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M. Sc. (Mathematics)

MAL-632

Partial Differential Equations

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Lesson 1: Preliminaries

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Structure

1.1 Introduction:

The reader must be familiar with the several notations and certain functional analysis concepts before studying the partial differential equations. Here, first we give brief description and then define partial differential equations.

1.2 Notations

(a) *Geometric notations*

(i) R^n = n-dimensional real Euclidean space

(ii) $R^1 = R$ = real line

(iii) e_i = unit vector in the i^{th} direction

$= (0, 0, 0, \ldots 1, \ldots 0)$

(iv) A point x in R^n is $x = (x_1, x_2, \ldots x_n)$

(v) $R^n_+ = \{|x = (x_1, x_2, \ldots x_n) \in R^n | x_n > 0\}$

= open upper half-space.

(vi) A point in R^{n+1} will be denoted as

$(x, t) = (x_1, x_2, \ldots, x_n, t)$

where t is time variable.

(vii) U, V, W denote open subsets of R^n. We write

$V \subset\subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact

i.e. V is compactly contained in U.

(viii) ∂U = boundary of U

$\bar{U}$ = closure of $U = U \cup \partial U$

(ix) $U_T = U \times (0, T]$

(x) $\Gamma_T = \bar{U}_T - U_T$

= parabolic boundary of U_T

(xi) $B^0(x, r) = \{y \in R^n \| x - y| < r\}$

= open ball in R^n with centre x and radius $r > 0$

(xii) $B(x, r) = \{y \in R^n \| x - y| \le r\}$

=closed ball in R^n with centre x and radius $r > 0$

(xiii) $\alpha(n)$ = volume of unit ball B (0, 1) in R^n

$$= \frac{r^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

$n\alpha(n)$ = surface area of unit sphere B (0, 1) in R^n

(xiv) If $a, b \in R^n$ s.t .

$a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ then

$$a.b = \sum_{i=1}^{n} a_i b_i \qquad \text{and}$$

$$|a| = \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}$$

(xv) $\square$ denotes n dimensional complex plane.

(b) *Notations for functions*

(i) If $u : U \to R$, we write

$u(x) = u(x_1, x_2, ..., x_n)$ where $x \in U$.

(ii) u is smooth if u is infinitely differentiable.

(iii) If u, v are two functions, we write

$u \equiv v$ if u, v agree for all arguments

$u := v$ means u is equal to v.

(iv) The support of a function u is defined as the set of points where the function is not zero and is denoted by spt u.

$$\text{spt } u = \left\{ \overline{x \in X \,\middle|\, f(x) \neq 0} \right\}$$

In other words, spt u is the closure of the set u where u does not vanish.

(v) The sign function is defined by

$$sgn\ x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$u^+ = \max(u, 0)$$

$$u^- = -\min(u, 0)$$

$$u = u^+ - u^-$$

$|u| = u^+ + u^-$

(vi) If $\boldsymbol{u} : U \to R^m$

$$\boldsymbol{u}(x) = \left(u^1(x), \ldots\ldots, u^m(x)\right)(x \in U)$$

where $\boldsymbol{u} = \left(u^1, u^2, \ldots, u^m\right)$

(vii) The symbol

$$\int_{\Sigma} f dS$$

denotes the integral of f over $(n - 1)$ dimensional surface Σ in R^n.

(viii) The symbol

$$\int_{C} f\, dl$$

denotes line integral of f over the curve C in R^n.

(ix) The symbol

$$\int_{V} f\, dx$$

denote the volume integral of S over $V \in R^n$ and $x \in V$ is an arbitrary point.

(x) Averages:

$$⨍_{B(x,r)} f\, dy = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} f\, dy$$

= average of f over ball $B(x, r)$

$$⨍_{\partial B(n,r)} f\, dS = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(n,r)} f\, dS$$

= average of f over surface of ball $B(x, r)$

(xi) A function $u : U \to R$ is called Lipschitz continuous if

$$\left|u(x) - u(y)\right| \le C\left|x - y\right|$$

for some constant C and all $x, y \in U$. We denote

$$\text{Lip}\,[u] = \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|}$$

(xii) The convolution of functions *f*, *g* is denoted by $f * g$.

(c) *Notations for derivatives*

Let $u : U \to R,\ \ x \in U$

(i) $$\frac{\partial u(x)}{\partial x_i} = \underset{h\to 0}{lt}\, \frac{u(x+he_i)-u(x)}{h}$$

provided that the limit exists.We denote $\dfrac{\partial u}{\partial x_i}$ by u_{x_i}

Similarly $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ by $u_{x_i x_j}$

$\dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$ by $u_{x_i x_j x_k}$ etc.

(ii) *Multi-index Notation*

(a) A vector α of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_i is a non-negative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

(b) Given multi-index α, define

$$D^\alpha u(x) = \frac{\partial^\alpha u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

(c) If *k* is a non-negative integer

$$D^k u(x) = \{D^\alpha u(x)|,\ |\alpha| = k\}$$

the set of all partial derivatives of order *k*

e.g. *for k = 1*

$$Du = \left(u_{x_1}, \ldots, u_{x_n}\right)$$

$= \text{grad}(u)$

for k = 2

$$D^2u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \frac{\partial^2 u}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

is called Hessian Matrix.

(d) $\left|D^k u(x)\right| = \left\{\sum_{|\alpha|=k} \left|D^\alpha u\right|^2\right\}^{1/2}$

(iii) $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$

= Laplacian of u

= trace of Hessian Matrix.

(iv) Let $x, y \in R^n$ i.e. $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

Then we write

$$D_x u = \left(u_{x_1}, \ldots, u_{x_n}\right)$$

$$D_y u = \left(u_{y_1}, \ldots, u_{y_n}\right)$$

the subscript *x or y* denotes the variable w.r.t. differentiation is being taken

(d) *Function Spaces*

(i) $C(U) = \{u : U \to R \mid u \text{ is continuous}\}$

(ii) $C(\bar{U}) = \{u \in C(u) \mid u$ is uniformly continuous

on bounded subsets of U$\}$

(iii) $C^k(U) = \{u : U \to R \mid u$ is k times continuous differentiable $\}$

(iv) $C^k(\bar{U}) = \{u : C^k(U) \mid D^{\alpha}$u is uniformly continuous

on bounded subsets of U, for all $|\alpha| \le k\}$

(v) $C^{\infty}(U) = \{u : U \to R | u$ is infinitly differentiable$\}$

(vi) $C_C(U)$ means $C(U)$ has compact support.

