Space of Analytic Functions

The space of continuous functions C(G, Ω)

Definition : If G is an open set in C and (Ω, d) is a complete metric space the set of all continuous functions from G to Ω, designated by C(G, Ω) is called space of continuous functions.

Compact set of a metric space

Definition : A subset K of a metric space X is compact if for every collection $\varsigma = \{G : G \text{ is open}; G \subset X\}$ of open sets in X,

....(1)

i.e. there is a finite number of sets $G_1, G_2, ..., G_n$ in ς such that

Cover of a compact set $\{x_n\}$ $K \subset \cup\{G : G \in \varsigma\}$ $K \subset G_1 \cup G_2 \cup ... \cup G_n$

A collection of set ς satisfying (1) is called cover of the compact set K.

If each member of is an open set then is called open cover of K.

e.*g*. empty set and all finite sets are compact.

Cauchy sequence : A sequence is called a Cauchy sequence if for every ε>0 there is an integer N such that $d(x_n, x_m) < \varepsilon,\ \forall n, m \geq N$

Complete metric space : A metric space (X,d) is called complete metric space if each Cauchy sequence has a limit in X.

Let $G \subset$ **C**, G is an open subset of **C** and H(G) the set of all analytic functioins defined on G, i.e. H(G) = {f : f is analytic function on G} then H(G) be a subset of space of continuous functons from G to **C.**

$H(G) \subset C(G, C)$

We denotes the set of analytic function on G by H(G) rather than A(G) because

g continuous function that are analytic in G}

Thus $A(G) \neq H(G)$

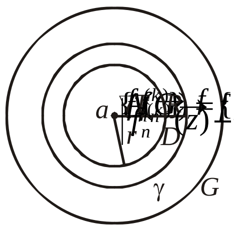
Theorem: If be a sequence in H(G) and $f \in C(G, \mathbf{C})$ such that

then f is analytic and $f_n^{(k)} \to f^{(k)}$ $\forall$ integer $k \geq 1$

Proof : To prove that f is analytic, we use Morera's theorem. If T be a triangle contained inside a disk D $\subset G$, then T is compact, and the sequence $\{f_n\}$ converges to f uniformly over T. Hence by Morera's theorem

$$\int_T f = \lim \int_T f = 0 \qquad ...(1)$$

since each f_n is analytic.



$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!}{2\pi} \int_\gamma \frac{|f_n(w) - f(w)|}{|w-z|^{k+1}} |dw|$$

Thus f must be analytic in every disk $D \subset G$.

Now we show that

Let $D = \overline{B}(a; r) \subset G$; then there is a number R > r such that $\overline{B}(a, R) \subset G$.

If we take a circle $\gamma \equiv |z - a| = R$ then by Cauchy's Integral formula

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \left[\frac{f_n(w)}{(w-z)^{k+1}} - \frac{f(w)}{(w-z)^{k+1}}\right] dw \qquad \forall \quad z \in D$$

$\Rightarrow$ ...(2)

Since $f_n \to f$ and f_n are continuous in C then $\exists\, M_n > 0$ where

$M_n = \sup\{|f_n(w) - f(w)| : |w - a| = R\}$ such that

$|f_n(w) - f(w)| \le M_n$, then from the equation (2)

$$\therefore \qquad \left|f_n^{(k)}(z) - f^{(k)}(z)\right| \le \frac{k!}{2\pi}\int_\gamma \frac{M_n}{(R-r)^{k+1}}|dw|$$

$$= \frac{k!M_n}{2\pi(R-r)^{k+1}}\int_\gamma |dw| = \frac{k!M_n}{2\pi(R-r)^{k+1}}.2\pi R$$

$$= \frac{k!M_nR}{2\pi(R-r)^{k+1}} \qquad \text{for } |z-a|<r \qquad ...(3)$$

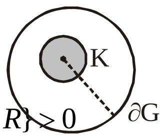
Since $f_n \to f$, and $\lim M_n = 0$ then (3) gives

$\Rightarrow \quad f_n^{(k)} \to f^{(k)}$ uniformly on $\bar{B}(a;r)$.

Now if K is an arbitrary compact subset of G and distance of each element of K from any of the boundary point of G is greater than r, *i.e.*

$0 < r < d(K, \partial G)$ then in K such that

$$K \subset \bigcup_{j=1}^{n} B(a_j;r)$$



Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $\bar{B}(a_j;r)$

$\Rightarrow \quad f_n^{(k)}$ converges uniformly to $f^{(k)}$ on K

Theorem : Hurwitz's Theorem

Let G be a region and suppose the sequence $\{f_n\}$ converges to f in $H(G)$. If $f \not\equiv 0$, $\bar{B}(a;R) \subset G$ and $f(z) \ne 0$ for $|z-a| = R$ then there is an integer N such that for $n \ge N$, f and f_n have the same number of zeros in $B(a;R)$.

Proof : Since for $|z-a| = R$, therefore we can define a positive number δ as

But f_n converges uniformly to f on . Therefore integer N such that if and then

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)|$$

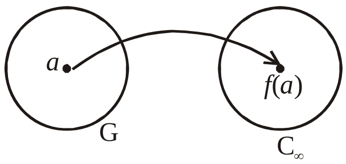
$$\le |f(z)| + |f_n(z)|$$

Thus from above it follows that f and f_n satisfies the condition of Rouche's theorem.

Thus f and f_n have the same number of zeros in $B(a;R)$.

Theorem: Let be a sequence in M(G) and suppose $f_n \to f$ in $C(G, \mathbf{C}_\infty)$ then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic or $f \equiv \infty$

Proof : Suppose there is a point 'a' in G with $f(a) \neq \infty$ and let $|f(a)| = M$



Now

therefore we can find a number $\rho > 0$ such that

....(1)

But $f_n \to f$ so there is an integer [illegible] $\forall \; z \in \bar{B}(a;r)$ *and* $n \geq n_0$

$$d(f_n(a);\ f(a)) < \frac{\rho}{2}, \quad \forall n \geq n_0$$

Also the set is compact in $C(G, C_\infty)$

$\Rightarrow$ it is equicontinuous

i.e. $\exists$ an r > 0 such that

$$\Rightarrow \quad d(f_n(z),\ f(z)) < \frac{\rho}{2}$$

$$\Rightarrow \quad d(f_n(z);\ f_n(a)) < \frac{\rho}{2}$$

This gives that $d\big(f_n(z),\ f(a)\big) \leq \rho$ for $|z - a| \leq r$ and for $n \geq n_0$

Now

$\Rightarrow$ ...(2)

In view of the ρ chosen in (1); the expression (2) can be written as

.....(3)

But $d\left(f_n(z)\,,\ f(z)\right) = \dfrac{2\,|\,f_n(z) - f(z)\,|}{\left\{\left[1+|\,f_n(z)\,|^2\right]\left[1+|\,f(z)\,|^2\right]\right\}^{1/2}}$

$\forall \quad z \in \overline{B}(a\,;\,r) \quad and \quad n \geq n_0$

$$\geq \frac{2\,|\,f_n(z) - f(z)\,|}{\left\{(1+4M^2)(1+4M^2)\right\}^{1/2}}$$

$\forall \quad z \in \overline{B}(a\,;\,r) \quad and \quad n \geq n_0$

Since $d\{f_n(z), f(z)\} \to 0$ uniformly for $z \in \overline{B}(a;r)$ this gives that $|\,f_n(z) - f(z)\,| \to 0$ uniformly for $z \in \overline{B}(a;r)$

From (3)

$\{f_n\}$ is bounded on $B(a;\,r)$

$\Rightarrow$ f_n has no poles and must analytic near $z = a$, $\forall\ n \geq n_0$.

$\Rightarrow$ f is analytic in a disc about a. .....(4)

Now suppose there is a **point 'a' in G with** $f(a) = \infty$. Then, if $g \in C(G, \mathbf{C}_\infty)$; define $1/g$ as follows:

$$\frac{1}{g}(z) = \frac{1}{g(z)} \quad \text{if } g(z) \neq 0 \text{ or } \infty;$$

$$\frac{1}{g}(z) = 0 \quad \text{if } g(z) = \infty; \text{ and}$$

$$\frac{1}{g}(z) = \infty \quad \text{if } g(z) = 0$$

Then it follows that $\dfrac{1}{g} \in C(G, \mathbf{C}_\infty)$

Also, since $f_n \to f$ in $C(G, \mathbf{C}_\infty)$ then by the property of metric over non zero complex number z_1 and z_2, i.e.

$$d(z_1, z_2) = d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) \text{ and } d(z, 0) = d\left(\frac{1}{z}, \infty\right)$$

It follows that $\dfrac{1}{f_n} \to \dfrac{1}{f}$ in $C(G, \mathbf{C}_\infty)$

Now each $\dfrac{1}{f_n}$ is meromorphic on G;

So, $\exists\ r > 0$ and an integer n_0 such that $\dfrac{1}{f}$ and $\dfrac{1}{f_n}$ are analytic on $B(a; r)$ for $n \geq n_0$

$\Rightarrow \quad \dfrac{1}{f_n} \to \dfrac{1}{f}$ uniformly on $B(a; r)$.

From Hurwitz theorem, either $\dfrac{1}{f} \equiv 0$ or $\dfrac{1}{f}$ has isolated zeros in $B(a; r)$.

$\therefore$ if $f \not\equiv \infty$ then $\dfrac{1}{f} \not\equiv 0$ and f must be meromorphic in $B(a; r)$

Combining this with the result (4) that f is meromorphic in G if f is not **identically infinite.**

If each f_n is analytic then $\dfrac{1}{f_n}$ has no zeros in $B(a; r)$. Then [corollary : $\{f_n\} \subset$ H(G) converges to f in $H(G)$ and each f_n never vanishes on G then either $f \equiv 0$ or never vanishes] then either $\dfrac{1}{f} \equiv 0$ or $\dfrac{1}{f}$ never vanishes.

But since $f(a) = \infty$ we have that $\dfrac{1}{f}$ has at least one zero; thus $f \equiv \infty$ in B(a;r)

$\sup\{|f(z)| : |z-a| < r,\ f \in \mathbf{F}\} < \infty$

Thus, either $f \equiv \infty$ or f is analytic

Locally bounded set of analytic functions

Definition : A set $\mathbf{F} \subset H(G)$ is locally bounded if $\forall\ a \in G,$ there are constants M and r >0, such that

$| f(z) | \leq M,$ for

Alternately

$\mathbf{F}$ is locally bounded if there is an r >0 such that

Montel's Theorem

A family $\mathbf{F}$ *in H(G) is normal iff is locally bounded*

Proof : To prove that is normal i.e. each sequence in has a subsequence which converges to a function *f* in H(*G*).

Necessary : Suppose is normal but *fails to be locally bounded.*

then there is a compact set $K \subset G$ such that

....(1)

[contradiction of locally bounded set]

i.e. there exists a sequence $\{f_n\}$ in $\mathbf{F}$ such that

Since $\mathbf{F}$ is normal therefore a function and a subsequence such that

$$f_{n_k} \to f$$

$\Rightarrow$ $\sup\{|f_{n_R}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$

If $|f(z)| \le M \;\; for \;\; z \in K$

$$n_k \le \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M$$

$\sup\{|f(z)| : z \in K, f \in \mathbf{F}\} = \infty$ $\sup\{|f_{n_k}(z)| : z \in K\} \ge n$

when $n_k \to \infty$ right hand side converges to M which cannot be true ($\infty \le M$)

Hence the assumption taken at start is wrong. Therefore if is normal it is locally bounded.

Corollary : is closed in $C(G, \mathbf{C}_\infty)$.

For this we prove the normality of M(*G*).

For this let us introduce the quality

$$\frac{2|f'(z)|}{1+|f(z)|^2} \qquad ....(1)$$

for each meromorphic function.

However if z is a pole of *f* then has no meaning because derivative increases the order of the pole. Therefore (1) is meaningless.

This can be rectify by taking limit of (1) as z approaches the pole. Now we show that the limit of (1) when z tends to pole. Let '*a*' be a pole of *f* of order ; then

f(z) can be expressed as

$$f(z) = g(z) + \frac{A_m}{(z-a)^m} + ... + \frac{A_1}{(z-a)}$$

for z in some disk about a and g(z) analytic in that disk.

