system under consideration may be written in the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + P_1(x, y), \\ \frac{dy}{dt} &= cx + dy + Q_1(x, y), \end{aligned} \tag{43}$$

where a, b, c, d, P_1, and Q_1 satisfy the requirements (1) and (2) above.

If P(x, y) and Q(x, y) in (4) can be expanded in power series about (0, 0), the system (4) takes the form

$$\frac{dx}{dt}=\left[\frac{\partial P}{\partial x}\right]_{(0,0)}x+\left[\frac{\partial P}{\partial y}\right]_{(0,0)}y+a_{12}x^2+a_{22}xy+a_{21}y^2+....,$$

$$\frac{dy}{dt}=\left[\frac{\partial Q}{\partial x}\right]_{(0,0)}x+\left[\frac{\partial Q}{\partial y}\right]_{(0,0)}y+b_{12}x^2+b_{22}xy+b_{21}y^2+...., \quad (44)$$

This system is of the form (43), where $P_1(x, y)$ and $Q_1(x, y)$ are the terms of higher degree in the right members of the equations. The requirements (1) and (2) will be met, provided the Jacobian.

$$\left.\frac{\partial(P,Q)}{\partial(x,y)}\right|_{(0,0)}\neq 0.$$

Observe that the constant terms are missing in the expansions in the right members of (44), since P(0, 0) = Q(0, 0) = 0

Example 3.4

The system

$$\frac{dx}{dt}=x+2y+x^2$$

$$\frac{dy}{dt}=-3x-4y+2y^2$$

is of the form (43) and satisfies the requirements (1) and (2). Here a = 1, b = 2, c = -3, d = - 4, and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}=2\neq 0,$$

Further $P_1(x, y) = x^2$, $Q_1(x, y) = 2y^2$, and hence

$$\lim_{(x,y)\to(0,0)}\frac{P_1(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2}{\sqrt{x^2+y^2}}=0$$

and

$$\lim_{(x,y)\to(0,0)}\frac{Q_1(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{2y^2}{\sqrt{x^2+y^2}}=0$$

By the requirement (2) the nonlinear terms $P_1(x, y)$ and $Q_1(x, y)$ in (43) tend to zero more rapidly than the linear terms ax + by and cx + dy. Hence one can say that the behaviour of the paths of the system (43) near (0, 0) would be similar to that of the paths of the related linear system

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy \tag{19}$$

obtained from (43) by neglecting the nonlinear terms. In other words, it would seem that the nature of the critical point (0, 0) of the nonlinear system (43) should be similar to that of the linear system (19). In general this is actually the case. We now state without proof the main theorem regarding this relation.

Theorem 3.7

Hypothesis Consider the non linear system

$$\begin{aligned}\frac{dx}{dt} &= ax + by + P_1(x, y),\\ \frac{dy}{dt} &= cx + dy + Q_1(x, y)\end{aligned} \tag{43}$$

where a, b, c, d, P_1, and Q_1 satisfy the requirements (1) and (2) above. Consider also the corresponding linear system

$$\begin{aligned}\frac{dx}{dt} &= ax + by,\\ \frac{dy}{dt} &= cx + dy,\end{aligned} \tag{19}$$

obtained from (43) by neglecting the nonlinear terms $P_1(x, y)$ and $Q_1(x, y)$. Both systems have an isolated critical point at (0, 0). Let λ_1 and λ_2 be the roots of the characteristic equation.

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0 \tag{22}$$

of the linear system (19).

Conclusions

(1) The critical point (0, 0) of the nonlinear system (43) is of the same type as that of the linear system (19) in the following cases:

(i) If λ_1 and λ_2 are real, unequal, and of the same sign, then not only is (0, 0) a node of (19), but also (0, 0) is a node of (43).

(ii) If λ_1 and λ_2 are real, unequal, and of the opposite sign, then not only is (0, 0) a saddle point of (19), but also (0, 0) is a saddle point of (43).

(iii) If λ_1 and λ_2 are real and equal and the system (19) is not such that $a = d \neq 0$, $b = c = 0$, then not only is (0, 0) a node of (19), but also (0, 0) is a node of (43).

(iv) If λ_1 and λ_2 are conjugate complex with real part not zero, then not only is (0, 0) a spiral point of (19), but also (0, 0) is a spiral point of (43).

2. The critical point (0, 0) of the nonlinear system (43) is not necessarily of the same type as that of the linear system (19) in the following cases:

(v) If λ_1 and λ_2 are real and equal and the system (19) is such that $a = d \neq 0$, $b = c = 0$, then although (0, 0) is a node of (19), the point (0, 0) may be either a node or a spiral point of (43).

(vi) If λ_1 and λ_2 are pure imaginary, then although (0, 0) is a center of (19), the point (0, 0) may be either a center or a spiral point of (43).

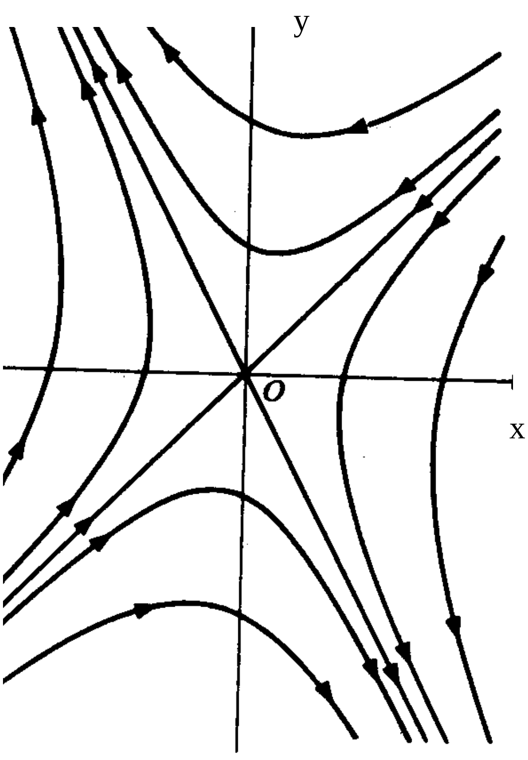
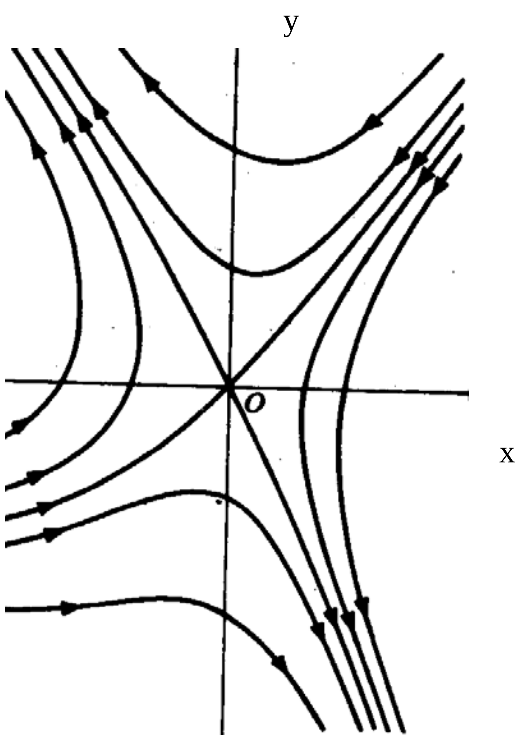


Figure 3.13a Linear System



Fgure 3.13b Nonlinear System

Although the critical point (0, 0) of the nonlinear system (43) is of the same type as that of the linear system (19) in cases (i), (ii), (iii) and (iv) of the conclusion, the actual appearance of the paths is somewhat different. For example, if (0, 0) is a saddle point of the linear system (19), then we know that there are four half-line paths entering (0, 0), two for $t \to +\infty$ and two for $t \to -\infty$. However, at the saddle point of the nonlinear system (43), in general we have four nonrectilinear curves entering (0, 0), two for $t \to +\infty$ and two for $t \to -\infty$ in place of the half-line paths of the linear case (see Figure 3.13).

Theorem 3.7 deals with the type of the critical point (0, 0) of the nonlinear system (43). Concerning the stability of this point, we state without proof the following theorem of Liapunov.

Theorem 3.8

Hypothesis is same as in Theorem 3.7.

Conclusion

1. If both roots of the characteristic equation (22) of the linear system (19) are real and negative or conjugate complex with negative real parts, then not only is (0, 0) an asymptotically stable critical point of (19) but also (0, 0) is an asymptotically stable critical point of (43).

2. If the roots of (22) are pure imaginary, then although (0, 0) is a stable critical point of (19), it is not necessarily a stable critical point of (43). Indeed, the critical point (0, 0) of (43) may be asymptotically stable, stable but not asymptotically stable or unstable.

3. If either of the roots of (22) is real and positive or if the roots are conjugate complex with positive real parts, then not only is (0, 0) an unstable critical point of (19) but also (0, 0) is an unstable critical point of (43).

Example 3.5

Consider

$$\frac{dx}{dt} = x + 4y - x^2, \qquad \frac{dy}{dt} = 6x - y + 2xy \tag{45}$$

This is of the form (43), where $P_1(x, y) = -x^2$ and $Q_1(x, y) = 2xy$. We see at once that the hypothesis of Theorems 3.7 and 3.8 are satisfied. Hence to investigate the critical point (0, 0) of (45), we consider the linear system.

$$\begin{aligned}\frac{dx}{dt} &= x + 4y,\\ \frac{dy}{dt} &= 6x - y\end{aligned} \tag{46}$$

of the form (19). The characteristic equation (22) of this system is

$$\lambda^2 - 25 = 0.$$

Hence the roots are $\lambda_1 = 5$; $\lambda_2 = -5$. Since the roots are real, unequal, and of opposite sign, we see from conclusion (ii) of Theorem 3.7 that the critical point (0, 0) of the nonlinear system (45) is a saddle point. From Conclusion (3) of Theorem 3.8 we further conclude that the point is unstable.

Eliminating dt from equations (45), we obtain the differential equation

$$\frac{dy}{dx} = \frac{6x - y + 2xy}{x + 4y - x^2}, \tag{47}$$

which gives the slope of the paths in the xy phase plane defined by the solutions of (45). The first order equation (47) is exact. Its general solution is readily found to be

$$x^2y + 3x^2 - xy - 2y^2 + c = 0 \tag{48}$$

where c is an arbitrary constant. Equation (48) is the equation of the family of paths in the xy phase plane. Several of these are shown in Figure (3.14).

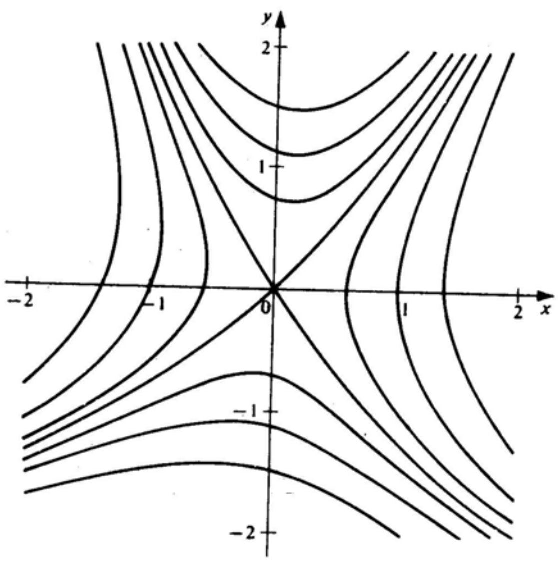


Figure 3.14

Example 3.6

Consider the nonlinear system

$$\frac{dx}{dt} = \sin x - 4y,$$
$$\frac{dy}{dt} = \sin 2x - 5y, \tag{49}$$

Using the expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \Lambda \ ,$$

we write this system in the form

$$\frac{dx}{dt} = x - 4y - \frac{x^3}{6} + \frac{x^5}{120} + ...,$$
$$\frac{dy}{dt} = 2x - 5y - \frac{4x^3}{3} + \frac{4x^5}{15} + ... \tag{50}$$

The hypotheses of Theorems 3.7 and 3.8 are satisfied. Thus to investigate the critical point (0, 0) of (49) [or 50], we consider the linear system

$$\frac{dx}{dt} = x - 4y,$$

$$\frac{dx}{dt} = 2x - 5y, \tag{51}$$

The characteristic equation of this system is

$$\lambda^2 + 4\lambda + 3 = 0.$$

Thus the roots are $\lambda_1 = -3$, $\lambda_2 = -1$. Since the roots are real, unequal and of the same sign, we see from conclusion (i) of Theorem 3.7 that the critical point (0,0) of the nonlinear system (49) is a node. From conclusion (1) of Theorem 3.8 we conclude that this node is asymptotically stable.

Example 3.7

In this example we shall find all the real critical points of the nonlinear system

$$\frac{dx}{dt} = 8x - y^2,$$

$$\frac{dy}{dt} = -6y + 6x^2, \tag{52}$$

and determine the type and stability of each of these critical points.

Clearly (0, 0) is one critical point of system (52). Also (52) is of the form (43) and the hypotheses of Theorems 3.7 and 3.8 are satisfied. To determine the type of critical point (0, 0), we consider the linear system.

$$\frac{dx}{dt} = 8x,$$
$$\frac{dy}{dt} = -6y,$$

of the form (19). The characteristic equation of this linear system is

$$\lambda^2 - 2\lambda - 48 = 0,$$

and thus the roots are $\lambda_1 = 8$, $\lambda_2 = -6$. Since the roots are real, unequal and of opposite sign, we see from conclusion (ii) of Theorem 3.7 that the critical point (0, 0) of the given nonlinear system (52) is a saddle point. From conclusion (3) of Theorem 3.8 we conclude that this critical point is unstable.

We now proceed to find all other critical points of (52). By definition, the critical points of this system must simultaneously satisfy the system of algebraic equations.

$$8x - y^2 = 0,$$

$$-6y + 6x^2 = 0 \tag{53}$$

From the second equation of this pair, $y = x^2$. Then substituting this into the first equation of the pair, we obtain

$$8x - x^4 = 0,$$

which factors into

$$x(2 - x)(4 + 2x + x^2) = 0.$$

This equation has only two real roots, x = 0 and x = 2. These are the abscissas of the real critical points of (52); the corresponding ordinates are determined from $y = x^2$. Thus we obtain the two real critical points (0 ,0) and (2, 4).

Since we have already considered the critical point (0, 0) and found that it is an (unstable) saddle point of the given system (52), we now investigate the type and stability of the other critical point (2, 4). To do this, we make the translation of coordinates

$$\xi = x - 2,$$

$$\eta = y - 4, \tag{54}$$

which transforms the critical point $x = 2$, $y = 4$ into the origin $\xi = \eta = 0$ in the $\xi\eta$ plane. We now transform the given system (52) into (ξ,η) coordinates. From (54), we have

$$x = \xi + 2, \quad y = \eta + 4;$$

and substituting these into (52) and simplifying, we obtain

$$\begin{aligned} \frac{d\xi}{dt} &= 8\xi - 8\eta - \eta^2, \\ \frac{d\eta}{dt} &= 24\xi - 6\eta + 6\xi^2. \end{aligned} \tag{55}$$

which is (52) in (ξ, η) coordinates. The system (55) is of the form (43) and the hypothesis of Theorem 3.7 and 3.8 are satisfied in these coordinates. To determine the type of the critical point $\xi = \eta = 0$ of (55), we consider the linear system.

$$\begin{aligned} \frac{d\xi}{dt} &= 8\xi - 8\eta, \\ \frac{d\eta}{dt} &= 24\xi - 6\eta. \end{aligned}$$

The characteristic equation of this linear system is

$$\lambda^2 - 2\lambda + 144 = 0.$$

The roots of this system are $1 \pm \iota\sqrt{143}$,which are conjugate complex with real part not zero. Thus by conclusion (iv) of Theorem 3.7, the critical point $\xi = \eta = 0$ of the nonlinear system (55) is a spiral point. From conclusion (3) of Theorem 3.8, we conclude that this critical point is unstable. But this critical point is the critical point x = 2, y = 4 of the given system (52). Thus the critical point (2, 4) of the given system (52) is an unstable spiral point.

In conclusion, the given system (52) has two real critical points, namely:

1. Critical point (0, 0); a saddle point; unstable;

2. Critical point (2, 4); a spiral point; unstable.

Example 3.8

Consider the two nonlinear systems

$$\frac{dx}{dt} = -y - x^2, \quad \frac{dy}{dt} = x \tag{56}$$

and

$$\frac{dx}{dt} = -y - x^3, \quad \frac{dy}{dt} = x \tag{57}$$

The point (0, 0) is a critical point for each of these systems. The hypotheses of Theorem 3.7 are satisfied in each case, and in each case the corresponding linear system to be investigated is

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x. \tag{58}$$

The characteristic equation of the system (58) is

$$\lambda^2 + 1 = 0$$

with the pure imaginary roots $\pm i$. Thus the critical point (0, 0) of the linear system (58) is a center. However, Theorem 3.7 does not give us definite information concerning the nature of this point for either of the nonlinear system (56) or (57). Conclusion (vi) of Theorem 3.7 tells us that in each case (0, 0) is either a center or a spiral point; but this is all that this theorem tells us concerning the two systems under consideration.

Summary

This chapter includes some basic definitions of non-linear systems to understand the desired concept in detail. The nature of critical points of various types is explained in detail with the help of five theorem and some suitable examples. The relationship between linear and non linear system is emphasized at the end of chapter.

Keywords Non-linear, critical point, stability, phase plane, path.

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Chapter-4

NON-LINEAR DIFFERENTIAL EQUATION -II

Objectives

This chapter provides an introduction to a method for studying the stability of more general autonomous systems. The student is made familiar with the methods to check stability and asymptotical stability of general autonomous system.

Dependence on a Parameter We briefly consider the differential equation of a conservative dynamical system in which the force F depends not only on the displacement x but also on a parameter λ. Specifically, we consider a differential equation of the form

$$\frac{d^2x}{dt^2} = F(x, \lambda). \tag{1}$$

where F is analytic for all values of x and λ. The differential equation (1) is equivalent to the nonlinear autonomous system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = F(x, \lambda) \tag{2}$$

For each fixed value of the parameter λ, the critical points of (2) are the points with coordinates $(x_c, 0)$, where the abscissas x_c are the roots of the equation $F(x, \lambda) = 0$, considered as an equation in the unknown x. In general, as λ varies continuously through a given range of values, the corresponding x_c vary and hence so do the corresponding critical points, paths, and solutions of (2). A value of the parameter λ at which two or more critical points coalesce into less than their previous number (or, vice versa, where one or more split up into more than their previous number) is called a critical value (or bifurcation value) of the parameter. At such a value the nature of the corresponding paths changes abruptly.

Example 4.1

Consider the differential equation

$$\frac{d^2x}{dt^2} = x^2 - 4x + \lambda \tag{3}$$

of the form (1), where

$$F(x, \lambda) = x^2 - 4x + \lambda$$

and λ is a parameter. The equivalent nonlinear system of the form (2) is

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= x^2 - 4x + \lambda \end{aligned} \tag{4}$$

The critical points of this system are the points $(x_1, 0)$ and $(x_2, 0)$, where x_1 and x_2 are the roots of the quadratic equation $F(x, \lambda)=0$; that is,

$$x^2 - 4x + \lambda = 0, \tag{5}$$

in the unknown x. We find

$$x = \frac{4 \pm \sqrt{16 - 4\lambda}}{2} = 2 \pm \sqrt{4 - \lambda}.$$

Thus the critical points of (4) are

$$\left(2 + \sqrt{4 - \lambda}, 0\right) \text{ and } \left(2 - \sqrt{4 - \lambda}, 0\right). \tag{6}$$

For $\lambda < 4$, the roots , of the quadratic equation are real and distinct, for $\lambda = 4$, the roots are real and equal, the common value being 2; and for $\lambda > 4$, they are conjugate complex. Thus for $\lambda < 4$, the critical points (6) are real and distinct. As $\lambda \to 4-$, the two critical points approach each other; and at $\lambda = 4$, they coalesce into the one single critical point (2, 0). For $\lambda > 4$, there simply are no real critical points. Thus we see that $\lambda = 4$ is the critical value of the parameter.

Liapunov's Direct Method

Russian Mathematician Liapunov obtained a method for studying the stability of more general autonomous systems. The procedure is known as Liapunov's direct (or second) method.

Consider the nonlinear autonomous system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y). \tag{7}$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y).

Definitions

Let E(x, y) have continuous first partial derivatives at all points (x, y) in a domain D containing the origin (0, 0).

1. The function E is called positive definite in D if E(0, 0) = 0 and E(x, y) > 0 for all other points (x, y) in D.
2. The function E is called positive semidefinite in D if E(0, 0) = 0 and E(x, y)≥0 for all other points (x, y) in D.
3. The function E is called negative definite in D if E(0, 0) = 0 and E(x, y) < 0 for all other points in D.
4. The function E is called negative semidefinite in D if E(0, 0) = 0 and E(x, y)≤0 for all other points (x, y) in D.

Example 4.2

The function E defined by E(x, y) = $x^2 + y^2$ is positive definite in every domain D containing (0, 0). Clearly, E(0, 0) = 0 and E(x, y) > 0 for all (x, y) ≠ (0, 0).

The function E defined by E(x, y) = x^2 is positive semidefinite in every domain D containing (0, 0). Note that E(0, 0) = 0, E(0, y) = 0 for all (0, y) such that y ≠ 0 in D, and E(x, y) > 0 for all (x, y) such that x ≠ 0 in D. There are no other points in D, and so we see that E(0, 0) = 0 and E(x, y) ≥ 0 for all other points in D.

Similarly, we see that the function E defined by E(x, y) = $-x^2 - y^2$ is negative definite in D and that defined by E(x, y) = $- x^2$ is negative semidefinite in D.

Definition

Let E(x, y) have continuous first partial derivatives at all points (x, y) in a domain D containing the origin (0, 0). The derivative of E with respect to the system (7) is the function $\dot{E}$ defined by

$$\dot{E}(x, y) = \frac{\partial E(x, y)}{\partial x} P(x, y) + \frac{\partial E(x, y)}{\partial y} Q(x, y) \tag{8}$$

Example 4.3

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -x + y^2, \\ \frac{dy}{dt} &= -y + x^2, \end{aligned} \tag{9}$$

and the function E defined by

$$E(x, y) = x^2 + y^2. \tag{10}$$

For the system (9), $P(x, y) = -x + y^2$, $Q(x, y) = -y + x^2$; and for the function E defined by (10),

$$\frac{\partial E(x, y)}{\partial x} = 2x, \qquad \frac{\partial E(x, y)}{\partial y} = 2y.$$

Thus the derivative of E defined by (10) with respect to the system (9) is given by

$$\dot{E}(x, y) = 2x(-x + y^2) + 2y(-y + x^2) = -2(x^2 + y^2) + 2(x^2y + xy^2). \tag{11}$$

Now let C be a path of system (7); let x = f (t), y = g(t) be an arbitrary solution of (7) defining C parametrically; and let E(x, y) have continuous first partial derivatives for all (x, y) in a domain containing C. Then E is a composite function of t along C; and using the chain rule, we find that the derivative of E with respect to t along C is

$$\begin{aligned} \frac{dE[f(t), g(t)]}{dt} &= E_x[f(t), g(t)]\frac{df(t)}{dt} + E_y[f(t), g(t)]\frac{dg(t)}{dt} \\ &= E_x[f(t), g(t)]P[f(t), g(t)] + E_y[f(t), g(t)]Q[f(t), g(t)] \\ &= \dot{E}[f(t), g(t)]. \end{aligned} \tag{12}$$

Thus we see that the derivative of E[f(t), g(t)] with respect to t along the path C is equal to the derivative of E with respect to the system (7) evaluated at x = f (t), y = g(t).

Definition

Consider the system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y). \tag{7}$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). Let E(x, y) be positive definite for all (x, y) in a domain D containing the origin and such that the derivative $\dot{E}(x, y)$ of E with respect to the system (7) is negative semidefinite for all $(x, y) \in D$. Then E is called a Liapunov function for the system (7) in D.

Example 4.4

Consider the system

$$\frac{dx}{dt} = -x + y^2,$$
$$\frac{dy}{dt} = -y + x^2, \tag{9}$$

and the function E defined by

$$E(x, y) = x^2 + y^2. \tag{10}$$

introduced in Example 4.3. Obviously the system (9) satisfies all the requirements of the immediately preceding definition in every domain containing the critical point (0, 0). Also, in Example 4.2 we observed that the function E defined by (10) is positive definite in every such domain. In Example 4.3, we found the derivative of E with respect to the system (9) as

$$\dot{E}(x, y) = -2(x^2 + y^2) + 2(x^2y + xy^2). \tag{11}$$

for all (x, y). If this is negative semidefinite for all (x, y) in some domain D containing (0,0), then E defined by (10) is a Liapunov function for the system (9).

Clearly $\dot{E}(0, 0) = 0$. Now observe the following: If $x < 1$ and $y \neq 0$, then $xy^2 < y^2$; if $y < 1$ and $x \neq 0$, then $x^2y < x^2$. Thus if $x < 1$, $y < 1$, and $(x, y) \neq (0, 0)$, $x^2y + xy^2 < x^2 + y^2$ and hence

$$-(x^2 + y^2) + (x^2y + xy^2) < 0.$$

Thus in every domain D containing (0, 0) and such that $x < 1$ and $y < 1$, $\dot{E}(x, y)$ given by (11) is negative definite and hence negative semidefinite. Thus E defined by (10) is a Liapunov function for the system (9).

We now state and prove two theorems on the stability of the critical point (0, 0) of system (7).

Theorem 4.1

Consider the system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y). \tag{7}$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). If there exists a Liapunov function E for the system (7) in some domain D containing (0, 0), then the critical point (0, 0) of (7) is stable.

Proof Let $K_\in$ be a circle of radius $\in > 0$ with center at the critical point (0, 0) where $\in > 0$ is small enough so that this circle $K_\in$ lies entirely in the domain D (see Fig. 4.1). From a theorem of real analysis, we know that a real valued fuction which is continuous on a closed bounded set assumes both a maximum and a minimum value on that set. Since the circle $K_\in$ is a closed bounded set in the plane and E is continuous in D and hence on $K_\in$, the real analysis theorem referred to in the preceding sentence applies to E on $K_\in$ and so, in particular, E assumes a minimum value on $K_\in$. Further, since E is also positive definite in D, this minimum value must be positive. Thus E assumes a positive minimum m on the circle $K_\in$. Next observe that since E is continuous at (0, 0) and $E(0, 0) = 0$, there exists a positive number δ satisfying $\delta < \in$ such that $E(x, y) < m$ for all (x, y) within or on the circle K_δ of radius δ and center at (0, 0). (see Figure 4.1).