Similarly ,$C_C^K(U)$ means $C^k(U)$ has compact support.

(vii) The function $u : U \to R$ is Lebsegue measurable over L^p if $\|u\|_{L^p(U)} < \infty$

The function $u : U \to R$ is Lebsegue measurable over L^{∞} if $\|u\|_{L^{\infty}(U)} < \infty$

where

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p \, dx\right)^{1/p} \qquad 1 \le p \le \infty$$

$$\|u\|_{L^{\infty}(U)} = \text{ess sup}_U \, | u |$$

(viii) $L^p(U) = \{u : U \to R \mid u$ is Lebsegue measurable over $L^p\}$

$L^{\infty}(U) = \{u : U \to R \mid u$ is Lebsegue measurable over $L^{\infty}\}$

(ix) $\|Du\|_{L^p(U)} = \| \, |Du| \, \|_{L^p(U)}$

Similarly $\|D^2u\|_{L^p(U)} = \| \, |D^2u| \, \|_{L^p(U)}$

(x) If $\underline{u} : U \to R^m$ is a vector

where $\underline{u} = (u^1, u^2, \ldots, u^m)$

then

$D^k \underline{u} = \left\{ D^\alpha \underline{u},\ |\alpha| = k \right\}$

Similarly other operators follow.

Big Oh (O) order

We say

$f = O(g)$ as $x \to x_O$ provided there exists a constant C such that

$|f(x)| \le C|g(x)|$

for all x sufficiently close to x_O

Little Oh (o) order

We say

$f = o(g)$ as $x \to x_o$ provided

$$\underset{x \to x_o}{lt} \frac{|f(x)|}{|g(x)|} \to 0$$

1.3 Inequalities

There are some fundamental inequalities

(a) *Cauchy's Inequality*

$$ab \le \frac{a^2}{2} + \frac{b^2}{2} (a,\ b \in R)$$

(b) *Holder's Inequality*

Let $1 \le p,\ q \le \infty$; $\frac{1}{p} + \frac{1}{q} = 1$

$u \in L^p(u), v \in L^q(u)$

$$\int_U |uv|\, dx \le \|u\|_{L^p(U)}\ \|v\|_{L^q(U)}$$

(c) *Minkowski's Inequality*

Let $1 \leq p \leq \infty$, and $u, v \in L^p(U)$

Then $\|u+v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}$

(d) *Cauchy Schwartz Inequality*

$|x \cdot y| \leq |x|\ |y| \qquad (x, y \in R^n)$

1.4 Calculus

(a) *Boundaries*

Let $U \subset R^n$ be open and bounded, $k = \{1, 2, \ldots, \}$

Definitions: (i) The boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exists $r > 0$ and a C^k function $\Upsilon : R^{n-1} \to R$ such that

$U \cap B(x^\circ, r) = \{x \in B(x^\circ, r) \mid x_n > \Upsilon(x_1, \ldots, x_{n-1})\}$

Similarly, ∂U is C^∞ if ∂U is C^k $(k = 1, 2, \ldots)$

Also ,∂U is analytic if Υ is analytic.

(ii) If U is C^1, then along ∂U, the outward unit normal at any point $x_0 \in \partial U$ is denoted by $\underline{v}(x^0) = (v_1, \ldots, v_n)$.

(iii) Let $u \in C^1(\bar{U})$ then normal derivative of u is denoted by

$$\frac{\partial u}{\partial v} = \underline{v} \cdot Du$$

(b) *Gauss- Green Theorem*

Let U be a bounded open subset of R^n and ∂U bes C^1. $u : U \to R^n$

also $u \in C^1(\bar{U})$ then

$$\int_U u_{x_i}\, dx = \int_{\partial U} u\nu^i \, dS \qquad (i = 1,\ 2,\ \dots,\ n)$$

(c) *Integration by parts formula*

Let $u,\ v \in C^1(\bar{U})$ then

$$\int_U u_{x_i} v \; dx = -\int_U u v_{x_i} \; dx + \int_{\partial U} u v \nu^i \;\; dS$$

Proof. By Gauss -Green's theorem

$$\int_U (uv)_{x_i} \; dx = \int_{\partial U} (uv) \;\; \nu^i dS$$

or $$\int_U u_{x_i} v \, dx + \int_U u v_{x_i} \; dx = \int_{\partial U} (uv)\nu^i \; dS$$

or $$\int_U u_{x_i} v \, dx = -\int_U u v_{x_i} \; + \int_{\partial U} (uv)\nu^i \; dS$$

(d) *Green's formulas*

Let $u,\ v \in C^2(\bar{U})$ then

(i) $$\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

Proof. $\int_U \Delta u dx = \int \left(u_{x_i}\right)_{x_i} \; dx$

Integrating by parts , taking the 2nd function as unity

$$= \int_{\partial U} u_{x_i} \nu^i dS$$

$$= \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

Hence the result.

(ii) $$\int_U Du.Dv\ dx = -\int_U u\Delta v\ dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u\ dS$$

Proof. $\int_U Du.Dv\ dx = -\int_U u\Delta v\ dx + \int_{\partial U} u\, Dv.\nu\, dS$

(integrating by parts)

$$= -\int_U u\Delta v\, dx + \int_{\partial U} u \frac{\partial v}{\partial \nu} dS$$

(iii) $$\int_U (u\Delta v - v\Delta u)\, dx = \int_{\partial U} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS$$

Proof. $\int_U u\Delta v\, dx = -\int_U Du\,.Dv\, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$

Similarly, $\int_U v\Delta u\, dx = -\int_U Du.\, Dv\, dx + \int_{\partial U} \frac{\partial u}{\partial \nu} u dS$

Subtracting, we get the result.