For $z \neq a$

$$\therefore \quad \frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2\left|\frac{mA_m}{(z-a)^{m+1}} + ... + \frac{A_1}{(z-a)^2} - g'(z)\right|}{1+\left|\frac{A_m}{(z-a)^m} + ... + \frac{A_1}{(z-a)} + g(z)\right|^2}$$

$$= \frac{2\,|z-a|^{m-1}|\, mA_m + ... + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}\,|}{|z-a|^{2m} + |\, A_m + ... + A_1(z-a)^{m-1} + g(z)(z-a)^m\,|^2}$$

Thus if $m \geq 2$

$$\lim_{z\to a}\frac{2|f'(z)|}{1+|f(z)|^2} = 0$$

$$f'(z) = g'(z) - \left[\frac{mA_m}{(z-a)^{m+1}} + ... + \frac{A_1}{(z-a)^2}\right]$$

If m = 1 then

$$\lim_{z\to a}\frac{2|f'(z)|}{1+|f(z)|^2} = \lim_{z\to a}\frac{2\left|\frac{A_1}{(z-a)^2} - g'(z)\right|}{1+\left|\frac{A_1}{(z-a)} + g(z)\right|^2}$$

$$= \lim_{z\to a}\frac{2\frac{1}{|z-a|^2}\left|A_1 - (z-a)^2 g'(z)\right|}{\frac{1}{|z-a|^2}\left[|z-a|^2 + |A_1 + g(z)(z-a)|^2\right]}$$

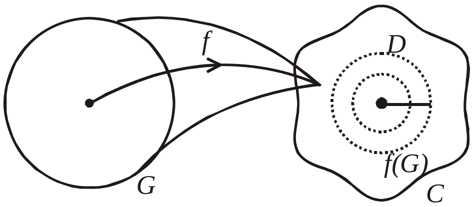
$$= \frac{2|A_1|}{|A_1|^2} = \frac{2}{|A_1|}$$

which shows that for $m \geq 1$ the limit of $\frac{2|f'(z)|}{1+|f(z)|^2}$ exits.

Theorem : Riemann Mapping Theorem

Let G be a simply connected region which is not the whole plane and let

Then there is a unique analytic function having the properties.



(a) $f(a) = 0$ and

(b) f is one-one

(c) $f(G) = \{z : |z| < 1\}$

Proof : First we prove uniqueness of f, let g be a function having the same properties like f, i.e.

$g(a) = 0$ and $g'(a) > 0$,

g is one-one and $|c| = 1 \Rightarrow C$

$g(G) = \{z : |z| < 1\}$ $\quad g : G \to D$

and $D = \{z : |z| < 1\}$ then

$g : G \underset{\text{onto}}{\overset{1-1}{\longrightarrow}} D$

and $fog^{-1} : D \underset{\text{onto}}{\overset{1-1}{\longrightarrow}} D$ and analytic.

Also $fog^{-1}(0) = f(g^{-1}(0)) = f(a) = 0$

Then by the theorem-

Let $f : G \underset{\text{onto}}{\overset{1-1}{\longrightarrow}} D$ analytic and $f(a) = 0 \ \exists$ a complex number c with $|c| = 1$

such that $f = c\varphi_{\alpha}$, $\varphi_{\alpha}(z) = \dfrac{z-\alpha}{1-\bar{\alpha}z}$ is Mobious transformation.

Implies that $\exists$ a constant c; and $fog^{-1}(z) = cz \quad \forall z$

$\Rightarrow \quad f(z) = cg(z)$

$\Rightarrow$

Since and $cg'(a) > 0$

$\Rightarrow$ c =1

$\Rightarrow$ *f* = *g*

Hence *f* is unique.

For the **existence of *f***, consider the family $\mathbf{F}$ of all analytic function f having properties (a) and (b) and satisfying for z in G.

The idea is to choose a member of $\mathbf{F}$ having property (c).

Suppose $\{K_n\}$ is a sequence of compact subsets of G such that

and $a \in K_n \quad \forall\ n$

Then $\{f(K_n)\}$ is sequence of compact subsets of $D = \{z : |z| < 1\}$.

Also, as n becomes larger $\{f(K_n)\}$ becomes larger and larger and tries to fill out the disk.

Choosing $f \in \mathbf{F}$ with the largest possible derivative at a, we choose the function which "starts out the fastest" at z =a.

$\{f \in H(G) : f \text{ is } \text{one} - \text{one},\ f(a) = 0,\ f'(a) > 0,\ f(G) \subset D\}$

$\bigcup_{n=1} K_n = G$

Thus

Lemma : Let G be a region which is not the whole plane and such that every non vanishing function on G has an analytic square root. If $a \in G$ then there is an analytic function *f* on G such that :

(a) *f(a)* = 0 and

(b) *f* is one-one

(c) $f(G) = D = \{z : |z| < 1\}$

Proof : Let we define $\mathbf{F}$ as

Since $f(G) \subset D$ then

$\sup\{|f(z)| : z \in G\} \leq 1$ for $f \in \mathbf{F}$

By Montel theorem $\mathbf{F}$ is normal if it is non-empty. Thus first of all it is to be proved that

....(1)

and $\mathbf{F}^{-} = \mathbf{F} \cup \{0\}$ ...(2)

Suppose (1) and (2) hold and consider the function

$f \to f'(a)$ of $H(G) \to \mathbf{C}$

This is a continuous function and since $\mathbf{F}^{-}$ is compact there is a f in with

$\mathbf{F}$.

Since (empty set) therefore $f \in \mathbf{F}$.

***f(G) = D* is remains to shows now.**

Suppose $w \in D$ such that $w \notin f(G)$, Then the function $\dfrac{f(z)-w}{1-\overline{w}f(z)}$ $\mathbf{F} \neq \phi$ $|f'(a)| \geq |g'(a)| \; \forall g \in$

is analytic in G and never vanishes.

then by the hypothesis there is an analytic function $h : G \to \mathbf{C}$ which is equal to the square root of a analytic function, i.e.

$$[h(z)]^2 = \frac{f(z) - w}{1 - \overline{w}f(z)} \qquad ....(3)$$

Since the Mobious transformation

$T\varsigma = \dfrac{\varsigma - w}{1 - \overline{w}\varsigma}$ maps *D* onto *D*, *h(G)* $\subset$ *D*.

Define $g : G \to C$ by

$$g(z) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h(z) - h(a)}{1 - \overline{h(a)}\,h(z)} \qquad ....(4)$$

Then $|g(G)| \leq 1 \Rightarrow$ *g*(*G*) $\subset$ *D*, which is obvious from (4)

and g is one-one, as f and hence h both are one-one.

Also

$$g'(a) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h'(a)[1-|h(a)|^2]}{[1-|h(a)|^2]^2}$$

$$= \frac{|h'(a)|}{1-|h(a)|^2}$$

But

$$|h(a)|^2 = \left|\frac{f(a)-w}{1-\overline{w}f(a)}\right| = \left|\frac{0-w}{1-0}\right| = |-w| = |w|$$

and derivative of (3) gives

$$2h(z)h'(z) = \frac{f'(z)(1-\overline{w}\,f(z)) - (f(z)-w)\,(-\overline{w}\,f'(z))}{[1-\overline{w}\,f(z)]^2}$$

$$2h(a)h'(a) = \frac{f'(a) - \overline{w}\,w\,f'(a)}{1}$$

$$2h(a)h'(a) = f'(a)(1-|w|^2) \qquad f(a) = 0$$

$$\therefore \qquad g'(a) = \frac{f'(a)(1-|w|^2)}{2\sqrt{|w|}} \cdot \frac{1}{1-|w|}$$

$$= f'(a)\left(\frac{(1+|w|)}{2\sqrt{|w|}}\right) \qquad \left[\because \frac{1+|w|}{2\sqrt{|w|}} > 1\right]$$

$$> f'(a)$$

This contradict that g is in $\mathbf{F}$ and the choice of f. Thus $f(G) = D$.

Theorem : Let $f(z)$ be analytic in a simply connected region R and suppose that $f(z)$ has no zeros in R. Then there is a analytic function $h(z)$ such that for all $z \in R$.

Proof : Since in R

then $\dfrac{f'(z)}{f(z)}$ is also analytic in R and therefore for any two points a and z in R the integral

....(1)

defines an analytic function of z in R

Let α be an argument of *f*(*a*) then *f*(*a*) can be expressed as

$f(a) = r\,e^{i\alpha}$ and set

$\beta = \log|\,f(a)\,| + i\alpha$

$e^{\beta} = e^{\log|f(a)|+i\alpha}$

$= e^{\log|\,f(a)\,|}\; e^{i\alpha} = e^{\log\, r}\; e^{i\alpha} = r\; e^{i\alpha}$

Then $e^{\beta} = f(a)$ ...(2)

and if we consider

$$h(z) = \beta + \int_a^z \frac{f'(\varsigma)}{f(\varsigma)}\, d\varsigma \quad ....(3)$$

We get $h(a) = \beta$, then from (2) we have $F'(z)\,f(z) - F(z)\,f'(z) = 0$ $h(z) = \int_a^z \frac{f'(\varsigma)}{f(\varsigma)}\, d\varsigma$ $h'(z) = \frac{f'(z)}{f(z)}$

$e^{h(a)} = e^{\beta} = f(a)$ ...(4)

$\Rightarrow$ h(z) is analytic in R for all $z \in R$

from (3)

Let $F(z) = e^{h(z)}$

then $F'(z) = e^{h(z)}\, h'(z) = F(z)\,\dfrac{f'(z)}{f(z)}$

$\Rightarrow \quad \dfrac{F'(z)}{F(z)} = \dfrac{f'(z)}{f(z)}$

$\Rightarrow$

$\Rightarrow \quad \dfrac{F'(z)\,f(z) - F(z)\,f'(z)}{[F(z)]^2} = 0$

$\because$ be non vanishing

$$\Rightarrow \qquad \frac{d}{dz}\left(\frac{f(z)}{F(z)}\right) = 0$$

$$\Rightarrow \qquad \frac{f(z)}{F(z)} = \kappa \quad \text{(a constant) for all } z \in R$$

Putting z =a, we obtain

$$\kappa = \frac{f(a)}{F(a)} = \frac{f(a)}{e^{h(a)}} = \frac{f(a)}{f(a)} = 1$$

$$\therefore \qquad f(z) = F(z) = e^{h(z)}$$

The analytic function $h(z)$ defined as above is called **logarithm of $f(z)$** in R and we write $h(z) = \log f(z)$.

Clearly, if $h(z)$ is a logarithm of $f(z)$, then $h(z)+2m\pi i$ (m an integer) is also logarithm of $f(z)$.

Convergence of logarithm series :

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$$

with radius of convergence 1.

If |z| < 1 then

$$\left|1 - \frac{\log(1+z)}{z}\right| = \left|1 - \left(1 - \frac{z}{2} + \frac{z^2}{3} - \ldots\right)\right|$$

$$= \left|\frac{z}{2} - \frac{1}{3}z^2 + \ldots.\right| \le \left(\frac{1}{2}|z| + \frac{1}{3}|z|^2 + \ldots.\right)$$

$$\le \frac{1}{2}(|z| + |z|^2 + \ldots..)$$

$$\le \frac{1}{2}\frac{|z|}{1-|z|} \qquad \ldots.(1)$$

Further if $|z| < \frac{1}{2}$ then

$$\left|1 - \frac{\log(1+z)}{z}\right| \le \frac{1}{2}$$

$$\Rightarrow \quad \left|\frac{z-\log(1+z)}{z}\right| \le \frac{1}{2}$$

$$\Rightarrow \quad |\log(1+z)| - |z| \le \frac{|z|}{2}$$

$$\Rightarrow \quad |\log(1+z)| \le \frac{3}{2}|z|$$

Since log (1+z) is converges to 1, then for $|z| < \frac{1}{2}$, we have

$$\frac{|z|}{2} \le |\log(1+z)| \le \frac{3}{2}|z| \qquad ....(2)$$

Proposition :

$\operatorname{Re} z_n > 0 \ \ \forall n \ge 1$. Then $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number iff. $\prod_{n=1}^{\infty} \log z_n$ converges.

Proof : Let $p_n = (z_1 \cdot z_2 \cdot z_3 \cdot .... \cdot z_n)$

$\lim_{n\to\infty} p_n = z = re^{i\theta}, \ k_n = k, \ n \ge n_0, \ -\pi < \theta < \pi$

and $\quad \ln(p_n) = \log|p_n| + i\theta_n$, $\qquad \theta - \pi < \theta_n < \theta + \pi$

If $\quad S_n = \log z_1 + \log z_2 + ... + \log z_n$

then $\quad \exp(S_n) = \exp[\log z_1 + \log z_2 + ... + \log z_n]$

$$= \exp[\log(z_1 . z_2 .... z_n)] = p_n$$

$\therefore \quad S_n = \ln(p_n) + 2\pi i k_n$ for some integer k_n.

Now suppose

, then

$$S_n - S_{n-1} = \log z_n \to 0$$

Also $\quad \ln(p_n) - \ln(p_{n-1}) \to 0$

Hence $(k_n - k_{n-1}) \to 0$ as $n \to \infty$

Since k_n is an integer, there exists integers n_0 and k such that

for

So $S_n \to \ln(z) + 2\pi i K$

Corollary :

If $\operatorname{Re} z_n > 0$ then the product. $\prod z_n$ converges absolutely iff the series $\sum (z_n - 1)$ converges absolutely.

Lemma :

Let X be a set and f_1, f_2, f_3 ... be functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for x in X. If there is a constant 'a' such that $\operatorname{Re} f(x) \leq a \;\; \forall x \in X$ then $\exp f_n(x) \to \exp f(x)$ uniformly for x in X.