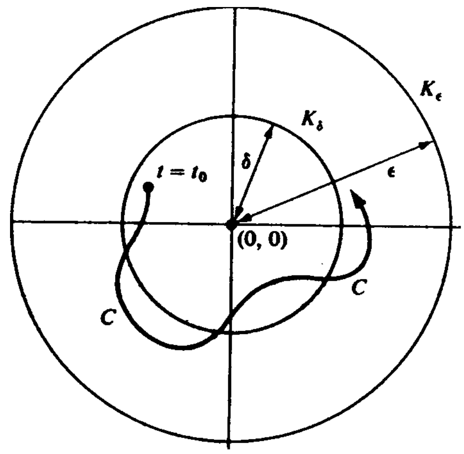

Figure 4.1

Now let C be any path of (7); let x = f(t), y = g(t) be an arbitrary solution of (7) defining C parametrically; and suppose C defined by [f(t), g(t)] is at a point within the "inner" circle K_δ at $t = t_0$. Then

$$E[f(t_0), g(t_0)] < m.$$

Since $\dot{E}$ is negative semidefinite in D we have

$$\frac{dE[f(t), g(t)]}{dt} \le 0$$

for [f(t), g(t)] $\in$ D. Thus E[f(t), g(t)] is a nonincreasing function of t along C. Hence

$$E[f(t), g(t)] \le E[f(t_0), g(t_0)] < m$$

for all $t > t_0$. Since E[f(t), g(t)] would have to be $\ge$ m on the "outer" circle K_ϵ, we see that the path C defined by x = f (t), y = g(t) must remain within K_ϵ for all $t > t_0$. Thus, from the definition of stability of the critical point (0, 0), we see that the critical point (0, 0) of (7) is stable.

Theorem 4.2

Consider the system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y). \qquad (7)$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and

Q have continuous first partial derivatives for all (x, y). If there exists a Liapunov function E for the system (7) in some domain D containing (0,0) such that E also has the property that $\dot{E}$ defined by (8) is negative definite in D, then the critical point (0,0) of (7) is asymptotically stable.

Proof As in the proof of the previous theorem, let $K_\in$ be a circle of radius $\in > 0$ with center at the critical point (0, 0) and lying entirely in D. Also, let C be any path of (7); let x = f(t), y = g(t) be an arbitrary solution of (7) defining C parametrically; and suppose C defined by [f (t), g(t)] is at a point within $K_\in$ at $t = t_0$ (see Figure 4.2).

Now since $\dot{E}$ is negative definite in D, using (12), we have

$$\frac{dE[f(t), g(t)]}{dt} < 0$$

for $[f(t), g(t)] \in D$. Thus E[f(t), g(t)] is a strictly decreasing function of t along C. Since E is positive definite in D, $E[f(t),g(t)] \geq 0$ for $[f(t), g(t)] \in D$. Thus $\lim_{t\to\infty} E[f(t), g(t)]$ exists and is some number $L \geq 0$. We shall show that L = 0.

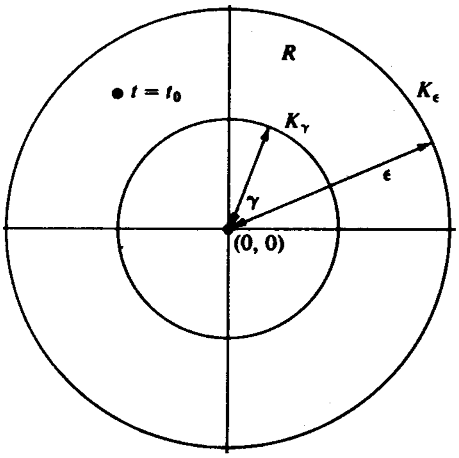

Figure 4.2

On the contrary assume that L > 0. Since E is positive definite, there exists a positive number γ satisfying $\gamma < \in$ such that E(x, y)< L for all (x, y) within the circle K of radius γ and center at (0,0). Now we can apply the same real analysis theorem on maximum and minimum values that we used in the proof of the preceding theorem to the continuous function $\dot{E}$ on the closed region R between and on the two circles $K_\in$ and K_γ. Doing so, since $\dot{E}$ is negative definite in D and hence in this region R which does not include (0, 0), we see that $\dot{E}$ assumes a negative maximum - k on R. Since

E[f(t), g(t)] is a strictly decreasing function of t along C and

$$\lim_{t\to\infty} E[f(t), g(t)] = L,$$

the path C defined by x = f(t), y = g(t) cannot enter the domain within K_γ for any t > t_0 and so remains in R for all t ≥ t_0. Thus we have $\dot{E}[f(t). g(t)] \leq -k$ for all t ≥ t_0. Then by (12) we have

$$\frac{dE[f(t), g(t)]}{dt} = \dot{E}[f(t), g(t)] \leq -k \tag{13}$$

for all t ≥ t_0. Now consider the identity

$$E[f(t), g(t)] - E[f(t_0), g(t_0)] = \int_{t_0}^{t} \frac{dE[f(t), g(t)]}{dt} dt\ . \tag{14}$$

Then (13) gives

$$E[f(t), g(t)] - E[f(t_0), g(t_0)] \leq -\int_{t_0}^{t} k\ dt\ .$$

and hence

$$E[f(t), g(t)] \leq E[f(t_0), g(t_0)] - k(t - t_0)$$

for all t ≥ t_0. Now let $t \to \infty$. Since - k < 0, this gives

$$\lim_{t\to\infty} E[f(t), g(t)] = -\infty.$$

But this contradicts the hypothesis that E is positive definite in D and the assumption that

$$\lim_{t\to\infty} E[f(t), g(t)] = L > 0$$

Thus L = 0; that is,

$$\lim_{t\to\infty} E[f(t), g(t)] = 0.$$

Since E is positive definite in D, E(x, y) = 0 if and only if (x, y) = (0, 0). Thus,

$$\lim_{t\to\infty} E[f(t), g(t)] = 0,$$

if and only if

$$\lim_{t\to\infty} f(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} g(t) = 0 .$$

But, from the definition of asymptotic stability of the critical point (0, 0), we see that the critical point (0, 0) of (7) is asymptotically stable.

Example 4.5

Consider the system

$$\frac{dx}{dt} = -x + y^2,$$
$$\frac{dy}{dt} = -y + x^2, \tag{9}$$

and the function E defined by

$$E(x, y) = x^2 + y^2. \tag{10}$$

previously studied in Examples 4.3 and 4.4. Before that, in Example 4.2, we noticed that the function E defined by (10) is positive definite in every domain containing (0, 0). In Example 4.3, we found the derivative of E with respect to the system (9) is given by

$$\dot{E}(x, y) = -2(x^2 + y^2) + 2(x^2y + xy^2). \tag{11}$$

Then, in Example 4.4, we found that $\dot{E}$ defined by (11) is negative semidefinite in every domain containing (0,0) and hence that E defined by (10) is a Liapunov function for the system (9) in every such domain. Now, applying Theorem 4.1,we see that the critical point (0, 0) of (9) is stable.

However, in Example 4.4, we actually showed that $\dot{E}$ defined by (11) is negative definite in every domain D containing (0, 0). Thus by Theorem 4.2, we see that the critical point (0, 0) of (9) is asymptotically stable.

Note

Liapunov's direct method is indeed "direct" in the sense that it does not require any previous knowledge about the solutions of the system (7) or the type of its critical point (0,0). Instead, if one can construct a Liapunov function for (7), then one can "directly" obtain information about the stability of the critical point (0, 0). However, there is no general method for constructing a Liapunov function, although methods for doing so are available for certain classes of equations.

Exercise

Determine the type and stability of the critical point (0, 0) of each of the nonlinear autonomous systems in questions 1-4.

1. $\frac{dx}{dt} = x + x^2 - 3xy, \qquad \frac{dy}{dt} = -2x + y + 3y^2$

2. $\frac{dx}{dt} = x + y + x^2 y, \qquad \frac{dy}{dt} = 3x - y + 2xy^3$

3. $\frac{dx}{dt} = (y+1)^2 - \cos x, \qquad \frac{dy}{dt} = \sin(x+y)$

4. Consider the autonomous system

$$\frac{dx}{dt} = ye^x,$$
$$\frac{dy}{dt} = e^x - 1.$$

(a) Determine the type of the critical point (0, 0).

(b) Obtain the differential equation of the paths and find its general solution.

Answers

1. Node, unstable

2. Saddle point, unstable

3. Saddle point, unstable

4(a) Saddle point (b) $y^2 = 2(x + e^{-x} + c)$

Limit Cycles and Periodic Solutions

A. Limit Cycles

We have already studied autonomous systems having closed paths. For example, in the neighborhood of center there is an infinite family of closed paths resembling ellipses. The closed paths about (0, 0) form a continuous family in the sense that arbitrarily near to any one of the closed paths of this family there is always another closed path of the family. Now we shall consider systems having closed paths which are isolated in the sense that there are no other closed paths of the system arbitrarily near to a given closed path of the system.

Now suppose the system (7) has a closed path C. Further, suppose (7) possesses a nonclosed path C_1 defined by a solution $x = f(t)$, $y = g(t)$ of (7) and having the following property: As a point R traces out C_1 according to the equations $x = f(t)$, $y = g(t)$, the path C_1 spirals and the distance between R and the nearest point on the closed path C approaches zero either as $t \to +\infty$ or as $t \to -\infty$. In other words, the nonclosed path C_1 spirals closer and closer around the closed path C either from the inside of C or from the outside either as $t \to +\infty$ or as $t \to -\infty$ (see Figure 4.3 where C_1 approaches C from the outside).

In such a case we call the closed path C a limit cycle, according to the following definition:

Definition

A closed path C of the system (7) which is approached spirally from either the inside or the outside by a nonclosed path C_1 of (7) either as $t \to +\infty$ or as $t \to -\infty$ is called a limit cycle of (7).

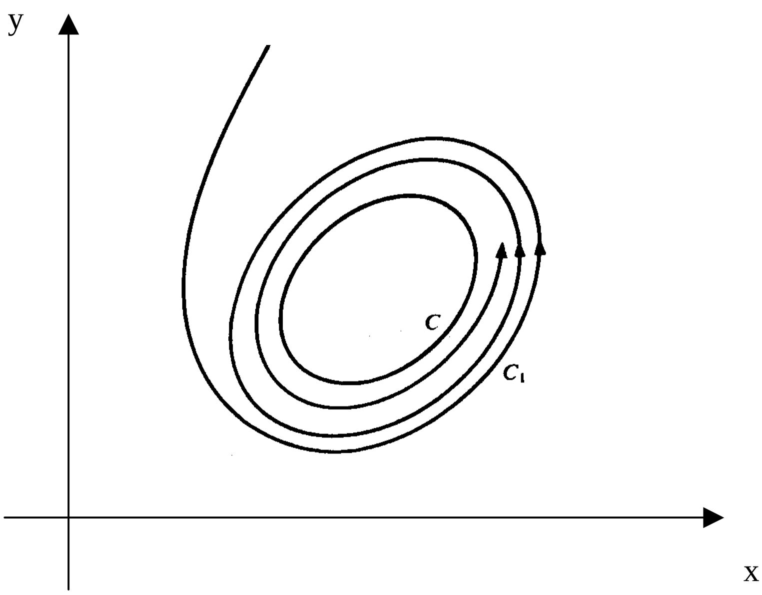

Figure 4.3

Example 4.6

The following example of a system having a limit cycle will illustrate the above discussion and definition.

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2),$$

$$\frac{dy}{dt} = -x + y(1 - x^2 - y^2) \tag{15}$$

To study this system we shall introduce polar coordinates (r, θ), where

$$x = r\cos\theta,$$

$$y = r\sin\theta. \tag{16}$$

From these relations we find that

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt},$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}. \tag{17}$$

Now, multiplying the first equation of (15) by x and the second by y and adding, we obtain

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2 + y^2)(1 - x^2 - y^2).$$

Introducing the polar coordinates defined by (16) and making use of (17), this becomes

$$r\frac{dr}{dt} = r^2(1 - r^2).$$

For $r \neq 0$, we may thus write

$$\frac{dr}{dt} = r(1 - r^2).$$

Now multiplying the first equation of (15) by y and the second by x and subtracting, we obtain

$$y\frac{dx}{dt} - x\frac{dy}{dt} = y^2 + x^2.$$

Again using (17), this becomes

$$-r^2 \frac{d\theta}{dt} = r^2,$$

and so for r ≠ 0 we may write

$$\frac{d\theta}{dt} = -1.$$

Thus in polar coordinates the system (15) becomes

$$\begin{aligned} \frac{dr}{dt} &= r(1-r^2), \\ \frac{d\theta}{dt} &= -1. \end{aligned} \qquad (18)$$

From the second of these equations we find at once that

$$\theta = -t + t_0,$$

where t_0 is an arbitrary constant. The first of the equations (18) is separable. Separating variables, we have

$$\frac{dr}{r(1-r^2)} = dt,$$

and an integration using partial fractions yields

$$\ln r^2 - \ln |1 - r^2| = 2t + \ln |c_0|.$$

After some calculations we obtain

$$r^2 = \frac{c_0 e^{2t}}{1 + c_0 e^{2t}}.$$

Thus we may write

$$r = \frac{1}{\sqrt{1 + ce^{-2t}}}, \qquad \text{where } c = \frac{1}{c_0}.$$

Thus, the solution of the system (18) may be written as

$$r = \frac{1}{\sqrt{1 + ce^{-2t}}},$$

$$\theta = -t + t_0,$$

where c and t_0 are arbitrary constants. We may choose $t_0 = 0$, then $\theta = -t$; using (16) the solution of the system (15) becomes

$$x = \frac{\cos t}{\sqrt{1+ce^{-2t}}},$$

$$y = \frac{-\sin t}{\sqrt{1+ce^{-2t}}}. \tag{19}$$

The solutions (19) of (15) define the paths of (15) in the xy plane. Examining these paths for various values of c, we note the following conclusions:

1. If c = 0. the path defined by (19) is the circle $x^2 + y^2 = 1$, described in the clockwise direction.

2. If $c \neq 0$, the paths defined by (19) are not closed paths but rather paths having a spiral behavior. If $c > 0$, the paths are spirals lying inside the circle $x^2 + y^2 = 1$. As $t \to +\infty$, they approach this circle; while as $t \to -\infty$, they approach the critical point (0, 0) of (15). If $c < 0$, the paths lie outside the circle $x^2 + y^2 = 1$. These "outer" paths also approach this circle as $t \to +\infty$; while as $t \to \ln\sqrt{|c|}$, both |x| and |y| become infinite.

Since the closed path $x^2 + y^2 = 1$ is approached spirally from both the inside and the outside by nonclosed paths as $t \to +\infty$, we conclude that this circle is a limit cycle of the system (15). (See Figure 4.4).

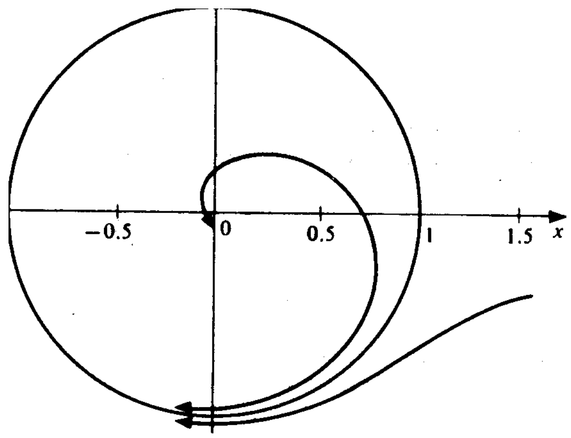

Figure 4.4

B. Existence and Nonexistence of Limit cycles

In Example 4.6 the existence of a limit cycle was ascertained by actually finding this limit cycle. In general such a procedure is, of course, impossible. Given the autonomous system (7) we need a theorem giving sufficient conditions for the existence of a limit cycle of (7). One of the few general theorems of this nature is the Poincare-Bendixson theorem, which we shall state below (Theorem 4.4). First, we shall state and prove a theorem on the nonexistence of closed paths of the system (7).

Theorem 4.3 Bendixson's Nonexistence Criterion

Hypothesis. Let D be a domain in the xy plane. Consider the autonomous system

$$\frac{dx}{dt} = P(x,y),$$

$$\frac{dy}{dt} = Q(x,y), \tag{7}$$

where P and Q have continuous first partial derivatives in D. Suppose that $\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}$ has the same sign throughout D.

Conclusion The system has no closed path in the domain D.

Proof Let C be a closed curve in D; let R be the region bounded by C; and apply Green's Theorem in the plane. We have

$$\int_C [P(x,y)dy - Q(x,y)dx] = \iint_R \left[\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} \right] ds,$$

where the line integral is taken in the positive sense. Now assume that C is a closed path of (7); let x = f(t), y = g(t) be an arbitrary solution of (7) defining C parametrically; and let T denotes the period of this solution. Then

$$\frac{df(t)}{dt} = P[f(t), g(t)],$$

$$\frac{dg(t)}{dt} = Q[f(t), g(t)],$$

along C and we have

$$\int_C [P(x,y)dy - Q(x,y)dx]$$

$$= \int_o^T \left\{ P[f(t), g(t)] \frac{dg(t)}{dt} - Q[f(t), g(t)] \frac{df(t)}{dt} \right\} dt$$

$$= \int_o^T \{ P[f(t), g(t)]\, Q[f(t), g(t)] - Q[f(t), g(t)]\, P[f(t), g(t)] \} dt$$

$$= 0$$

Thus $$\iint_R \left[\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} \right] ds = 0.$$

But this double integral can be zero only if $\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}$ changes sign. This is a contradiction. Thus C is not a path of (7) and hence (7) possesses no closed path in D.

Example 4.7

$$\frac{dx}{dt} = 2x + y + x^3,$$

$$\frac{dy}{dt} = 3x - y + y^3. \tag{20}$$

Here $P(x, y) = 2x + y + x^3$,

$Q(x, y) = 3x - y + y^3$,

and $$\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} = 3(x^2 + y^2) + 1.$$

Since this expression is positive throughout every domain D in the xy plane, the system (20) has no closed path in any such domain. In particular, then, the system (20) has no limit cycles and hence no periodic solutions.

Having considered this nonexistence result, we now turn to the Poincare-Bendixson existence theorem. We shall merely state the theorem and indicate its significance.

Definition

Let C be a path of the system (7) and let x = f(t), y = g(t) be a solution of (7) defining C. Then we shall call the set of all points of C for $t \geq t_0$, where t_0 is some value of t, a half-path of (7). In other words, by a half-path of (7) we mean the set of all points with coordinates $[f(t), g(t)]$ for $t_0 \leq t < +\infty$. We denote a half-path of (7) by C^+.

Definition

Let C^+ be a half path of (7) defined by $x = f(t)$, $y = g(t)$ for $t \geq t_0$. Let (x_1, y_1) be a point in the xy plane. If there exists a sequence of real numbers $\{t_n\}$, n=1,2,…….. such that $t_n \to +\infty$ and $[f(t_n), g(t_n)] \to (x_1, y_1)$ as $n \to +\infty$, then we call (x_1, y_1) a limit point of C^+. The set of all limit points of a half-path C^+ will be called the limit set of C^+ and will be denoted by $L(C^+)$.

Theorem 4.4 Poincare-Bendixson Theorem

Hypothesis

1. Consider the autonomous system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y), \tag{7}$$

where P and Q have continuous first partial derivatives in a domain D of the xy plane. Let D_1 be a bounded subdomain of D, and let R denote D_1 plus its boundary.

2. Let C^+ defined by $x = f(t)$, $y = g(t)$, $t \geq t_0$, be a half-path of (7) contained entirely in R. Suppose the limit set $L(C^+)$ of C^+ contains no critical points of (7).

Conclusion Either (1) the half-path is itself a closed path [in this case C^+ and $L(C^+)$ are identical] or (2) $L(C^+)$ is a closed path which C^+ approaches spirally from either the inside or the outside [in this case $L(C^+)$ is a limit cycle]. Thus in either case, there exists a closed path of (7) in R.

C. The Index of a Critical Point

We again consider the system (7), where P and Q have continuous first partial derivatives for all (x, y), and assume that all of the critical points of (7) are isolated. Now consider a simple closed curve (By a simple closed curve we mean, a closed curve having no double points; for example, a circle is a simple closed curve, but a figure of digit eight is not) [not necessarily a path of (7)] which passes through no critical points of (7). Consider a point (x_1, y_1) on C and the vector $[P(x_1, y_1), Q(x_1, y_1)]$ defined by (7) at the point (x_1, y_1). Let θ denotes the angle from the positive x direction to this vector (see Figure 4.5).

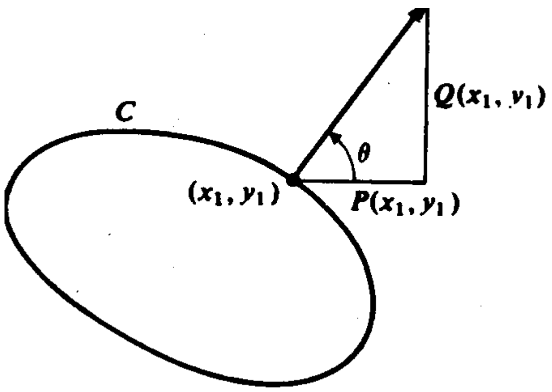

Figure 4.5

Now let (x_1, y_1) describes the curve C once in the counterclockwise direction and returns to its original position. As (x_1, y_1) describes C, the vector $[P(x_1, y_1), Q(x_1, y_1)]$ changes continuously, and consequently the angle θ also varies continuously. When (x_1, y_1) reaches its original position, the angle θ will have changed by an amount $\Delta\theta$. We will now define the index of the curve C.

Definition

Let θ denotes the angle from the positive x direction to the vector $[P(x_1, y_1), Q(x_1, y_1)]$ defined by (7) at (x_1, y_1). Let $\Delta\theta$ denote the total change in θ as (x_1, y_1) describes the simple closed curve C once in the counterclockwise direction. We call the number

$$I = \frac{\Delta\theta}{2\pi}$$

the index of the curve C with respect to the system (7).

Clearly $\Delta\theta$ is either equal to zero or a positive or negative integral multiple of 2π and hence I is either zero or a positive or negative integer. If $[P(x_1, y_1), Q(x_1, y_1)]$ merely oscillates but does not make a complete rotation as (x_1, y_1) describes C, then I is zero. If the net change $\Delta\theta$ in θ is a decrease, then I is negative.

Now let (x_0, y_0) be an isolated critical point of (7). It can be shown that all simple closed curves enclosing (x_0, y_0) but containing no other critical point of (7) have the same index. This leads us to make the following definition.

Definition

By the index of an isolated critical point (x_0, y_0) of (7) we mean the index of a simple closed curve C which encloses (x_0, y_0) but no other critical points of (7).

From an examination of Figure 4.6 we may reach the following conclusion intuitively: The index of a node, a center, or a spiral point is + 1, while the index of a saddle point is -1.

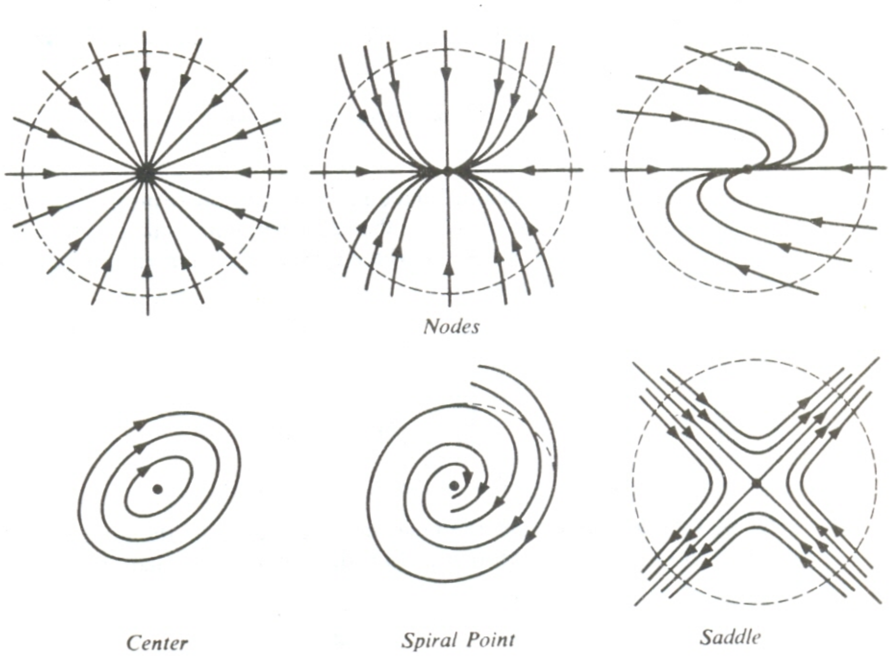

Figure 4.6

We now list some interesting results concerning the index of a simple closed curve C and then point out several important consequences of these results. In each case when we say index we shall mean the index with respect to the system (7) where P(x, y) and Q(x, y) have continuous first partial derivatives for all (x, y) and (7) has only isolated critical points.

1. The index of a simple closed curve which neither passes through a critical point of (7) nor has a critical point of (7) in its interior is zero.

2. The index of a simple closed curve which surrounds a finite number of critical points of (7) is equal to the sum of the indices of these critical points.

3. The index of a closed path of (7) is + 1.

From these results the following conclusions follow at once.

(a) A closed path of (7) contains at least one critical point of (7) in its interior [for otherwise, by (1), the index of such a closed path would be zero; and this would contradict (3)].

(b) A closed path of (7) may contain in its interior a finite number of critical points of (7), the sum of the indices of which is + 1 [this follows at once from (2) and (3)].

Summary

Topic discussed in this chapter includes Liapunov's direct method to check the stability of general autonomous systems and some theorems to check stability and asymptotical stability of such systems. Pioncare Bendixson theorem giving the sufficient condition for the existence of limit cycle of general autonomous systems is stated. A theorem on nonexistence of limit cycles along with the index of critical points are presented at the end of the chapter.

Keywords Liapunov direct method, limit cycle, index, Bendixson non-existence criterion.