(e) Conversion of *n*-dimensional integrals into integral over sphere

(i) *Coarea formula*

Let $u : R^n \to R$ be Lipschitz continuous and assume that for $a.\ e\ .\ r \in R$, the level set

$\{x \in R^n \mid u(x) = r\}$

is a smooth and $n - 1$ dimensional surface in R^n. Suppose also $f : R^n \to R$ is smooth and summable. Then

$$\int_{R^n} f\,|Du|\, dx = \int_{-\infty}^{\infty} \left(\int_{\{u=r\}} f\ dS \right) dr$$

Cor. Taking $u(x) = |x - x_0|$

Let $f : R^n \to R$ be continuous and summable then

$$\int_{R^n} f\,dx = \int_0^\infty \left(\int_{\partial B(x_0,r)} f dS \right) dr$$

for each point $x_0 \in R^n$. or we can say

$$\frac{d}{dr}\left(\int_{B(x_0,r)} f\,dx \right) = \int_{\partial B(x_0,r)} f\,dS$$

for each $r > 0$.

(f) To construct smooth approximations to given functions

Def If $U \subset R^n$ is open, given $\in > 0$. We define

$U_\in := \{ x \in U \,|\, dist(x, \partial U) > \in \}$

Def. Standard Mollifier

Let $\eta \in C^\infty(R^n)$ such that

$$\eta(x) := \begin{Bmatrix} c \exp\left(\dfrac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{Bmatrix}$$

the constant c is chosen so that $\int_{R^n} \eta\,dx = 1$

Def. We define

$$\eta_\in(x) := \frac{1}{\in^n} \eta\left(\frac{x}{\in} \right)$$

for every $\in > 0$.

Properties. (i) The functions $\eta_\in$ are C^∞ since $\eta(x)$ are C^∞.

(ii) $\int_{R^n} \eta_\in \, dx = \frac{1}{\in^n} \int_{R^n} \eta\left(\frac{x}{\in}\right) dx$

$= \int_{R^n} \eta(x)\, dx$ (by definition of n-tuple integral)

$= 1.$

(g) Mollification of a function

If $f : U \to R$ is locally integrable

We define the mollification of f

$f^\in := \eta_\in * f$ in $U_\in$

$= \int_U \eta_\in (x-y) f(y) \, dy$

$= \int_{B(0,\in)} \eta_\in (y) \, f(x-y) \, dy$ (by definition)

Properties. (i) $f^\in \in C^\infty(U_\in)$

(ii) $f^\in \to f$ almost everywhere. (a.e.) as $\in \to 0$

(iii) If $f \in C(U)$ then $f^\in \to f$ uniformly on compact subset of *U*.

almost everywhere.

1.5 Function Analysis Concepts

(i) ***L^P* space**. Assume U to be a open subset of R^n and $1 \le p \le \infty$. If $f : U \to R$ is measurable, we define

$$\|f\|_{L^p(U)} := \begin{cases} \left(\int_U |f|^p \, dx\right)^{1/p} & \text{if } 1 \le p < \infty \\ \text{ess sup}_U |f| & \text{if } p = \infty \end{cases}$$

Transformation from Ball $B(x, r)$ **to unit Ball** $B(0, 1)$

Let $B(x, r)$ be a ball with centre x and radius r and B (0, 1) be an arbitrary point of $B(x, r)$ and z be an arbitrary point of B (0, 1) then relation between y and z is

$y = x + rz$.

1.6 Definition and classification of Partial Differential Equation

Many physical, geometric and probabilistic problems can be modelled by partial differential equations. In this section, we define the partial differential equation, system of partial differential equations, their classifications and the classical and weak solutions etc

Defination: **Partial Differential Equation**

A partial differential equation is an equation involving an unknown function of two or more variables and its partial derivatives i.e. Let U be an open subset of R^{n}4. An expression of the type

$$F\left(D^k u(x),\, D^{k-1} u(x),\, \ldots, Du(x), u(x), x\right) = 0 \qquad (x \in U) \qquad (1)$$

is called a k^{th}-order partial differential equation, where

$F : R^{n^k} \times R^{n^{k-1}} \times \ldots R^{n^x} \times R \times U \to R$ is given and

$u : U \to R$ is the unknown function.

Note that $D^k u(x) \in R^{n^k}$,

$D^{k-1} u(x) \in R^{n\,k-1}$,

$Du(x) \in R^n$,

$u(x) \in R$

Exp. Let

$\theta = \theta(x, y, z)$ where $(x, y, z) \in R^3$

then

$$f\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial x^2}, \frac{\partial^2\theta}{\partial y^2}, \frac{\partial^2\theta}{\partial x\partial y}\right) = 0$$

defines a 2nd order Partial Differential Equation over R^3, θ is the unknown function and *f* is prescribed

Classification of Partial Differential Equations

Partial Differential Equation can be classified into four categories

(a) Linear (b) Semi-linear (c) Quasi-linear (d) Non-linear.

(a) *Linear Partial Differential Equation*: A Partial Differential Equation of k^{th} order is called linear if it has the form

$$\sum_{|\alpha|\le k} a_0(x) D^\alpha u = f(x) \qquad (2)$$

for given functions $a_0\left(|\alpha| \le k\right), f$.

i.e. coefficient of derivatives are only functions of *x*.

Exp. (i) $u_t + b.Du = 0$

where $b \in R^n$ is a constant.

(ii) $\Delta u = 0$

(b) *Semi-linear Partial Differential Equation*: A Partial Differential Equation is called semi-linear if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1}u, \ldots, Du, u, x\right) = 0 \qquad (3)$$

i.e. coefficient of highest order derivative is a function of *x* only.

Exp. $\phi(x)\Delta u - u_x u_y = 0$

(c) *Quasi-linear Partial Differential Equation*: A Partial Differential Equation is called quasi-linear if it is of the form

$$\sum_{|\alpha|=k} \left(a_a D^{k-1}u, \ldots, Du, u, x\right) D^\alpha u + a_0\left(D^{k-1}u, \ldots, Du, u, x\right) = 0 \qquad (4)$$

i.e. coefficient of highest order derivative are lower order derivative and function of x but not same order derivatives.

Exp. $u_{xx}\,u_x + u_{yy}\,u_y + u = 0$

is quasi linear partial differential equation.

(d) *Non-linear Partial Differential Equation*: A Partial Differential Equation is non-linear in the highest order derivatives.