Proof : For given $\varepsilon > 0$ choose such that

whenever $|z| < \delta$ ....(1)

Now choose n_0 such that

$|f_n(x) - f(x)| < \delta \quad \forall x \in X$ whenever $n \geq n_0$ $\varepsilon e^{-a} |\exp f(x)| \leq \varepsilon$

Thus, from (1)

$$\Rightarrow \quad \left| \frac{\exp f_n(x)}{\exp f(x)} - 1 \right| < \varepsilon e^{-a}$$

$$\Rightarrow \quad |\exp f_n(x) - \exp f(x)| < \varepsilon e^{-a} \, |\exp f(x)|$$

It follow that for any $x \in X$ and

$$\Rightarrow \quad \exp f_n(x) \to \exp f(x) \text{ uniformly for } x \in X.$$

Lemma :

Let (X, d) be a compact metric space and let be a sequence of continuous function from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in

X.

Then the product $f(x) = \prod_{n=1}^{\infty} (1 + g_n(x))$ converges absolutely and uniformly for x in X. Also there is an integer n_0 such that $f(x) = 0$ *iff* $g_n(x) = -1$ for some $n,\ 1 \le n \le n_0$

Proof : Since $\sum g_n(x)$ converges uniformly for $x \in X$ therefor their exists an integer n_0 such that and $n \ge n_0$

$\Rightarrow$ and $|\log(1+g_n(x))| \le \frac{3}{2}|g_n(x)|;\quad \forall\, n \ge n_0$ and $x \in X$

Thus $h(x) = \sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))$ converges uniformly for $x \in X$

Now

$\therefore$ is also continuous.

Hence $h(x)$ is continuous.

$\because$ X is compact $\Rightarrow h(x)$ must be bounded. $|g_n(x)| < \frac{\quad}{2}\ \forall\ x \in X$ $[1+g_2(x)]......[1+g_n(x)].\exp h(x)$

In particular, there is a constant a such that

Then

$$\exp h(x) = \exp\left[\sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))\right]$$

$$= \prod_{n=n_0+1}^{\infty}(1+g_n(x)),$$ converges uniformly for $x \in X$ and

for any $x \in X$.

Now if we take

then

$$f(x) = \prod_{n=1}^{\infty}(1+g_n(x)),$$

converges uniformly for $x \in X$.

Definition : An **Elementary factor** is one of the following function or $E(z;\, p)$

for p = 0, 1, 2, 3,... defined as

$$E(z;0) \text{ or } E_0(z) = 1 - z$$

$$E(z;\, p) \text{ or } E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right),\ \ p \geq 1$$

$$E(\frac{z}{a};\ p) \text{ or } E_p\left(\frac{z}{a}\right) = \left(1 - \frac{z}{a}\right)\exp\left(\frac{z}{a} + \frac{1}{2}\left(\frac{z}{a}\right)^2 + ...\frac{1}{p}\left(\frac{z}{a}\right)^p\right),\ \ p \geq 1$$

$\Rightarrow$ $E_p\left(\frac{z}{a}\right)$ has a simple zero at $z = a$ and no other zero.

Also if b is a point in $\mathbb{C} - G$ then

$$E_p\left(\frac{a-b}{z-b}\right) = \left(1 - \frac{a-b}{z-b}\right)\exp\left[\frac{a-b}{z-b} + \frac{1}{2}\left(\frac{a-b}{z-b}\right)^2 + ... + \frac{1}{p}\left(\frac{a-b}{z-b}\right)^p\right]$$

has a simple zero at $z = a$, and is analytic in G.

Lemma

If $|z| \leq 1$ and $p \geq 0$ then $|1 - E_p(z)| \leq |z|^{p+1}$

Proof : We restrict for $p \geq 1$.

For a fixed p let the power series expansion of $E_p(z)$ about z =0 is

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \qquad ....(1)$$

differentiating (1), we have

$$E'_p(z) = 1 + \sum_{k=1}^{\infty} k\, a_k z^{k-1}$$

From the definition of $E_p(z)$

$$E_p'(z) = -1\exp\left(z+\frac{z^2}{2}+....+\frac{z^p}{p}\right)+(1-z)\exp\left(z+\frac{z^2}{2}+....+\frac{z^p}{p}\right)(1+z+z^2+....+z^{p-1})$$

$$=\exp\left(z+\frac{z^2}{2}+....+\frac{z^p}{p}\right)\left[-1+(1-z)(1+z+z^2+.....z^{p-1})\right]$$

On simplifying expressiion in the square bracket only z^p remains and all other terms will cancelled out.

$$E_p' = -z^p \exp\left(z+\frac{z^2}{2}+....+\frac{z^p}{p}\right)$$

$$\therefore \quad \sum_{k=1}^{\infty} k\, a_k\, z^{k-1} = -z^p \exp\left(z+\frac{z^2}{2}+.....+\frac{z^p}{p}\right)$$

On comparing coefficients of like power of z, we find that

$$a_1 = a_2 = ... = a_p = 0$$

Also the coefficient of the expression of $\exp\left(z+\frac{z^2}{2}+.....+\frac{z^p}{p}\right)$ are all positive

, therefore $\quad a_k \le 0$ for $k \ge p+1$

Thus $|a_k| = -a_k$ for $k \ge p+1$

$$\Rightarrow \quad E_p(1) = 0 = 1 + \sum_{k=p+1}^{\infty} a_n$$

or $\quad \sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1$

Hence for $|z| \le 1$

$$|E_p(z)-1| = \left|\sum_{k=p+1}^{\infty} a_k z^k\right|$$

$$= |z|^{p+1}\left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right|$$

$$\le |z|^{p+1}\left|\sum_{k=p+1}^{\infty} |a_k|\right|$$

$$= |z|^{p+1}$$

Proposition

Let $\operatorname{Re} z_n > -1$ then the series $\sum \log(1+z_n)$ converges absolutely iff $\sum z_n$ converges absolutely.

Proof : If $\sum |z_n|$ converges then $z_n \to 0$

$\Rightarrow \quad |z_n| < \frac{1}{2}$

Since $\frac{1}{2}|z| \le |\log(1+z)| \le \frac{3}{2}|z|$ for $|z| < \frac{1}{2}$

Therefore for $|z_n| < \frac{1}{2}$ the series $\sum |\log(1+z_n)|$ is convergent.

Conversely if, $\sum |\log(1+z_n)|$ converges then

$|z_n| < \frac{1}{2}$

$\therefore \quad \sum |z_n|$ is converges

Absolute convergence of an infinite product.

$$\prod |z_n| \text{ converges } \not\Rightarrow \prod z_n \text{ converges}$$

Example: Let $Z_n = -1 \ \forall \ n$ $\qquad \prod z_n$

then $\quad |z_n| = 1 \ \ \forall \ n$

$\Rightarrow \quad \prod |z_n|$ converges to 1.

However $\prod_{k=1}^{n} z_k$ is ± 1 depending on whether n is even or odd,

$\Rightarrow$ does not converges.

Definition : If $\operatorname{Re} z_n > 0$, $\forall n$ then the infinite product $\prod z_n$ is said to converges absolutely if the series $\sum \log z_n$ converges absolutely.

Theorem : Weirestrass factorization theorem

Given an infinite sequence of complex numbers $a_0 = 0, a_1, a_2, ..., a_n$ with no finite point of accumulation, the most general entire function having zeros at those points only (a zero a_n $n \ge 1$ of multiplicity α being repeated α times in the sequence) is given by

$$F(z) = e^{h(z)} z^m \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k_n\right) \qquad ...(1)$$

where $h(z)$ is an arbitrary entire function, $m \geq 0$ is the order of multiplicity of $a_0 = 0$, and the k_n are non negative integers such that the series

$$\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^{k_n+1} \qquad ...(2)$$

Converges for each finite value of z.

Proof : Arranging the given zeros in increasing order of modulus, so that

$$0 < |a_1| \leq |a_2| \leq |a_3| \leq ....$$

with $|a_n| \to \infty$

The sequence $\{k_n\}$ of non-negative integers can always be found such that the series (2) is convergent for each z.

We may take $k_{n+1} = n$, since

$$\left|\frac{z}{a_n}\right|^n \leq \frac{1}{2^n} \quad \text{or} \quad \left|\frac{z}{a_n}\right| \leq \frac{1}{2} \qquad ...(3)$$

as soon as $|a_n| \geq 2|z|$ which for any given z holds for sufficiently large value of n , say $n>N$

$\Rightarrow$ N depends on |z| $\Rightarrow$ pointwise convergence.

Since we are not interested in uniform convergence, only point wise convergence.

From the already proved result that

if $|z| \leq \frac{1}{p} < 1$ then

$$|\log E(z;k)| \leq \frac{p}{p-1} |z|^{k+1} \qquad ...(4)$$

then for p =2

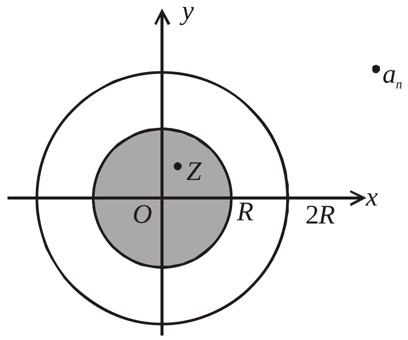
$$|\log E(z;k)| \leq 2|z|^{k+1}$$

provided

$$|z| \leq \frac{1}{2}$$

We have

$$\left|\log E\left(\frac{z}{a_n}; k\right)\right| \leq 2\left|\frac{z}{a_n}\right|^{k+1}$$



provided

$$\left|\frac{z}{a_n}\right| \leq \frac{1}{2}$$

$\Rightarrow \quad |a_n| \geq 2|z|$

Let R be an arbitrary positive number and consider two circles with centres at the origin and radii R and 2R.

If we take |z| < R and n large enough so that $|a_n| > 2R$, we have $|a_n| > 2|z|$

Hence the series

$$\sum_{|a_n|>2R} \log E\left(\frac{z}{a_n}; k_n\right)$$

is absolutely and uniformly convergent for all z, such that |z| < *R*, and it follows that the product

$$\prod_{|a_n|>2R} E\left(\frac{z}{a_n}; k_n\right)$$

is also absolutely and uniformly convergent for |*z*| < *R*.

Since the product

$$\prod_{|a_n|\leq 2R} E\left(\frac{z}{a_n}; k_n\right)$$

contain a finite number of factors each of which is an analytic function, it follows that

$$f_1(z) = \prod_{|a_n|\leq 2R} E\left(\frac{z}{a_n}; k_n\right)$$

is analytic in |*z*| < *R* and vanishes in the disc only at those points of the sequence $a_1, a_2,$ ... which lie in |*z*| < *R*.

Then from the theorem : If $f_k(z); \quad k = 1, 2, \ldots$ are analytic in open set

$A \subset \mathbf{C}$, $\sum_{n=1}^{\infty} | f_k(z) |$ is uniformly convergent on every compact set $K \subset A$, then

converges absolutely in A to F(z) which is analytic in A.

We have

$$f_2(z) = \prod_{|a_n| > 2R} E\left(\frac{z}{a_n}; k_n\right)$$

is analytic and different from zero in |z| < R.

Hence the product

$$f(z) = f_1(z) f_2(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k_n\right)$$

is also analytic in |z| <R and vanishes in this region only at those points of the sequence $a_1, a_2 ....$ which lie in there.

R was arbitrary chosen positive number, so that *f*(*z*) is analytic in the whole finite place.

$$\prod_{k=1}^{\infty} (1 + f_k(z))$$

$$\sin \pi z = e^{g(z)} z \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}; \qquad n \neq 0$$

$\Rightarrow$ It is an entire function and has less zero precisely at the points $a_1, a_2 ...., a_n ...$

$\therefore$ $F(z) = e^{h(z)} z^m f(z)$

is most general entire function with the prescribed zeros.

Factorization of sinπz :

$\sin \pi z$ is an entire function with simple zeros at $0, \pm 1, \pm 2, ...$

Then by Weierstrass factorization theorem the most general form of this entire function be

$\therefore$

$$= e^{g(z)} z . \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n}\right) e^{-z/n} \left(1 + \frac{z}{n}\right) e^{z/n} \right]$$

$$= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \text{...(1)}$$

Taking logarithmic differentiation of (1)

$$\pi \cot \pi z = \frac{d}{dz}\left[g(z) + \log z + \prod_{n=1}^{\infty} \left(\log(n^2 - z^2) - \log n^2 \right) \right]$$

$$= h'(z) + \frac{1}{z} + \prod_{n=1}^{\infty} \left(\frac{2z}{z^2 - n^2} \right) \qquad z \neq \pm n \qquad \text{...(2)}$$

But ...(3)

Hence $h'(z) = 0$ and it follows that $h(z) = C$ (a constant)

Let $e^C = C_1$ then (1) gives

$$\sin \pi z = C_1 z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{\sin \pi z}{\pi z} = \frac{C_1}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right); \qquad z \neq 0$$

$$\pi \cot \pi z = \frac{1}{z} + \prod_{n=1}^{\infty} \left(\frac{2z}{z^2 - n^2} \right)$$

Taking $z \to 0$

$$1 = \frac{C_1}{\pi} \lim_{z \to 0} \lim_{n \to \infty} \prod_{k=1}^{n} \left(1 - \frac{k^2}{n^2}\right)$$

Since $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ is uniformly converges, therefore the order of limit can be changed

$$= \frac{C_1}{\pi} \lim_{n \to \infty} \lim_{z \to 0} \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right)$$

$$\therefore \quad 1 = \frac{C_1}{\pi} \lim_{n \to \infty} \prod_{K=1}^{n} (1) = \frac{C_1}{\pi}$$

$$\Rightarrow \quad C_1 = \pi$$

$$\Rightarrow \quad \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Gamma function

According to Weierstrass (the weierstrass fact. th) the Γ function can be defined as the reciprocal of a particular entire function with simple zeros at the points 0, -1, -2,.. namely

$$F(z) = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

$$\Rightarrow \quad \frac{1}{F(z)} = \Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n} \qquad ...(1)$$

where the constant γ (Euler or Mascheroni constant) is chosen so that $\Gamma(1)=1$

From (1) we see that $\Gamma(z)$ is a meromorphic function on C with simple poles at z =0, -1, -2,...