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Chapter-5

CALCULUS OF VARIATIONS -I

Objectives

In this chapter, it is shown that the variational problems give rise to a system of differential equations, the Euler-Lagrange equations. Furthermore, the minimizing principle that underlies these equations leads to direct methods for analyzing the solutions to these equations. These methods have far reaching applications and will help students in developing problem solving techniques. Student will be able to formulate variational problems and analyze them to deduce key properties of system behaviour.

Introduction

The calculus of variations is one of the oldest subjects of mathematics, yet it remains very active and is still evolving fast. Besides its mathematical importance and its links to other branches of mathematics including geometry and partial differential equations, it is widely used in engineering, physics, economics and biology.

The calculus of variations concerns problems in which one wishes to find the minima or extrema of some quantity over a system that has functional degrees of freedom. Many important problems arise in this way across pure and applied mathematics and physics. They range from the problem in geometry of finding the shape of a soap bubble, a surface that minimizes its surface area, to finding the configuration of a piece of elastic that minimises its energy. Perhaps most importantly, the principle of least action is now the standard way to formulate the laws of mechanics and basic physics. These days, calculus of variations attracts attention of mathematics and provides new tools to find the best possible solution, and to understand the essence of optimality.

The calculus of variations seeks to optimize (often minimize) a special class of functions called functionals. Its aim is to explore methods for finding maximum or minimum of a functional defined over a class of functions. The usual form of functional is

$$I[y] = \int_a^b F(x, y, y')dx\,. \tag{1}$$

Here I[y] is not a function of x because x disappears when definite integral is evaluated. The argument y of I[y] is not a simple variable but a function y = y(x). The square bracket in I[y] emphasize this fact. A functional can be thought of function of functions. The value of r.h.s. of equation (1) will change as the function y(x) is varied, but when y(x) is fixed, it evaluates to a scalar quantity (a constant). We seek the y(x) that minimizes I[y].

Function

By a function we mean a correspondence between the elements of sets A and B s.t. to each element of A there corresponds exactly one element of set B.

Functional

Let A be a class of functions. A correspondence between the functions of class A and the set of real numbers s.t. to each function belonging to A, there corresponds exactly one real number, is called a functional. Its input is vector and output is scalar.

or

A functional is a correspondence which assigns a unique real number to each function belonging to some class. We can say that a functional is a kind of function, where the independent variable is itself a function (or curve).

A functional is denoted by capital letter I or [J]. If y(x) represents the function in the class of functions of a functional J, then we write J = J[y(x)].

Domain

The class of functions y(x) on which the functional J[y(x)] is defined is called the domain of the functional J, rather than a region of coordinate space.

Some examples of functionals

1. A simple example of a functional is the arc length λ between two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a curve $y = y(x)$.

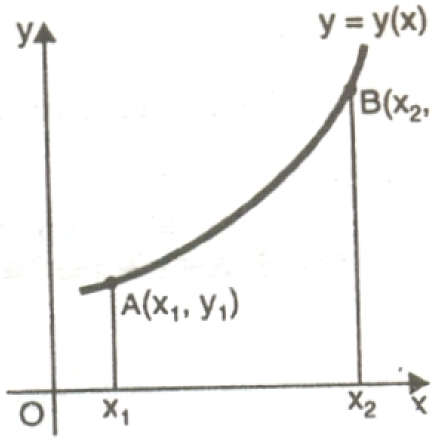

Figure 5.1

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = dx\sqrt{1+(y'(x))^2}, \qquad y' = \frac{dy}{dx}.$$

This length is given by

$$\lambda[y(x)] = \int_{x_1}^{x_2} \sqrt{1+[y'(x)]^2} \quad dx$$

or $\lambda[y(x)] = \int_{x_1}^{x_2} \left(f(x, y(x), y'(x)\right)dx$.

Here functional is the integral of the distance along any of these curves, as in figure 5.1. We are to choose among $y_1(x)$, $y_2(x)$, $y_3(x)$ which makes $I[y]$ minimum.

Thus a definite number is associated with each such curve, namely, its length. Thus, the length of a curve is a functional defined on set of such curves as length of the arc is determined by the choice of functions.

2. Another example is the area S of a surface bounced by a given curve C because this area is determined by the choice of the surface z = z(x, y)

$$S\,[z(x,y)] = \iint_D \left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dxdy\ .$$

Here D is the projection of the area bounded by the curve C on the xy-plane.

3. Let y(x) be an arbitrary continuously differentiable function, defined on the interval [a, b].

Then the formula

$J[y] = \int_a^b y'^2\,(x)dx$ defines a functional on the set of all such functions y(x).

4. As a more general example, let F(x, y, z) be a continuous function of three variables. Then the expression

$$J[y] = \int_a^b F(x, y(x), y'(x)]dx \qquad (1)$$

where y(x) ranges over the set of all continuously differentiable functions defined on the interval [a, b], defines a functional. By choosing different values of F(x, y, z), we obtain different functionals e.g. if F(x, y, z) = $\sqrt{1+z^2}$.

Then J[y] is the length of the curve y = y(x), as in the first example, while if

F(x, y, z) = z^2 , case (3) is obtained. Further, we shall be concerned mainly with functionals of the form (1)

Note What types of functions are allowed in the domain of functional?

The integral $I[y] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx \qquad (1)$

is well defined real number if (i) the integrand is a continuous function of x and for this it is sufficient to assume that y'(x) is continuous. Thus first, we will always assume that the function F(x, y, y') has continuous partial derivatives of second order w.r.t. x, y and y' and satisfy the given boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$. Functions of this kind will be called admissible functions.

Some typical examples of variational problems i.e. problems involving the determination of maxima and minima of functionals.

Besides Brachistochrone problem (details will be provided later on), three other problems which exerted great influence on the subject, are

(i) In the problem of **geodesics**, if is required to determine the line of shortest length connecting two given points (x_0, y_0, z_0) and (x_1, y_1, z_1) on a surface S given by $\phi(x, y, z) = 0$. This is a typical variational problem with a constraint, since here we are required to minimize the arc length λ joining the two points on S given by the functional

$$\lambda = \int_{x_0}^{x_1} \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

subject to the constraint $\phi(x, y, z) = 0$. This problem was first solved by Jacob Bernoulli in 1698, but a general method of solving such problems was given by Euler.

(ii) In the problem of **minimal surface of revolution**, a curve $y = y(x) \geq 0$ is rotated about the x-axis through an angle 2π. The resulting surface bounded by the planes x = a and x = b has the area

$$S = 2\pi \int_a^b y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx\,.$$

Clearly, determination of the particular curve y = y(x) which minimizes S constitutes a variational problem.

(iii) In the **isoperimetric problem**, it is required to find a closed line of given length which encloses a maximum area S. The solution of this problem is the circle. The problem consists of the maximization of the area A bounded by the closed curve r = r(θ) of given length λ. This mean that the functional A given by $A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$ is maximum subject to $\lambda = \int_0^{2\pi}\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta$

Thus in **calculus of variations** we consider some quantity (arc length, surface area, time of descent) that depends upon the entire curve, and we seek the curve that minimizes the quantity in question. In following part, emphasis will be given to the maxima or minima of the following functionals

$$\int_{x_0}^{x_1} F[x, y(x), y'(x)]\, dx$$

$$\int_{x_0}^{x_1} F[x, y(x), y'(x) \dots y^{(n)}(x)]\, dx$$

$$\int_{x_0}^{x_1} F[x, y_1(x), y_2(x) \dots y_n(x), y_1'(x), y_2'(x) \dots\dots, y_n'(x)]\, dx$$

in which function F is given, and the functions y(x), $y_1(x)$, .., $y_n(x)$ are the arguments of the functionals. (' ⇒ derivative w.r.t. x)

Maximum and Minimum values of functionals

A functional J [y(x)] is said to have a maximum value on y = $y_0(x)$ if the value of the functional on any curve close to y = $y_0(x)$ is not greater that J [$y_0(x)$] i.e. J[y(x)]≤ J [$y_0(x)$] ∀ curves y(x) close to $y_0(x)$. [i.e. $\Delta J = J[y(x)] - J[y_0(x)] \le 0$]. A functional J[y(x)] is said to have a minimum value on y = $y_0(x)$ if the values of the functional on any curve close to y= $y_0(x)$ is not less than J[$y_0(x)$] i.e. J[y(x)] ≥ J[$y_0(x)$] ∀ curves y(x) close to $y_0(x)$.

The maximum and minimum values of a functional are called its **extremum** values.

Definition

Extremal is the value of y(x) from which the value of I(y) is either maximum or minimum in the field of calculus of variations.

Definition

The definite integral $I(y) = \int_a^b y(x)dx$ is a functional defined on a class of continuous functions on the interval [a, b]

Definition

If $F(x, y(x), y'(x))$ is a continuous function with respect to all its arguments, then the integral

$$I(y) = \int_a^b F(x, y(x), y'(x))\, dx$$

defines a functional on a set of continuous differentiable functions defined on the closed interval [a, b].

Theorem 5.1

If a functional I [y(x)] attains a maximum or minimum on $y = y_0(x)$, where the domain of definition belongs to certain class, then at $y = y_0(x)$, $\delta I = 0$,

where $\delta I = \frac{\partial}{\partial \alpha} I[y(x) + \alpha \delta y(x)]$ at $\alpha = 0$ for fixed y and δy and different values of parameter, is variation of the functional I[y(x)].

Proof For fixed $y_0(x)$ and δy, $y_0(x) + \alpha\delta y$ defines class of functions.

For $\alpha = 0$, we get a function $y_0(x)$

Clearly $I[y_0(x) + \alpha\delta y] = \psi(\alpha)$ (say) is a function of α, which attains the maximum or minimum value at $y_0(x)$ (i.e. at $\alpha = 0$).

Then $\psi'(\alpha) = 0$ at $y_0(x)$

$\psi'(\alpha) = 0$ for $\alpha = 0$ [$\alpha = 0$ gives $y_0(x)$]

or $$\frac{\partial}{\partial \alpha} I[y_0(x) + \alpha\delta y] = 0 \qquad at\ \alpha = 0.$$

$\Rightarrow$ $\delta I = 0$. Thus the variation of a functional is zero on curves on which an extermum of the functional is achieved.

The Simplest Variational Problem

Euler's Equation

Euler's equation for functionals containing first order derivative and one independent variable.

Theorem 5.2: Obtain the necessary condition for

$$I(y) = \int_a^b F(x, y(x), y'(x))\, dx \tag{1}$$

to be extremum, satisfying the boundary conditions y(x) = A, y(b) = B .

or

Let I(y) = $\int_a^b F(x, y(x), y'(x))dx$ be a functional defined on the set of functions y(x) which has continuous first order derivative in the interval [a, b] and satisfies the boundary conditions y(a) = A, y(b) = B, where A and B are prescribed at the fixed boundary points a and b. Also, F is differentiable three times w.r.t. all its arguments. Then a necessary condition for I[(y, x)] to be an extremum (for a given function y(x)] is that $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ i.e. if the functional I[y(x)] has an extremum on a function y(x), then y(x) satisfies the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Note Brachistochrone problem and example of shortest distance between two points are variational problems of this type.

Proof

Let y = y(x) be the curve which extremizes (i.e. maximizing or minimizing) the functional

$$I[y(x)] = \int_a^b F(x, y(x), y'(x))\, dx \qquad (1)$$

and satisfying the boundary conditions

$$\left.\begin{array}{l} y(a) = A \\ y(b) = B \end{array}\right\} \qquad (2)$$

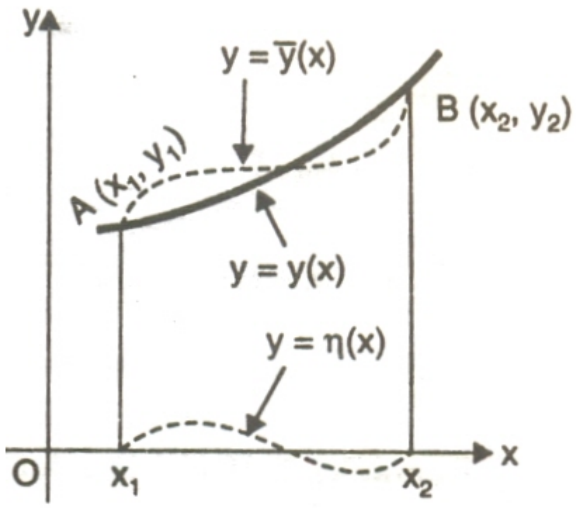

Figure 5.2

[How we find this function y(x)? We shall obtain a differential equation for y(x) by comparing the values of I that correspond to neighbouring admissible functions. Since y(x) gives a minimum value to I, I will increase if we 'disturb' y(x) slightly. These disturbed functions are constructed as follows].

Here we assume that extremizing curve admits continuous first order derivatives and is differentiable twice.

Let $\eta'(x)$ be any continuous, differentiable, arbitrary (but fixed) function s.t. $\eta''(x)$ is continuous and

$$\eta(a) = \eta(b) = 0 \ . \qquad (3)$$

If α is a small parameter,

then $\qquad [y(x, \alpha)] = \bar{y}(x) = y(x) + \alpha\, \eta(x) \qquad (4)$

(represents a one parameter family of admissible functions) and will satisfy the same boundary conditions (2) as satisfied by y(x).

The vertical deviation of a curve in this family from the minimizing curve y(x) is $\alpha\eta(x)$, as shown in fig. 5.2. The significance of (4) lies in the fact that for each family of this type, i.e., for each choice of the function $\eta(x)$, the minimizing function y(x) belongs to the family and corresponds to the value of parameter $\alpha = 0$. The difference $\bar{y} - y = \alpha\eta$ is called variation of the function y denoted by δy.

Now, with $\eta(x)$ fixed, we substitute

$\bar{y}(x) = y(x) + \alpha \ \eta(x)$ and

$\bar{y}'(x) = y'(x) + \alpha \ \eta'(x)$ into functional (1) and get a function of α, i.e.

$$I[\bar{y}(x)] = I(\alpha) = \int_a^b I[x, y(x,\alpha), y'(x,\alpha)]\,dx$$

$$= \int_a^b F[x, y(x) + \alpha\eta(x), \quad y'(x) + \alpha\eta'(x)]dx \ . \tag{5}$$

When $\alpha = 0$, (4) yields $\bar{y}(x) = y(x)$; and since y(x) minimizes the integral, $\therefore I(\alpha)$ must have a minimum when $\alpha = 0$. By elementary calculus, we known that a necessary condition for the extremum of a functional is that its variation must be zero i.e. $I'(\alpha) = 0$ when $\alpha = 0$ i.e. $I'(0) = 0$

i.e. $$\frac{\partial}{\partial\alpha} I[\alpha]_{\alpha=0} = 0 \qquad \left[\text{i.e.} \frac{\partial}{\partial\alpha} I[y(x) + \alpha\eta(x)]_{\alpha=0} = 0\right]$$

or $$\frac{dI}{d\alpha} = 0 \quad at \quad \alpha = 0 \ .$$

The derivative $I'(\alpha)$ can be computed by differentiating (5) under the integral sign, i.e.

i.e. $$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} F[x, \bar{y}, \bar{y}']dx \quad \text{as} \quad \left[\frac{db}{d\alpha} = 0, \frac{da}{d\alpha} = 0\right] \tag{6}$$

[Leibnitz's Rule for differentiation under the integral sign

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}\Bigg]$$

Now, by the chain rule for differentiating functions of several variables, we have

$$\frac{\partial}{\partial \alpha} F[x,\bar{y},\bar{y}'] = \frac{\partial F}{\partial x}.\frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \alpha} \ ,$$

x being independent of α $\therefore \ \frac{\partial x}{\partial \alpha} = 0$.

Also $\frac{\partial \bar{y}}{\partial \alpha} = \eta(x) \quad and \quad \frac{\partial \bar{y}'}{\partial \alpha} = \eta'(x)$.

So from (6) , we have

$$\frac{dI}{d\alpha} = I'(\alpha) = \int_a^b \left[\frac{\partial F}{\partial \bar{y}}\eta(x) + \frac{\partial F}{\partial \bar{y}'}\eta'(x)\right]dx \ . \tag{7}$$

Now I' (0) = 0. Putting α = 0 in (7) yields $\int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right]dx = 0$ (8)

$$[\because for \alpha = 0 \quad \bar{y} = y]$$

[In this equation the derivative η'(x) appears along with the function η(x).] we can eliminate η'(x) by integrating the 2nd term by parts, which gives,

$$\int_a^b \frac{\partial F}{\partial y'}\eta'(x)dx = \left[\eta(x)\frac{\partial F}{\partial y'}\right]_a^b - \int_a^b \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx$$

$$= \quad -\int_a^b \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx \qquad [\because \eta(a) = \eta(\mathrm{b}) = 0] \ .$$

We can therefore write (8) in the form

$$\int_a^b \eta(x)\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]dx = 0 \ . \tag{9}$$

Now our reasoning upto this point is based on a fixed choice of the function $\eta(x)$. However, since the integral in (9) must vanish for every such function, $\therefore$ the expression in brackets must also vanish. (The integrand being a continuous function on [a, b]). or

Using fundamental lemma of calculus of variations, which states that if

$\int_a^b \eta(x)H(x)dx = 0$ for any sufficiently differentiable function $\eta(x)$ within the integration range, that vanishes at the end points of the interval, then it follows that H(x) is identically zero on its domain i.e. $H(x) \equiv 0$.

This yields

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{10}$$

which is known as Euler's equation.

Conclusion If y(x) is an admissible function that minimizes the integral (1), then y satisfies Euler's equation but converse is not necessarily true i.e. if y can be found that satisfies this equation, y need not minimize I

Second term in equation (10) can be written in expanded form as

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$$

$\therefore$ Euler's equation (10) becomes

$$F_{y'y'}\frac{d^2y}{dx^2} + F_{y'y}\frac{dy}{dx} + \left(F_{y'x} - F_y\right) = 0 \tag{11}$$

which is a second order differential equation unless $F_{y'y'} = 0$.

Special Cases These particular cases can be obtained either directly from some identity or from Euler's equation.

Case A If x and y are missing from function F, then (11) reduces to

$$F_{y'y'} \frac{d^2 y}{dx^2} = 0$$

and if $F_{y'y'} \neq 0$, we have $\frac{d^2 y}{dx^2} = 0$ and so $y = c_1 x + c_2$. $\Rightarrow$ extremals are all straight lines.

Case B If y is missing from the function F,

then Euler's equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\Rightarrow \quad \frac{\partial F}{\partial y'} = c_1$ which is a first order equation for the extremals. This differential equation is solved to get extremals.

Case C If x is missing from the function F, then Euler's equation can be integrated to

$$\frac{\partial F}{\partial y'} y' - F = c_1 .$$

This follows from the identity

$\frac{d}{dx}\left[\frac{\partial F}{\partial y'} y' - F\right] = y'\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) - \frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x}$, since $\frac{\partial F}{\partial x} = 0$ and expression in brackets on right is zero by Euler's equation.

$$\Rightarrow \frac{d}{dx}\left[\frac{\partial F}{\partial y'} y' - F\right] = 0 \qquad \Rightarrow \frac{\partial F}{\partial y'}\ y' - F = c_1 .$$

This case is also called Beltrami identity.

Note The solutions of Euler's equation satisfying the boundary conditions are called stationary functions. The value of the functional at a stationary function is called a stationary value of the functional.

Case D If y' is missing from the function F i.e. functional is of the form

$$\int_a^b F(x,y)dx\,.$$

Then Euler's equation reduces to

$$F_y - \frac{d}{dx}(0) = 0$$

$\Rightarrow F_y = 0$, this is not a differential equation but a finite equation. This finite equation when solved for y, does not involve any arbitrary constant. Thus, in general it is not possible to find y satisfying the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ and as such this variatonal problem does not in general, admit a solution.

Case E When the functional is of the form

$$I(y) = \int_a^b f(x,y)\sqrt{1+y'^2}\,dx\,.$$

Here $F(x, y, y') = f(x, y)\sqrt{1+y'^2}$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial f}{\partial y}\sqrt{1+y'^2} - \frac{d}{dx}\left[f(x,y)\frac{1}{2}.\frac{2y'}{\sqrt{1+y'^2}}\right]$$

$$= \quad f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f(x,y)\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]$$

$$= \quad f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f(x,y)\left[\frac{y''}{\sqrt{1+y'^2}} + y'\left(-\frac{1}{2}\right)\left(1+y'^2\right)^{-3/2}\right]2y'y^n$$

$$= \quad f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f(x,y)\left[\frac{y''}{\sqrt{1+y'^2}} - y'^2\,y''\left(1+y'^2\right)^{-3/2}\right]$$

$$= \quad f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f(x,y)\left[\frac{y''}{\left(1+y'^2\right)^{1/2}} - \frac{y'^2\,y''}{\left(1+y'^2\right)^{3/2}}\right]$$

$$= \quad f_y\sqrt{1+y'^2} - f_x \frac{y'}{\sqrt{1+y'^2}} - f(x,y)\left[\frac{\left(1+y'^2\right)y''-y'^2\,y''}{\left(1+y'^2\right)^{3/2}}\right]$$

$$= \quad f_y\sqrt{1+y'^2} - \frac{f_x\,y'}{\sqrt{1+y'^2}} - f(x,y)\frac{y''}{\left(1+y'^2\right)^{3/2}}$$

$$= \quad \frac{1}{\sqrt{1+y'^2}}\left[f_y\,(1+y'^2) - f_x\,y' - \frac{f(x,y)y''}{\left(1+y'^2\right)}\right].$$

So Euler equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$ becomes as

$$\frac{1}{\sqrt{1+y'^2}}\left[f_y\,(1+y'^2) - f_x\,y' - \frac{f(x,y)y''}{\left(1+y'^2\right)}\right] = 0$$

$$\Rightarrow \quad f_y\,(1+y'^2) - f_x\,y' - \frac{f(x,y)y''}{\left(1+y'^2\right)} = 0\ .$$

Applications

Example 5.1 Find the plane curve of shortest length joining the two points A (a_1, b_1) and B (a_2, b_2). or

Show that shortest distance between two points in a plane is a straight line.

Let y(x) be a function whose curve passes through A (a_1, b_1) and B(a_2, b_2). The length of the arc between A and B is given by $I[y] = \int_{a_1}^{a_2} ds$. Now, as $(ds)^2 = dx^2 + dy^2$

$$\Rightarrow ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = dx\sqrt{1+y'^2}$$

$$\therefore I[y] = \int_{a_1}^{a_2} \sqrt{1+y'^2}\,dx\ .$$

Comparing it with $I[y] = \int_a^b F[x,y,y']\,dx;$

y(a) =A, y(b) = B ,

we have

$$F[x, y, y'] = [1 + y'^2]^{1/2} \qquad (1)$$

Let then length of the curve between given points be minimum for the curve y(x).

$\therefore$ the functional $\int_{a_1}^{a_2} F[x, y, y']\,dx$ has a minimum value at the function y(x).

$\therefore$ y(x) satisfies Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (2)$$

As from (1), F is independent of y

$$\therefore \quad \frac{\partial F}{\partial y} = 0 \qquad (3)$$

and $\dfrac{\partial F}{\partial y'} = \dfrac{y'}{\sqrt{1+y'^2}}$. $\qquad (4)$

$\therefore$ equation (2) becomes

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = c_1 \quad \Rightarrow \quad \frac{y'^2}{1+y'^2} = c^2$$

$$y'^2\,(1-c^2) = c^2 \quad \Rightarrow \quad y'^2 = \frac{c^2}{1-c^2} = D^2\,(say)$$

$$\Rightarrow \quad y'^2 = D^2 \quad \Rightarrow \quad y = Dx + E \qquad (5)$$

[OR x and y variables are missing in case A, $\Rightarrow$ This problem falls under case A.

Because

$$f_{y'y'} = \frac{\partial^2 f}{\partial y'^2} = \frac{1}{\left[1+(y')^2\right]^{3/2}} \neq 0$$

so, according to case A, we have $\frac{d^2y}{dx^2} = 0 \Rightarrow$ extermals are the two parameter family of straight lines $y = c_1x + c_2$].

This is the equation of extremals. We are to find that extremal which passes through the points (a_1, b_1) and (a_2, b_2).

$\therefore$ from (5) $y(a_1) = Da_1 + E$

$$\Rightarrow b_1 = Da_1 + E \tag{6}$$

Now

$$y(a_2) = Da_2 + E \qquad \Rightarrow \qquad b_2 = Da_2 + E \tag{7}$$

Subtracting (6) and (7)

$$b_1 - b_2 = D\,(a_1 - a_2)$$

$$\Rightarrow \qquad D = (b_1 - b_2)/(a_1 - a_2) \tag{8}$$

Substituting this in (6) , we have

$$b_1 = \frac{b_1 - b_2}{a_1 - a_2}a_1 + E$$

$$\Rightarrow E = b_1 - a_1\left(\frac{b_1 - b_2}{a_1 - a_2}\right) = \frac{a_1b_2 - a_2b_1}{a_1 - a_2} \tag{9}$$

Substituting (8) and (9) in (5) ,

$$y = \frac{b_1 - b_2}{a_1 - a_2}\,x + \frac{a_1b_2 - a_2b_1}{a_1 - a_2}\ . \tag{10}$$

This is a first degree equation and is of course the straight line joining the two points A and B.

[This analysis shows only that if I has a stationary value, then the corresponding stationary curve must be the straight line (10). Also, it is clear from the geometry that I has no maximizing curve but does have a minimizing curve. Thus we conclude that (10) actually is the shortest curve joining our two points. A more interesting problem is that of finding the shortest curve joining two fixed points on a given surface and lying entirely on that surface. These curves are called geodesics. In this case solution is less obvious and possibly many solutions may exist.]

Example 5.2 Find the extremum (extremals) of the functional

$$I[y] = \int_1^2 \sqrt{\frac{1+y'^2}{x}}\,dx \tag{1}$$

where y(1) = 0,   y(2) = 1.

Proof We know that the necessary condition for the functional

$$I[y] = \int_a^b F(x, y, y')\,dx \tag{2}$$

to be extremum is that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{(Euler's equation)} \tag{3}$$

satisfying the conditions y(a) =A,   y(b) = B.