Exp. $u_{xx}\,u_{yy} + b\,Du = 0\,, b \in R^n$

is non-linear.

1.7 Definition and classification of system of Partial Differential Equation

Definition: **System of Partial Differential Equations**

An expression of the form

$$\underline{F}\left(D^k\,\underline{u}(x), D^{k-1}\,\underline{u}(x), \ldots, D\,\underline{u}(x), x\right) = 0, \qquad x \in U \qquad (5)$$

is called a k^{th} order system of partial differential equations in $\underline{u}$ where

$\underline{F} : R^{mn^k} \times R^{mn^{k-1}} \times \ldots R^{m^n} \times R^m \times U \to R^m$

is given.and

$\underline{u} = \left(u^1, u^2, \ldots, u^m\right)$ be the unknown function s.t $\underline{u} : U \to R^m$.

Exp. Navier ‘s equations of equilibrium in linear elasticity.

$\mu\,\Delta\,\underline{u} + (\lambda + \mu)D\;\; div\,\underline{u} = 0$

form a system of partial differential equation in $\underline{u} = \left(u^1, u^2, u^3\right)$.

Classification of System of Partial Differential Equations

Note: System of partial differential equations are classified in the same way as partial differential equations are classified.

Examples of Linear Partial Differential Equations

There are some well-known linear equations

(i) Laplace's equation

$$\Delta u = 0 \text{ or } \sum_i u_{x,x} = 0$$

(ii) Linear transport equation

$$u_t + \underline{b}.Du = 0, b \in R^n$$

$$Du = \left(u_{x_1}, \ldots, u_{x_n}\right)$$

(iii) Heat (Diffusion) equation

$$u_t - \Delta u = 0$$

(iv) Wave equation

$$u_{tt} - \Delta u = 0$$

These will be studied in detail later on.

1.8 Solutions of Partial Differential Equation

Solution. An expression of u which satisfies the given PDE is called a solution of the Partial Differential Equation.

Well posed problem. A given problem in Partial Differential Equation is well posed if

(i) the problem has a solution

(ii) solution is unique

(iii) solution depends continuously on the data given problem.

Classical Solution. If a solution of a given problem satisfies the above three conditions i.e. the solution of k^{th} order partial differential equation exists, is unique and is at least k times differentiable, then the solution is called classical solution. Solution of wave equation, Lalpace equation etc. are classical solutions.

Weak Solution. If a solution of a given problem exists and is unique but does not satisfy the conditions of differentiability, such solution is called weak solution.

Exp. The gas conservation equation

$$u_t + F(u)_n = 0$$

models a shock wave in particular situation. So solutions exists, is unique, but not continuous. Such solution is known as weak solution.

Remark. There are several physical phenomenon in which the problem has a unique solution, but does not satisfy the condition of differentiability. In such cases, we cannot claim that we are not able to find the solution rather such solutions are called weak solutions.

1.9 Suggested References

1. L.C. Evans, "Partial Differential Equations," American Mathematical Society, Rhade.
2. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum Outline Series, McGraw Hill Series.

Lesson 2

Solution of Linear Partial Differential Equations

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Structure

2.1 Introduction

In this lesson, we shall consider the solution of single linear equations ,namely, Transport equation,Laplace equation and Poisson Equation. Also,we discuss the properties of harmonic functions, such as Strong Maximum principle , estimates of derivatives, Harnack's Inequality etc.

2.2 Transport equation

2.2.1 Homogeneous problem

The simplest partial differential equation is the transport equation with constant coefficient, which is

$$u_t + b.Du = 0 \text{ in } R^n \times (0, \infty) \tag{1}$$

where b is a fixed vector in R^n ,i.e.

$b = (b_1, b_2, \dots, b_n)$ and

$u : R^n \times [0, \infty) \to R$

is the unknown function such that $u = u(x, t)$.

$x = (x_1 \dots\dots x_n) \in R^n$

denotes a spatial variable and $t \geq 0$ is the time variable.To solve (1), we observe the L.H.S. of equation (1) carefully, we find that it denotes the dot product of $(u_{x_1}, u_{x_2}, \dots, u_{x_n}, u_t)$ with $(b_1, \dots, b_n, 1)$. So, L.H.S. of equation (1) tells that the derivative of u in the direction of $(b, 1)$ is zero in R^{n+1} dimensional space. So ,if $(x, t) \in R^n \times (0, \infty)$, we define the parametric equation of line in the direction $(b, 1)$

$$z(s) := u(x + sb, t + s) \tag{2}$$

where $s \in R$, is the parameter.

Or $\quad z(s) = u(\theta, \phi)$

where $\theta = x + sb,\ \phi = t + s$

Differentiating w.r.t. s

$\dot{z}(s) = u_\theta b + u_\phi .1$

$= 0$ (using eqn 1)

$\Rightarrow z(s)$ = constant for each s, i.e., u is constant on the line through $(x,\ t)$ in the direction of $(b,\ 1)$.

Hence, we conclude that if we know the value of $u(x,\ t)$ at any point on each such line, we know $u(x,\ t)$ everywhere in $R^n \times [0,\ \infty)$.

2.2.2 Initial value problem

Consider the initial value problem

$u_t + b.Du = 0$ in $R^n \times (0,\ \infty)$, (3a)

$u = g$ on $R^n \times \{t = 0\}$ (3b)

where $b \in R^n$ and g is the prescribed function.

Solution. As above, L.H.S. of eq. (3a) represents the directional derivative of u in the direction of $(b,\ 1) \in R^n \times (0,\ \infty)$. Hence, the parametric equation of line through $(x,\ t)$ in the direction of $(b,\ 1)$ is given by

$z(s) := u(x + sb,\ t + s)$

where s is the parameter.

Also, $\dot{z}(s) = 0$ (using 3a)

Therefore, z (s) is constant on this line i.e. u $(x,\ t)$ is constant on the line through $(x,\ t)$ in the direction of $(b,\ 1)$. This line touches the plane $R^n \times \{t = 0\}$ for $s = -t$ i.e. the point $(x - tb,\ 0)$

where

$u(x,\ t) = g\ (x - tb)$ (using 3b)

Since u is constant on this line, so

$$u\,(x,\,t) = g\,(x - tb) \qquad \textit{for all } x \in R^n,\; t \geq 0 \tag{4}$$

is required solution of initial value problem.