Existence of γ

Now we show that there exists such γ.

substituting z =1 in the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$, we get

$$\Gamma(1) = e^{-1\log C}\cdot\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1}e^{1/n}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1}e^{1/n} = C \quad \text{a finite positive number} \qquad ...(2)$$

Let $\gamma = \log C$ then on substituting z =1 in (1), we get

$$= \frac{1}{C}.C = 1$$

$\therefore$

From (2), we see that the constant γ satisfies the equation

$$...(3)$$

Hence the existance of γ is acertained.

Lemma :

Let $H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}$ then $\gamma = \lim\limits_{n\to\infty}(H_n - \log n)$

Proof : Taking logarithm on both sides of

$$e^{\gamma} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{-1} e^{1/k}$$

We have

$$\gamma = \sum_{k=1}^{\infty}\log\left[\left(1+\frac{1}{k}\right)^{-1} e^{1/k}\right]$$

$$= \sum_{k=1}^{\infty}\log\left[\left(\frac{k}{k+1}\right)e^{1/k}\right]$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left[\log k - \log(k+1) + \frac{1}{k}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\underbrace{\{\log k - \log(k+1)\}}_{\left\{\begin{array}{l}\log 1 = 0 \textit{ and all intermediate} \\ \textit{terms will cancelled out except} \\ \log(n+1)\end{array}\right\}} + \sum_{k=1}^{n}\frac{1}{k}\right]$$

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!.n^{z}}{z(z+1)(z+2).....(z+n)}$$

$$= \lim_{n\to\infty}\left[-\log(n+1) + \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\log n - \log(n+1) + \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \log n\right]$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \log n - \log\left(\frac{(n+1)}{n}\right)\right]$$

$$\Rightarrow \quad \gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \log n - 0\right] \qquad \left[\because \lim_{n\to\infty}\log\left(\frac{(n+1)}{n}\right) = 0\right]$$

$$\Rightarrow \quad \gamma = \lim_{n\to\infty}(H_n - \log n)$$

Euler's formula for $\Gamma(z)$ or Gauss formula

From the definition of $\Gamma(z)$

$$= \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

$$= \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k e^{z/k}}{(z+k)}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}(1.2.3....n).\exp\left(z+\frac{z}{2}+\frac{z}{3}+....+\frac{z}{n}\right)}{z(z+1)(z+2).....(z+n)}$$

$$= \lim_{n\to\infty} \frac{e^{-z(H_n-\log n)}.n!.\exp\left\{z\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}\right)\right\}}{z(z+1)(z+2).....(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!.e^{-zH_n}.e^{-z\log n}.e^{zH_n}}{z(z+1)(z+2).....(z+n)}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!.n^z}{z(z+1)(z+2).....(z+n)}$$

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!.n^{z+1}}{(z+1)(z+2).....(z+n+1)} \quad ...(5)$$

Functional equation $\Gamma(z+1) = z\Gamma(z)$

Replacing z by (z+1) in the equation (5)

$$= \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2).....(z+n)} \frac{n\, z}{(z+n+1)}$$

$$= \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2).....(z+n)} \lim_{n\to\infty} \frac{z}{(\frac{z+1}{n}+1)}$$

$$= \Gamma(z)\,.\,z$$

Reflection formula

Proof : Using Euler's formula

$$= \lim_{n\to\infty}\left[\frac{(n!)^2 .n}{z(1-z^2)(2^2-z^2).....(n^2-z^2)(n+1-z)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{z(1-z^2)\left(1-\frac{z^2}{2^2}\right).....\left(1-\frac{z^2}{n^2}\right)\left(\frac{n+1-z}{n}\right)}\right]$$

$$= \frac{1}{z\lim_{n\to\infty}\left\{(1-z^2)\left(1-\frac{z^2}{2^2}\right).....\left(1-\frac{z^2}{n^2}\right)\right\}.1}$$

$$= \frac{1}{z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}$$

$$\frac{1}{n^s} = e^{-s\log n}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \quad z \neq 0, \pm 1, \pm 2, \ldots$$

$$\Gamma(z)\,\Gamma(1-z) = \lim_{n\to\infty}\left[\frac{n!.n^z}{z(z+1)(z+2).....(z+n)}\right].\lim_{n\to\infty}\left[\frac{n!.n}{(1-z)(2-z}\right.$$

$$= \frac{1}{z.\frac{\sin \pi z}{\pi z}}$$

$$= \frac{\pi}{\sin \pi z}$$

Ordinary Dirichlet Series

Definition : The series of the form

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} \qquad ....(1)$$

called ordinary Dirichlet series, where the a_n are given constants, $s = \sigma + it$ is a complex variable and .

Zeta function of Riemann

If $a_n = 1, \forall n$ then the series (1) becomes

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s = \sigma > 1$$

represents Zeta function of Riemann.

Theorem : The series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely and uniformly for $\sigma \geq 1+\varepsilon$ ($\varepsilon > 0$ arbitrary)

Proof : We have

$$\left|\frac{1}{n^s}\right| = \frac{1}{|n^{\sigma+it}|} = \frac{1}{|n^{\sigma}||n^{it}|} = \frac{1}{n^{\sigma}} \leq \frac{1}{n^{1+\varepsilon}}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$ converges, by Weiestrass M-test

Theorem : For $\operatorname{Re} s > 1$

$$\Gamma(s)\ \varsigma(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Proof : The integral $\int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$ converges at both the lower and upper limits whenever $\operatorname{Re} s > 1$

Since

$$\lim_{t \to 0+} \frac{t}{e^t - 1} = 1$$

so that by the definition of limit there is a $\delta > 0$ such that for $0 < t \leq \delta$ the inequality

$$\left|\frac{t}{e^t - 1} - 1\right| < \varepsilon = \frac{1}{2}$$

$$\Rightarrow \quad \left|\frac{t}{e^t - 1}\right| - 1 < \frac{1}{2}$$

$$\Rightarrow \quad \left|\frac{t}{e^t-1}\right| < \frac{3}{2} \text{ holds.}$$

Hence for $0 < \delta_1 < \delta$ and $\sigma = \operatorname{Re} s > 1$

$$\int_{\delta_1}^{\delta}\left|\frac{t}{e^t-1}\right| t^{\sigma-2}dt \le \frac{3}{2}\int_{\delta_1}^{\delta} t^{\sigma-2}dt = \frac{3}{2}\frac{1}{\sigma-1}\left(\delta^{\sigma-1}-\delta_1^{\sigma-1}\right)$$

$$\Rightarrow \int_{\delta_1}^{\delta}\left|\frac{t}{e^t-1}\right| t^{\sigma-2}dt \to \frac{3}{2}\frac{\delta^{\sigma-1}}{\sigma-1} \ as \ \delta_1 \to 0$$

on the other hand, we have

$$\lim_{t\to+\infty}\frac{t^m}{e^t-1} = 0$$

So that there is a b_1, such that for $t \ge b_1$ the ineuqality $\left|\frac{t^m}{e^t-1}\right| < \frac{1}{2}$ is satisfied

Hence for $0 < b_1 < b$ and choosing $m > \sigma$ we get

$$\int_{b_1}^{b}\left|\frac{t^m}{e^t-1}\right| t^{\sigma-m-1}dt \le \frac{1}{2}\int_{b_1}^{b} t^{\sigma-m-1}dt = \frac{1}{2}\frac{1}{\sigma-m}\left(b^{\sigma-m}-b_1^{\sigma-m}\right)$$

$$\to \frac{1}{2}\frac{1}{m-\sigma}b_1^{\sigma-m} \ as \ b \to \infty$$

Now in $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$

which is valid for $\sigma > 0$, we make the substitution $t = nu$ to obtain

$$\Gamma(s) = n^s\int_0^\infty e^{-nu}u^{s-1}du$$

and $$\Gamma(s)\sum_{n=1}^{N}\frac{1}{n^s} = \int_0^\infty\left(\underbrace{\sum_{n=1}^{N}e^{-nu}}_{(sum\ of\ G.P)}\right)u^{s-1}du$$

$$= \int_0^\infty\left(\frac{e^{-u}(1-e^{-Nu})}{(1-e^{-u})}\right)u^{s-1}du$$

$$= \int_0^\infty\frac{u^{s-1}}{e^u-1}du - \int_0^\infty\frac{u^{s-1}e^{-Nu}}{e^u-1}du$$

Now it is required to prove that

$$\int_0^\infty \frac{u^{s-1}e^{-Nu}}{e^u-1}\,du \quad \to 0 \ \ as \ \ N \to \infty$$

$$\int_0^\infty \frac{u^{s-1}e^{-Nu}}{e^u-1}\,du = \int_0^\delta \frac{u^{s-1}e^{-Nu}}{e^u-1}\,du + \int_\delta^\infty \frac{u^{s-1}e^{-Nu}}{e^u-1}\,du; \qquad \delta > 0$$

$$\Rightarrow \qquad \left| \int_0^\infty \frac{u^{s-1}e^{-Nu}}{e^u-1}\,du \right| < \int_0^\delta \frac{u^{\sigma-1}}{e^u-1}\,du + e^{-N\delta}\int_\delta^\infty \frac{u^{\sigma-1}}{e^u-1}\,du$$

For a given ε >0, we choose δ sufficiently small so as to make the first integral on right hand side less than $\frac{\varepsilon}{2}$.

Fixing that δ we can now take N large enough so as to make the second term on the right less than

$\therefore$ for $N \to \infty$

$$\Gamma(s)\,\varsigma(s) = \int_0^\infty \frac{u^{s-1}}{e^u-1}\,du \qquad\qquad \text{valid for Re s >1}$$

Note:B(E) denotes a closed algebra of $C(K,D)$ that contains every rational function with a pole in E

$|b-a|<r\,d(a,K)$ $\frac{\varepsilon}{2}$

Lemma

If $a \in \mathbf{C}-k$ then $(z-a)^{-1} \in B(E)$

Proof : Case I : $\infty \notin E$

Let and let

$$V = \{a \in \mathbf{C} : (z-a)^{-1} \in B(E)\}$$

so $E \subset V \subset U$

If and then ...(1)

$\Rightarrow$ V is open

$\Rightarrow$ a number such that

$$|b-a| < r\,|z-a|; \qquad \forall\, z \in K. \qquad\qquad ...(2)$$

But

$$= (Z-1)^{-1}\left[1-\frac{b-a}{Z-a}\right]^{-1} \qquad \text{...(3)}$$

From (2)

$|b-a|\,|z-a|^{-1} < r < 1; \quad \forall z \in K$ which implies that

$$\left[1-\frac{b-a}{z-a}\right]^{-1} = \sum_{n=0}^{\infty}\left(\frac{b-a}{z-a}\right)^{n} \qquad \text{...(4)}$$

By the Weirestrass M-Test series on right hand side of (4) converges uniformly on K.

If $\quad Q_n(z) = \sum_{k=0}^{n}\left(\frac{b-a}{z-a}\right)^{k}$ then

$$(z-a)^{-1}Q_n(z) \in B(E)$$

Since $a \in V$ and B(E) is an algebra (3) shows that B(E) is closed and uniformly convergence of (4) implies that $(z-b)^{-1} \in B(E)$

$$(z-b)^{-1} = \left[\frac{(z-a)-(b-a)}{z-a}\right]^{-1}(z-a)^{-1}$$

$\Rightarrow \quad b \in V$

Then (1) implies that V is open

If $b \in \partial V$ then let a_n be a sequence in V with

$$b = \lim_{n\to\infty} a_n$$

Since $b \notin V$ then from (1) it follows that

$|b-a_n| \geq d(a_n, K)$ then for $n \to \infty; \quad a_n \to b;$ we get

$$0 = d(b, K) \quad or \quad b \in K$$

Thus $\partial V \cap U = \phi \ (null \ \ set)$

If H is a component of $U = \mathbf{C} - K$ then $H \cap K \neq \phi$

So $\quad H \cap V \neq \phi$

$\therefore \quad H \subset V$

But H was arbitrary so or V = U

Case 2 :

Let d = the metric on

Let a_0 is in the unbounded components of $U = \mathbf{C} - K$ such that

$$d(a_0, \infty) \le \frac{1}{2} d(\infty, K)$$

and $|a_0| > 2\max\{|z|: \quad z \in K\}$ ...(5)

Let so E_0 meets each components of

If $a \in \mathbf{C} - K$ then case-I gives that $(z-a)^{-1} \in B(E_0)$

Now $|z/a_0| \le \frac{1}{2}$; $\forall z \in K$ [from (5)]

Therefore

$$\frac{1}{z-a_0} = \frac{1}{-a_0(1 - z/a_0)} = -\frac{1}{a_0}\sum_{n=0}^{\infty} (z/a_0)^n$$

$(E - \{\infty\})\, U\, \{a_0\}$

Converges uniformly on K.