[It may be noticed that the given functional I may be named as t[y(x)] i.e. as the time spent on translation along the curve y = y(x) from one point to another, if the rate of motion v = (ds/dt) is equal to x. As ds = $(1+ y'^2)^{1/2}$ dx

and $\quad \frac{ds}{dt} = x \qquad$ then dt $= \frac{ds}{x}$

and $\qquad t = \int_{x_0}^{x_1} \frac{ds}{x} = \int \frac{\sqrt{1+y'^2}}{x}dx.$

Comparing (1) and (2) , we have

$$F(x, y, y') = \frac{(1+y'^2)^{1/2}}{x}\,. \tag{4}$$

Since it is independent of y $\Rightarrow \frac{\partial F}{\partial y} = 0$ (5)

and $\dfrac{\partial F}{\partial y'} = \dfrac{y'}{x\sqrt{1+y'^2}}$. (6)

Making use of (5) and (6) in (3) , we find

$$0 - \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \qquad \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \qquad \frac{y'}{x\sqrt{1+y'^2}} = c \tag{7}$$

$\Rightarrow y'^2 = c^2x^2\,(1+y'^2)$

$\Rightarrow \qquad (1- c^2x^2)\, y'^2 = c^2x^2$

$$\Rightarrow \qquad y'^2 = \frac{c^2x^2}{1-c^2x^2}$$

$$\Rightarrow \qquad y' = \frac{cx}{\sqrt{1-c^2x^2}}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-1}{2c}\,\frac{(-2c^2x)}{\sqrt{1-c^2x^2}} .$$

Integrating , we get

$$y = \frac{-1}{2c}\,\frac{(1-c^2x^2)^{-1/2+1}}{\frac{1}{2}} + D$$

$$= -\frac{1}{c}\sqrt{1-c^2x^2} + D,$$ where D is the constant of integration.

$$y(x) - D = -\frac{1}{c}\sqrt{1-c^2x^2} .$$

$$\Rightarrow (y(x) - D)^2 = \frac{1}{c^2}\left(1-c^2x^2\right)$$

$$[y(x) - D]^2 + x^2 = \frac{1}{c^2}$$

$$\text{i.e. } x^2 + [y(x) - D]^2 = \frac{1}{c^2} \tag{8}$$

which is a family of circles, with centers on the axis of co-ordinates i.e. centres (0,D).
[2nd method of solving equation (7) is by introducing a parameter y' = tan t then

$$x = \frac{1}{c}\ \frac{y'}{\sqrt{1+y'^2}} = \frac{1}{c}\sin t$$

or $\quad x = \bar{c}\ \sin t$, where $\bar{c} = \frac{1}{c}$

$$\frac{dy}{dx} = \tan t \Rightarrow dy = \tan t \quad dx,$$

$\Rightarrow \quad dy = \tan t\ \ \bar{c}\ \cos t\, dt = \bar{c}\ \sin dt,$

$\Rightarrow \quad y = -\ \bar{c}\ \cos t + c_1$

Thus $x = \bar{c}\ \sin t$ and $y - c_1 = -\ \bar{c}\ \cos t$.

Eliminating t $\Rightarrow x^2 + (y - c_1)^2 = \bar{c}^{\,2}$, which is a family of circles].

Now using boundary conditions

$y(1) = 0, \quad y(2) = 1$, we get

$$[y(1) - D]^2 + 1^2 = \frac{1}{c^2}$$

$$\Rightarrow D^2 + 1 = \frac{1}{c^2}\text{, and} \tag{9}$$

$$[y(2) - D]^2 + 2^2 = \frac{1}{c^2} \Rightarrow [1 - D]^2 + 4 = \frac{1}{c^2} \tag{10}$$

Equating (9) and (10)

$$D^2 + 1 = D^2 - 2D + 5 \qquad \Rightarrow \quad D = 2$$

Now (10) $\Rightarrow$ $c^2 = \frac{1}{5}$ $\Rightarrow$ $c = \pm \frac{1}{\sqrt{5}}$

So the final solution is

$$[y(x) - 2]^2 + x^2 = 5.$$

Minimal Surface of Revolution

Example 5.3 Find the curve with fixed boundary points such that its rotation about the axis of abscissa gives rise to a surface of revolution of minimum surface area.

OR

Find the curve passing through (x_0, y_0) and (x_1, y_1) which generates the surface of minimum area when rotated about x-axis.

Proof Let $A(x_0, y_0)$ and $B(x_1, y_1)$ be two given points on the same side of x-axis. Let $y = y(x)$ be any continuous curve joining these two points.

i.e. $y(x_0) = y_0$ and $y(x_1) = y_1$ (1)

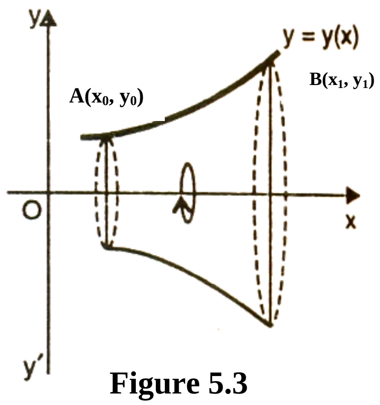

Figure 5.3

We know that, the area of surface of revolution generated by rotating the curve $y = y(x)$ about the x-axis is

$$I(y) = \int_{x_0}^{x_1} 2\pi\, y ds$$

where $(ds)^2 = (dx)^2 + (dy)^2 = (dx)^2 (1+y'^2)$.

$\Rightarrow$ $ds = dx\,(1+y'^2)^{1/2}$.

$$\therefore \text{ I(y) } = 2\pi \int_{x_0}^{x_1} y(1+y'^2)^{1/2}dx \quad . \tag{2}$$

Comparing it with

$$\text{I(y)} = \int_{x_0}^{x_1} F(x, y, y')dx \quad . \tag{3}$$

Now, I has to be minimum.

Also, necessary condition for this functional to be extremum satisfying the condition

y(a) = A and y(b) = B is that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad . \tag{4}$$

Here $\quad F(x, y, y') = y\,[1 + y'^2]^{1/2}$ (5)

Since the integrand F does not depend explicitly an x, so Euler's equation (4) reduces to (Using Case C)

$$F - y'\frac{\partial F}{\partial y'} = c \qquad \text{i.e.}$$

$\Rightarrow \quad y(1+y'^2)^{1/2} - y\,y'^2\,(1+y'^2)^{-1/2} = c,$

$\Rightarrow \quad y(1+y'^2) - y\,y'^2 = c\,(1+y'^2)^{1/2},$

$\Rightarrow \quad y = c\,(1+y'^2)^{1/2}.$

Squaring, we get

$y^2 = c^2(1+y'^2) \Rightarrow y^2 - c^2 = c^2 y'^2$

$$\Rightarrow \quad \frac{y^2 - c^2}{c^2} = y'^2 \qquad \Rightarrow \quad y' = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$. Separating variables

$$\Rightarrow \quad dx = \frac{c\,dy}{\sqrt{y^2 - c^2}} \quad .$$

Integrating , we get

$$x + c_1 = c \cosh^{-1} \frac{y}{c}$$

$$\Rightarrow \quad \frac{x + c_1}{c} = \cosh^{-1} \frac{y}{c}$$

$$\Rightarrow \quad \cosh\left(\frac{x + c_1}{c}\right) = \frac{y}{c}$$

$$\Rightarrow \quad y = c \cosh\left(\frac{x + c_1}{c}\right), \tag{6}$$

which constitutes a two parameters family of catenaries. The constants c and c_1 are determined by the conditions $y(x_0) = y_0$, $y(x_1) = y_1$. Thus (6) is a catenary passing through two given points. The surface generated by rotation of the catenary is called a catenoid.

Theorem 5.3

Find out the necessary condition for a functional I(y) = $\int_a^b F(x, y, y')dx$ (1)

to have an extremum, when y is not prescribed at the ends.

OR

Given the functional I[y] = $\int_a^b F(x, y, y')dx$ where the value of the unknown function y(x) is not assigned at one or both the ends at x = a & x = b. Then find the continuous differentiable curve for which the given functional attains extremum values.

Proof

The given functional is

$$I[y] = \int_a^b F(x, y, y')dx \quad . \tag{1}$$

Let y(x) be the actual extremal of (1) and let η(x) is a function defined on [a, b] with its continuous first order derivatives, then for assigned y(x) and η(x), the functional

$I[y + \alpha\eta] = \int_a^b F\left[x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)\right] dx$ is a function of α (a real no.) which attains extremum at α = 0.

$$\therefore \qquad \frac{dI}{d\alpha} = 0 \quad \text{when} \quad \alpha = 0$$

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_a^b F\left[(x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)\right] dx$$

$$= \int_a^b \frac{\partial}{\partial\alpha} F[x, y(x) + \alpha\eta(x), \quad y'(x) + \alpha\eta'(x)] dx$$

$$+ \ F[x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)]_{x=b} \frac{db}{d\alpha}$$

$$- F[x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)]_{x=a} \frac{da}{d\alpha}$$

$$= \int_a^b \frac{\partial}{\partial\alpha}\left[F(x, y(x), y'(x)) + \alpha\eta\,(x)\frac{\partial F}{\partial y} + \alpha\eta'(x)\frac{\partial F}{\partial y'} + \text{higher order terms are neglected,}\right]$$

$$= \int_a^b \frac{\partial}{\partial\alpha}\left[\alpha\eta(x)\frac{\partial F}{\partial y} + \alpha\eta'(x)\frac{\partial F}{\partial y'}\right] \mathrm{dx}$$

$$= \int_a^b \eta(x)\frac{\partial F}{\partial y} dx + \eta(x)\frac{\partial F}{\partial y'}\Big|_a^b - \int_a^b \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \int_a^b \eta(x)\frac{\partial F}{\partial y} dx + \left(\frac{\partial F}{\partial y'}\right)_{\mathrm{x=b}} \eta(b) - \left(\frac{\partial F}{\partial y'}\right)_{x=a} \eta(a) - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x) dx$$

$$= \int_a^b \eta(x)\frac{\partial F}{\partial y} dx - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x) dx + \left[\left(\frac{\partial F}{\partial y'}\right)\eta(x)\right]_{x=b} - \left(\frac{\partial F}{\partial y'}\eta(x)\right)_{\mathrm{x=a}} . \qquad (2)$$

Here η(x) does not necessarily vanishes at the end points.

The equation (2) must be satisfied by all permissible values of η(x)

$$\Rightarrow \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx = 0$$

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x=b} - \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x=a} = 0$$

$$\therefore \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

(Second order diff equation will consist of two arbitrary constants which will be determined by using natural boundary conditions).

Also $\left(\frac{\partial F}{\partial y'}\eta(x)\right)_{x=b} = 0$

and $\left(\frac{\partial F}{\partial y'}\eta(x)\right)_{x=a} = 0$

$\Rightarrow \quad \left(\frac{\partial F}{\partial y'}\right)_{x=b} = 0 \quad$ and $\left(\frac{\partial F}{\partial y'}\right)_{x=a} = 0$

are two conditions. [as $\eta(b) \neq 0$, $\eta(a) \neq 0$].

Case I Suppose that the value of y(x) at one point is given i.e. $y(x)|_{x=a}$ is given,

$\Rightarrow \qquad \eta(a)=0.$

Then calculate $\left(\frac{\partial F}{\partial y'}\right)_{x=b} = 0$ (from natural boundary condition).

Case II When the value of y(x) at upper end is given i.e. $y(x)|_{x=b}$ is given,

$\Rightarrow \eta(b) = 0$

Then calculate $\left(\frac{\partial F}{\partial y'}\right)_{x=a} = 0$ (from natural boundary condition).

Case III When neither $y(x)\big|_{x=b}$ nor $y(x)\big|_{x=a}$ is given, Then use the natural boundary conditions $\left(\frac{\partial F}{\partial y'}\right)_{x=a} = 0$ and $\left(\frac{\partial F}{\partial y'}\right)_{x=b} = 0$.

Example 5.4 Test for extremal the functional

$$I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)\,dx \tag{1}$$

when

(i) y is defined at the end points by

$$y(0) = 0, \quad y(\pi/2) = 1.$$

(ii) y is not prescribed at the end points.

Solution We know that the necessary condition for the functional

$$I[y] = \int_a^b F(x, y, y')dx \tag{2}$$

to have extremal is that y should satisfy the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \ . \tag{3}$$

Comparing (1) and (2), we get

$$F(x, y, y') = y'^2 - y^2 \tag{4}$$

$$\frac{\partial F}{\partial y} = -2y, \quad \frac{\partial F}{\partial y'} = 2y' \tag{5}$$

Making use of (5) in (3), we get

$$-2y - \frac{d}{dx}(2y') = 0$$

$$y + \frac{d}{dx}(y') = 0$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + y = 0$$

The Auxiliary equation is

$$D^2 + 1 = 0 \qquad \therefore\ D^2 = -1, \text{ where } D = \frac{d}{dx}$$

$\Rightarrow \qquad D = \pm i = 0 \pm i$

$\therefore$ solution is given by

$$y = e^{0x}(c_1 \cos 1.x + c_2 \sin 1.x)$$

i.e. $y = c_1 \cos x + c_2 \sin x$ (6)

This is the equation of the extremals.

Case I $\quad y(0) = 0, \qquad y\left(\frac{\pi}{2}\right) = 1$ are the boundary conditions.

$$y(0) = c_1 \cos 0 + c_2 \sin = 0$$

$$\Rightarrow \qquad c_1 = 0$$

and $y\left(\frac{\pi}{2}\right) = c_1 \cos\frac{\pi}{2} + c_2 \sin\frac{\pi}{2} = 1 \qquad \Rightarrow c_2 = 1$

Thus we find $c_1 = 0,\ c_2 = 1$

Thus the extremum can be achieved only on the curves $y = \sin x$.

Case II $\quad$ When y is not prescribed at the end points. Then

$$\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0 \qquad \text{and} \qquad \left(\frac{\partial F}{\partial y'}\right)_{x=\frac{\pi}{2}} = 0$$

Now,

$$\left(\frac{\partial F}{\partial y'}\right)_{x=0} = (2y')_{x=0} = 0 \qquad \Rightarrow \quad (y')_{x=0} = 0\ .$$

As $y = c_1 \cos x + c_2 \sin x$

$\Rightarrow \quad y' = -c_1 \sin x + c_2 \cos x$

$y'\big|_{x=0} = 0 \Rightarrow -c_1 . 0 + c_2 .1 = 0 \qquad\qquad c_2 = 0$

and $\frac{\partial F}{\partial y'}\Big|_{x=\pi/2} = 0 \quad \Rightarrow \quad (2y')_{x=\pi/2} = 0$

$\Rightarrow (y')\Big|_{x=\frac{\pi}{2}} = 0$

$\Rightarrow \quad (-c_1 \sin x + c_2 \cos x)_{x=\pi/2} = 0$

$\Rightarrow \quad -c_1 \sin \frac{\pi}{2} + c_2 .0 = 0$

$\Rightarrow \quad -c_1 = 0 \qquad\qquad \Rightarrow c_1 = 0 \qquad (8)$

$\therefore$ solution is $y(x) = 0$

Ex(i) If value of y at one end point is given $y(0) = 0$, $\left(\frac{\partial F}{\partial y'}\right)_{x=\pi/2} = 0$

(ii) $y(\pi/2) = 1$, $\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0$

(iii) If value at both end points is not given. Then

$$\left(\frac{\partial F}{\partial y'}\right)_{x=0} = 0, \quad \& \quad \left(\frac{\partial F}{\partial y'}\right)_{x=\pi/2} = 0$$

Brachistochrone Problem

Theorem 5.4

State the Brachistochrone problem and solve it.

Proof The Brachistochrone problem was passed by John Bernoulli in 1696, in which he advanced the problem of the line of quickest descent. The name ‘Brachistochrone’ is derived from the Greek words ‘brachisto’ meaning shortest and ‘chrone’ meaning time.

Let A and B be two fixed points. In this problem, it is required to find the curve (line) connecting two given points A and B, that does not lie on a vertical line, such that a particle sliding down this curve (line) under gravity (in the absence of any resistance or friction) from the point A reaches point B in the shortest time. This time depends upon the choice of the path (curve) and hence is a functional. This curve s.t. the particle takes the least time to go from A to B is called the brachistochrone.

[It is easy to see that the line of quickest descent will not be the straight line connecting A and B, though this is the shortest distance between the two points. The reason is that, the velocity of motion in a straight line will build up rather (comparatively) slowly. However, if we imagine a curve that is steeper near A, even though the path becomes longer, a considerable portion of the path will be covered at a greater speed. It turns out that the required path i.e. the line of quickest descent is a cycloid.]

Fix the origin at A with x-axis horizontal and y-axis vertically downwards, so that the point B is in the xy plane.

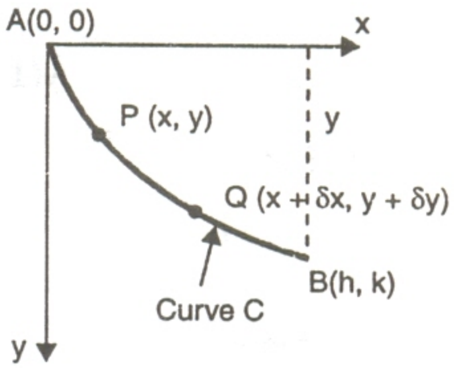

Figure 5.4

Let (h, k) be the co-ordinates of B. Let the particle is traversed from A to B along the curve C. let m be the mass of the particle. Let velocity of the particle be v when the particle is at the point P(x, y) on C and at A(origin) its velocity be zero.

By the principle of work and energy ,

K. E. at P – K.E. at A = work done in moving particle from A to P

$$\frac{1}{2}mv^2 - \frac{1}{2}m(0)^2 = mgy$$

$\Rightarrow \qquad v = \sqrt{2gy} \qquad .$ (1)

Let Q(x + δx, y + δy) be a neighbouring point on C r.t. $\acute{P}Q = \delta s$

$$\therefore \ \delta s = \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \sqrt{1 + y'^2}\,dx \ .$$

$\therefore$ time of descent from P to Q

$$= \frac{\delta s}{v} \quad \text{(distance travelled/velocity)}$$

$$= \qquad \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\delta x = \frac{1}{\sqrt{2g}} \cdot \frac{\sqrt{1+y'^2}}{\sqrt{y}}\delta x$$

$\therefore$ total time of taken by the particle from A(0, 0) to B (h, k) is

$$t[y(x)] = \frac{1}{\sqrt{2g}}\int_0^h \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx, \qquad (2)$$

with y(0) = 0 and y(h) = k .

$\therefore$ in order to find the shortest time of descent, we have to minimize the functional

$\int_0^h \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx$, subject to the fixed boundaries y(0) = 0, y(h) =k.

(From physical considerations, the functional has no maximum value)

$$\text{Let } F(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}} \ .$$

Since the integrand is independent of x, the Euler's equation reduces to

$$F - F_{y'}\frac{dy}{dx} = C$$

$$\Rightarrow \qquad \frac{\sqrt{1+y'^2}}{\sqrt{y}} - \left(\frac{1}{\sqrt{y}}.\frac{2y'}{2\sqrt{1+y'^2}}\right)y' = C$$

$$\Rightarrow \qquad \frac{1}{\sqrt{y}} \frac{1}{\sqrt{1+y'^2}} \left(1+y'^2 - y'^2\right) = C$$

$$\Rightarrow \qquad y(1+y'^2) = \frac{1}{C^2} = C_1 \text{ (say)} \tag{3}$$

I method

Introducing the parameter t by putting

$$y' = \cot t \text{ in (3)}$$

$$\Rightarrow \qquad y = C_1 \sin^2 t = \frac{C_1}{2}\left(1-\cos 2t\right) \tag{4}$$

Now dx = dy/y'

$= \dfrac{1}{\cot t} \dfrac{C_1}{2}(2\sin 2t)dt$ (using y from (4) find dy)

$= 2 C_1 \sin^2 t \, dt.$

$= C_1 (1 - \cos 2t)\, dt$

$$x = C_1 \int \left(1-\cos 2t\right)dt + C_2$$

$$x = \frac{C_1}{2}\left(2t - \sin 2t\right) + C_2 .$$

$\therefore$ The equation of the extremals i.e. the equation of the desired line in parametric form is

$$x = \frac{C_1}{2}\left(u - \sin u\right) + C_2$$

and $\quad y = \dfrac{C_1}{2}\left(1-\cos u\right)$. (putting u = 2t) (5)

The boundary conditions are

$y(0) = 0, \quad y(h) = k.$

$$\Rightarrow \frac{C_1}{2}(1-\cos u) = 0 \qquad \Rightarrow C_1 \text{ is arbitrary.}$$

$$\Rightarrow \frac{C_1}{2}(0-\sin 0) + C_2 = 0 \Rightarrow C_2 = 0$$

$\therefore$ The required curve is

$$x = \frac{C_1}{2}(u-\sin u), \qquad y = \frac{C_1}{2}(1-\cos u)$$

which is the standard form of cyclic curve and is called cycloid curve.

Thus we get a family of cycloids with $\frac{C_1}{2}$ as the radius of the rolling circle. The value of C_1 is found by the fact that the cycloid passes through B (h, k) i.e. y(h) = k.

Thus, the brachistochrone is a cycloid.

II method

Equation (3) is $y(1+y'^2) = C$

$$\Rightarrow y'^2 = \frac{C}{y} - 1 = (C-y)/y$$

$$y' = \frac{\sqrt{C-y}}{y} \qquad \text{i.e.} \quad \frac{dy}{dx} = \frac{\sqrt{C-y}}{\sqrt{y}}$$

Separating the variables and integrating

$$\int_0^y \frac{\sqrt{y}}{\sqrt{C-y}}\,dy = \int_0^x dx$$

On L.H.S. put $y = C\sin^2\theta \Rightarrow$

$$dy = 2C\cos\theta\sin\theta\,d\theta$$

$$\therefore \int_0^\theta \frac{\sqrt{C}\sqrt{\sin^2\theta}}{\sqrt{C-C\sin^2\theta}} 2C\cos\theta\sin\theta = x$$

or $$x = \int_0^\theta 2C\sin^2\theta \; d\theta = 2C\int_0^\theta \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{2C}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{C}{2}\left[2\theta - \sin 2\theta\right]$$

x = b $\left[2\theta - \sin 2\theta\right]$,

and y = C $\sin^2\theta$ $= \frac{C}{2}\left[1-\cos 2\theta\right] = b[1-\cos 2\theta]$.

Taking $2\theta = \phi$, we have

$x = b\,[\phi - \sin\phi]$,

$y = b\,(1 - \cos\phi)$,

which is the cycloid. The value of b can be found from the fact that curve passes through (h, k).

Exercise

1. Show that the curve of shortest distance between any two points on a circular cylinder of radius a is a helix.

2. Find the extremal of the functional $\int_0^1 (x + y'^2)dx$ that satisfies the boundary conditions y(0) = 1, y(1) = 2 .

3. Show that the Euler's equation for the functional

   $\int_{x_1}^{x_2} (a(x)y'^2 + 2b(x)yy' + c(x)y^2)dx$ is a second order linear differential equation.

4. Find the extremals and the stationary function of the functional $\int_0^\pi (y'^2 - y^2)dx$ that satisfy the boundary conditions y(0) = 1, y(π) = -1.

Answers

2. $y = x + 1$ 3. $a(x)y'' + a'(x)y' + (b'(x) - c(x))\,y = 0$

4. $y = C_1 \cos x + C_2 \sin x$, where C_1 and C_2 are arbitrary constants ;

   $y = \cos x + C_2 \sin x$, where C_2 is an arbitrary constant.

The Fundamental Lemma of the Calculus of Variations

Theorem 5.5

If a function $\Phi(x)$ is continuous on the closed interval $[x_0, x_1]$ and if

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\ dx = 0, \tag{1}$$

for an arbitrary continuous function $\eta(x)$ subject to some conditions of general character only then $\Phi(x) \equiv 0$ on $[x_0, x_1]$.

Note The conditions such as (i) $\eta(x)$ should be a first or higher order differentiable function (ii) $\eta(x)$ should vanish at the end points i.e. $\eta(x_0) = \eta(x_1) = 0$, $\eta(x) \in C^k$ and (iii)($|\eta(x)| < \varepsilon$ and $|\eta'(x)| < \varepsilon$.

are called the conditions of general character.

Proof Assume that $\Phi(x) \neq 0$ (say positive) at a point $x = \bar{x}$ in $x_0 \leq x \leq x_1$. (By assuming this we will arrive at a contradiction).

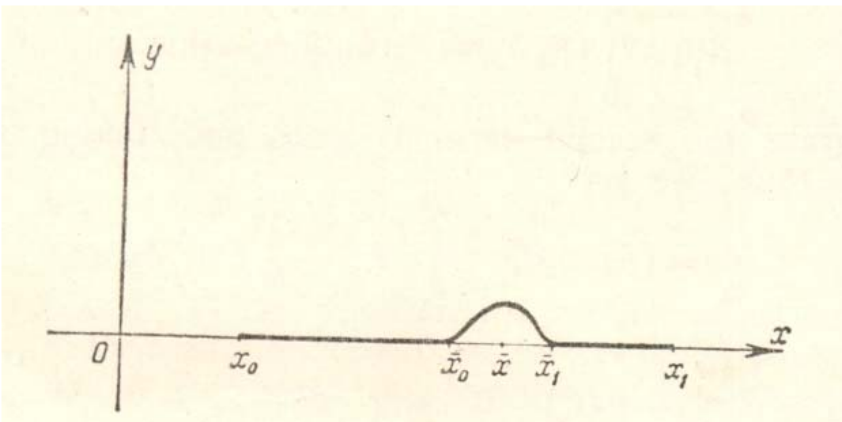

Figure 5.5 **A continuous function which is positive in an interval but vanishes outside.**

Since Φ(x) is continuous and Φ(x) ≠ 0 it follows that Φ(x) maintains positive sign in a certain neighbourhood $(\bar{x}_0 \le x \le \bar{x}_1)$ of the point $\bar{x}$. Since η(x) is an arbitrary continuous function, we might choose η(x) s.t. η(x) remains positive in $\bar{x}_0 \le x \le \bar{x}_1$ but vanishes outside this interval

Then, from equation (1) it follows that

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)dx = \int_{x_0}^{\bar{x}_1} \Phi(x)\eta(x)dx + \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\eta(x)dx + \int_{\bar{x}_1}^{x_1} \Phi(x)\eta(x)dx$$

$$= \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\eta(x)dx > 0, \tag{2}$$

since the product Φ (x) η(x) remains positive everywhere in $[\bar{x}_0, \bar{x}_1]$ and vanishes outside this interval. This contradiction between (1) and (2) shows that our original assumption Φ(x) ≠ 0 at some point $\bar{x}$ must be wrong and hence $\Phi(x) \equiv 0 \forall x \in [a,b]$.