Note. If the function g (x) is C^1 then Eq. (4) gives the classical solution of problem.

2.2.3 Non-homogenous problem

Consider the non-homogeneous case of transport equation

$$u_t + b.Du = f\,(\mathrm{x,t}) \;\text{ in }\; R^n \times (0,\,\infty) \tag{5}$$

with initial condition

$$u = g \text{ on } R^n \times \{t = 0\} \tag{6}$$

Solution. Fix a point $(x,\,t) \in R^{n+1}$, as discussed before, the equation of line passing through $(x,\,t)$ in the direction of (b, 1) is given by

$$z(s) = u\,(x + sb,\, t + s) \tag{7}$$

where s is the parameter.

Differentiating w.r.t. s

$$\dot{z}(s) = u_\theta\,(\theta,\,\phi).b \;+\; u_\phi\,(\theta,\,\phi)$$

where $\theta = x + sb, \phi = t + s$

$$\dot{z}(s) = f\,(x + sb,\, t + s) \qquad\qquad \text{(using 5)}$$

Integrating w.r.t. s from -t to 0

$$\int_{-t}^{0} \dot{z}(s)\,ds = \int_{-t}^{0} f\;(x + sb,\, t + s)\;ds$$

$$\left|z(s)\right|_{-t}^{0} \;=\; \int_{-t}^{0} f\;(x + sb,\, t + s)\;ds$$

Substitute $t + s = \psi,\; ds = d\psi$

$$z(0) - z(-t) = \int_0^t f\ (x + b\ (\psi - t),\ \psi)\ d\psi$$

$$u\ (x,\ t) - u\ (x - bt,\ 0) = \int_0^t f\ (x + b(s - t),\ s)\ ds \qquad \text{(using 7)}$$

$$\Rightarrow \qquad u\ (x,\ t) = u(x - bt,\ 0) + \int_0^t f\ (x + b\ (s - t),\ s)\ ds$$

$$= g\ (x - bt) + \int_0^t f\ (x + b\ (x - t),\ s)\ ds \qquad (8)$$

Equation (8) gives the solution for each $x \in R^n$ and $t \geq 0$. It is the required solution of initial value problem for non-homogeneous Transport equation.

2.3 Laplace's Equation

Let U be an open subset of R^n and $u : U \to R$, then the equation

$\Delta u = 0 \qquad x \in U$

or

$\sum_{i=1}^{n} u_{x_i x_i} = 0$

defines the Laplace's equation in *u*.

Harmonic function

A C^2 function satisfying the Laplace's equation is called Harmonic function.

Physical occurrence

We get the Laplace's equation in several physical phenomenon such as irrotational flow of incompressible fluid, diffusion problem ,conduction problem etc.

2.3.1 Fundamental solution

Consider the equation

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 \qquad (1)$$

where, $u : U \to R, \quad U \subset R^n, \; x \in U$.

Eq. (1) is a linear partial differential equation. To solve (1), we rotate the original coordinate system $Ox_1 \ldots x_n$ to $Ox'_1 x'_2 \ldots x'_n$ at angle θ about O (Fig. 1).

Let $l_{ij} = \cos\left(x'_i, \, x_j\right)$

We have $x_i{}' = \sum_j l_{ij} x_j$

Similarly, $x_i = \sum_j l_{ji} \, x'_j$

Table of direction cosines

	x_1	x_2		x_n
$x_1{}'$	l_{11}	l_{12}	...	l_{1n}
$x_2{}'$	l_{21}	l_{22}	...	l_{2n}
	...			
$x_n{}'$	l_{n1}	l_{n2}		l_{nn}

$$u = u\left(x_1, \, x_2, \, \ldots, \, x_n\right)$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x'_1} \frac{\partial x_1{}'}{\partial x_i} + \frac{\partial u}{\partial x_2{}'} \frac{\partial x_2{}'}{\partial x_i} + \ldots + \frac{\partial u}{\partial x_n{}'} \frac{\partial x_n{}'}{\partial x_i}$$

$$= \frac{\partial u}{\partial x'_1} l_{1i} + \frac{\partial u}{\partial x'_2} l_{2i} + \ldots + \frac{\partial u}{\partial x'_n} l_{ni}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = l_{1i} \left[\frac{\partial^2 u}{\partial x_1'^2} l_{1j} + \frac{\partial^2 u}{\partial x'_1 \, \partial x'_2} l_{2j} + \ldots + \frac{\partial^2 u}{\partial x'_1 \, \partial x'_n} l_{nj} \right]$$

$$+l_{2i}\left[\frac{\partial^2 u}{\partial x_1'\ \partial x_2'}l_{1j}+\frac{\partial^2 u}{\partial x_2'^2}l_{2j}+\ldots+\frac{\partial^2 u}{\partial x_2'\ \partial x_n'}l_{nj}\right]$$

$$+l_{ni}\left[\frac{\partial^2 u}{\partial x_1'\ \partial x_n'}\ l_{1j}+\frac{\partial^2 u}{\partial x_2'\ \partial x_n'}\ l_{2j}+\ldots+\frac{\partial^2 u}{\partial x_n'^2}\ l_{nj}\right]$$

Taking $i = j$

$$\frac{\partial^2 u}{dx_1^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}=\frac{\partial^2 u}{\partial x_1'^2}+\frac{\partial^2 u}{\partial x_2'^2}+\ldots+\frac{\partial^2 u}{\partial x_n'^2}$$

This shows that Laplace's equation remains invariant w.r.t. the transformation of coordinate axes. To find a solution of eq. (1) in R^n, we seek a radial solution, i.e.

$$u(x) := V(r)$$

where

$$r=|x|=\sqrt{x_1^2+x_2^2+\ldots+x_n^2} \tag{2}$$

$$\frac{\partial r}{\partial x_i}=\frac{1}{2}\left(x_1^2+\ldots+x_n^2\right)^{-1/2}2x_i=\frac{x_i}{r} \qquad x\neq 0$$

$$u_{x_i x_j}=V''\ (r)\frac{x_i x_j}{r^2}+V'\ (r)\frac{\delta_{ij}}{r}-\frac{V'\ (r)\,x_i x_j}{r^3}$$