Then

$$Q_n(z) = -\frac{1}{a_0}\sum_{k=0}^{n} (z/a_0)^k \text{ is a polynomial}$$

and $(z-a_0)^{-1} = u - \lim Q_n$ on K.

Since Q_n has its only pole at ∞ ,

Thus $(z-a_0)^{-1} \in B(E)$

$\Rightarrow \quad B(E_0) \subset B(E)$

$\Rightarrow \quad (z-a)^{-1} \in B(E)$ for each $a \in \mathbf{C} - K$

Definition : Function element is a pair (*f*,*G*) where *G* is a region and *f* is an analytic function on *G*.

Definition : Germ of *f* at *a* : $[f]_a$ **.**

For a given function element (*f*, *G*) the germ of *f* at *a*, denoted by $[f]_a$, is the collection of all function elements (g, D) such that $a \in D$ and in a neighbourhood of a.

$\Rightarrow [f]_a$ = collection of function elements and it is not a function element itself.

Note: The equivalence of two germs of f i.e. $[f]_a = [f]_b$ is meaning less until $a = b$.

Definition : Analytic continuation along a path.

Let $\gamma : [0,1] \to \mathbf{C}$ be a path and suppose that for each t in [0,1] there is a function element (f_t, D_t) such that

(a) $\gamma(t) \in D$

(b) for each t in [0,1] there is a $\delta > 0$ such that $|s - t| < \delta$

$\Rightarrow$ $\gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$ (i.e. germ of f_s and f_t at $\gamma(s)$ are equal)

Then (f, D) is the analytic continuous of (f_0, D_0) along the path γ; or (f_1, D_1) is obtained from (f_0, D_0) by analytic continuation along γ.

Power series method of analytic continuation

$f(z) = g(z) \ \forall z$ $\quad R(t) \equiv \infty$

Lemma : Let $\gamma : [0,1] \to \mathbf{C}$ be a path and let $\{(f_t, D_t) : 0 \le t \le 1\}$ be an analytic continuation along γ. For $0 \le t \le 1$ let R(t) be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either or $R : [0,1] \to (0, \infty)$ is continuous.

Proof : If $R(t) = \infty$ for some value of t then it is possible to extend f_t to an entire function.

Since

$$f_s(z) = f_t(z) \quad \forall z \in D_s$$

$\Rightarrow$ $R(s) = \infty \quad \forall s \in [0,1]$

i.e. $R(s) \equiv \infty$

Suppose that $R(t) < \infty \ \forall t$

For a particular $t \in [0,1]$ let $\tau = \gamma(t)$

let

be the power series expansion of f_t about τ.

Let $\delta_1 > 0$ be such that $|s-t| < \delta_1$

$\Rightarrow \quad \gamma(s) \in D_t \cap B(\tau; R(t))$

and $\quad [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$

Now let $\sigma = \gamma(s)$ for a fix s;

Now f_t can be extended to an analytic function on $B(\tau; R(t))$.

Since f_s agrees with f_t on a neighbourhood of σ (by definition germs), f_s can be extended so that it is also analytic on .

If f_s has a power series expansion

$$f_s(z) = \sum_{n=0}^{\infty} \sigma_n (z-\sigma)^n \quad \text{about } z = \sigma$$

then the radius of convergence R(s) must be at least as big as the distance from σ to the circle

$$\ldots \geq R(t) - |\tau - \sigma|$$

$$f_t(z) = \sum_{n=0} \tau_n (z-\tau)^n$$

i.e.

$\Rightarrow \quad R(t) - R(s) \leq |\gamma(t) - \gamma(s)|$

Similarly it can be shown that

Hence for

Since $\gamma : [0,1] \to \mathbf{C}$ is continuous.

$\Rightarrow \quad$ R must be continuous at t.

Rung's Theorem : Let K be a compact subset of **C** and let E be a subset of $\mathbf{C}_\infty - K$ that meets each component of $\mathbf{C}_\infty - K$. If *f* is analytic in an open set containing K and ε>0 then there is a rational function R(Z) whose only poles lie in E and such that

$$|f(z) - R(z)| < \varepsilon \qquad \forall\, z \in K$$

Proof : By the fact that if K be a compact subset of the region G, then there are straight line segments $\gamma_1, \gamma_2, ...\gamma_n$ in $G - K$such that for every function *f* in H(G).

$$f(z) = \sum_{K=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw \quad \forall z \in K$$

The line segments form a finite number of closed polygons.

Also If γ be a rectifiable curve and let K be compact set such that $K \cap \{\gamma\} = \phi$. If f is continuous function on $\{\gamma\}$ and ε>0 then there is a rational function R(z) having all its poles on and such that

Definition : Unrestricted analytic continuation :

Let (*f, D*) be a function element and let *G* be a region which contains *D*: then (*f, D*) admits unrestricted analytic continuation in *G* if for any path γ in G with initial point in *D* there is an analytic continuation of (*f, D*) along γ.

$$\left| \int_{\gamma} \frac{f(w)}{w-z} dw - R(z) \right|; \quad \forall z \in K$$

Fixed-End-Point (FEP) homotopic

Definition : If $\gamma_0, \gamma_1 : [0,1] \to G$ are two rectifiable course in G such that $\gamma_0(0) = \gamma_1(0) = 0$ and $\gamma_0(0) = \gamma_1(0) = b$ then γ_0 and γ_1 are FEP homotopic if there is a continuous map

$\Gamma : I^2 \to G$ such that

$\Gamma(s,0) = \gamma_0(s)$

$\Gamma(s,1) = \gamma_1(s)$

$\Gamma(0,t) = a$

$\Gamma(1,t) = b$

for $0 \le s,\ t \le 1$

$I^2 = [0,1] \times [0,1]$

Theorem : Monodromy : Let (f, D) be a function element and let G be a region containing D such that (f, D) having unrestricted continuation in G. Let $a \in D, b \in G$ and let γ_0 and γ_1 be paths in G from a to b; let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, D_t) : 0 \le t \le 1\}$ be analytic continuation of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed-end-point homotrpic in G then

$$[f_1]_b = [g_1]_b$$

Proof : Since γ_0 and γ_1 are FEP homotopic in G

$\Rightarrow$ there is a continuous function

$\Gamma : [0,1] \times [0,1] \to G$ such that

$$\Gamma(t,0) = \gamma_0(t) \quad \Gamma(t,1) = \gamma_1(t)$$

$$\Gamma(0,u) = a; \quad \Gamma(1,u) = b$$

for all t and u in [0, 1]

Let $\gamma_u(t) = \Gamma(t,u)$ for a fix u, $0 \le u \le 1$, from a to b. $[f_1]_b = [h_{1,0}]_b$ ....(1)

By hypothesis there is an analytic continuation $\{(h_{t,u}, D_{t,u}) : 0 \le t \le 1\}$ of (f, D) along γ_u

By the result that if $\gamma : [0.1] \to C$ be a path from a to b and $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then

$$[f_1]_b = [g_1]_b$$

Then it follows that

$$[g_1]_b = [h_{1,1}]_b \text{ and}$$

Now it is sufficient to show that

...(2)

To show this

Let $U = \{u \in [0,1] : [h_{1,u}]_b = [h_{1,0}]_b\}$ ...(3)

and we show that U is non-empty open and closed subset of [0,1].

Since

Now it is remains to show that U is both open and closed.

Let us consider for

$u \in [0,1]$ $\exists$ a $\delta > 0$ such that if $|u - v| \leq \delta$

then $[h_{1,u}]_b = [h_{1,v}]_b$ ....(4)

For a fixed there is an ε > 0 such that if σ is any path from a to b with $|\gamma_u(t) - \sigma(t)| < \varepsilon$, $\forall\, t$ and if to any continuation of (f,D) along σ, then

$[h_{1,u}]_b = [K_1]_b$ ....(5)

Now Γ is a uniformly continuous function, so there is δ >0

such that if then

$|\gamma_u(t) - \gamma_v(t)| = |\Gamma(t,u) - \Gamma(t,v)| < \varepsilon \;\; \forall\; t.$ ....(6)

Suppose $u \in U$ and let δ > 0 be the number given in (4). By definition of U, , this implies U is open. If [illegible] and δ is as choosen above, then $\exists$ a such that

But form (4)

$[h_{1,u}]_b = [h_{1,v}]_b$; and

since

$\Rightarrow$

$\Rightarrow$ U is closed

Mean Value Theorem: Let be a harmonic function and let $\bar{B}(a;r)$ be a closed disk contained in G. If γ is the circle centred at a and of radius r i.e. $\gamma : |z - a| = r$ then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,d\theta$$

Proof : Let D be a disk such that

$\overline{B}(a;r) \subset D \subset G,$ and

Let f be analytic on D such that $u = \text{Re}f$

Then by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}dz$$

$$|z-a| = r$$

$$z = a + re^{i\theta}$$

$$dz = ir^{i\theta}$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} f(a+re^{i\theta})d\theta$$

$$f(a) = \frac{1}{2\pi}\int_{0}^{2\pi} f(a+re^{i\theta})d\theta$$

Then $\text{Re } f(a) = \frac{1}{2\pi}\int_{0}^{2\pi}\text{Re } f(a+re^{i\theta})d\theta$

$$A = \{Z \in G : u(z) = u(a)\}$$

$\Rightarrow \quad u(a) = \frac{1}{2\pi}\int_{0}^{2\pi} u(a+re^{i\theta})d\theta$

Mean Value Property (MVP)

Definition : A continuous function $u : G \to R$ has the MVP if whenever $\overline{B}(a;r) \subset G$

$$u(a) = \frac{1}{2\pi}\int_{0}^{2\pi} u(a+re^{i\theta})d\theta$$

Maximum Principle : Let G be a region and let u is a continuous real valued function on G with the MVP. If there is a point *a* in G such that $u(a) \geq u(z) \quad \forall z \in G,$ then u is constant function.

Proof : Let the set A be defined by

Since u is continuous, the set A is closed in G.

If let r be chosen such that $\overline{B}(Z_0; r) \subset G$.

Suppose $b \in B(z_0; r)$ such that $u(b) \neq u(a)$; then .

$\therefore$ By continuity in the neighbouhood of b

If $|z_0 - b| = \rho$ and $b = z_0 + \rho e^{i\theta}$ $0 \leq \theta \leq 2\pi$

then there is a proper interval I of [0, 2π] such that $\theta \in I$ and

Hence by MVP

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\phi})d\phi < u(z_0)$$

which is a contradiction.

So $B(z_0; r) \subset A$ and A is also open.

by connectedness of G, A = G.

Harmonic function on a disk : $u(b) < u(a)$ $u(z) < u(z_0)$ $z \in A$ $u(z_0 + \rho e^{i\phi}) < u(z_0)$ $\forall \phi \in I$

$$P_r(\theta) = \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

Study of harmonic function on a disk, open unit disk {z ; |z| < 1 }

Definition : Poisson Kernel : The function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$$

for $0 \leq r < 1$ and $-\infty < \theta < \infty$ is called the **Poisson kernel**.

Theorem :

Let $z = re^{i\theta}$, $0 \leq r < 1$ then

$$\Rightarrow \quad \frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{1+z}{1-z} = (1+z)(1-z)^{-1}$$

$$= (1+z)(1+z+z^2+....)$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}$$

$$\Rightarrow \quad \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n \frac{(e^{in\theta} + e^{-in\theta})}{2}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

$$= P_r(\theta) \qquad \text{....(1)}$$

Also

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{(1+re^{i\theta})(1-re^{-i\theta})}{|1-re^{i\theta}|^2} = \frac{1+re^{i\theta} - re^{-i\theta} - r^2}{|1-re^{i\theta}|^2}$$

$$= \frac{(1-r^2) + r(i2\sin\theta)}{|1-re^{i\theta}|^2}$$

$$\Rightarrow \quad \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta + r^2} \qquad \text{....(2)}$$

$$[\because |1-re^{i\theta}|^2 = 1 - 2r\cos\theta + r^2]$$

Combining (1) and (2), we get the result.

Prop. 2.3

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$

(b) $P_r(\theta) > 0 \ \forall\theta,\ P_r(-\theta) = P_r(\theta)$ and Pr is periodic in θ with period 2π.