[**For example** η(x) = k$(x-\bar{x}_0)^{2n}$ $(x-\bar{x}_1)^{2n}$ on the interval $(\bar{x}_0 \le x \le \bar{x}_1)$, where n is a positive integer and k is a constant number. It is obvious that the function η(x) satisfies the above condition i.e. η(x) vanishes at the end points and may be made arbitrarily small in absolute values together with its derivatives by reducing the absolute values of the constant k. Also η(x) is continuous and has continuous derivative upto order 2n-1.]

Euler's equation for functionals of the form

I[y_1,……, y_n] = $\int_a^b F(x, y_1, ..., y_n, y_1', y_2' ......, y_n')dx$ **i.e.**

Euler's equation for n dependent functions.

Theorem 5.6 A necessary condition for the curve $y_i = y_i(x)$ (i = 1, …n) to be an extremal of the functional

$\int_a^b F(x, y_1, ..., y_n, y_1^{'}, y_2^{'} ...., y_n^{'}) dx$ is that the function $y_i(x)$ satisfy the Euler equation. $F_{y_i} - \frac{d}{dx} F_{y'_i} = 0 \qquad (i = 1 ..., n)$.

or

A necessary condition for the functional $\int_a^b F(x, y_1, ..., y_n, y_1^{'}, y_2^{'} ......, y_n^{'}) dx$ to be an extremum is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1^{'}}\right) = 0, \qquad i = 1, ..., n$$

Proof Consider the functional

$$I[y_1, ... y_n] = \int_a^b F(x, y_1, ..., y_n, y_1^{'}, y_2^{'} ...., y_n^{'}) d \tag{1}$$

(depending upon one independent and several (n) dependent variables and their first derivatives when ends are fixed), where

x is one independent variable; $y_1, \ldots, y_n$ are n dependent variables depending on x and satisfying the conditions

$$y_i(a) = A_i \quad \text{and } y_i(b) = B_i, \qquad i = 1, ...n$$

A_i and B_i are constants.

[In other words, we are looking for an extremum of the functional (1) defined on the set of smooth curves joining two fixed points in (n+1) dimensional Euclidean space. The problem of finding geodesics i.e. shortest curve joining two points of some mainfold is of this type].

We know that a necessary condition for a functional (1) to attain extremal is that $\delta I = 0$,

i.e. $$\delta \int_a^b F(x, y_1, ..., y_n, y_1^{'}, y_2^{'} ......, y_n^{'}) dx = 0$$

i.e. $$\int_a^b \delta F \; dx = 0$$

or $$\int_a^b \left[\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_2}\delta y_2 + ... + \frac{\partial F}{\partial y_n}\delta y_n + \right.$$

$$\left. \frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2'}\delta y_2' + ... + \frac{\partial F}{\partial y_n'}\delta y_n'\right] dx = 0 \qquad (2)$$

Taking the general term

$\int_a^b \left(\frac{\partial F}{\partial y_i'}\right)\delta y_i' \, dx$ and integrating it by parts, we have

$$= \left[\frac{\partial F}{\partial y_i'}\delta y_i\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)\delta y_i \quad dx \qquad (3)$$

Thus, using (3) in equation (2), we get

$$\Rightarrow \int_a^b \left[\left\{\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right)\right\}\delta y_1 + \left\{\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right)\right\}\delta y_2 + \ldots \text{upto n terms}\right] dx$$

$$+ \sum_{i=1}^{n}\left[\frac{\partial F}{\partial y_i'}\delta y_i\right]_a^b = 0$$

Now variation of y_i at the end points must vanish i.e. $\delta y_i = 0$ at the ends. Therefore second term in above is zero, $\forall i = 1,....2.,.....,n.$

$\therefore$ the necessary condition implies that

$$\int_a^b \left\{\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)\right\}\delta y_i \, dx = 0 \quad \text{for every i } = 1, \ldots, \text{n}$$

Also δ_{y_i} are arbitrary

$$\Rightarrow \frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0, \quad \mathrm{i} = 1, \ldots \mathrm{n} \;, \qquad (4)$$

which forms system of n second order differential equations called Euler's equations (one for each function involved in functional). Its general solution contains 2n arbitrary constants, which are determined from the given boundary conditions.

A problem for optics

Example 5.5 Propagation of light in an inhomogeneous medium.

OR

Find the differential equation of the lines of propagation of light in an optically nonhomogenous medium in which the speed of light is v(x, y, z) .

Solution [This is an illustration of above principle].

Suppose that three-dimensional space is filled with an optically inhomogenous medium s.t. the velocity of propagation of light at each point is some function v(x, y, z) of the co-ordinates of the point.

According to well known Fermat's law, light propagates from one point $A(x_0, y_0)$ to another $B(x_1, y_1)$ along the curve, for which, the time T of passage of light will be least.

If the equation of the desired curve i.e. equation of the path of light ray be y = y(x) and z = z(x), then the time taken by the light to traverse the curve equals

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2+z'^2}}{v(x,y,z)}dx \quad \text{i.e. } T = \int_{x_0}^{x_1} \frac{ds}{v}$$

where ds is a line element on the path.

Writing Euler's equations for this functional i.e.

$$\frac{\sqrt{1+y'^2+z'^2}}{v^2} \quad \frac{\partial v}{\partial y} + \frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2+z'^2}}\right] = 0,$$

$$\frac{\sqrt{1+y'^2+z'^2}}{v^2} \quad \frac{\partial v}{\partial z} + \frac{d}{dx}\left[\frac{z'}{v\sqrt{1+y'^2+z'^2}}\right] = 0 \ .$$

These differential equations determine the path of the light propagation.

Example 5.6 Use calculus of variation to find the curve joining points (0, 0, 0) & (1, 2, 4) of shortest length. Also find the distance between these two points.

Solution Given points are (0, 0, 0) and (1, 2, 4). Suppose C be the curve y = y(x) and

I[y, z] = length of the curve joining (0, 0, 0) & (1, 2, 3)

In this case, length of small segment is

$$= \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \ .$$

$$\text{So total length} = \int_{(0,0,0)}^{(1,2,4)} \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \ ,$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx.$$

Hence $\quad I[y, z] = \int_0^1 \sqrt{1 + y'^2 + z'^2} \ dx.$

Boundary conditions are

y(0) = 0, y(1) = 2 ,

z(0) = 0 , z(1) =4 .

Now, we know that if

$$I[y_1, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y_1', y_2' \ldots\ldots, y_n')\,dx \ ,$$

then the condition for this functional to be extermal is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \ .$$

Here $I[y, z] = \int_0^1 \sqrt{1 + y'^2 + z'^2} \ dx \ ,$

$$F = \sqrt{1 + y'^2 + z'^2} \ ,$$

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \ .$$

$\therefore$ The Euler's equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \& \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

now become

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \& \quad -\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

The first Euler's equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \qquad \Rightarrow \left(\frac{\partial F}{\partial y'}\right) = A \tag{1}$$

Second Euler's equation is $\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \ \Rightarrow \ \left(\frac{\partial F}{\partial z'}\right) = B$ (2)

Now

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2 + z'^2}} = A \qquad \text{and} \tag{3}$$

$$\frac{\partial F}{\partial z'} = \frac{z'}{\sqrt{1 + y'^2 + z'^2}} = B \ . \tag{4}$$

Dividing (3) and (4), we get

$$\frac{y'}{z'} = \frac{A}{B} \quad \Rightarrow y' = z' \frac{A}{B} = z'C \ .$$

Putting value of y' in (3)

$$\frac{z'C}{\sqrt{1 + y'^2 + z'^2}} = A$$

or $\qquad z'^2 C^2 = A^2 (1 + y'^2 + z'^2)$, where $y' = z'C$.

$\Rightarrow \qquad z'^2 = \dfrac{A^2}{C^2 - A^2C^2 - A^2}$

$\Rightarrow \qquad z' = D$ (say) $\qquad \Rightarrow \; z = Dx + E$.

Similarly $y = D'x + E'$.

Now given conditions are

$y(0) = 0, \quad z(0) = 0$,

$y(1) = 2, \quad z(1) = 4$.

$\Rightarrow \qquad E' = 0, \quad E = 0 ; \quad D = 4, \; D' = 2$.

$\Rightarrow \qquad z = 4x, \quad y = 2x$.

These are two surfaces and

$$\Rightarrow \qquad \frac{z}{4} = \frac{y}{2} = \frac{x}{1}$$

which are the equations of a straight line.

Now using geometry , equation of a straight line passing through (0, 0, 0), (1, 2, 4) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

$$\frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{4-0} ,$$

or $\qquad \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{4}$.

Distance = $\sqrt{1+4+16} = \sqrt{21}$

Using variation = $\displaystyle\int_0^1 \sqrt{1 + y'^2 + z'^2}\, dx$

$$= \int_0^1 \sqrt{1 + 2^2 + 4^2}\, dx = \int_0^1 \sqrt{21}\, dx = \sqrt{21} \qquad Ans.$$

Hence result is verified.

Example 5.7 Test for extremum the functional

$$I[y,z] = \int_0^1 \left(y'^2 - z'^2\right) dx \tag{1}$$

$$y(0) = 0,\ y(1) = 1,\ z(0) = 0,\ z(1) = 2\ . \tag{2}$$

Solution System of Euler's equations for (1) is $y'' = 0$, $z'' = 0$.

Their solutions are

$$y(x) = C_1 + C_2x\ ,$$

$$z(x) = C_3 + C_4x\ , \tag{3}$$

C_1, C_2, C_3 and C_4 are arbitrary constants .

$\Rightarrow$ $C_1 = 0$, $C_2 = 1$, $C_3 = 0$, $C_4 = 2$.

(By using (2) in (3))

Therefore, desired solutions are

$$y(x) = x \quad , \quad z(x) = 2x\ ,$$

i.e. $2y = 2x = z$, which is a straight line passing through origin.

Example 5.8 Find the extremals of the functional

$$v[y(x), z(x)] = \int_0^{\pi/2} \left[y'^2 + z'^2 + 2yz\right] dx, \text{ subject to boundary conditions}$$

$$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1, \quad z(0) = 0, \quad z\left(\frac{\pi}{2}\right) = -1\ .$$

Solution The system of Euler's differential equations is of the form

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad i.e. \quad 2z - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow \quad y'' - z = 0\ .$$

Similarly $\quad\Rightarrow\quad z'' - y = 0.$

Eliminating one of the unknown functions say z we get $y^{(iv)} - y = 0 \Rightarrow (D^4 - 1)\, y = 0$

Solution of this differential equation can be written as

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x \,.$$

Now $\quad z = y'' \quad \Rightarrow z = C_1 e^x + C_2 e^{-x} - C_3 \cos x - C_4 \sin x \,.$

Using the boundary conditions, we find

$$C_1 = 0,\quad C_2 = 0,\quad C_3 = 0,\quad C_4 = 1;$$

Hence

$y = \sin x$, $z = -\sin x$ are the required extremals.

Example 5.9 Show that the functional $\int_0^1 \left(2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) dt$ such that $x(0) = 1$, $y(0) = 1$, $x(1) = 1.5$, $y(1) = 1$ is stationary for $x = \frac{2+t^2}{2}, y = 1$.

Solution Given functional is $\int_0^1 \left(2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) dt \quad i.e. \int_0^1 (2x + x'^2 + y'^2)\,dt$

Let $\quad F(t, x, y, x', y') = 2x + x'^2 + y'^2$.

The Euler's equations are

$$F_x - \frac{d}{dt} F_{x'} = 0 \tag{1}$$

and

$$F_y - \frac{d}{dt} F_{y'} = 0 \quad . \tag{2}$$

Here $F_x = 2$, $F_y = 0$, $F_{x'} = 2x'$, $F_{y'} = 2y'$.

$\therefore \quad (1) \;\Rightarrow\; 2 - \frac{d}{dt}(2x') = 0 \;\Rightarrow\; \frac{d^2x}{dt^2} = 1 \;\Rightarrow\; \frac{dx}{dt} = t + c_1$

$$\Rightarrow \quad x = \frac{t^2}{2} + c_1 t + c_2 \quad . \tag{3}$$

$$(2) \quad \Rightarrow \quad 0 - \frac{d}{dt}(2y') = 0 \quad \Rightarrow \quad \frac{dy'}{dt} = 0 \quad \Rightarrow \quad y' = c_3$$

$$\Rightarrow \quad y = c_3 t + c_4 \quad . \tag{4}$$

$\therefore$ (3) and (4) are the equations of the extremals.

The boundary conditions are x(0) = 1, y(0) = 1, x(1) = 1.5, y(1) = 1 .

$$y(0) = 1 \quad \Rightarrow \quad 0 + c_1(0) + c_2 = 1 \quad \Rightarrow \quad c_2 = 1 \ .$$

$$y(0) = 1 \quad \Rightarrow \quad c_3(0) + c_4 = 1 \quad \Rightarrow \quad c_4 = 1 \ .$$

$$x(1) = 1.5 \Rightarrow \quad \frac{1}{2} + c_1(1) + c_2 = 1.5 \ \Rightarrow \ \frac{1}{2} + c_1 + 1 = 1.5 \Rightarrow c_1 = 0 \ .$$

$$y(1) = 1 \quad \Rightarrow \quad c_3(1) + c_4 = 1 \quad \Rightarrow \quad c_3 + 1 = 1 \Rightarrow c_3 = 0 \ .$$

$\therefore \quad c_1 = 0, \quad c_2 = 1, \quad c_3 = 0, \quad c_4 = 1$.

$$\therefore (3) \quad \Rightarrow \quad x = \frac{t^2}{2} + 0.t + 1 \quad i.e., \quad x = \frac{2 + t^2}{2} \quad .$$

$$(4) \quad \Rightarrow \quad y = 0.t + 1 \quad \text{i.e.,} \quad y = 1 \ .$$

$\therefore$ The stationary functions are $x = \dfrac{2 + t^2}{2}, y = 1$.

Remark Since, the function $2x + x'^2 + y'^2$ is not a homogeneous functions of x' and y', we have treated the given functional as dependent on two functions.

Example 5.10 Find the extremals of the functionals

$$J[y(x), z(x)] = \int_0^{\pi} (2yz - 2y^2 + y'^2 - z'^2)\, dx.$$

Also find that extermal which satisfies the boundary conditions

$$y(0) = 0, \ y(\pi) = 1, \ z(0) = 0, \ z(\pi) = 1.$$

Solution We have $J[y(x), z(x)] = \int_0^{\pi} (2yz - 2y^2 + y'^2 - z'^2)\,dx$

Let $F(x, y, z, y', z') = 2yz - 2y^2 + y'^2 - z'^2$

The Euler's equations are

$$F_y - \frac{d}{dx}F_{y'} = 0 \tag{1}$$

and

$$F_z - \frac{d}{dx}F_{z'} = 0 \quad . \tag{2}$$

Here $F_y = 2z - 4y$, $F_z = 2y$, $F_{y'} = 2y'$ $F_{z'} = -2z'$.

$\therefore$ (1) $\Rightarrow$ $2z - 4y - \frac{d}{dx}(2y') = 0 \Rightarrow \frac{d^2y}{dx^2} + 2y - z = 0$. (3)

(2) $\Rightarrow$ $2y - \frac{d}{dx}(-2z') = 0 \Rightarrow \frac{d^2z}{dx^2} + y = 0$. (4)

(3) $\Rightarrow$ $(D^2 + 2)\,y - z = 0$. (5)

(4) $\Rightarrow$ $y + D^2 z = 0$. (6)

Operating (5) by D^2 and adding to (6), we get

$$D^2(D^2 + 2)\,y + y = 0 .$$

$$\Rightarrow (D^4 + 2D^2 + 1)\,y = 0 \Rightarrow (D^2 + 1)^2\,y = 0 .$$

The A.E. is $(D^2+1)^2 = 0$. $\therefore$ $D = \pm i, \pm i$

$$y = e^{0x}(c_1 + c_2x)\cos 1.x + (c_3 + c_4x)\sin 1.x).$$

$$y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x \ . \tag{7}$$

(3) $\Rightarrow$ $z = \frac{d^2y}{dx^2} + 2y$ (8)

(7) $\Rightarrow$ $\frac{dy}{dx} = c_2\cos x - (c_1 + c_2x)\sin x + c_4\sin x + (c_3 + c_4x)\cos x$

$= (c_2 + c_3 + c_4x) \cos x + (c_4 - c_1 - c_2x) \sin x$

$\therefore \qquad \dfrac{d^2y}{dx^2} = c_4 \cos x - (c_2 + c_3 + c_4x)\sin x - c_2 \sin x + (c_4 - c_1 - c_2x)\cos x$

$= (2c_4 - c_1 - c_2x) \cos x - (2c_2 + c_3 + c_4x) \sin x$.

$\therefore$ (8) $\Rightarrow$ $z = (2c_4 - c_1 - c_2x) \cos x - (2c_2 + c_3 + c_4x) \sin x$

$+ 2(c_1 + c_2x) \cos x + 2(c_3 + c_4x) \sin x$

$= (2c_4 + c_1 + c_2x) \cos x + (c_3 - 2c_2 + c_4x) \sin x$.

$\therefore$ The family of extremals is

$$y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x ,$$

$$z = (2c_4 + c_1 + c_2x) \cos x + (c_3 - 2c_2 + c_4x) \sin x,$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

The boundary conditions are:

$y(0) = 0,\ y(\pi) = 1,\ z(0) = 0,\ z(\pi) = 1$.

$y(0) = 0 \Rightarrow (c_1 + c_2.0) \cos 0 + (c_3 + c_4.0) \sin 0 = 0 \Rightarrow c_1 = 0$.

$y(\pi) = 1 \Rightarrow (c_1 + \pi c_2) \cos \pi + (c_3 + \pi c_4) \sin \pi = 1$.

$\Rightarrow (0 + \pi c_2)(-1) = 1 \Rightarrow c_2 = - 1/\pi$.

$z(0) = 0 \Rightarrow (2c_4 + c_1 + c_2.0) \cos 0 + (c_3 - 2c_2 + c_4.0) \sin 0 = 0$

$\Rightarrow (2c_4 + 0 + 0) .1 + \left(c_3 + \dfrac{2}{\pi} + 0\right)0 = 0 \Rightarrow c_4 = 0$.

$z(\pi) = 1 \Rightarrow (2(0) + 0 + \pi (-1/\pi)) \cos \pi + (c_3 - 2(-1/\pi) + 0.\pi) \sin \pi = 1$

$\Rightarrow$ (-1) (-1) =1, which is true.

$\therefore$ $c_1 = 0,\ c_2 = -1/\pi,\ c_4 = 0$

$\therefore \qquad y = \left(0 - \dfrac{x}{\pi}\right)\cos x + (c_3 + 0x)\sin x = c_3 \sin x - \dfrac{x}{\pi}\cos x$.

$$z=\left(2(0)+0-\frac{x}{\pi}\right)\cos x+\left(c_3-2\left(-\frac{1}{\pi}\right)+0.x\right)\sin x$$

$$= c_3 \sin x+\frac{1}{\pi}(2\sin x-x\cos x) \ .$$

$\therefore$ The required extremals are $y = c_3 \sin x-\frac{x}{\pi}\cos x$

$$z = c_3 \sin x+\frac{1}{\pi}(2\sin x - x\cos x),$$

where c_3 is any arbitrary constant.

Example 5.11 Find the extremals of the functional $\int_0^1 \sqrt{1+y'^2+z'^2}\,dx$ that satisfy the boundary conditions $y(0) = 0$, $y(1) = 2$, $z(0) = 0$, $z(1) =4$.

Sol. Given functional is $\int_0^1 \sqrt{1+y'^2+z'^2}\,dx$

Let $\quad F(x, y, z, y', z') = \sqrt{1+y'^2+z'^2}$.

The Euler's equations are $F_y-\frac{d}{dx}F_{y'}=0 \qquad (1)$

and $$F_z-\frac{d}{dx}F_{z'}=0 \ . \qquad (2)$$

Here $F_y = 0$, $F_z = 0$, $F_{y'} = \dfrac{y'}{\sqrt{1+y'^2+z'^2}}, F_{z'}=\dfrac{z'}{\sqrt{1+y'^2+z'^2}}$.

$\therefore \quad (1) \Rightarrow \quad 0-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right)=0$,

$$\Rightarrow \quad \frac{y'}{\sqrt{1+y'^2+z'^2}}=c_1 \ . \qquad (3)$$

$$(2) \quad \Rightarrow \quad 0-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right)=0 \ ,$$

$$\Rightarrow \quad \frac{z'}{\sqrt{1+y'^2+z'^2}}=c_2 \ . \tag{4}$$

Dividing (3) by (4), we get $\frac{y'}{z'}=\frac{c_1}{c_2}=c_3 \quad say$

$\therefore \qquad y' = c_3z'$.

Putting the value of y' in (4), we get $\frac{z'}{\sqrt{1+(1+c_3^2)z'^2}}=c_2$

$$\Rightarrow \quad z'^2=c_2^2(1+(1+c_3^2))z'^2 \Rightarrow z'=\frac{c_2}{\sqrt{1-c_2^2-c_2^2\ c_3^2}}=c_4, say$$

$$\Rightarrow \quad z = c_4x + c_5 \tag{5}$$

$\therefore \qquad y' = c_3z' \ \Rightarrow \ y' = c_3c_4 = c_6$, say

$$\Rightarrow \quad y = c_6x + c_7 \tag{6}$$

$\therefore$ The extremals are given by (5) and (6)

The boundary conditions are

$y(0) = 0, \ y(1) = 2, \ z(0) = 0, \ z(1) = 4$.

$\therefore \quad c_6(0) + c_7 = 0, \ c_6(1) + c_7 = 2, c_4(0) + c_5 = 0, \ c_4(1) + c_5 = 4$.

Solving these equations, we get

$c_6 = 2, \ c_7 = 0, c_4 = 4, c_5 = 0$.

(6) $\Rightarrow \quad y = 2.x + 0 \Rightarrow y = 2x$.

(5) $\Rightarrow \quad z = 4.x + 0 \Rightarrow z = 4x$.

$\therefore$ The required extremals are $y = 2x$, $z = 4x$.

Exercise

1. Find the Euler-Ostrogradsky equation for the functional J[z(x,y)] = $\iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial x}\right)^2\right] dxdy$ where the values of z are prescribed on the boundary of the domain D.

2. Find the Euler-Ostrogradsky equation for the functional J[z(x,y)] = $\iint_D \left[\left(\frac{\partial z}{\partial x}\right)^4 + \left(\frac{\partial z}{\partial y}\right)^4 + 12zf(x,y)\right] dxdy$, where the values of z are prescribed on the boundary of the domain D.

3. Find the stationary function of functional $\int_{(-1,0)}^{(1,0)} \left(k\sqrt{\dot{x}^2 + \dot{y}^2} - x\dot{y}\right)dt, k > 0$

4. Find the extremals of the functional $\int_0^1 \left(y'^2 + z'^2\right)dx$ that satisfy the boundary conditions y(0) =0, z(0)= 0, y(1) =1, z(1) = 2.

5. Find the extremals of the functional $\int_0^{\pi/2} \left(y'^2 + z'^2 + 2yz\right)dx$ that satisfy the boundary conditions y(0) = 0, y(π/2) = -1, z(0) = 0, z(π/2) = 1.

Answers

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ 2. $\left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial y^2} = f(x,y)$

3. Arc of circle joining (-1, 0) to (1, 0) and of radius k.

4. y = x, z = 2x. 5. y = - sin x, z = sin x.

Summary

The calculus of variations, which plays an important role in both pure and applied mathematics, dates from the time of Newton. Development of the subject started mainly with the work of Euler and Lagrange in the eighteenth century and still continues. This chapter develops the theory of the calculus of variations and its application to various practical problems. Many of the simple applications of calculus of variations are described and, where possible, the historical context of these problems is discussed.

Keywords Calculus of variations, Euler equation, brachistochrone problem, shortest length.

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Chapter-6

CALCULUS OF VARIATIONS -II

Objectives

This course introduces clear and elegant methods of calculus of variations to solve large number of problems in Science and Engineering. In these problems, the extremal property is attributed to an entire curve (function). A group of methods aimed to find 'optimal' functions is called calculus of variations. Theory originated by Bernoulli, Newton and Euler has been used to study Isoperimetric problems and problems with some integral constraints. This theory still attracts attention of mathematicians and it helps scientists and engineers.

Functionals dependent on higher order derivatives (Euler's equation).

Theorem 6.1

A necessary condition for the extremum of a functional of the form

$$I[y(x)] = \int_a^b F\left[x, y(x), y'(x), \ldots\ldots., y^{(n)}(x)\right] dx,$$

where we assume F to be differentiable n+2 times w.r.t. all its arguments, is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \ldots. + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0$$

[This is a variational problem depending upon one independent variable, one dependent variable, and its derivatives upto order of n]

Proof Let y = y(x) be the curve which extremizes the functional

$$I\left[y(x)\right] = \int_a^b F\left[x, y(x), y'(x), \ldots., y^{(n)}(x)\right] dx, \tag{1}$$

satisfying the boundary conditions

$$
\begin{aligned}
&y(a) = A, \qquad y(b) = B \\
&y'(a) = A', \qquad y'(b) = B' \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&y^{(n-1)}(a) = A^{n-1}, \quad y^{(n-1)}(b) = B^{n-1}.
\end{aligned}
\tag{2}
$$

Since at the boundary points the values of y together with all their derivatives upto the order (n-1) (inclusive) are given, we assume that, extremizing curve is differentiable 2n times.