Taking $i = j$

$$u_{x_i x_j}=V''\ (r)\frac{x_i x_i}{r^2}+V'\ (r)\frac{\delta_{ii}}{r}-\frac{V'\ (r)\,x_i x_i}{r^3}$$

$$\Delta u = V''\ (r)+\frac{(n-1)}{r}V'\ (r)$$

$$V''\ +\frac{n-1}{r}V'\ =0 \tag{3}$$

$$\frac{V''}{V'}=-\ \frac{n-1}{r}$$

Integrating w.r.t. r

$$\log V'\ =-\ (n-1)\ \log\ r\ +\ \log\ a$$

where log a is a constant, or

$$V' = \frac{a}{r^{n-1}}$$

Again integrating

$$V(r) = \begin{cases} a' \log r + b & n = 2 \\ \dfrac{a'}{r^{n-2}} + b & n \geq 3 \end{cases}$$

where a' and b are constants.

Hence if $r > 0$, the solution of Laplace's equation (1) is

$$u(x) = \begin{cases} a' \log|x| + b & x = 2 \\ \dfrac{a'}{|x|^{n-2}} + b & x \geq 3 \end{cases}$$

Without loss of generality we take $b = 0$. To find a', we normalize the solution, i.e.

$$\int_{R^n} u(x)\, dx = 1$$

which yields a'.Thus

$$u(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)\, |x|^{n-2}} & n \geq 3 \end{cases} \tag{4}$$

for each $x \in R^n$, $x \neq 0$.

The solution (4) is known as Fundamental solution of Laplace equation. We denote it by $\Phi(x)$.

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)\, |x|^{n-2}} & n \geq 3 \qquad x \neq 0 \end{cases}$$

Remark. The solution (4) is not valid for $x = 0$, since $u\,(x)$ is singular for $x = 0$.

2.3.2 Mean value theorem

Theorem. If $u \in C^2(U)$ is harmonic then

$$u(x) = \fint_{\partial B(x,r)} u\,ds = \fint_{B(x,r)} u\,dy \tag{1}$$

for each ball $B(x,r) \subset U$.

Proof.

$$\phi(r) := \fint_{\partial B(x,r)} u(y)\,ds(y) \tag{2}$$

Shifting the integral to unit ball, if z is an arbitrary point of unit ball then

$$\phi(r) := \fint_{\partial B(x,r)} u(x+rz)\,ds(z)$$

Differentiating w.r.t. r

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz).z\,ds(z)$$

$$= \fint_{\partial B(x,r)} Du(y).\frac{y-x}{r}\,ds(y)$$

$$= \fint_{\partial B(x,r)} Du(y).\upsilon\,ds(y)$$

where υ is unit outward normal to $\partial B(x,r)$.

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\,ds(y)$$

$$= \frac{1}{n\alpha(x)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \upsilon}\,ds(y)$$

$$= \frac{r}{n} \fint_{B(x,r)} \Delta u\,dy \qquad \text{(by Green's formula)}$$

$= 0$ since u is harmonic.

Hence $\phi(r)$ is independent of r.

So $\phi(r) = \underset{t \to 0}{\text{limit}} \fint_{\partial B(x,t)} u(y)\, ds(y)$

$= u\,(x)$ (3)

From (2) and (3)

$$u(x) = \fint_{\partial B(x,r)} u(y)\, ds(y) \tag{4}$$

For the second equality,

$$\int_{B(x,r)} u\ dy = \int_0^r \left(\int_{\partial B(x,t)} u\ ds \right) dt \quad \text{(using coarea formula)}$$

$$= \int_0^r u(x)\, n\,\alpha(n)\, t^{n-1}\ dt \qquad \text{(using 4)}$$

$$= u(x)\ \alpha(n) r^n$$

$$\Rightarrow \qquad u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u\, dy$$

$$= \fint_{B(x,r)} u\ dy \tag{5}$$

Combining (4) and (5), we get the result.

Note. The above formula is known as mean value theorem. Converse of this result is also true.

Converse of Mean Value Theorem

If $u \in C^2(U)$ satisfies the relation

$$u(x) = \fint_{\partial B(x,r)} u\ ds$$

for each ball $B(x,r) \subset U$. Then, u is harmonic in U.

Proof. Suppose that u is not harmonic, so

$$\Delta u \neq 0 .$$

Hence there exists a ball $B(x,r) \subset U$ such that $\Delta u > 0$ within $B(x,r)$

Preceeding ,as above,

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u \ dy > 0$$

Also $\phi(r) = u(x)$

$$\phi'(r) = 0$$

which is contrary to $\phi'(r) > 0$. So u is harmonic.

2.4 Poisson's equation

We observe that the solution of Laplace's equation $\Delta u = 0$, $\Phi(x)$ is harmonic for $x \neq 0$. Shifting the origin to a new point y, the Laplace equation remains unchanged. So $\Phi(x-y)$ is harmonic for $x \neq y$.

If $f : R^n \to R$

is harmonic;then $\Phi(x-y) \ f(y)$ is harmonic for each $y \in R^n$ and $x \neq y$.

If we take the sum of all different points y over R^n ,then,

$$\int_{R^n} \Phi(x-y) \ f(y) \ dy$$

is harmonic.No, since $\Delta u(x) = \int_{R^n} \Delta_x \Phi(x-y) \ f(y) \ dy$

is not valid near the singularity $x = y$. We must proceed more carefully.