Theorem : Let $D = \{z : |z| < 1\}$ and suppose that $f : \partial D \to R$ is a continuous function. Then there is a continuous function $u : D^- \to R$ such that

(a) $u(z) = f(z) \ \ \forall \ z \in \partial D$

(b) *u* is harmonic in *D*.

Moreover *u* is unique and is defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt \quad \text{for } 0 \le r < 1,\ 0 \le \theta \le 2\pi$$

Proof : Define $u : \overline{D} \to R$ as

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt \qquad \text{for } \ 0 \le r < 1 \qquad ...(1)$$

and let $u(e^{i\theta}) = f(e^{i\theta})$ ...(2)

Then u satisfies part (a).

Now we have to show that u is continuous on D and harmonic in D.

(i) Proving u is harmonic in D.

If $0 \le r < 1$ then from (1)

$$u(re^{i\theta}) = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} \text{Re}\left[\frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}}\right]}_{\text{by definition of } P_r(\theta)} f(e^{it})dt$$

$$= \text{Re}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}}\right]dt\right\}$$

$$= \text{Re}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\right]dt\right\} \qquad ...(3)$$

∂D $e^{i\alpha}$ $\varepsilon > 0$ $\exists$ a ρ, $0 < \rho < 1$

Let us define $g : D \to \mathbf{C}$ by

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{e^{it} + z}{e^{it} - z}\right]dt \qquad ...(4)$$

Then g is analytic and

Re $g = u$ ...(5)

$\therefore \quad \nabla^2 u = 0 \Rightarrow u$ is harmonic.

(ii) Continuity of *u* on D$^-$

Since u is harmonic on D therefore it remains to prove that u is continuous at each point of the boundary of D.

If $\alpha \in [-\pi, \pi]$ and and an arc A of about such that for $\rho < r < 1$ and $e^{i\theta}$ in A

$$|u(re^{i\theta} - f(e^{i\alpha})| < \varepsilon \qquad ....(6)$$

for particular α =0 we show that (6) holds.

Since f is continuous at $z=1$ $\exists$ a $\delta>0$ such that

$$|f(e^{i\theta})-f(1)|<\frac{\varepsilon}{3} \quad \text{if } |\theta|<\delta \qquad \text{...(7)}$$

Let $\quad M=\max\{|f(e^{i\theta})|:|\theta|\le\pi\}$

Then from the result on Poisson kernel $P_r(\theta)<P_r(\delta)$ if $0<\delta<|\theta|\le\pi$

There exists a $\rho, 0<\rho<1$ such that

$$\text{...(8)}$$

for $\quad \rho<r<1$ and $|\theta|\ge\frac{\delta}{2}$

Let A be the arc $\left\{e^{i\theta}:|\theta|<\frac{\delta}{2}\right\}$ then if

$e^{i\theta}\in A$ and $\rho<r<1$

$$u(re^{i\theta})-f(1)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f(e^{it})dt-f(1)$$

$$P_r(\theta)<\frac{\varepsilon}{3M}$$

$$=\frac{1}{2\pi}\int_{|t|<\delta}P_r(\theta-t)[f(e^{it})-f(1)]dt+\frac{1}{2\pi}\int_{|t|\ge\delta}P_r(\theta-t)[f(e^{it})-f(1)]dt$$

if $\quad |t|\ge\delta$ and $|\theta|\le\frac{\delta}{2}$ then $|t-\theta|\ge\frac{\delta}{2}$

$$|u(re^{i\theta})-f(1)|\le\frac{1}{2\pi}\int_{|t|<\delta}P_r(\theta-t)|f(e^{it})-f(1)|dt+\frac{1}{2\pi}\int_{|t|\ge\delta}P_r(\theta-t)|f(e^{it})-f(1)|dt$$

$$\le\frac{\varepsilon}{3}+2M.\frac{\varepsilon}{3M} \qquad \text{[from (7) and (8)]}$$

$\Rightarrow \quad |u(re^{i\theta})-f(1)|<\varepsilon$

Hence for general value of α.

$$|u(re^{i\theta})-f(e^{i\alpha})|<\varepsilon \qquad \text{...(9)}$$

Show that u is continuous on D^{-}.

(iii) For *u* is unique

Suppose $v \in D^{-}$ which is harmonic on D and $v(e^{i\theta}) = f(e^{i\theta}) \ \forall \ \theta$.

then (u-v) is harmonic is D and $(u-v)(z) = 0. \quad \forall z \in \partial D$

$\Rightarrow \quad u - v = 0$

$\Rightarrow \quad u$ is unique.

Harnack inequality

If $u : \overline{B}(a;R) \to R$ is continuous, harmonic in $B(a;R)$ and $u \geq 0$ then for $0 \leq r < R$ and all θ

$$\frac{R-r}{R+r}u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$$

Harnack's theorem : Let G be a region with

(a) The metric space Har(G) is complete.

(b) If $\{u_n\}$ is a sequence in Har(G) such that $u_1 \leq u_2 \leq ....$ then either $u_n(z) \to \infty$ uniformly on compact subset of G or $\{u_n\}$ converges in Har(G) to a harmonic function.

Proof : (a) Har(G) is complete

Let $\{u_n\}$ be a sequence in Har(G) such that

$u_n \to u \quad$ in $C(G, R)$

Then u has the MVP

$\Rightarrow \quad$ u is harmonic

[by the theorem if $u : G \to R$ which has MVP then u is harmonic.]

(b) Assuming $u_1 \geq 0$

Let $\quad u(z) = \sup\{(u_n(z) : n \geq 1\}; \quad \forall \ z \in G$

$\Rightarrow \quad$ Either $u(z) = \infty$ or $u(z) \in \mathbf{R}$ and $u_n(z) \to u(z), \ \forall \ z \in G$

Define

$A = \{z \in G : u(z) = \infty\}$ ....(1)

$B = \{z \in G : u(z) < \infty\}$ ...(2)

Then $A \bigcup B = G$ and $A \bigcap B = \phi$

Now we show that both A and B are open

If $a \in G,$ and R be chosen such that

Then by Harnack's inequality

$$\frac{R - |z-a|}{R + |z-a|} u_n(a) \le u_n(z) \le \frac{R + |z-a|}{R - |z-a|} u_n(a); \quad \forall\, z \in B(a;R) \text{ and } \forall\, n \ge 1.$$

...(3)

If $a \in A$ then

so that from (3)

$u_n(z) \to \infty, \quad \forall\, z \in B(a;R)$ [left half of (3)]

$\Rightarrow \quad B(a;R) \subset A$

$\Rightarrow$ A is open.

Similarly : If $a \in B$ then right half of (3) gives that $B(a;R) \in G$

for $|z-a| < R$

$\Rightarrow \quad u_n(z) < \infty \qquad \forall\, z \in B(a;R)$

$\Rightarrow$

$\Rightarrow$ B is open

Since G is connected, either A =G or B =G

Suppose A = G; that is $u \equiv \infty$

If and $0 < \rho < R$ then

$$M = \frac{(R-\rho)}{R+P} > 0 \text{ and}$$

(3) gives that

$Mu_n(a) \le u_n(z)$ for $|z-a| \le \rho$

$\Rightarrow \quad u_n(z) \to \infty$ uniformly for $z \in \overline{B}(a;\rho)$

$\Rightarrow \quad \forall a \in G \;\; \exists \;$ a $\rho > 0$ such that $u_n(z) \to \infty$ uniformly for $|z-a| \le \rho$

$\Rightarrow \quad u_n(z) \to \infty$ uniformly for z in any compact set.

Now suppose B= G

i.e. $\quad u(z) < \infty \quad \forall\, z \in G$

If $\rho < R$ then there is a constant N, which depends only on a and ρ such that

for $|z-a| \le \rho$ and $\forall\ n$

So if

$$\le C[u_n(a) - u_m(a)]$$

for some constant C.

$\Rightarrow \quad \{u_n(z)\}$ is a uniformly Cauchy sequence on $\bar{B}(a;\rho)$

$\Rightarrow \quad \{u_n\}$ is a Cauchy sequence in Har(G) and must converge to a harmonic function [by part (a)]

Since $u_n(z) \to u(z)$

$\Rightarrow \quad u$ is the required harmonic function. $u_n(z) \le N u_n(a) - M u_m(a)$

Definition :

Subrarmonic function- If φ be a continuous function and if such that

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})\,d\theta$$

Definition:

Superharmonic- If φ be a continuous function and if

such that

$$\varphi(a) \ge \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})\,d\theta$$

Definition :

If G is a region and f be a continuous function $f : \partial_\infty G \to R$ then the **Parron family** denoted by $\mathbf{P}(f, G)$ consists of all *subharmonic function* $\varphi\colon G \to \mathbf{R}$ such that

Dirichlet Problem : It consists in determining all regions G such that for any continuous function there is a continuous function $u : G^- \to R$ such that $u(z) = f(z)$ for and u is harmonic in G.

Definition : Barrier for G at a

Let G be a region and . A barrier for G at a is a family $\{\psi_r : r > 0\}$ of functions such that

(a) ψ_r is defined and superharmonic on $G(a; r)$ with $0 \le \psi_r(z) \le 1$;

(b) $\lim\limits_{z \to a} \psi_r(z) = 0$ $\quad = r\} \quad \forall a \in \partial_\infty G$

(c) for

If we define $\hat{\psi}_r$ by letting

$\hat{\psi}_r = \psi_r$ on $G(a; r)$

$\hat{\psi}_r(z) = 1$ for

then is superharmonic.

$\Rightarrow$ $\hat{\psi}_r$ approaches the function which is one everywhere but at $z=a$, $\hat{\psi}_r$ is zero.

Theorem : Let G be a region and let $a \in \partial_\infty G$ such that there is a barrier for G at a.

If $f : \partial_\infty G \to \mathbf{R}$ is continuous and u is the Perron function associated with f then

$$\lim_{z \to a} u(z) = f(a)$$

Proof : Let $\{\psi_r : r > 0\}$ be a barrier for G at a. Assuming $a \neq \infty$ and (otherwise consider the function f- f(a)}

Let ε > 0 and choose δ > 0 such that

$|f(w)| < \varepsilon$ whenever $w \in \partial_\infty G$ and

$|w-a| < 2\delta;$

Let $\psi = \psi_\delta$

Let $\hat{\psi} : G \to R$ defined by

$\hat{\psi}(z) = \psi(z)$ for $z \in G(a;\delta)$ and

for $z \in G - B(a;\delta)$ ...(1)

Then is superharmonic.

If $|f(w)| \leq M$ for all w in $\partial_\infty G$ ...(2)

then

$-M\hat{\psi} - \varepsilon$ is subharmonic.

Consider $-M\hat{\psi} - \varepsilon$ in $\mathbf{P}(f,G)$ ....(3)

If $w \in \partial_\infty G - B(a;\delta)$ then from (1) and (2) $\mathbf{P}(f,G) \leq u(z)$; $\forall z \in G$

$$\limsup_{z \to w} [-M\hat{\psi}(z) - \varepsilon] = -M - \varepsilon < f(w) \qquad ...(4)$$

Because

$$\Rightarrow \quad \limsup_{z \to w} [-M\hat{\psi}(z) - \varepsilon] \leq -\varepsilon \quad \forall\ w \in \partial_\infty G \qquad ....(5)$$

If particular if $w \in \partial_\infty G \cap B(a;\delta)$ then

$\limsup_{z \to w} [-M\hat{\psi}(z) - \varepsilon] \leq -\varepsilon < f(w)$ by the choice of δ.

$\Rightarrow$ the consideration (3) is valid.

Hence

...(6)

Similarly

$\liminf_{z \to w} [M\hat{\psi}(z) + \varepsilon] \geq \limsup_{z \to w} (\varphi(z))$ and w in .

By the Maximum principle for subharmonic and superharmonic we have

$\forall \varphi$ in and $z \in G$

Hence

....(7)

(6) and (7)

$\Rightarrow \quad -M\hat{\psi}(z) - \varepsilon \le u(z) \le M\hat{\psi}(z) + \varepsilon \quad \forall z \in G$ ....(8)

But (is Parron function)

Since ε is arbitrary (8) implies

$$\lim_{Z \to a} u(z) = 0 = f(a)$$

Harmonic function

Definition : Let G be an open set and $G \subset C$, $u : G \to \mathbf{R}$ is harmonic if u has continuous partial derivatives and satisfying the Laplace equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{or} \quad \nabla^2 u = 0$$

Harmonic conjugate

Definition : Let f be an analytic funcion defined as $f : G \to \mathbf{C}$ then $u = \operatorname{Re} f$ and are called harmonic conjugates.

Theorem : If $|z| \le \frac{1}{p} < 1$ then $|\log E(z;k)| \le \frac{p}{p-1} |z|^{k+1}$.