The curve y = y(x) is extremal of (1). Then y(x) satisfies $\delta I = 0$

i.e. $\int_a^b \delta F\left[x, y, y', \ldots\ldots, y^{(n)}\right] dx = 0$

$$
\Rightarrow \quad \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \ldots\ldots\ldots + \frac{\partial F}{\partial y^n}\delta y^n\right] dx = 0 \tag{3}
$$

Integrating the second term on the right once term-by-term, we have

$$
\int_a^b F_{y'}\,\delta y' = \left[F_{y'}\,\delta y\right]_a^b - \int_a^b \frac{d}{dx}F_{y'}\,\delta y dx
$$

Integrating the third term twice

$$
\int_a^b F_{y''}\,\delta y''\,\mathrm{dx} = \left[F_{y''}\,\delta y'\right]_a^b - \left[\frac{d}{dx}F_{y''}\,\delta y\right]_a^b + \int_a^b \frac{d^2}{dx^2}F_{y''}\,\delta y dx
$$

and so forth, the last term n times

$$
\int_a^b F_{y^{(n)}}\,\delta y^{(n)}\,\mathrm{dx} = \left[F_{y^{(n)}}\,\delta y^{(n-1)}\right]_a^b - \left[\frac{d}{dx}F_{y^{(n)}}\,\delta y^{(n-2)}\right]_a^b
$$

$$
+ \ldots\ldots + (-1)^n \int_a^b \frac{d^n}{dx^n}F_{y^{(n)}}\,\delta y dx
$$

Now taking into account the boundary conditions, according to which the variations

$\delta y = \delta y' = \delta y'' = \ldots\ldots = \delta y^{(n-1)} = 0$ for x = a and for x = b, we get from equation (3)

$$\delta I = \int_a^b \left[F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \dots\dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} \right] \delta y \, dx = 0 \tag{4}$$

Now since δy is arbitrary and the first factor under the integral sign is a continuous function of x on the same curve y = y(x), therefore by the fundamental lemma, the first factor is identically zero.

$$\therefore \qquad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \dots\dots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

or
$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \dots\dots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0 \tag{5}$$

Thus the function y = y(x), which extremizes the functional

$$I[y(x)] = \int_a^b F\left[x, y, y', \dots, y^{(n)}\right] dx,$$

must be a solution of the equation (5).

This differential equation of order 2n is called the Euler-Poisson equation. The general solution of this equation contains 2n arbitrary constants, which may be determined from 2n boundary conditions and hence we get the solutions called extremals.

Example 6.1. Find the extremal of the functional

$$I[y] = \int_0^1 (1 + y''^2) dx;$$

y(0) = 0, y'(0) = 1, y(1) = 1, y'(1) = 1.

Solution Let F(x, y, y', y'') = $(1 + y''^2)$

Here $\qquad \dfrac{\partial F}{\partial y} = 0, \ \dfrac{\partial F}{\partial y'} = 0, \ \dfrac{\partial F}{\partial y''} = 2y''$

∴ Corresponding Euler's equation is

$$\frac{d^2}{dx^2}(2y'') = 0 \qquad \Rightarrow \qquad y^{IV} = 0$$

Integrating four times, we obtain

$$y^{III} = A \qquad\qquad y^{II} = Ax + B,$$

$$y^{I} = A\frac{x^2}{2} + Bx + C$$

$$y = A\frac{x^3}{6} + B\frac{x^2}{2} + Cx + D, \qquad (1)$$

which is the general solution.

Now using the boundary conditions

$$y(0) = 0 \Rightarrow D = 0,$$

$$y'(0) = 1 \Rightarrow C = 1,$$

$$y'(1) = 1 \Rightarrow 1 = \frac{A}{2} + B + C \Rightarrow 1 = \frac{A}{2} + B + 1 \Rightarrow B = -\frac{A}{2}$$

$$\Rightarrow A + 2B = 0$$

$$y(1) = 1 \Rightarrow 1 = \frac{A}{6} + \frac{B}{2} + C + D \Rightarrow 1 = \frac{A}{6} + \frac{B}{2} + 1 \Rightarrow \frac{A}{6} = -\frac{B}{2}$$

$$\Rightarrow A + 3B = 0 \Rightarrow A = B = 0$$

$\therefore$ equation (1) becomes y = x. Thus the extremum can be attained only on the straight line y = x.

Example 6.2 Find the extremal of the functional

$$\int_0^{\pi/2} \left(y''^2 - y^2 + x^2\right)dx \text{ that satisfies the conditions}$$

$$y(0) = 1, \; y'(0) = 0, \; y\left(\frac{\pi}{2}\right) = 0, \; y'\left(\frac{\pi}{2}\right) = -1$$

Solution Euler's Poisson equation is

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0 \tag{1}$$

Here $f = y''^2 - y^2 + x^2$

$\therefore \quad \dfrac{\partial f}{\partial y} = -2y,\ \dfrac{\partial f}{\partial y'} = 0,\ \dfrac{\partial f}{\partial y''} = 2y''$

$\therefore$ equation (1) becomes

$$-2y+\frac{d^2}{dx^2}(2y'')=0$$

$$-y+\frac{d^4}{dx^4}y=0 \qquad \Rightarrow \qquad (D^4-1)y=0$$

Auxiliary equation is $m^4 - 1 = 0 \quad \Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow \quad m = \pm 1, \pm i$

Its solution is

$$y(x) = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x \tag{2}$$

$$y'(x) = c_1e^x - c_2e^{-x} - c_3 \sin x + c_4 \cos x \tag{3}$$

To find c_1, c_2, c_3, c_4 use boundary conditions

As $y(0) = 1 \quad \therefore \quad 1 = c_1 + c_2 + c_3$ (4)

As $y'(0) = 0 \quad \Rightarrow \quad 0 = c_1 - c_2 + c_4$ (5)

As $y\left(\dfrac{\pi}{2}\right) = 0 \quad \Rightarrow \quad 0 = c_1e^{\pi/2} + c_2e^{-\pi/2} + c_4$ (6)

As $y'\left(\dfrac{\pi}{2}\right) = -1 \quad \Rightarrow \quad -1 = c_1e^{\pi/2} - c_2e^{-\pi/2} - c_3$ (7)

Solving (4) – (7), we get

$c_1 = c_2 = c_4 = 0 \quad$ and $\quad c_3 = 1$

Hence solution is y(x) = cos x. So the extremum can be achieved only on the curve y = cos x.

Example 6.3 Find the extremal of the functional

$\int_{-a}^{a}\left(\rho y+\frac{1}{2}\mu\, y''^2\right)dx$ which satisfies the boundary conditions

y(-a) = 0, y'(-a) = 0, y(a) = 0, y' (a) = 0

Solution Euler's Poisson equation is

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0 \tag{1}$$

Here $f=\rho y+\frac{1}{2}\mu\, y''^2$

This is the variational problem to which is reduced the problem of finding the axis of a flexible bent cylindrical beam fixed at the ends. If the beam is homogeneous, then ρ and μ are constants.

Now $\frac{\partial f}{\partial y}=\rho, \frac{\partial f}{\partial y'}=0, \ \frac{\partial f}{\partial y''}=\mu y''$

∴ Equation (1) reduces to

$$\rho+\frac{d^2}{dx^2}(\mu y'')=0 \qquad \text{or} \qquad \mu\frac{d^4y}{dx^4}+\rho=0$$

or $\frac{d^4y}{dx^4}+\frac{\rho}{\mu}=0 \qquad \Rightarrow \qquad \frac{d^4y}{dx^4}=-\frac{\rho}{\mu}$

Integrating

$$\frac{d^3y}{dx^3}=-\frac{\rho}{\mu}x+c_1$$

Integrating again and again

$$\frac{d^2y}{dx^2} = -\frac{\rho}{2\mu}x^2 + c_1x + c_2$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{\rho}{2\mu}\frac{x^3}{3} + \frac{c_1x^2}{2} + c_2x + c_3 \tag{2}$$

$$\Rightarrow \qquad y = -\frac{\rho}{24\mu}x^4 + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \tag{3}$$

Now applying the boundary conditions, we have

$$\text{y(-a) = 0} \Rightarrow \; 0 = -\frac{\rho}{24\mu}a^4 - \frac{c_1a^3}{6} + \frac{c_2a^2}{2} - c_3a + c_4 \tag{4}$$

$$\text{y'(-a) = 0} \Rightarrow \; 0 = \frac{\rho}{6\mu}a^3 + \frac{c_1a^2}{2} - c_2a + c_3 \tag{5}$$

$$\text{y(a) = 0} \Rightarrow \; 0 = -\frac{\rho}{24\mu}a^4 + \frac{c_1a^3}{6} + \frac{c_2a^2}{2} + c_3a + c_4 \tag{6}$$

$$\text{y' (a) = 0} \Rightarrow \; 0 = -\frac{\rho}{6\mu}a^3 + \frac{c_1a^2}{2} + c_2a + c_3 \tag{7}$$

Subtracting (4) from (6), we get

$$\frac{c_1}{3}a^3 + 2c_3a = 0 \tag{8}$$

Subtracting (7) from (5), we get

$$\frac{2\rho}{6\mu}a^3 - 2c_2a = 0 \qquad \Rightarrow \qquad c_2 = \frac{\rho}{6\mu}a^2$$

Putting the value of c_2 in (5), we have

$$0 = \frac{\rho}{6\mu}a^3 + \frac{c_1a^2}{2} - \frac{\rho}{6\mu}a^3 + c_3$$

$$\text{or} \qquad 0 = \frac{c_1a^2}{2} + c_3 \qquad \Rightarrow \qquad c_3 = -\frac{c_1a^2}{2} \tag{9}$$

Putting the value of c_2 and c_3 in (4) and (6), we have

$$0 = -\frac{\rho}{24\mu}a^4 - \frac{c_1 a^3}{6} + \frac{\rho}{12\mu}a^4 + \frac{c_1 a^2}{2} + c_4 \qquad (10)$$

and $$0 = -\frac{\rho}{24\mu}a^4 + \frac{c_1 a^3}{6} + \frac{\rho}{12\mu}a^4 - \frac{c_1 a^3}{2} + c_4 \qquad (11)$$

Adding (10) and (11),

$$0 = -\frac{\rho}{12\mu}a^4 + \frac{\rho}{6\mu}a^4 + 2c_4$$

or $$2c_4 = -\frac{\rho}{12\mu}a^4 \qquad \Rightarrow \qquad c_4 = -\frac{\rho}{24\mu}a^4$$

solving further we have

$c_1 = 0$, $c_3 = 0$

Thus the solution is

$$y = -\frac{\rho}{24\mu}x^4 + \frac{c_2}{2}x^2 + c_4$$

$$y = -\frac{\rho}{24\mu}x^4 + \frac{\rho a^2}{12\mu}x^2 - \frac{\rho a^4}{24\mu}$$

or $$y = -\frac{\rho}{24\mu}(x^4 - 2a^2x^2 + a^4)$$

or $$y = -\frac{\rho}{24\mu}(x^2 - a^2)^2$$

Example 6.4 Find the extremal and the stationary function of the functional $\frac{1}{2}\int_0^1 (y'')^2\,dx$. The boundary conditions are $y(0) = 0$, $y(1) = \frac{1}{2}$, $y'(0) = 0$ and $y'(1) = 1$.

Solution Let $F = (x, y, y', y'') = \frac{1}{2}\,y''^2$

$\therefore$ Euler's Poisson equation is

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \tag{1}$$

Here $F_y = F_{y'} = 0$, $F_{y''} = y''$

$\therefore$ (1) $\Rightarrow$ $\quad 0 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(y'') = 0 \;\Rightarrow\; \frac{d^4 y}{dx^4} = 0$

Integrating again and again, we get

$$\frac{d^3 y}{dx^3} = c_1$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = c_1 x + c_2, \qquad \frac{dy}{dx} = c_1\frac{x^2}{2} + c_2 x + c_3,$$

$$\Rightarrow \quad y = c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4.$$

$$\Rightarrow \quad y = c_5 x^3 + c_6 x^2 + c_3 x + c_4 \text{ (putting } \frac{c_1}{6} = c_5,\ \frac{c_2}{2} = c_6) \tag{2}$$

This is the equation of extermals .

The boundary conditions are

$$y(0) = 0,\ y(1) = \frac{1}{2},\ y'(0) = 0,\ y'(1) = 1$$

$$(2) \Rightarrow \quad y' = 3c_5 x^2 + 2c_6 x + c_3 \tag{3}$$

$$y(0) = 0 \Rightarrow c_4 = 0 \tag{4}$$

$$y(1) = \frac{1}{2} \Rightarrow c_5 + c_6 + c_3 + c_4 = \frac{1}{2} \tag{5}$$

$$y'(0) = 0 \Rightarrow c_3 = 0 \tag{6}$$

$$y'(1) = 1 \Rightarrow 3c_5 + 2c_6 + c_3 = 1 \tag{7}$$

$$\therefore (5) \Rightarrow c_5 + c_6 = \frac{1}{2} \tag{8}$$

$(7) \Rightarrow 3c_5 + 2c_6 = 1$ (9)

Solving (8) and (9), we get

$$c_5 = 0, \; c_6 = \frac{1}{2}$$

$\therefore$ (2) $\Rightarrow y = 0x^3 + \frac{1}{2}x^2 + 0\,x + 0$

i.e. $\quad y = \frac{1}{2}x^2$

This is the equation of stationary function.

Example 6.5 Find the extremal of the functional $\int_a^b (y + y'')dx$ that satisfies the boundary conditions $y(a) = y_0$, $y(b) = y_1$, $y'(a) = y_0'$ and $y'(b) = y_1'$.

Solution Given functional is $\int_a^b (y + y'')dx$

Let $\quad F = (x, y, y', y'') = y + y''$

The Euler's Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$ (1)

Here $\quad F_y = 1, F_{y'} = 0, F_{y''} = 1$

$\therefore$ (1) $\Rightarrow \quad 1 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(1) = 0$

$\Rightarrow \quad 1 - 0 + 0 = 1 = 0$, which is impossible.

$\therefore \quad$ The problem has no solution, because it does not admit of extremals.

Example 6.6 Find the extremal of the functional $J[y(x)] = \int_{-1}^{0} \left(240y + y'''^2\right)dx$ subject to the conditions $y(-1) = 1$, $y(0) = 0$, $y'(-1) = -4.5$, $y'(0) = 0$, $y''(-1)=16$, $y''(0)=0$.

Solution We have $J[y(x)] = \int_{-1}^{0} \left(240y + y'''^2\right) dx$

Let F (x, y, y', y'', y''') = $240y + y'''^2$

The Euler's Poisson equation is $F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \frac{d^3}{dx^3}F_{y'''} = 0$ (1)

Here $F_y = 240$, $F_{y'} = 0$, $F_{y''} = 0$, $F_{y'''} = 2y'''$

$\therefore$ (1) $\Rightarrow$ $240 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(0) - \frac{d^3}{dx^3}(2y''') = 0$ $\Rightarrow$ $\frac{d^6 y}{dx^6} = 120$

$\Rightarrow$ $\frac{d^5 y}{dx^5} = 120x + c_1$ $\Rightarrow$ $\frac{d^4 y}{dx^4} = 60x^2 + c_1 x + c_2$

$\Rightarrow$ $\frac{d^3 y}{dx^3} = 20x^3 + c_1 \frac{x^2}{2} + c_2 x + c_3$

$\Rightarrow$ $\frac{d^2 y}{dx^2} = 5x^4 + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$

$\Rightarrow$ $\frac{dy}{dx} = x^5 + c_1 \frac{x^4}{24} + c_2 \frac{x^3}{6} + c_3 \frac{x^2}{2} + c_4 x + c_5$

$\Rightarrow$ $y = \frac{x^6}{6} + c_1 \frac{x^5}{120} + c_2 \frac{x^4}{24} + c_3 \frac{x^3}{6} + c_4 \frac{x^2}{2} + c_5 x + c_6$

$\therefore$ $y = \frac{x^6}{6} + ax^5 + bx^4 + cx^3 + dx^2 + c_5 x + c_6$ (2)

(Putting $a = \frac{c_1}{120}, b = \frac{c_2}{24}, c = \frac{c_3}{6}, d = \frac{c_4}{2}$)

This is the equation of extremals.

The boundary conditions are

y(-1) = 1, y(0) = 0, y'(-1) = - 4.5, y'(0) = 0, y''(-1)=16, y''(0)=0.

(2) $\Rightarrow$ $y' = x^5 + 5ax^4 + 4bx^3 + 3cx^2 + 2dx + c_5$

$\Rightarrow \qquad y'' = 5x^4 + 20ax^3 + 12bx^2 + 6cx + 2d$

$$y(-1) = 1 \;\Rightarrow\; \frac{1}{6} - a + b - c + d - c_5 + c_6 = 1 \tag{3}$$

$$y(0) = 0 \;\Rightarrow 0 + a.0 + b.0 + c.0 + d.0 + c_5.0 + c_6 = 0 \Rightarrow c_6 = 0 \tag{4}$$

$$y'(-1) = -4.5 \Rightarrow \; -1 + 5a - 4b + 3c - 2d + c_5 = -\frac{9}{2} \tag{5}$$

$$y'(0) = 0 \Rightarrow \; 0 + 5a.0 + 4b.0 + 3c.0 + 2d.0 + c_5 = 0 \Rightarrow c_5 = 0 \tag{6}$$

$$y''(-1) = 16 \Rightarrow \; 5 - 20a + 12b - 6c + 2d = 16 \tag{7}$$

$$y''(0) = 0 \Rightarrow 5(0) + 20a.0 + 12b.0 - 6c.0 + 2d = 0 \Rightarrow d = 0 \tag{8}$$

$\therefore \qquad c_6 = 0,\ c_5 = 0,\ d = 0.$

$$\therefore (3) \Rightarrow a - b + c = -\frac{5}{6} \tag{9}$$

$$(5) \Rightarrow \; 5a - 4b + 3c = -\frac{7}{2} \tag{10}$$

$$(7) \Rightarrow \; 20a - 12b + 6c = -11 \tag{11}$$

Solving (9), (10), (11), we get a = 0, b = 1, c = 1/6

$$\therefore (2) \Rightarrow \qquad y = \frac{x^6}{6} + 0.x^5 + 1.x^4 + \frac{1}{6}x^3 + 0.x^2 + 0.x + 0$$

or $$y = \frac{x^6}{6} + x^4 + \frac{x^3}{6} \qquad or \qquad y = \frac{x^3}{6}(x^3 + 6x + 1).$$

This is the equation of the required extremal.

Isoperimetric Problems

Such type of problems involve one or more constraint conditions.

Definition Isoperimetric problems/problems with constraints of integral type

Here the problem is of finding the closed plane curve of given length ℓ and enclosing (bounding) the largest area. This is (obviously) a circle. Thus if the curve is expressed in perimetric form by x = x(t), y = y(t) and is traversed once counterclockwise as t increases from t_1 to t_2, then the enclosed area is known to be

$$S = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt, \tag{1}$$

which is an integral depending upon two unknown functions. In this problem one has to find the extremum of the functional S with the auxiliary condition that the length of the curve must be constant; i.e. the functional

$$\ell = \int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{2}$$

retains a constant value. Conditions of this type are called Isoperimetric. Euler elaborated the general methods for solving problems with Isoperimetric conditions.

Length of the curve is given by (2). The problem is to maximize (1) subject to the side condition that (2) must have a constant value.

Theorem 6.2 If y(x) is the extremal of the functional

$$I[y] = \int_a^b F(x, y, y')dx \tag{1}$$

subject to the conditions

$$y(a) = A,\ y(b) = B,\ J[y] = \int_a^b G(x, y, y')dx = \ell \tag{2}$$

where J[y] is another functional. Then, if y = y(x) is not an extremal of J[y], there exists a constant λ such that y = y(x) is an extremal of the functional

$$\int_a^b (F + \lambda G)dx,$$

i.e. y = y(x) satisfies the differential equation

$$F_y - \frac{d}{dx}F_{y'} + \lambda\left(G_y - \frac{d}{dx}G_{y'}\right) = 0 \tag{3}$$

or $$\frac{\partial}{\partial y}(F+\lambda G)-\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F+\lambda G)\right)=0$$

(F + λG is called the Auxiliary function)

Proof Suppose that y(x) is the actual extremising (stationary) function of the functional (1) subject to the conditions (2).

Consider the variation in y as

$$\delta y = \varepsilon_1\eta_1(x) + \varepsilon_2\eta_2(x),$$

where ε_1 and ε_2 are scalars, η_1 and η_2 are arbitrary functions, but fixed choosen in such a way that $\eta_1(a) = 0$, $\eta_1(b)=0$, $\eta_2(a) = 0$, $\eta_2(b) = 0$ and have continuous second derivatives, so that y + δy also satisfies the same conditions as satisfied by y. Thus $\bar{y} = (y+\delta y)$ is a two parameter family of neighbouring functions.

$$I[y+\varepsilon_1\eta_1(x)+\varepsilon_2\eta_2(x)] = \int_a^b F\big(x, y+\varepsilon_1\eta_1(x)+\varepsilon_2\eta_2(x), y'+\varepsilon_1\eta_1'(x)+\varepsilon_2\eta_2'(x)\big)\,dx$$

$$= \int_a^b F(x,\bar{y},\bar{y}')dx$$

which for assigned η_1, η_2 and y behaves as a function of ε_1 and ε_2.

$$I[\varepsilon_1,\varepsilon_2] = \int_a^b F\big(x, y+\varepsilon_1\eta_1(x)+\varepsilon_2\eta_2(x), y'+\varepsilon_1\eta_1'(x)+\varepsilon_2\eta_2'(x)\big)\,dx \tag{4}$$

Similarly

$$J[\varepsilon_1,\varepsilon_2] = \int_a^b G\big(x, y+\varepsilon_1\eta_1+\varepsilon_2\eta_2, y'+\varepsilon_1\eta_1'+\varepsilon_2\eta_2'\big)\,dx = \ell \tag{5}$$

Now we want to find the necessary conditions for the function (4) to have a stationary value at $\varepsilon_1 = \varepsilon_2 = 0$, where ε_1 and ε_2 satisfy equation (5). Using the method of Lagrange's multiplier, we introduce the function

$$K(\varepsilon_1, \varepsilon_2, \lambda) = I(\varepsilon_1, \varepsilon_2) + \lambda J(\varepsilon_1, \varepsilon_2) = \int_a^b \overline{F}(x,\bar{y},\bar{y}')dx$$

where $\overline{F} = F + \lambda G$.

The necessary condition for the given functional to have an extremum is that

$$\frac{\partial K}{\partial \varepsilon_1} = \frac{\partial}{\partial \varepsilon_1}(I + \lambda J) = 0 \qquad \text{at } \varepsilon_1=0,\ \varepsilon_2 = 0$$

$$\frac{\partial K}{\partial \varepsilon_2} = \frac{\partial}{\partial \varepsilon_2}(I + \lambda J) = 0 \qquad \text{at } \varepsilon_1=0,\ \varepsilon_2 = 0$$

$$\int_a^b \left\{ \left(\frac{\partial F}{\partial y}\eta_1 + \frac{\partial F}{\partial y'}\eta_1' \right) + \lambda \left(\frac{\partial G}{\partial y}\eta_1 + \frac{\partial G}{\partial y'}\eta_1' \right) \right\} dx = 0 \tag{6}$$

Similarly

$$\int_a^b \left\{ \left(\frac{\partial F}{\partial y}\eta_2 + \frac{\partial F}{\partial y'}\eta_2' \right) + \lambda \left(\frac{\partial G}{\partial y}\eta_2 + \frac{\partial G}{\partial y'}\eta_2' \right) \right\} dx = 0 \tag{7}$$

Integrating by parts equation (6), we get

$$\frac{\partial F}{\partial y'}\eta_1 \Big|_a^b - \int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) - \left(\frac{\partial F}{\partial y} \right) \right] \eta_1 dx + \lambda \frac{\partial G}{\partial y'}\eta_1 \Big|_a^b - \lambda \int_a^b \left[\frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) - \left(\frac{\partial G}{\partial y} \right) \right] \eta_1 dx = 0$$

$$\Rightarrow \quad \int_a^b \left[\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F + \lambda G) \right) - \left(\frac{\partial}{\partial y}(F + \lambda G) \right) \right] \eta_1 dx = 0 \ \ [\text{as } \eta_1(a) = \eta_1(b) = 0]$$

or
$$\int_a^b \left[\left(F_y - \frac{d}{dx}F_{y'} \right) + \lambda \left(G_y - \frac{d}{dx}G_{y'} \right) \right] \eta_1 dx = 0 \tag{8}$$

Similarly integrating equation (7) by parts, we get

$$\int_a^b \left[\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F + \lambda G) \right) - \frac{\partial}{\partial y}(F + \lambda G) \right] \eta_2 dx = 0 \tag{9}$$

Taking η_2 in such a way that

$$\int_a^b \left(\frac{d}{dx}G_{y'} - \frac{\partial G}{\partial y} \right) \eta_2 dx \neq 0$$

we can take from (9)

$$\lambda = \frac{\int\limits_a^b \left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} \right) \eta_2 dx}{\int\limits_a^b \left(\frac{d}{dx}\frac{\partial G}{\partial y'} - \frac{\partial G}{\partial y} \right) \eta_2 dx}$$

This equation ensures the existence of λ.

As η_1 and η_2 are arbitrary, two conditions (8) and (9) are embodied in (10) as only one condition

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F + \lambda G) \right) - \frac{\partial}{\partial y}(F + \lambda G) = 0 \tag{10}$$

$\Rightarrow$ (8) is the extremal of $F+\lambda G$, which proves the theorem.

Note To use above theorem, i.e. to solve a given isoperimetric problem, we first write general solution of (10), which will contain two arbitrary constants in addition to the parameter λ. We then determine these three quantities from the boundary conditions $y(a) = A$, $y(b) = B$ and the subsidiary condition $J[y]= \ell$.

Everything said above generalizes immediately to the case of functionals depending on several functions $y_1, \ldots, y_n$ and subject to subsidiary conditions of the form $J[y] = \int\limits_a^b G(x, y, y')dx$. Suppose we are looking for an extremum of the functional

$$J[y_1, \ldots, y_n] = \int\limits_a^b F(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n)dx \tag{1}$$

subject to the conditions

$$y_i(a) = A_i, \qquad y_i(b) = B_i \qquad (i = 1, \ldots, n) \tag{2}$$

and

$$\int\limits_a^b G_j(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n)dx = \ell_j \quad (j = 1, \ldots, k) \tag{3}$$

where $k < n$. In this case a necessary condition for an extremum is that

$$\frac{\partial}{\partial y_i}\left(F + \sum_{j=1}^{n} \lambda_j G_j \right) - \frac{d}{dx}\left\{ \frac{\partial}{\partial y'_i}\left(F + \sum_{j=1}^{n} \lambda_j G_j \right) \right\} = 0 \tag{4}$$

The 2n arbitrary constants appearing in the solution of (4) and the values of k parameters $\lambda_1, \ldots, \lambda_k$ sometimes called Lagrange's multipliers are determined from the boundary conditions (2) and the subsidiary conditions (3). The proof of above theorem is exactly on same lines.

Note The solutions of the equation (10) above (the extremals of our problem) involve three undetermined parameters, two constants of integration and the Lagrange multiplier λ. The stationary function is then selected from these extremals by imposing the two boundary conditions and giving the integral J its prescribed value ℓ.