Theorem. Suppose $f \in C_c^2\left(R^n\right)$ i.e. f is twice differentiable with compact support. Let

$$u(x) = \int_{R^n} \Phi(x-y) \ f(y) \ dy$$

$$= \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int_{R^n} \log |x-y| \; f(y)\, dy & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{R^n} \dfrac{f(y)}{|x-y|^{n-2}}\, dy & n \geq 3 \end{cases} \qquad (1)$$

is a solution of Poisson's equation

$$\Delta u = -f \text{ in } R^n \qquad (2)$$

Proof. To show that, u (x) represented by eq. (1), is a solution of eq. (2), we need to prove

(i) $u \in C^2\left(R^n\right)$

(ii) $-\Delta u = f$ in R^n

(i) We have

$$u(x) = \int_{R^n} \Phi(x-y) f(y)\, dy$$

By change of variable $x - y = z$

$$= u(x) = \int_{R^n} \Phi(z)\; f\;(x-z)\, dz$$

$$= \int_{R^n} \Phi(y)\; f\;(x-y)\, dy \qquad (3)$$

Let us calculate u_{x_i}. By definition

$$\frac{u(x+he_i)-u(x)}{h} = \int_{R^n} \Phi(y) \frac{\left[f(x+he_i-y)-f(x-y)\right]}{h}\, dy$$

where $h \neq 0$ and $e_i = (0, 0, ..., 1, ..., 0)$ (in ith place)

Taking the limit $h \to 0$, since $f \in C_c^2\left(R^n\right)$ so

$$\underset{h\to 0}{lt} \frac{f(x+he_i-y)-f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y)$$

Hence

$$\frac{\partial u(x)}{\partial x_i} = \int_{R'} \Phi(y)\, \frac{\partial f}{\partial x_i}(x-y)\, dy \qquad (i = 1, 2, \ldots, n)$$

$$< \infty$$

Similarly

$$\frac{\partial^2 u(x)}{\partial x_i\, \partial x_j} = \int_{R^n} \Phi(y) \frac{\partial^2 f(x-y)}{\partial x_i \partial x_j}\, dy \qquad (i, j = 1, 2, \ldots, n)$$

is continuous.So $u \in C^2(R^n)$.

(ii) By part (i)

$$\Delta u(x) = \int_{R^n} \Phi(y) \Delta_x f(x-y)\, dy$$

Since $\Phi(y)$ is singular at $y = 0$,so we include it in small ball $B(0, \in)$, where $\in > 0$.

Hence

$$\Delta u(x) = \int_{B(0,\, \in)} \Phi(y) \Delta_x f(x-y)\, dy + \int_{R^n - B(0, \in)} \Phi(y)\, \Delta_x f(x-y)\, dy$$

$$= I_{\in} + J_{\in} \tag{4}$$

where

$$I_{\in} := \int_{B(0,\, \in)} \Phi(y) \Delta_x f(x-y)\, dy \tag{5}$$

$$J_{\in} := \int_{R^n - B(0, \in)} \Phi(y)\, \Delta_x f(x-y)\, dy \tag{6}$$

$$|I_{\in}| = \left| \int_{B(0,\, \in)} \Phi(y)\, \Delta_x f(x-y)\, dy \right|$$

$$\leq |\Phi(y)|\, |\Delta_x f(x-y)| \left| \int_{B(0,\, \in)} dy \right|$$

$$|I_\in| \le \begin{cases} C \log\in.\in^2 & n=2 \\ C\,\dfrac{1}{\in^{n-2}}.\in^n & n\ge 3 \end{cases} \tag{7}$$

since $f \in C_C^2\left(R^n\right)$, so f is bounded.

Now

$$J_\in = \int_{R^n - B(0,\,\in)} \Phi(y)\ \Delta_x f\left(x-y\right)\, dy$$

$$= \int_{R^n - B(0,\,\in)} \phi(y) \Delta_y f\left(x-y\right)\, dy \qquad \left(\because \frac{\partial}{\partial x} = -\frac{\partial}{\partial y},\ \Delta_x = \Delta_y\right)$$

Integrating by parts

$$J_\in = -\int_{R^n - B(0,\in)} D\Phi(y).D_y f\left(x-y\right) dy + \int_{\partial B(0,\in)} \Phi(y) \frac{\partial f}{\partial \upsilon}(x-y)\, ds(y)$$

$$=:\ K_\in + L_\in \tag{8}$$

$$|L_\in| = \left| \int_{\partial B(0,\in)} \Phi(y)\ \frac{\partial f}{\partial \upsilon}(x-y)\ ds(y) \right|$$

$$\le \|Df\|_{L^\infty\left(R^n\right)}\ |\Phi(y)| \left| \int_{\partial B(0,\in)} ds(y) \right|$$

$$\le \begin{cases} C\ \in.\log\in & \text{for } n=2 \\ C\,\dfrac{1}{\in^{n-2}}.\in^{n-1} & \text{for } n\ge 3 \end{cases} \tag{9}$$

$$K_\in = -\int_{R^n - B(0,\in)} D\Phi(y)\ D_y f\left(x-y\right)\, dy$$

Integrating by parts

$$K_\in = \int_{R^n - B(0,\in)} \Delta\Phi(y) f\left(x-y\right)\, dy - \int_{\partial B(0,\in)} f\left(x-y\right) \frac{\partial \Phi}{\partial \upsilon}\, ds(y)$$

$$= -\int_{\partial B(0,\in)} f(x-y)\,\frac{\partial \Phi}{\partial \nu}\,ds \qquad \text{(since } \Delta\Phi = 0\text{)} \qquad (10)$$

But $\frac{\partial \phi}{\partial \nu} = D\Phi.\nu$

$$= -\frac{1}{n\alpha(n)}\,\frac{y}{|y|^n}.\frac{-y}{|y|} \qquad \text{(since normal is in opposite direction)}$$

$$= \frac{1}{n\,\alpha(n)}\,\frac{\in^2}{\in^{n+1}} \qquad \text{on } \partial B(0,\in)$$

$$= \frac{1}{n\,\alpha\,(n)\in^{n-1}}$$

Substituting in equation (10)

$$K_{\in} = -\frac{1}{n\,\alpha(x)\in^{n-1}} \int_{\partial\beta(0,\in)} f(x-y)\,ds(y)$$

$$= -\fint_{\partial B(0,\in)} f(x-y)\,ds(y)$$

Shifting the centre of $B(0,\in)$ to $\mathrm{B}(x,\in)$

$$K_{\in} = -\fint_{\partial B(0,\in)} f(y)\,ds(y)$$

$$= -f(x) \qquad \text{as} \quad \in \to 0 \qquad (11)$$

Using eq. (4), (7), (8), (9) and (11), and taking the limit as $\in \to 0$

$$\Delta u(x) = -f(x)$$

Hence the result.