Proof : Let k > 0

$$\log E(z;k) = Log(1-z) + \left(z + \frac{z^2}{2} + \ldots.. + \frac{z^k}{k} \right)$$

$$Log(1-z) = -z - \frac{z^2}{2} - \frac{z^k}{k} - \frac{1}{k+1} z^{k+!} \ldots.$$

for $|z| < 1$

$$\therefore \quad \log E(z;k) = -\frac{1}{k+1} z^{k+1} - \frac{1}{k+2} z^{k+2} - \ldots.. \qquad \text{for } |z| \le \frac{1}{p} < 1$$

$$\therefore \quad |Log\, E(z;k)| \le |\log E(z;k)| \le |z|^{k+1}\left(\frac{1}{k+1}+\frac{1}{k+2}|z|+...\right)$$

$$\le |z|^{k+1}\,(1+|z|+|z|^2+.....) \qquad [\because k>0]$$

$$\le |z|^{k+1}\left(1+\frac{1}{p}+\frac{1}{p^2}+...\right)$$

$$=\frac{p}{p-1}|z|^{k+1}$$

for k =0

$$|\log E(z;0)| = |\log(1-z)| \le |z|\left(1+\frac{|z|}{2}+...\right)$$

$$\le |z|\left(1+\frac{1}{p}+\frac{1}{p^2}+...\right)$$

$$|\log E(z;0)| = \frac{p}{p-1}|z|$$

for p =2

$$|Log\, E(z;k)| \le \frac{2}{2-1}|z|^{k+1} = 2|z|^{k+1}$$

provided $|z| \le \frac{1}{2}$

Example : construct an entire function with simple zeros at the point $0, 1, 2^p, 3^p, \ ... \ , n^p, \ \ ...(p>1)$

Solution : We may take $k_n = 0$ for every *n*. The series

$$\sum_{n=1}^{\infty}\left|\frac{z}{n^p}\right| = |z|\sum_{n=1}^{\infty}\frac{1}{n^p}$$

converges for $\forall z$ when p >1

$\therefore$ by Weierstrass factorization Theorem m=1 (simple zero at $z=0$), $k_n = 0, \quad a_n = \dfrac{1}{n^p}$

$$\Rightarrow \qquad F(z) = e^{h(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^p}\right)$$

Example : Express $\sin \pi z$ as an infinite product.

Solution : since $\sin \pi z$ is an entire function with simple zeros at $0, \pm 1, \pm 2, \pm 3, \ldots$

Then by Weierstrass factorization Theorem

....(1)

provided $\displaystyle\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^{k_n+1}$ converges for $\forall$z where $a_n,\ n \geq 1$ are zeros.

since $\displaystyle\sum_{n=-\infty}^{\infty} \left|\frac{r}{a_n}\right|^{k_n+1} < \infty; \ n \neq 0, \ \forall\, r > 0$ $\quad F(z) = e^{h(z)} z^m \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k_n\right)$ $\quad \sin \pi z = e^{h(z)} z \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k_n\right)$

Then it is sufficient to choose $k_n = 1, \ \forall\, n,$

then $\displaystyle\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left|\frac{z}{a_n}\right|^2 = |z|^2 \left(\frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2^2} + \ldots\right)$

$$= 2|z|^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{that conveyes for each z}$$

$\therefore$ in (1) we may put $a_n = \pm n, \ k_n = 1$ to get

$$\sin \pi z = e^{h(z)} z \prod_{n=1}^{\infty} E\left(\frac{z}{\pm n}; 1\right)$$

$$\sin \pi z = e^{h(z)} z \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)\left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n}\right) \exp\left(-\frac{z}{n}\right)\right]$$

$$\sin \pi z = e^{h(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Example : Construct an entire function with simple zero at the point 0, -1, -2, ..., *-n*.

Solution : Given that $a_n = -n$ then the series

$$\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^{k_n+1} \quad \text{be convergent if } k_n = 1$$

i.e. $$\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^{k_n+1} = |z|^2 \sum_{n=1}^{\infty} \frac{1}{|-n|^2} = |z|^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Then Weierstrass factorization Theorem gives

$$F(z) = e^{h(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{(-n)}\right) e^{-\frac{z}{n}}$$

$$= e^{h(z)} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

Definition : Rank of infinite product

The *smallest non negative integer* k for which the series $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}}$ converges is said to be rank of the infinite product. $\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k_n\right)$

Definition Canonical (or regular) product :

If k denotes the rank of infinite product and if we take $k_n = k \;\; \forall\, n$, then the infinite product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; k\right) \quad \text{is called canonical product.}$$

If no such integer k exists, the infinite product is said to be of **infinite rank.**

Definition : Exponent of convegence of zeros of an entire function

If $\quad F(z) = e^{h(z)}.z^m P(z)$ ...(1)

where P(z) is canonical product of rank k and A a non-empty set of non negative **real members** α such that

converges.

Then the number ρ = glb A is called the exponent of convegence of the zeros of the entire function given by (1)

Theorem : Theorem for the computation of the exponent of convegence.

Let $|a_n| = r_n (0 < r_1 \le r_2 \le ... \le r_n \le ....; r_n \to \infty)$ then the exponent of convergence is given by

$$\frac{1}{\rho_c} = \lim_{n\to\infty} \frac{\log r_n}{\log n}$$

Example : If (i) $|a_n| = n^{\frac{1}{2}} \log_n$ (ii) $|a_n| = |a|^n, |a| > 1$ then find $\rho_{c.}$

Ans. (i) 2 (ii) 0

$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\alpha}}$ $F(z) = z^2 e^{z^2}$

Definition : Genus and Exponential degree of an entire function :

If $F(z) = e^{h(z)} z^m P(z)$ is an entire function with canonical product P(z) of rank *k* and *h*(*z*) is a polynomial of degre $q \ge 0$ then non-negative integer $p = \max(k, q)$ is called the genus of the entire function and *q* is said to be the **exponential degree** of *F*. If P(z) is not of finite rank *k*, or if *h*(*z*) is not a polynomial, then *F* is said be of infinite genus.

Example : Find genus of the entire function

$F(z) = e^{z^2} z^2 P(z)$

$h(z) = z^2$ is a polynomial of degree 2.

$P(z) = 1$

$\therefore \quad k_n = k = 0$

$\therefore \quad \max(k, q) = \max(0,2) = 2$ is the genus of *F*(*z*).

Definition : Order of an entire function

If F(z) be an entire function and A is a positive constant such that

$$\max_{|z|=r} |F(z)| = M(r) < e^{r^A}$$

for all sufficiently large value of $r = |z|$, then $F(z)$ is caled an entire function of finite order.

Alternatively

F(z) is of finite orer if $\exists\ A > 0$ such that

$$|F(z)| = 0(e^{r^A})$$

or $\quad |F(z)| < K\, e^{r^A}; \qquad K>0$

Definition : Order ρ of an entire function of finite order :

Let $S = \{A : |F(z)| < e^{r^A}, r > r_0\}$ then the order ρ of an entire function F of finite order is defined by $\rho = \inf\{A : |f(z)| < e^{r^A}\} M(r) < e^{Br^\rho} \quad \forall$

If there is not a positive constant A such that

$$|F(z)| < e^{r^A} \quad \forall\ r \text{ large enough}$$

F(z) is said be of **infinite order**, $\rho = \infty$.

Definition : Type of entire function:

If F(z) is an entire function of finite order ρ and there exists a constant $B > 0$ such that

r large enough

then $F(z)$ is said to be of *finite type* and

$$\sigma = \inf\{B : M(r) < e^{Br^\rho},\ r > R\}$$

is called the *type of F*.

If $\sigma > 0$, F is said to be of *normal type*.

If $\sigma = 0$, F is called *minimum type*.

If there is no B such that $M(r) < e^{Br^{\rho}}$, then F is called of *infinite type* (or *maximum type*).

exponential type σ : An entire function F is said to be of expontial type $\sigma(\sigma < \infty)$ if either the function is of order ρ =1 and type , or the function is of **order less then 1.**

Theorem : The order ρ of an entire function is given by the formula

Proof : Suppose $\rho < \infty$ then

$\forall\, r$ large enough ...(1)

then for given ε >0 we have

$$M(r) < e^{r^{\rho+\varepsilon}} \qquad \forall\, r \text{ large enough} \qquad ...(2)$$

Also, there are some z with arbitrary modulus for which

$$M(r) < e^{r^{\rho-\varepsilon}} \qquad ..(3)$$

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} \qquad \frac{\log\log M(r)}{\log r} < \rho + \varepsilon \qquad (r = |z|)$$

Then (2) implies

$$\log\log M(r) < (\rho + \varepsilon)\log r$$

i.e. $\forall r < \text{Re}$ ...(4)

and (3) implies

$$\frac{\log\log M(r)}{\log r} > \rho - \varepsilon \qquad ...(5)$$

$$\therefore \quad \rho - \varepsilon < \frac{\log\log M(r)}{\log r} < \rho + \varepsilon$$

$$\Rightarrow \quad \rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r}$$

Theorem : The type σ of an entire function of finite order ρ is given by

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^p}$$

Proof : Since the entire function $f(z)$ of finite order ρ is of type σ

Therefore

$$M(r) < e^{\sigma r^{\rho}} \qquad ...(1)$$

then for given $\varepsilon > 0$, we have

$$M(r) < e^{(\sigma+\varepsilon)r^{\rho}} \qquad \forall\ r \text{ large enough} \qquad ...(2)$$

and $$M(r) > e^{(\sigma-\varepsilon)r^{\rho}} \qquad ...(3)$$

for infinity many values of r

Then (2) implies

$$\frac{\log M(r)}{r^p} < \sigma + \varepsilon \qquad ...(4)$$

and (3) implies

$$\frac{\log M(r)}{r^p} > \sigma - \varepsilon \qquad ....(5)$$

$|z| \le R$

$$\Rightarrow \quad \sigma - \varepsilon < \frac{\log M(r)}{r^p} < \sigma + \varepsilon$$

$$\Rightarrow \quad \sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^p}$$

Theorem : Jensen Formula

If f(z) is analytic in the disc $|z| \le R$ and if $a_k \ne 0$ $(1 \le k \le n)$ are the zero of $f(z)$ in those zeros being repeated according to their multiplicities, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})|\,d\theta = \log|f(0)| + \sum_{k=1}^{n} \log\frac{R}{|a_k|}$$

Proof : Since $f(z)$ is analytic in $|z| \le R$, $\exists$ an open disc $|z| < R' = R + \delta$ $(\delta > 0)$, where f(z) is analytic and has no other zero than the a_k

Then if the function

$$g(z) = \frac{a_1 a_2 ... a_n f(z)}{(a_1 - z)(a_2 - z)...(a_n - z) f(0)} \qquad ...(1)$$

is dfined at the point $a_1, a_2, ..., a_n$ then it becomes analytic in $|z| < R'$ and does not vanish analytic in this disc. Then there exists a function $h(z)$ analytic in R'such that $e^{h(z)} = g(z)$ an analytic branch $h(z)$ of $\log g(z)$ in the disc R'.

From (1) g(0) =1, we may coose h(0) =0 [from (2)]

Then $\dfrac{h(z)}{z}$ can be made analytic in $|z| < R'$

If we consider $C: z = \mathrm{Re}^{i\theta}$, $0 \le \theta \le 2\pi$ then by the Cauchy-Gousrat theorem

$$\frac{1}{2\pi i}\int_{C+} \frac{h(z)}{z} dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{h(\mathrm{Re}^{i\theta})}{\mathrm{Re}^{i\theta}} Rie^{i\theta} d\theta = 0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\mathrm{Re}^{i\theta}) d\theta = 0 \qquad ...(3)$$

But $\quad \mathrm{Re}\, h(z) = \log|g(z)|$

Taking real part of (3)

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{a_1.a_2......a_n f(\mathrm{Re}^{i\theta})}{(a_1 - \mathrm{Re}^{i\theta})...(a_n - \mathrm{Re}^{i\theta}) f(0)}\right| d\theta = 0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|f(0)| d\theta + \frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi} \log|a_k| d\theta$$

$$- \frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi} \log|a_k - \mathrm{Re}^{i\theta}| d\theta = 0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})| d\theta - \log|f(0)| + \frac{1}{2\pi}\sum_{k=1}^{n} \log|a_k| \times 2\pi$$

$$+ \frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi} \log\frac{1}{|a_k - \mathrm{Re}^{i\theta}|} d\theta = 0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})| d\theta - \log|f(0)| + \frac{1}{2\pi}\sum_{k=1}^{n} \log|a_k| \times 2\pi$$

$$+\frac{1}{2\pi}\sum_{k=1}^{n}\int_{0}^{2\pi}\log\left(\frac{1}{|\mathrm{R}e^{i\theta}|\left|\frac{a_k e^{-i\theta}}{R}-1\right|}\right)d\theta=0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\theta})|\,d\theta-\log|f(0)|+\frac{1}{2\pi}\sum_{k=1}^{n}\log|a_k|\times 2\pi$$

$$+\frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi}\left[-\log|R|-\log\left|1-\frac{a_k e^{-i\theta}}{R}\right|\right]d\theta=0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\theta})|\,d\theta-\log|f(0)|+\frac{1}{2\pi}\sum_{k=1}^{n}\log|a_k|\times 2\pi$$

$$+\frac{1}{2\pi}\sum_{k=1}^{n}2\pi(-\log R)-\frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi}\log\left|1-\frac{a_k e^{-i\theta}}{R}\right|d\theta=0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\theta})|\,d\theta-\log|f(0)|+\frac{1}{2\pi}\sum_{k=1}^{n}\log|a_k|\times 2\pi$$

$$+(-n)\log R-\frac{1}{2\pi}\sum_{k=1}^{n}\underbrace{\int_0^{2\pi}\log\left|1-\frac{a_k e^{-i\theta}}{R}\right|d\theta}_{(=0\text{ when }|a_k/R|\le 1)}=0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\theta})|\,d\theta=\log|f(0)|-\sum_{k=1}^{n}\log|a_k|+\sum_{k=1}^{n}\log R$$

$$=\log|f(0)|+\sum_{k=1}^{n}\log\left(\frac{R}{|a_k|}\right)$$

Corollary : If n(r) denotes the number of zeros of the entire function F(z) in the disc $|z|\le r$, and $F(0)\ne 0,$ then

$$\int_0^R \frac{n(t)}{t}\,dt\le \log M(R)-\log|F(0)|$$

Proof : Let $a_1, a_2, ..., a_n$ are zeros of F(z) in |z| <R such that $|a_1|<|a_2|< .... <|a_n|$