In the case of integrals that depend upon two or more functions, this result can be extended in the same way as in the previous chapter. For example, if

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx$$

has a stationary value subject to the side condition

$$J = \int_{x_1}^{x_2} g(x, y, z, y', z')dx = \ell$$

Then the stationary function y(x) and z(x) must satisfy the system of equations

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \quad \text{and} \quad \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0$$

where $F = f + \lambda g$. Reasoning is similar.

Lagrange's Multiplier

Some problems in elementary calculus are quite similar to isoperimetric problems. For example, suppose we want to find the point (x, y) that yields stationary values for a function

$$z = f(x, y) \tag{1}$$

where the variables x and y are not independent but are constrained by a side condition

$$g(x, y) = 0 \tag{2}$$

The usual procedure is to arbitrarily designate one of the variable x and y in (2) as independent say x, and other as dependent on it, so that dy/dx can be found out from

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0 \tag{3}$$

Also, since z is now a function of x alone, $\frac{dz}{dx} = 0$ is a necessary condition for z to have a stationary value, so

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

or
$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial g / \partial x}{\partial g / \partial y} = 0 \tag{4}$$

On solving (2) and (4) simultaneously, we obtain the required points (x, y).

One drawback of this approach is that the variables x and y occur symmetrically but are treated unsymmetrically. It is possible to solve the same problem by a different and more elegant method that also has many practical advantages. We form the function

$$F(x, y, \lambda) = f(x, y) + \lambda\, g(x, y)$$

and investigate its unconstrained stationary values by means of the necessary conditions

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \tag{5}$$

$$\frac{\partial F}{\partial \lambda} = g(x, y) = 0$$

If λ is eliminated from the first two of these equations, then the system clearly reduces to

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial g/\partial x}{\partial g/\partial y} = 0 \text{ and } \quad g(x, y) = 0$$

and this is the system obtained in above paragraph. This technique (solving (5) for x and y) does not disturb the symmetry of the problem by making an arbitrary choice of independent variable and it remains the side condition by introducing λ as another variable. The parameter λ is called Lagrange multiplier and this method is known as the method of Lagrange multiplier.

Note In theorem 6.2, we consider a two parameter family of neighboring functions

$$\bar{y}(x) = y(x) + \varepsilon_1\eta_1(x) + \varepsilon_2\eta_2(x),$$

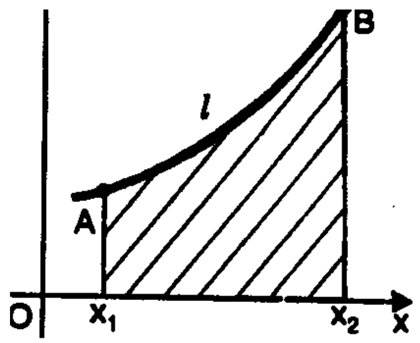
as by considering one parameter family of function $\bar{y}(x) = y(x) + \alpha\eta(x)$, these will not maintain the second integral J at the constant value c or ℓ.

Example 6.7 Find the plane curve of fixed perimeter so that the area covered by the curve, ordinates and x-axis is maximum. OR

Given two points x_1 and x_2 on the x-axis and an arc length ℓ. Find the shape of curve of length ℓ joining the given points which with the x-axis encloses with largest area.

Solution Let y = y(x) be a curve of given length ℓ between given points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\therefore \qquad \ell = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$$



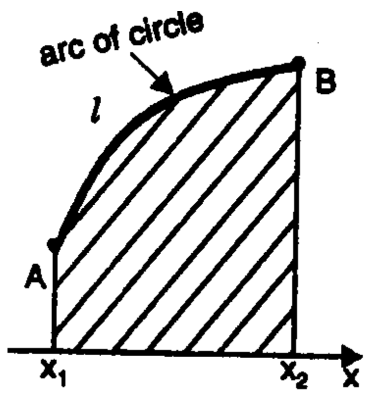


Figure 6.1

Also, area enclosed by the curve, ordinate and x-axis is $\int_{x_1}^{x_2} y\,dx$. We want to maximize the functional $\int_{x_1}^{x_2} y\,dx$ under the condition

$$\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = \ell$$

F(x, y, y') = y, G(x, y, y') = $\sqrt{1+y'^2}$

The Euler equation is

$$F_y - \frac{d}{dx}F_{y'} + \lambda\left(G_y - \frac{d}{dx}G_{y'}\right) = 0 \tag{1}$$

Here $F_y = 1$, $F_{y'} = 0$, $G_y = 0$, $G_{y'} = \frac{y'}{\sqrt{1+y'^2}}$

$$\therefore (1) \Rightarrow \quad 1 - \frac{d}{dx}(0) + \lambda\left(0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\right) = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\lambda\, y'}{\sqrt{1+y'^2}}\right) = 1$$

$$\Rightarrow \quad \frac{\lambda\, y'}{\sqrt{1+y'^2}} = x + c_1$$

$$\Rightarrow \quad \lambda^2 y'^2 = (x+c_1)^2(1+y'^2)$$

$$\Rightarrow \quad y'^2 = (x+c_1)^2 \big/ \left(\lambda^2 - (x+c_1)^2\right)$$

$$\frac{dy}{dx} = y' = \frac{x+c_1}{\sqrt{\lambda^2 - (x+c_1)^2}}$$

$$dy = -\frac{1}{2}\frac{-2(x+c_1)}{\sqrt{\lambda^2 - (x+c_1)^2}}\,dx$$

$$\Rightarrow \qquad y = -\frac{1}{2}\frac{\left(\lambda^2 - (x+c_1)^2\right)^{\frac{1}{2}}}{\frac{1}{2}} \quad -c_2$$

$$\Rightarrow \qquad (y+c_2)^2 = \lambda^2 - (x+c_1)^2$$

$$\Rightarrow \qquad (x+c_1)^2 + (y+c_2)^2 = \lambda^2 \tag{2}$$

Now $\qquad \ell = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = \int_{x_1}^{x_2} \sqrt{1 + \frac{(x+c_1)^2}{\lambda^2 - (x+c_1)^2}}\,dx$

$$= \int_{x_1}^{x_2} \frac{\lambda}{\sqrt{\lambda^2 - (x+c_1)^2}}\,dx \quad = \quad \lambda \sin^{-1}\frac{x+c_1}{\lambda}\Bigg|_{x_1}^{x_2}$$

$$\therefore \qquad \lambda\left[\sin^{-1}\frac{x_2+c_1}{\lambda} - \sin^{-1}\frac{x_1+c_1}{\lambda}\right] = \ell \tag{3}$$

(since $\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = \ell$)

Equation (3) is solved to find the value of λ.

∴ The required curve is an arc of the circle (2), with centre ($-c_1$, $-c_2$) and radius λ. The values of c_1 and c_2 are found by using the fact that A and B are on this arc. The position of the arc is shown in the figure 6.1.

Example 6.8 Prove that sphere is a solid figure of revolution, which for a given surface area, has a maximum volume.

Sol. Let curve OPA rotates about x-axis as shown in the figure. Let coordinates of A be (a, 0). Let S be the surface area of the solid figure of revolution.

$$\therefore \qquad \int_0^a 2\pi y \; ds = S$$

$$\Rightarrow \qquad \int_0^a 2\pi y \sqrt{1+y'^2} \quad dx = S$$

The volume of the solid figure of revolution is $\int_0^a \pi\, y^2 \; dx$.

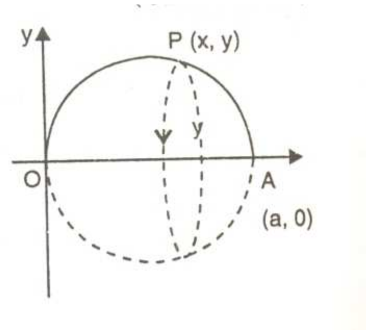


Figure 6.2

$\therefore$ We have the maximize the functional $\int_0^a \pi y^2 \; dx$ with boundary conditions y(0) = y(a) = 0 and under the condition $\int_0^a 2\pi y\sqrt{1+y'^2} \;\; dx = S$.

Let $\quad F(x, y, y') = \pi y^2$ and $G(x, y, y') = 2\pi y\sqrt{1+y'^2}$.

The Euler's equation is $\quad F_y - \dfrac{d}{dx}F_{y'} + \lambda\left(G_y - \dfrac{d}{dx}G_{y'}\right) = 0$

Here x is missing from both functions F and G.

$\therefore$ The Euler's equation reduces to

$$F + \lambda G - \left(F_{y'} + \lambda G_{y'}\right)\frac{dy}{dx} = c \qquad (1)$$

Here $F_{y'} = 0, \quad G_{y'} = \dfrac{2\pi y y'}{\sqrt{1+y'^2}}$

$$\therefore (1) \Rightarrow \quad \pi y^2 + 2\pi\lambda y\sqrt{1+y'^2} - \left(0 + \frac{2\pi\lambda y y'}{\sqrt{1+y'^2}}\right)y' = c$$

$$\Rightarrow \quad \pi y^2 + \left(\frac{2\pi\lambda y(1+y'^2) - 2\pi\lambda y y'^2}{\sqrt{1+y'^2}}\right) = c$$

$$\Rightarrow \qquad \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = c \tag{2}$$

Since the curve passes through (0, 0) we have 0 + 0 = c i.e. c = 0

$$\Rightarrow \quad \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = 0 \qquad \Rightarrow \quad y + \frac{2\lambda}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \quad \sqrt{1+y'^2} = -\frac{2\lambda}{y} \qquad \Rightarrow y'^2 = \frac{4\lambda^2}{y^2} - 1 = \frac{4\lambda^2 - y^2}{y^2}$$

$$\Rightarrow \quad y' = \frac{\sqrt{4\lambda^2 - y^2}}{y} \qquad \Rightarrow \quad \frac{y}{\sqrt{4\lambda^2 - y^2}} dy = dx$$

$$\Rightarrow \quad -\frac{1}{2} \cdot \frac{\left(4\lambda^2 - y^2\right)^{1/2}}{1/2} = x + c_1 \qquad \Rightarrow \quad -\sqrt{4\lambda^2 - y^2} = x + c_1 \tag{3}$$

(3) passes through (0, 0) $\therefore$ $-2\lambda = c_1$

$$\therefore \quad -\sqrt{4\lambda^2 - y^2} = x - 2\lambda$$

$$\Rightarrow \quad 4\lambda^2 - y^2 = (x-2\lambda)^2 \qquad \Rightarrow \qquad (x-2\lambda)^2 + y^2 = 4\lambda^2 \tag{4}$$

(4) passes through (a, 0)

$$\therefore \quad (a-2\lambda)^2 + 0 = 4\lambda^2$$

$$\Rightarrow \quad a^2 + 4\lambda^2 - 4a\lambda = 4\lambda^2$$

$$\Rightarrow \quad a^2 - 4a\lambda = 0 \Rightarrow \lambda = \frac{a}{4}$$

$$\therefore (4) \Rightarrow \qquad \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} \tag{5}$$

(5) represents a circle with centre at (a/2, 0) and radius a/2.

$\therefore$ The curve OPA is a semi-circle

$\therefore$ The solved figure of revolution is a sphere.

Hence the result holds.

Remark In particular, if $l = \pi\left(\frac{AB}{2}\right)$, then AB would be a diameter of the maximizing circle.

Example 6.9 Among all the curves of length = ℓ (>2a) in the upper half plane passing through the points (-a, 0) and (a, 0), find the one which together with interval [-a, a] encloses the largest area.

or

Solve the problem I[y] = $\int_{-a}^{a} y dx$ = maximum,

subject to the conditions

$$y(-a) = y(a) = 0 \quad \text{and} \quad J[y] = \int_{-a}^{a} \sqrt{1+y'^2}\, dx = \ell$$

Solution Let y = y(x) be a curve of length ℓ between the points (-a, 0) and (a, 0).

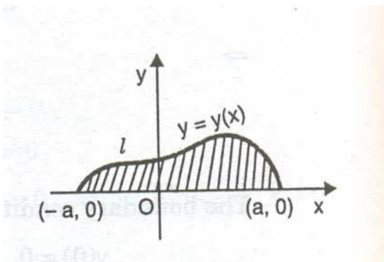


Fig. 6.3

$$\therefore \qquad \ell = \int_{-a}^{a} \sqrt{1+y'^2}\, dx$$

Also the area enclosed by the curve and x-axis is $\int_{-a}^{a} y dx$.

We are to maximize the functional $\int_{-a}^{a} y dx$ under the condition $\int_{-a}^{a} \sqrt{1+y'^2}\, dx = \ell$.

Let F(x, y, y') = y and G(x, y, y') = $\sqrt{1+y'^2}$ (1)

Exactly same as in example 6.7, we get equation (2) as

$$(x + c_1)^2 + (y + c_2)^2 = \lambda^2 \qquad (2)$$

The boundary conditions are

$$y(-a) = 0, \qquad y(a) = 0$$

$\therefore$ $$(a + c_1)^2 + c_2^2 = \lambda^2 \qquad (3)$$

$$(-a + c_1)^2 + c_2^2 = \lambda^2 \qquad (4)$$

$(3) - (4) \Rightarrow$ $(a + c_1)^2 - (-a + c_1)^2 = 0$

$\Rightarrow$ $2ac_1 + 2ac_1 = 0 \Rightarrow 4ac_1 = 0 \Rightarrow c_1 = 0$

$\therefore (3) \Rightarrow$ $a^2 + c_2^2 = \lambda^2 \quad \Rightarrow c_2 = \sqrt{\lambda^2 - a^2}$

$\therefore (2) \Rightarrow$ $(x - 0)^2 + (y + \sqrt{\lambda^2 - a^2})^2 = \lambda^2$

i.e. $$x^2 + (y + \sqrt{\lambda^2 - a^2})^2 = \lambda^2 \qquad (5)$$

$$y' = \frac{x + c_1}{\sqrt{\lambda^2 - (x + c_1)^2}} \quad \Rightarrow \quad y' = \frac{x}{\sqrt{\lambda^2 - x^2}}$$

$\therefore$ $$\ell = \int_{-a}^{a} \sqrt{1 + y'^2}\, dx = 2\int_0^a \sqrt{1 + \frac{x^2}{\lambda^2 - x^2}}\, dx = 2\int_0^a \frac{\lambda}{\sqrt{\lambda^2 - x^2}} dx$$

$$= 2\lambda \sin^{-1} \frac{x}{\lambda}\bigg|_0^a \quad = 2\lambda \left(\sin^{-1}\frac{a}{\lambda} - \sin^{-1} 0\right) = 2\lambda \sin^{-1}\frac{a}{\lambda}$$

$\therefore$ $$2\lambda \sin^{-1}\frac{a}{\lambda} = \ell$$

$\Rightarrow$ $$\sin\frac{\ell}{2\lambda} = \frac{a}{\lambda}.$$

let $\lambda = \lambda_0$ be a solution of this transcendental equation.

$\therefore (5) \Rightarrow$ $x^2 + (y + \sqrt{\lambda_0^2 - a^2})^2 = \lambda_0^2$

Therefore, the required curve is an arc of the circle

$$x^2 + (y + \sqrt{\lambda_0^2 - a^2}\,)^2 = \lambda_0^2$$

This is the circle with centre at $(0, -\sqrt{\lambda_0^2 - a^2}\,)$ and radius λ_0.

Remark In particular if $\ell = \pi a$ then $\ell > 2a$ and $\sin\frac{\ell}{2\lambda} = \frac{a}{\lambda}$ reduces to $\sin\frac{\pi}{2}\left(\frac{a}{\lambda}\right) = \frac{a}{\lambda}$.

This equation yields $\lambda = \lambda_0 = a$.

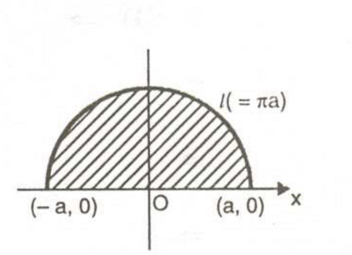


Fig. 6.4

$\therefore$ In this case, the required arc is a part of the circle $x^2 + y^2 = a^2$.

Example 6.10 Find a curve C having a given length, which encloses a maximum area.

Solution Area bounded by the curve C is

$$A = \frac{1}{2}\int (xdy - ydx)$$

$$= \frac{1}{2}\int\left(x\frac{dy}{dx} - y\frac{dx}{dx}\right)dx = \frac{1}{2}\int (xy' - y)dx \tag{1}$$

The length of C is given.

$$\therefore \qquad \ell = \int \sqrt{1 + y'^2}\,dx \tag{2}$$

Using Lagrange's multiplier

$$I = A + \lambda\,\ell$$

$$= \frac{1}{2}\int_C (xy' - y)dx + \lambda\int_C \sqrt{1 + y'^2}\,dx$$

$$= \int \left(\frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2} \right) dx$$

$$\text{H} = \frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}$$

$$\therefore \qquad \frac{\partial H}{\partial y} = -\frac{1}{2}, \qquad \frac{\partial H}{\partial y'} = \frac{1}{2}x + \frac{\lambda}{2} \cdot \frac{1}{\sqrt{1 + y'^2}} 2y' = \frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}}.$$

By the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'} \right) = 0$$

i.e. $$-\frac{1}{2} - \frac{d}{dx}\left[\frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}} \right] = 0$$

or $$\frac{d}{dx}\left[\frac{x}{2} + \frac{\lambda y'}{\sqrt{1 + y'^2}} \right] + \frac{1}{2} = 0$$

$$\frac{1}{2} + \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}} \right) + \frac{1}{2} = 0$$

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}} \right) = -1.$$

Integrating this we obtain

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = -x + c_1, \qquad c_1 \text{ is constant of integration.}$$

$$\frac{\lambda^2 y'^2}{1 + y'^2} = (x - c_1)^2 \quad \Rightarrow \quad \lambda^2 y'^2 = (x - c_1)^2 + (x - c_1)^2 y'^2$$

or $$y'^2[\lambda^2 - (x - c_1)^2] = (x - c_1)^2$$

$$y'^2 = \frac{(x - c_1)^2}{\lambda^2 - (x - c_1)^2}$$

$$y' = \frac{(x-c_1)}{\sqrt{\lambda^2 - (x-c_1)^2}} \quad or \quad \frac{dy}{dx} = \frac{(x-c_1)}{\sqrt{\lambda^2 - (x-c_1)^2}}$$

Integration gives

$$y - c_2 = \pm\left(-\frac{1}{2}\right)\frac{\sqrt{\lambda^2 - (x-c_1)^2}}{1/2} \qquad \text{(}-c_2 \text{ is the constant of integration)}$$

or $$(y - c_2)^2 = \lambda^2 - (x - c_1)^2$$

or $$(x - c_1)^2 + (y - c_2)^2 = \lambda^2 \; .$$

This equation represents a circle.

The area enclosed will be maximum if within the given length we form the circles.

Exercise

1. Find the extremal of the functional $\int_0^1 y'^2 dx$, y(0) = 1, y(1) = 6 subject to the condition $\int_0^1 y\, dx = 3$.

2. Find the extremal of the functional $\int_0^{\pi} \left(y'^2 - y^2\right) dx$ under the boundary conditions y(0) = 0, y(π) = 1 and subject to additional condition $\int_0^{\pi} y dx = 1$.

3. Find the extremal of the functional $\int_0^1 \left(x^2 + y'^2\right) dx$, y(0) = 0, y(1) = 0, subject to the condition $\int_0^1 y^2 dx = 2$.

4. Find the extremal of the functional $\int_0^1 y'^2\, dx$, y(0) = 0, y(1) = $\frac{1}{4}$, subject to the condition $\int_0^1 \left(y - y'^2\right) dx = \frac{1}{12}$.

Answers

1. $y = 3x^2 + 2x + 1$

2. $y = -\frac{1}{2}\cos x + \frac{2-\pi}{4}\sin x + \frac{1}{2}$

3. $y = \pm 2 \sin n\pi x$

4. $y = -\frac{x^2}{4} + \frac{x}{2}$

Variational Problems with Geometric Constraints

We now consider a problem of different type like: Find the functions $y_i(x)$ for which the functional

$$J[y_1, \ldots, y_n] = \int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\,dx. \qquad (1)$$

has an extremum where the admissible functions satisfy the boundary conditions

$$y_i(a) = A_i, \; y_i(b) = B_i \qquad (i = 1, \ldots, n) \qquad (2)$$

and k finite subsidiary conditions ($k < n$)

$$g_j(x, y_1, \ldots, y_n) = 0 \qquad (j = 1, \ldots, k) \qquad (3)$$

In other words, the functional (1) is not considered for all curves satisfying the boundary conditions (2), but only for those which lie in the $(n - k)$ dimensional manifold defined by the system (3).

Theorem 6.3 Consider the functional

$$I[y, z] = \int_a^b F(x, y, z, y', z')\,dx \qquad (1)$$

subject to the boundary conditions

$$y(a) = A_1, \qquad y(b) = B_1$$

$$z(a) = A_2, \qquad z(b) = B_2 \qquad (2)$$

and which lies on the surface

$$g(x, y, z) = 0 \tag{3}$$

If $$y = y(x) \text{ and } z = z(x) \tag{4}$$

are the extremals of (1) and if g_y and g_z do not vanish simultaneously at any point of the surface (3); there exists a function $\lambda(x)$ s.t. (4) is an extremal of the functional

$$\int_a^b [F + \lambda(x)g]dx \qquad \text{i.e. satisfy the differential equations}$$

$$\left.\begin{aligned} &F_y + \lambda g_y - \frac{d}{dx}F_{y'} = 0, \\ &F_z + \lambda g_z - \frac{d}{dx}F_{z'} = 0, \qquad (\lambda \text{ may be constant}). \end{aligned}\right\} \tag{5}$$

Proof Let y(x) and z(x) are the extremals of (1), subjected to the conditions (2) and (3). For variations δy and δz respectively in y and z, we must have

$$\delta I[y, z] = \delta\int_a^b F(x, y, z, y', z')dx = 0$$

$$\int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z'}\delta z'\right)dx = 0$$

$$\int_a^b \frac{\partial F}{\partial y}\delta y dx + \int_a^b \frac{\partial F}{\partial y'}\delta y' dx + \int_a^b \frac{\partial F}{\partial z}\delta z dx + \int_a^b \frac{\partial F}{\partial z'}\delta z' dx = 0$$

$$\int_a^b \frac{\partial F}{\partial y}\delta y dx + \left[\frac{\partial F}{\partial y'}\delta y\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx + \int_a^b \frac{\partial F}{\partial z}\delta z dx$$

$$+\left[\frac{\partial F}{\partial z'}\delta z\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\delta z\, dx = 0$$

$$\Rightarrow \qquad \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx + \int_a^b \left[\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)\right]\delta z dx = 0. \tag{6}$$

Also, $g(x, y, z) = 0$,

$\Rightarrow$ $g_y\delta y + g_z\delta z = 0$ (relation between δy and δz)

$\Rightarrow$ $\lambda g_y\delta y + \lambda g_z\delta z = 0$,

where we assume here λ as a scalar $\lambda = \lambda(x)$ continuous, differentiable function on [a, b] but arbitrary.

$$\Rightarrow \qquad \int_a^b \left(\lambda g_y \delta y + \lambda g_z \delta z\right) dx = 0$$

$$\int_a^b \lambda g_y \delta y dx + \int_a^b \lambda g_z \delta z dx = 0 \tag{7}$$

Adding (6) and (7),

$$\int_a^b \left[F_y - \frac{d}{dx}F_{y'} + \lambda g_y\right]\delta y dx + \int_a^b \left[F_z - \frac{d}{dx}F_{z'} + \lambda g_z\right]\delta z dx = 0 .$$

Select λ so that

$$F_y - \frac{d}{dx}F_{y'} + \lambda g_y = 0$$

and $$F_z - \frac{d}{dx}F_{z'} + \lambda g_z = 0$$

This completes the proof of the theorem.

Example 6.11 Find the extremal in the isoperimetric problem

$$I[y(x), z(x)] = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)dx \tag{1}$$

when $y(0) = z(0) = 0$

$$y(1) = z(1) = 1 \tag{2}$$

and $$\int_0^1 (y'^2 - xy' - z'^2)dx = 2 \tag{3}$$

Solution $f = (y'^2 + z'^2 - 4xz' - 4z)$; $g = y'^2 - xy' - z'^2$

Then two Euler's equations are

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0 \tag{4}$$

and

$$\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)-\frac{\partial F}{\partial z}=0 \tag{5}$$

where $\quad F = f + \lambda g.$

Here $\quad F = f + \lambda g = y'^2 + z'^2 - 4xz' - 4z + \lambda(y'^2 - xy' - z'^2)$

Then (4) and (5) become as

$$\frac{d}{dx}[2y' + 2\lambda y' - \lambda x] = 0 \quad \text{and} \tag{6}$$

$$\frac{d}{dx}[2z' - 4x - 2\lambda z'] + 4 = 0 \tag{7}$$

Integrating (6)

$$2y'(1+\lambda) = c_1 + \lambda x$$

$$y' = \frac{c_1 + \lambda x}{2(1+\lambda)}$$

Again integrating

$$y = \frac{c_1}{2(1+\lambda)}x + \frac{\lambda x^2}{4(1+\lambda)} + c_2 \tag{8}$$

Similarly from (7) (by integrating)

$$2z' - 4x - 2\lambda z' + 4x = c_3$$

$$2z'(1-\lambda) = c_3$$

Integrating again

$$z = \frac{c_3 x}{2(1-\lambda)} + c_4 \tag{9}$$

Using the boundary conditions (2) , we have

$$0 = c_2 , \qquad 1 = \frac{c_1}{2(1+\lambda)} + \frac{\lambda}{4(1+\lambda)},$$

$$\Rightarrow \qquad c_1 = \frac{4+3\lambda}{2} \ ,$$

$$1 = \frac{c_3}{2(1-\lambda)}, \quad c_4 = 0$$

$$c_3 = 2(1 - \lambda)$$

Using these values of c_1, c_2, c_3 and c_4 in (8) and (9)

$$y = \frac{(4+3\lambda)}{4(1+\lambda)} x + \frac{\lambda x^2}{4(1+\lambda)} \tag{10}$$

$$= \frac{1}{4(1+\lambda)}[(4+3\lambda)x + \lambda x^2]$$

and $\qquad z = \dfrac{2(1-\lambda)}{2(1-\lambda)} x \qquad \Rightarrow \qquad z = x$

Now we find λ.