Then

$$\sum_{k=1}^{n}\log\left(\frac{R}{|a_k|}\right)=n\log R-\sum_{k=1}^{n}\log|a_k|$$

$$= n\log R + \sum_{k=1}^{n-1} k[\log|a_{k+1}| - \log|a_k|] - n\log|a_n|$$

$$= \sum_{k=1}^{n-1}\int_{|a_k|}^{|a_{k+1}|} k\frac{1}{t}dt + n(\log R - \log|a_n|)$$

$$= \sum_{k=1}^{n-1}\int_{|a_k|}^{|a_{k+1}|} \frac{k}{t}dt + n\int_{|a_n|}^{R}\frac{1}{t}dt \qquad ....(1)$$

$n(t) = 0$ for $0 \le t \le |a_1|$

$n(t) = k$ for $|a_k| \le t < |a_{k+1}|,\ k = 1,2,....,(n-1)$

$n(t) = n$ when $|a_n| \le t < R$

We have (from 1)

$$\sum_{k=1}^{n}\log\frac{R}{|a_k|} = \int_{|a_1|}^{|a_2|}\frac{1}{t}dt + \int_{|a_2|}^{|a_3|}\frac{2}{t}dt + \int_{|a_3|}^{|a_4|}\frac{3}{t}dt + ... + \int_{|a_{n-1}|}^{|a_n|}\frac{(n-1)}{t}dt + n\int_{|a_n|}^{R}\frac{1}{t}dt$$

$$= \int_0^R \frac{n(t)}{t}dt$$

Then by Jesens formula $\qquad M_c(2r) < \exp\{(2r)^{\rho+\varepsilon/3}\}$

$$\int_0^R \frac{n(t)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi}\log|F(\mathrm{Re}^{i\theta})d\theta - \log|F(0)|$$

$$\le \log M(R) - \log F(0)$$

Theorem : (Hadamard) The exponent of convergence of the zeros of an entire function of finite order is no greater then ρ. i.e. $\rho_c \le \rho$

Proof : $\{a_n\}^\infty$ sequence of zeros of $F(z)$, $|a_k| \le |a_{k+1}|$ and real fucntion $n(t)$ is monotonic increasing with t

$$\therefore \quad \int_0^{2r}\frac{n(t)}{t}dt \ge \int_r^{2r}\frac{n(t)}{t}dt \ge n(r)\int_r^{2r}\frac{dt}{t} = n(r)\log 2$$

$$\Rightarrow \quad n(r) \le \frac{1}{\log 2}\int_0^{2r}\frac{n(t)}{t}dt \qquad ....(1)$$

For given $\in > 0$ we have . For r large enough or $\quad \log M(2r) < (2r)^{\rho+\varepsilon/3} < r^{\rho+2\varepsilon/3}$ ...(2)

Replacing R by $2r$ in last result of the corollary and using (2)

$$\int_0^{2r} \frac{n(t)}{t}\,dt < r^{\rho+2\varepsilon/3} - \log|F(0)| < r^{\rho+\varepsilon} \qquad \text{...(3)}$$

(1) and (3)

$$n(r) < \frac{1}{\log 2} r^{\rho+\varepsilon} \qquad \text{...(4)}$$

$\Rightarrow \quad n(r) = o(r^{\rho+\varepsilon})$ as $r \to \infty$

Now we show that

converges whatever be $\delta > 0$

Let ε be such that and $r = r_n$

$$n = n(r_n) < \frac{1}{\log 2} r_n^{\rho+\varepsilon} \qquad \text{....(5)}$$

$0 < \varepsilon < \delta$ $\rho_c \le \rho$ $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho+\varepsilon}}$

for all r large enough

from (5)

$$\frac{1}{r_n^{\rho+\varepsilon}} < \frac{1}{n}\frac{1}{\log 2}$$

and $\quad \dfrac{1}{r_n^{\rho+\varepsilon}} < A\dfrac{1}{n^{(\rho+\delta)/(\rho+\varepsilon)}} \qquad A = \left(\dfrac{1}{\log 2}\right)^{(\rho+\delta)/(\rho+\varepsilon)}$

$\dfrac{(\rho+\delta)}{(\rho+\varepsilon)} > 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(\rho+\delta)/(\rho+\varepsilon)}}$ converges

Hence $\displaystyle\sum_{n=1}^{\infty} r_n^{-(\rho+\varepsilon)}$

$\Rightarrow \quad$ Exponent of convergence ρ_c of zeros is $\rho + \delta$

$\Rightarrow$

Theorem : Suppose that about each zero $a_n, |a_n| > 1,$ of a cononical product

$P(z)$, a dix of radius $\frac{1}{r_n^p}$ is described where $r_n = |a_n|$ and = order $P(z)$. Then in the region R complementary to the union of all those discs, the inequality

holds an infinitely many circles of radii arbitrarily large.

Proof : It is obivious that the sum $2\sum_{n=1}^{\infty}\frac{1}{r_n^p}$ infinite as $(p > \rho = \rho_c)$ the union of the intervals $\bigcup_n [r_n - r_n^{-p}, r_n + r_n^{-p}]$ on the real axis each of length $2r_n^{-p}$ does not cover the entire positive real axis.

There are infinitely many circles with center at the origin and radii arbitrarily large which lie in R.

k = rank of $P(z)$, then

$$|P(z)| = \prod_{r_n \le 2r}\left|E\left(\frac{z}{a_n};k\right)\right|\prod_{r_n > 2r}\left|E\left(\frac{z}{a_n};k\right)\right| \qquad \text{....(1)}$$

$$p > \rho$$

$$|P(z)| > e^{-r^{\rho+\varepsilon}} \qquad (\varepsilon > 0)$$

$$\Rightarrow \log|P(z)| = \sum_{r_n \le 2r}\log\left|E\left(\frac{z}{a_n};k\right)\right| + \sum_{r_n > 2r}\log\left|E\left(\frac{z}{a_n};k\right)\right|$$

$$\ge \sum_{r_n \le 2r}\log\left|1-\frac{z}{a_n}\right| - \sum_{r_n \le 2r}\left|\frac{z}{a_n} + \dots + \left(\frac{z}{a_n}\right)^k \cdot \frac{1}{k}\right| - \sum_{r_n > 2r}\log\left|E\left(\frac{z}{a_n};k\right)\right| \qquad \text{....(2)}$$

[on using triangle inequally $|z_1 + z_2| \ge |z_1| - |z_2|$]

For $|z| = r$ and $r_n \le 2r$, we obtain

$$\left|\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k\right| \le \left|\left(\frac{z}{a_n}\right)^k\left\{\frac{1}{k} + \frac{1}{k-1}\left(\frac{a_n}{z}\right)^{k-1} + \dots + \left(\frac{a_n}{z}\right)^{k-1}\right\}\right|$$

$$\le \left(\frac{r}{r_n}\right)^k \underbrace{\left\{\frac{1}{k} + \dots + \left|\frac{a_n}{z}\right|^{k-1}\right\}}_{(A_1 \text{ which does not depend on } r)}$$

$$= A_1\left(\frac{r}{r_n}\right)^k \qquad \text{....(3)}$$

$$\therefore \quad \sum_{r_n \le 2r} \left| \frac{z}{a_n} + \ldots\ldots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k \right| \le A_1 \sum_{r_n \le 2r} \left(\frac{r}{r_n}\right)^k$$

$$= A_1 \sum_{r_n \le 2r} \frac{2^{\rho+\varepsilon/2-k} . r^{\rho+\varepsilon/2+k}}{2^{\rho+\varepsilon/2-k} . r^{\rho+\varepsilon/2} (r_n)^k}$$

$$= A_1 \sum_{r_n \le 2r} \frac{2^{\rho+\varepsilon/2-k} . r^{\rho+\varepsilon/2}}{(2r)^{\rho+\varepsilon/2-k} . (r_n)^k}$$

$$= A_1 2^{\rho+\varepsilon/2-k} . r^{\rho+\varepsilon/2} \sum_{r_n \le 2r} \frac{1}{(r_n)^{\rho+\varepsilon/2-k} . (r_n)^k}$$

$$= r^{\rho+\varepsilon/2} \underbrace{\left(A_1 2^{\rho+\varepsilon/2-k} \sum_{n=1} \frac{1}{r_n^{\;\rho+\varepsilon/2}} \right)}_{(A_2 \text{ which does not depend on } r)}$$

$$= A_2 r^{\rho+\varepsilon/2} \qquad \ldots.(4)$$

If $k=0$ the sum in (4) does not appear in (2). For z outside every circle $|z-a_n| = r_n^{-p}$ with $r_n \le 2r$.

We have

$$\left| 1 - \frac{z}{a_n} \right| = \frac{|a_n - z|}{|a_n|} = \frac{r_n^{-p}}{r_n} = r_n^{-p-1} \ge (2r)^{-p-1}$$

Hence for all circles $|z| = r$ in the region R, with r sufficiently large,

$$\log |P(z)| > -r^{\rho+\varepsilon} \Rightarrow |P(z)| > e^{-r^{\rho+\varepsilon}}$$

Hadanard's Factorization theorem

If F(z) is an entire function of finite order ρ, then the factorization $F(z) = e^{h(z)} z^m P(z)$ is always possible where $h(z)$ is a polynomial of degree $\le \rho$, $m \ge o$ is the multiplicity of $z = 0$ and $P(z)$ is a canonical product of rank .

Proof : According to Weierstrass factorization theorem an entire function $F(z)$ can be factorize in the following from

...(1)

When $h(z)$ is an entire function and $P(z)$ *a product which may or may not be canonical.*

By the previous theorem $\rho_c \le \rho$, so that $P(z)$ is of finite rank $k \le \rho$.

Also since $|P(z)| > e^{-r^{\rho+\varepsilon}}$

replacing ε by we have

$\le \rho$ $\quad \frac{\varepsilon}{2}$ $\quad |P(z)| > e^{-r^{\rho+\varepsilon/2}}$ $\quad F(z) = e^{h(z)} z^m P(z)$

on infinitely many circles $|z| = r$ of arbitrarily large radius.

Aslo $|F(z)| < e^{r^{\rho+\varepsilon/2}}$ is satisfied for all values of r sufficiently large, since F is suppose to be of order ρ. Then it follows that

$$e^{\operatorname{Re} h(z)} = |e^{h(z)}| = \frac{|F(z)|}{|z^m P(z)|} < \frac{e^{r^{\rho+\varepsilon/2}}}{e^{-r^{\rho+\varepsilon/2}}}$$

$$= e^{2r^{\rho+\varepsilon/2}} < e^{r^{\rho+\varepsilon}} \qquad \text{on circles of arbitrarily large radius.}$$

Hence on such circles

$$\operatorname{Re} h(z) < r^{\rho+\varepsilon}$$

$\Rightarrow \quad h(z)$ is a polynomial of degree not greater then ρ.

Example : Show $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ by Hadaward Factorization Theorem.

Solution : Let $F(z) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n} z^n}{(2n+1)!}$ ....(1)

$$= 1 - \frac{\pi^2 z}{3!} + ...$$

when $F(z) = \sum_{n=0}^{\infty} c_n z^n$ then by the formula

$$\frac{1}{\rho} = \lim_{n \to \infty} \frac{\log(1/|c_n|)}{n \log n} \qquad ...(2)$$

$$\frac{1}{\rho} = 2 \ \text{ i.e. } \rho = \frac{1}{2} \qquad ...(3)$$

Since zeros of $F(z)$ are z =1, 4, ..., n^2, ... we have

$$F(z) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right) \qquad ...(4)$$

But according to **Hadamard Factorization theorem**, $h(z)$ must be a polynomial of degee $\leq$ order of $F(z)$ i.e. degree of

$\Rightarrow \quad h(z) = C$ (constant)

$$\therefore \quad F(z) = e^C \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right) \qquad h(z) \leq \frac{1}{2}$$

Since $F(0) = 1$

$\Rightarrow \quad$ C =0

$$\therefore \quad \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$$

Replacing z by z^2

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$