Differentiating (10),

$$y' = \frac{1}{4(1+\lambda)}[4 + 3\lambda + 2\lambda x]$$

$$\Rightarrow \qquad y'^2 = \frac{1}{16(1+\lambda)^2}[4 + 3\lambda + 2\lambda x]^2$$

and $\qquad z' = 1$

Putting the values of y' and z' in (3), we get

$$\int_0^1 \left[\frac{1}{16(1+\lambda)^2}[4+3\lambda+2\lambda x]^2 - \frac{x}{4(1+\lambda)}[4+3\lambda+2\lambda x] - 1 \right] dx = 2$$

$$\Rightarrow \qquad \int_0^1 \frac{1}{16(1+\lambda)^2}[16 + 9\lambda^2 + 4\lambda^2 x^2 + 16\lambda x + 12\lambda^2 x + 24\lambda - 16x - 16\lambda x$$

$$-12\lambda x - 12\lambda^2 x - 8\lambda x^2 - 8\lambda^2 x^2]dx - \int_0^1 1dx = 2$$

$$\Rightarrow \quad \frac{1}{16(1+\lambda)^2}\int_0^1 \left[x^2(-4\lambda^2 - 8\lambda) + x(-12\lambda - 16) + 16 + 24\lambda + 9\lambda^2\right]dx - 1 = 2$$

$$\Rightarrow \quad \frac{1}{16(1+\lambda)^2}\left[\frac{-4\lambda^2 - 8\lambda}{3} + \left(\frac{-12\lambda - 16}{2}\right) + 16 + 24\lambda + 9\lambda^2\right] = 3$$

$$\Rightarrow \quad \frac{-4\lambda^2 - 8\lambda}{3} - 6\lambda - 8 + 16 + 24\lambda + 9\lambda^2 = 3(16)(1+\lambda)^2$$

$$\Rightarrow \quad -4\lambda^2 - 8\lambda - 18\lambda - 24 + 48 + 72\lambda + 27\lambda^2 = 48(3)(1+\lambda)^2$$

$$\Rightarrow \quad 23\lambda^2 + 46\lambda + 24 - 144 - 144\lambda^2 - 288\lambda = 0$$

$$\Rightarrow \quad 121\lambda^2 + 242\lambda + 120 = 0$$

$$\lambda = \frac{-242 \pm \sqrt{(242)^2 - 4(121)(120)}}{2(121)}$$

$$= \frac{-242 \pm \sqrt{58564 - 58080}}{242}$$

$$= \frac{-242 \pm 22}{242} = -.9090 \quad or \quad -1.0909 .$$

Example 6.12 Among all curves lying on the sphere $x^2 + y^2 + z^2 = a^2$ and passing through two given points (x_0, y_0, z_0) and (x_1, y_1, z_1), find the one which has the least length.

Solution The length of the curve $y = y(x)$ and $z = z(x)$ is given by the integral

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx$$

Using theorem 6.3, we form the auxiliary functional

$$\int_{x_0}^{x_1}\left[\sqrt{1+y'^2+z'^2}+\lambda(x)(x^2+y^2+z^2)\right]dx$$

Then the corresponding auxiliary equations are

$$2y\lambda(x)-\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2+z'^2}}=0$$

$$2z\lambda(x)-\frac{d}{dx}\frac{z'}{\sqrt{1+y'^2+z'^2}}=0$$

Solving these equations, we obtain a family of curves depending on four constants, whose values are determined from the boundary conditions

$$y(x_0) = y_0, \qquad y(x_1) = y_1$$

$$z(x_0) = z_0, \qquad z(x_1) = z_1$$

Example 6.13 Find the shortest distance between the points A(1, -1, 0) and B(2,1, -1) lying on the surface 15x – 7y + z – 22 = 0.

Solution Formulation

$$I[y] = \int_1^2 \sqrt{1+y'^2+z'^2}\,dx$$

$$y(1) = -1, \qquad z(1) = 0$$

$$y(2) = 1, \qquad z(2) = -1$$

Curve lies on the surface 15x – 7y + z – 22 = 0. Consider the Auxiliary function

$$F = f + \lambda g = \sqrt{1+y'^2+z'^2} + \lambda(x)\,(15x - 7y + z - 22) \tag{1}$$

Extremal will be the solution of

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \tag{2}$$

$$\frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)=0 \tag{3}$$

$$\frac{\partial F}{\partial y}=-7\lambda(x), \qquad \frac{\partial F}{\partial y'}=\frac{y'}{\sqrt{1+y'^2+z'^2}},$$

$$\frac{\partial F}{\partial z} = \lambda(x), \qquad \frac{\partial F}{\partial z'} = \frac{z'}{\sqrt{1+y'^2+z'^2}}$$

The corresponding Euler's equations (2) and (3) become as

$$-7\lambda(x) - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \tag{4}$$

$$\lambda(x) - \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0 \tag{5}$$

[Now we determine three functions y, z and λ]

Using value of λ(x) from (5) in (4), we get

$$-7\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{y'+7z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\frac{y'+7z'}{\sqrt{1+y'^2+z'^2}} = \text{c} \tag{6}$$

Now from equation of surface, differentiating, we get

$$15 - 7y' + z' = 0 \qquad \Rightarrow \; z' = 7y' - 15$$

Using this value of z' in equation (6) , we get

$$(y' + 49y' - 105)^2 = c^2[1 + y'^2 + (7y' - 15)^2] \tag{7}$$

Solving this equation (7) for y'

$$y' = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \text{B (say) (constant)}$$

$\Rightarrow$ $\qquad$ y = Bx + C

Now $\qquad$ z' = 7B – 15 = B'

$z = B'x + C'$

Using the given boundary conditions,

$-1 = B + C;\ 0 = B' + C';\ 1 = 2B + C;\ -1 = 2B' + C'$

$\Rightarrow \qquad B = 2;\ C = -3;\ B' = -1;\ C' = 1$

$\therefore \qquad y(x) = 2x - 3 \ \ \&\ \ z(x) = -x + 1.$

Put y and z in equation (4)

$\Rightarrow \qquad -7\lambda(x) - 0 = 0 \Rightarrow 7\lambda(x) = 0 \Rightarrow \lambda(x) = 0$

Now we are giving a name to such type of problems.

Geodesic

In the problems of geodesics, it is required to find the shortest curve connecting two given (fixed) points (x_0, y_0, z_0) and (x_1, y_1, z_1) on a given surface S given by $\phi(x, y, z) = 0$, and lying entirely on that surface. This is a typical variational problem with a constraint since here we are required to minimize the arc length ℓ joining the two fixed points on S given by the functional

$$\ell = \int_{x_0}^{x_1} \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]^{1/2} dx$$

subject to the constraint $\phi(x, y, z) = 0$. This problem was first solved by Jacob Bernoulli in 1698, but a general method of solving such problem was given by Euler. (The study of properties of geodesics is one of the focal points of the branch of mathematics known as differential geometry).

Thus a geodesic on a surface is a curve along which the distance between any two points of a surface is minimum.

To solve problems on geodesics, we must first study invariance of Euler's equation.

Invariance of Euler's Equation

Consider the function

$$I[y(x)] = \int_a^b F(x, y, y')dx \qquad (1)$$

Let u and v be two variables. Suppose the equation is transformed by the replacement of the independent variable and the function y(x) as

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

where $$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$$

x_u, x_v, y_u and y_v denotes the partial derivatives of x w.r.t. u, v and partial derivatives of y w.r.t. u, v.

Then the curve given by the equation y = y(x) in the xy-plane corresponds to the curve v = v(u) in the uv-plane.

Now $$\frac{dx}{du} = \frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}\frac{dv}{du} = x_u + x_v \frac{dv}{du}$$

$$\frac{dy}{du} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\frac{dv}{du} = y_u + y_v \frac{dv}{du}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{y_u + y_v v'}{x_u + x_v v'}$$

Thus the functional (1), changes into the functional

$$J_1[u(v)] = \int_{u_1}^{u_2} F\left[x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right](x_u + x_v v')du$$

which can be written as

$$J_1[u(v)] = \int_{u_1}^{u_2} G(u, v, v')\, du \ \text{(say)} \qquad (2)$$

Now extremal of (1) can be obtained from (2). If y = y(x) satisfies the Euler's equation $F_y - \frac{d}{dx}F_{y'} = 0$ corresponding to the original functional J[y(x)], then it can be proved that the functional v = v(u) satisfies the Euler's equation $G_u - \frac{d}{dx}G_{v'} = 0$

corresponding to the new functional $J_1[v(u)]$ or v(u) is an extremal of $J_1[v(u)]$ if y(x) is an extremal of J[y(x)]. Therefore, the extremals of functional I[y(x)] can be found by solving Euler's equation of the transformed functional $J_1[v(u)]$. This is called the Principle of Invariance of Euler's equation under coordinates transformations.

Example 6.14 Find the extremals of

$$I[y(x)] = \int_0^{\log 2} (e^{-x} y'^2 - e^x y^2) dx$$

Solution If we write Euler's equation as such it may not be simple to solve it. Therefore we make use of the substitution

$$x = \log u \text{ and } y = v$$

We have

$$e^{-x} = e^{-\log u} = \frac{1}{u}$$

$$e^{x} = e^{\log u} = u$$

$$y^2 = v^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}.\frac{dv}{du}}{\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}.\frac{dv}{du}} = \frac{0 + 1v'}{\frac{1}{u} + 0} = uv' \qquad \text{[we denote } \frac{dv}{du} \text{ by v']}$$

Also $$dx = \left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}.\frac{dv}{du}\right) du = \frac{1}{u} du$$

$\therefore$ Given functional reduces to

$$I[v] = \int_1^2 \left(\frac{1}{u} u^2 v'^2 - uv^2\right) \frac{1}{u} du$$

$$= \int_1^2 (v'^2 - v^2) du \qquad (1)$$

$\therefore$ Extremal can be found from equation (1)

Its Euler equation is

$$\frac{\partial f}{\partial v} - \frac{d}{du}\left(\frac{\partial f}{\partial v'}\right) = 0$$

or $$-2v - \frac{d}{du}(2v') = 0$$

or $$2v + 2\frac{d^2 v}{du^2} = 0$$

or $$\frac{d^2 v}{du^2} + v = 0$$

or $$(D^2 + 1)v = 0$$

Auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

Thus the solution is

$$v = c_1 \cos u + c_2 \sin u$$

or $$y = c_1 \cos(e^x) + c_2 \sin(e^x) \quad [\text{since } x = \log u \text{ and } u = e^x]$$

which is the required extremal.

Values of c_1 and c_2 can be determined from the boundary conditions.

Example 6.15 Find the extremals of the functional

$$\int_{\theta_1}^{\theta_2} \sqrt{(r^2 + r'^2)}\,d\theta \qquad \text{where } r = r(\theta).$$

Solution Let $I[r(\theta)] = \int_{\theta_1}^{\theta_2} \sqrt{(r^2 + r'^2)}\,d\theta$ (1)

Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $$r^2 = x^2 + y^2$$

i.e. $$r = \sqrt{x^2 + y^2}$$

and $$\tan\theta = \frac{y}{x}$$

Now $$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$

and $$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

Squaring and adding, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + r'^2$$

$$\therefore \quad \int \sqrt{r^2 + r'^2}\, d\theta = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int \sqrt{(dx)^2 + (dy)^2}$$

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int \sqrt{1 + y'^2}\, dx$$

Suppose at $\theta = \theta_1$, $x = x_1$

and at $\theta = \theta_2$, $x = x_2$

$$\therefore \quad \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

$$\therefore \quad I[r(\theta)] = I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

The Euler equation is

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\therefore \quad \frac{y'}{\sqrt{1 + y'^2}} = c$$

or $\quad y'^2 = c^2\ (1 + y'^2) \quad$ [Squaring] or $\qquad y'^2(1- c^2) = c^2$

or $\qquad y' = \dfrac{c}{\sqrt{1-c^2}} = c_1$ (say)

$\therefore \qquad \dfrac{dy}{dx} = c_1$

Integrating, $\quad y = c_1x + c_2$

$\therefore$ Extremals are $r \sin\theta = c_1 r \cos \theta + c_2$.

Exercise Find the extremal of the functional $\int_{\theta_1}^{\theta_2} r\sin\theta\sqrt{(r^2 + r'^2)}d\theta$ by using the transformation $x = r \cos\theta$, $y = r \sin\theta$.

Answer $\log[r^2 \sin\theta + \sqrt{r^2 \sin^2\theta - c_1^2}\,] = c_1 r\cos\theta + c_2$.

Differential equation governing the Geodesic on a surface

Geodesics

Suppose we have a surface σ (with coordinates u and v) specified by a vector equation

$$\vec{r} = \vec{r}(u,v) \tag{1}$$

The shortest curve lying on σ and connecting two points of σ is called the geodesics. Clearly the equations for the geodesics of σ are the Euler equations of the corresponding variational problem i.e. the problem of finding the minimum distance (measured along σ) between two points of σ.

A curve lying on the surface (1) is specified by the equations

$$u = u(t), \qquad v = v(t)$$

The arc length between the points corresponding to the values t_1 and t_2 of the parameter t equals

$$I[u,v] = \int_{t_1}^{t_2} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt, \tag{2}$$

where E, F and G are the coefficients of the first quadratic (fundamental) form of (1), given by

$$E = \vec{r}_u . \vec{r}_u, \qquad (\vec{r}_u = \text{partial derivative of } \vec{r} \text{ w.r.t. u})$$

$$F = \vec{r}_u . \vec{r}_v,$$

$$G = \vec{r}_v . \vec{r}_v$$

Let $\quad L = \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt,$

The Euler's equations are

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) = 0$$

$$\frac{\partial L}{\partial v} - \frac{d}{dt}\left(\frac{\partial L}{\partial v'}\right) = 0$$

$$\frac{\partial L}{\partial u} = \frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{2\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}$$

$$\frac{\partial L}{\partial u'} = \frac{2Eu' + 2Fv'}{2\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}$$

Similarly, we can find $\dfrac{\partial L}{\partial v}$ and $\dfrac{\partial L}{\partial v'}$.

$\therefore$ Euler's equations are

$$\frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\left(\frac{2(Eu' + Fv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}\right) = 0$$

and
$$\frac{E_v u'^2 + 2F_v u'v' + G_v v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\left(\frac{2(Fu' + Gv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}\right) = 0.$$

These are differential equations governing the geodesics.

Example 6.16 Find the geodesics on a right circular cylinder of radius a.

Solution Let the axis of the cylinder be taken along z-axis. Let A and B be any two points on the given cylinder and let (a, θ_1, z_1) and (a, θ_2, z_2) be their cylindrical coordinates respectively. Let z(θ) be a function where curve passes through A and B and lying itself on the surface of the given cylinder.

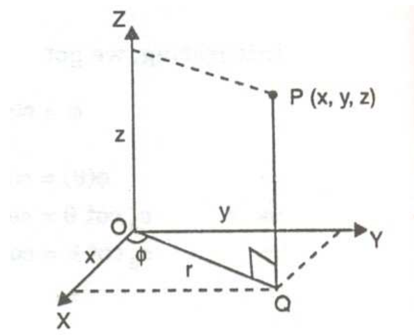


Fig. 6.5

The length of the arc between A and B is $\int_{\theta_1}^{\theta_2} ds$ .We use cylindrical polar coordinates r, θ, z and r = a, where a is radius of cylinder which is constant.

The element of arc on a cylinder of radius a is given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= (dr)^2 + (a\, d\theta)^2 + (dz)^2$$

$$= 0 + (a\, d\theta)^2 + dz^2 \qquad (\because r = a \Rightarrow dr = 0)$$

$$(ds)^2 = a^2(d\theta)^2 + \left(\frac{dz}{d\theta}\right)^2 (d\theta)^2 = (a^2 + z'^2)(d\theta)^2$$

$$ds = \sqrt{a^2 + z'^2}\, d\theta$$

$$s = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'^2}\, d\theta \qquad (1)$$

The variational problem, here, associated with geodesics is $\int ds$ = minimal, where ds is a line element of curve on surface of a right circular cylinder.

∴ Variational problem becomes

$$\int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta = \text{min.} \qquad (2)$$

For extremal, Euler's equation is satisfied by z(θ).

$$\frac{\partial f}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = 0 \tag{3}$$

The integrand here does not depend upon z = z(θ).

∴ Euler's equation is

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = 0 \tag{4}$$

i.e. $$\frac{\partial f}{\partial z'} = c \text{ (constant)} \tag{5}$$

and $$\frac{\partial f}{\partial z'} = \frac{1}{2}\frac{1}{\sqrt{a^2 + z'^2}}2z' = \frac{z'}{\sqrt{a^2 + z'^2}}$$

∴ equation (5) becomes

$$\frac{z'}{\sqrt{a^2 + z'^2}} = c \quad \text{or} \quad z'^2 = c^2(a^2 + z'^2)$$

or $$z'^2(1 - c^2) = a^2c^2$$

$$\frac{dz}{d\theta} = z' = \frac{ac}{\sqrt{1 - c^2}}$$

or $$z = \frac{ac}{\sqrt{1 - c^2}}\theta + c'$$

or $$\theta = \frac{z.\sqrt{1 - c^2}}{ac} - \frac{\sqrt{1 - c^2}}{ac}c'$$

∴ θ = mz + b form, which is a circular helix,

or $z = c_2\theta + c_3$. The values of c_2 and c_3 are found by using the fact that this curve is to pass through A and B. This curve is called a Helix.

Example 6.17 Find the geodesics on the surface of a sphere. OR

Among all curves on a sphere of radius R that joins the two points, find the shortest curve. OR

Show that the geodesics on a sphere of radius a are its great circles.

Solution Spherical co-ordinates are

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, where

$r \ge 0$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. Let A and B be any two points on the given sphere and let (a, θ_1, ϕ_1) and (a, θ_2, ϕ_2) be their spherical coordinates respectively. Let $\phi(\theta)$ be a function whose curve passes through A and B and lying itself on the surface of the given sphere.

$\therefore$ The length of the arc between A and B is $\int_{\theta_1}^{\theta_2} ds$.

In spherical co-ordinates, we have

$$(ds)^2 = h_1^2 (dr)^2 + h_2^2 (d\theta)^2 + h_3^2 (d\phi)^2$$

$$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$$

$$(ds)^2 = 1^2.0 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \qquad \text{(since } r = a,\ dr = 0)$$

$$= a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$$

$$ds = \sqrt{a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2}$$

$$= a\, d\theta \sqrt{1 + \sin\theta^2 \phi'^2}$$

$$I[\phi] = \int a\sqrt{1 + \sin^2\theta\, \phi'^2}\, d\theta$$

Let $F(\theta, \phi, \phi') = a\sqrt{1 + \sin^2\theta\, \phi'^2}$

Let the length of the curve between A and B has a minimum value for this curve or let $\phi(\theta)$ be a geodesic between A and B.

$\therefore$ The functional $\int_{\theta_1}^{\theta_2} F(\theta, \phi, \phi')\, d\theta$ has a minimum value on the function $\phi(\theta)$.

$\therefore$ $\phi(\theta)$ satisfies the Euler's equation.

The Euler's equation is

$$F_\phi - \frac{d}{d\theta} F_{\phi'} = 0 \qquad (1)$$

Here $\quad F_\phi = 0, \qquad F_{\phi'} = \dfrac{a\phi' \sin^2\theta}{\sqrt{1+\phi'^2 \sin^2\theta}}$

$\therefore$ (1) $\Rightarrow$ $\quad 0 - \dfrac{d}{d\theta}\left(\dfrac{a\phi' \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\right) = 0$

$\Rightarrow \quad \dfrac{d}{d\theta}\left(\dfrac{\phi' \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\right) = 0$

$\Rightarrow \quad \dfrac{\phi' \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}} = c_1$

$\Rightarrow \quad \phi'^2 \sin^4\theta = c_1^2(1+\phi'^2\sin^2\theta)$

$\Rightarrow \quad \phi' = \dfrac{c_1}{\sin\theta\sqrt{\sin^2\theta - c_1^2}} = \dfrac{c_1}{\sin^2\theta\sqrt{1-c_1^2\cos ec^2\theta}} = \dfrac{(\cos ec^2\theta)\, c_1}{\sqrt{(1-c_1^2) - c_1^2\cot^2\theta}}$

$$= \frac{-d(c_1\cot\theta)}{\sqrt{(1-c_1^2) - c_1^2\cot^2\theta}}$$

Integrating, we get

$$\phi = \cos^{-1}\frac{c_1\cot\theta}{\sqrt{(1-c_1^2)}} + c_2$$

$\Rightarrow \quad \phi(\theta) = \cos^{-1}(c_3 \cot\theta) + c_2$, say

$\quad c_3 \cot\theta = \cos(\phi(\theta) - c_2)$

$\Rightarrow \quad c_3 \cot\theta = \cos\phi(\theta)\cos c_2 + \sin\phi(\theta)\sin c_2$

$\Rightarrow \quad \cot\theta = \dfrac{\cos c_2}{c_3}\cos\phi(\theta) + \dfrac{\sin c_2}{c_3}\sin\phi(\theta)$,

$\Rightarrow \quad \cot\theta = c_4\cos\phi(\theta) + c_5\sin\phi(\theta)$, say

Multiplying by a $\sin\theta$, we get

$$a\cos\theta = c_4(a\cos\phi(\theta)\sin\theta) + c_5\, a(\sin\phi(\theta)\sin\theta)$$

$$z = c_4 x + c_5 y \quad \text{(using cartesian coordinates)}$$

The values of c_4 and c_5 are found by using the fact that this plane is to pass through A and B. $z = c_4 x + c_5 y$ represents a plane passing through the centre (0, 0, 0) of the sphere (definition of great circle).

$\therefore$ The geodesics is the arc of the great circle passing through given point.

Example 6.18 Find the geodesics on a right circular cone of semi-vertical angle α.

Solution Let the vertex of the cone be at the origin and its axis along z-axis.

Let A and B be any two points on the given cone and let (r_1, α, ϕ_1) and (r_2, α, ϕ_2) be their spherical coordinates respectively. Let $r(\phi)$ be a function whose curve passes through A and B and lying itself on the surface of the given cone.

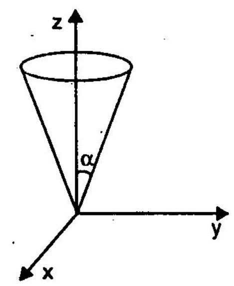


Figure 6.6

$\therefore$ The length of the arc between A and B is given by $\int_{\phi_1}^{\phi_2} ds$.

Since we are dealing with spherical coordinates, we have

$$ds = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta\, d\phi)^2}$$

$$\Rightarrow \quad ds = \sqrt{(dr)^2 + r^2.0 + r^2\sin^2\alpha\,(d\phi)^2} \qquad (\because \theta = \alpha \Rightarrow d\theta = 0)$$

$$= \sqrt{r'^2 + r^2\sin^2\alpha}\, d\phi$$

$$\therefore \quad \text{Length of arc} = \int_{\phi_1}^{\phi_2} \sqrt{r'^2 + r^2\sin^2\alpha}\, d\phi$$

Let $\quad F(\phi, r, r') = \sqrt{r'^2 + r^2 \sin^2 \alpha}$

Let the length of the curve between A and B has a minimum value for this curve i.e. r(ϕ) is a geodesic between A and B.

∴ The functional $\int_{\phi_1}^{\phi_2} F(\phi, r, r')\, d\phi$ has a minimum value on the function r(ϕ).

∴ r(ϕ) satisfies the Euler's equation.

The Euler's equation is $F_r - \frac{d}{d\phi} F_{r'} = 0$ (1)

Since ϕ is missing from the function F, the Euler equation reduces to

$$F - F_{r'} \frac{dr}{d\phi} = c \tag{2}$$

$$\Rightarrow \quad \sqrt{r'^2 + r^2 \sin^2 \alpha} - \frac{r'}{\sqrt{r'^2 + r^2 \sin^2 \alpha}} . r' = c$$

$$\Rightarrow \quad r'^2 + r^2 \sin^2 \alpha - r'^2 = c\sqrt{r'^2 + r^2 \sin^2 \alpha}$$

$$\Rightarrow \quad r^4 \sin^4 \alpha = c^2 (r'^2 + r^2 \sin^2 \alpha)$$

$$\Rightarrow \quad r' = \frac{r \sin \alpha}{c} \sqrt{r^2 \sin^2 \alpha - c^2} \Rightarrow \quad \frac{cdr}{r \sin \alpha \sqrt{r^2 \sin^2 \alpha - c^2}} = d\phi$$

$$\Rightarrow \quad \frac{c}{\sin \alpha} \int \frac{d(r \sin \alpha)}{r \sin \alpha \sqrt{r^2 \sin^2 \alpha - c^2}} = \phi + c_1$$

$\Rightarrow \quad \frac{c}{\sin \alpha} \int \frac{dt}{t\sqrt{t^2 - c^2}} = \phi + c_1$, where t = r sinα

$\Rightarrow \quad \frac{c}{\sin \alpha} \int \frac{c \sec u \tan u du}{r \sec u \sqrt{c^2 \sec^2 u - c^2}} = \phi + c_1$, where t = c sec u

$$\Rightarrow \quad \frac{1}{\sin \alpha} \int du = \phi + c_1 \quad \Rightarrow \quad u = \phi \sin \alpha + c_1 \sin \alpha$$

$\Rightarrow \quad$ u = ϕ sinα + c_2, say

$$\Rightarrow \qquad \sec u = \sec(\phi \sin\alpha + c_2) \Rightarrow \frac{t}{c} = \sec(\phi \sin\alpha + c_2)$$

$$\Rightarrow \qquad \frac{r \sin\alpha}{c} = \sec(\phi \sin\alpha + c_2) \Rightarrow \qquad c_3 r = \sec(\phi \sin\alpha + c_2)$$

The value of c_2 and c_3 are found by using the fact that this curve is to pass through A and B.

$\therefore$ The required curve is $c_3 r = \sec(\phi \sin\alpha + c_2)$.

Summary

From ancient times, geometers noticed extremel properties of symmetric figures and bodies. The circle has maximum area among all figures with fixed perimeter, the right triangular and square have maximal area among all triangles and quadrangles with fixed perimeters, respectively, etc. Extremal problems are attractive due to human's natural desire to find perfect solutions, they also root in natural laws of physics. This chapter covers classical techniques of calculus of variations, discusses natural variational principles in classical and continuum mechanics and introduces modern applications.

Keywords Euler-Poisson equation, isoperimetric problems, integral constraints, geodesics.

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