2.5 Properties of Harmonic Functions

2.5.1 Strong Maximum principle

Suppose $U \subset R^n$ is open and bounded. Let $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U. Also U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u$$

then u is constant within U.

Proof. Let $u(x_0) = \max_{\bar{U}} = M$

Take $0 < r < dist(x_0, \partial U)$,

since u is harmonic,, for the ball $B(x_0, r)$

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dy \qquad \text{(By Mean Value theorem)}$$

$$\leq M .$$

Equality holds only if $u = M$ within $B(x_0, r)$.

Hence $u(y) = M$ for all $y \in B(x_0, r)$.

To show that this result holds for the set U, consider the set

$$X = \{x \in U \mid u(x) = M\}$$

We prove that X is both open and closed.

X is closed since if x is the limit point of set X, then $\exists$ a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$

Since u is continuous so $\{u(x_n)\} \to u(x)$

So $\quad u(x) = M$

$\Rightarrow \quad x \in X$

$\Rightarrow \quad X$ is closed.

To show that X is open, i.e. X is neighbourhood of each of its points .Let $x \in X$, there exists a ball $B(x,r) \subset U$ such that

$$u(x) = \fint_{B(x,r)} u \, dy$$

So $\quad x \in B(x,r) \subset X$

Hence X is open.

But U is connected. The only set which is both open and closed in U is U. So $U = X$. Hence $u(x) = M \;\forall\; x \in U$ .So u is constant in U.

Maximum Principle

Suppose $U \subset R^n$ is open and bounded. Let $u \in C^2(u) \cap C(\bar{U})$ is harmonic within U then

$$\max_{\bar{U}} u = \max_{\partial U} u$$

Proof. Suppose there exists a point $x_0 \in U$ such that $u(x_0) = \max_{\partial U} u = M$

So $u(y) \le u(x_0)$ for some y and suppose $x_0 \notin \partial U$ since U is harmonic, so by mean value theorem, there exists a ball $B(x_0, r) \subset U$ such that

$$u(x_0) = \fint_{\partial B(x_0, r)} u \, dS(y)$$

$$M \le \frac{1}{n\alpha(n) r^{n-1}} |u(y)| \left| \int_{\partial B(x_0, r)} dS(y) \right|$$

$$\le |u(y)|$$

Maximum value is less than $|u(y)|$ which is a contradiction. Hence $x_0 \in \partial U$. Hence the result.

Cor. If U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$\Delta u = 0$ in U

$u = g$ on ∂U

where $g \geq 0$ then u is positive everywhere in *U*.

Uniqueness of solution

There exists a unique solution of $C^2(U) \cap C(\bar{U})$ of the boundary value problem

$-\Delta u = f$ in U

$u = g$ on ∂U (1)

where $g \in C(\partial U),\ f \in C(U)$

Proof. Let u and $\bar{u}$ be two solutions of problem (1) then

$-\Delta u = f$ in U

$u = g$ on ∂U

$-\Delta \bar{u} = f$ in U

$\bar{u} = g$ on ∂U

Let $w := \pm(u - \bar{u})$

$\Delta w = 0$ in U

$w = 0$ on ∂U

i.e. w is harmonic in U and w attains maximum value on boundary which is zero. If U is connected then w is constant. So $w = 0$ in U

So $u = \bar{u}$ in *U*.

2.5.2 Regularity property

Regularity property states that if $u \in C^2(U)$ is harmonic then $u \in C^\infty(U)$ i.e. Harmonic functions are regular function or smooth functions.

Theorem. If $u \in C(U)$ satisfies the mean value property

$$u(x) = \fint_{B(x,r)} u\, dy = \fint_{\partial B(x,r)} u\, dS \qquad (1)$$

for every ball $B(x,r) \subset U$ then $u \in C^{\infty}(U)$.

Proof. Consider the set

$U_{\in} = \{x \in U \mid dist(x, \partial u) > \in\}$ and η be the standard mollifier.

$$u^{\in} :\equiv \eta_{\in} * u \text{ in } U_{\in} \qquad (2)$$

We first show that $u^{\in} \in C^{\infty}(U_{\in})$

Fix $x \in U_{\in}$ where $x = (x_1, x_2, \ldots, x_n)$.

Let h be very small such that $x + he_i \in U_{\in}$

$$u^{\in}(x) = \eta_{\in} * u$$

$$= \frac{1}{\in^n} \int_{U_{\in}} \eta\left(\frac{x-y}{\in}\right) u(y)\, dy \qquad (3)$$

$$u^{\in}(x + he_i) = \frac{1}{\in^n} \int_{U_{\in}} \eta\left(\frac{x-y+he_i}{\in}\right) u(y)\, dy \qquad (4)$$

Subtracting (3) and (4)

$$\frac{u^{\in}(x+he_i) - u^{\in}(x)}{h} = \frac{1}{\in^n} \int_{U_{\in}} \left[\frac{\eta\left(\frac{x-y+he_i}{\in}\right) - \eta\left(\frac{x-y}{\in}\right)}{h}\right] u(y)\, dy$$

Taking the limit as $h \to 0$

$$\frac{\partial u^{\in}}{\partial x_i} = \frac{1}{\in^{n+1}} \int_{U_{\in}} \frac{\partial \eta(\frac{x-y}{\in})}{\partial x_i} u(y)\, dy = \int_{U_{\in}} \frac{\partial \eta_{\in}(x-y)}{\partial x_i} u(y)\, dy$$

since $\eta \in C^{\infty}(R^n)$

so $\dfrac{\partial u^{\epsilon}}{\partial x_i}$ exists.

Similarly $D^{\alpha} u^{\epsilon}$ exists for each multiindex α

So $u^{\epsilon} \in C^{\infty}(U_{\epsilon})$

We now show that $u \equiv u^{\epsilon}$ on U_{ϵ}

Let $x \in U_{\epsilon}$ then

$$u^{\epsilon}(x) = \int_U \eta_{\varepsilon}(x-y)\,u(y)\,dy$$

$$= \int_{B(x,\epsilon)} \frac{1}{\epsilon^n}\eta\left(\frac{|x-y|}{\epsilon}\right)u(y)\,dy \qquad \text{(by definition)}$$

$$= \frac{1}{\epsilon^n}\int_0^{\epsilon} \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B(x,r)} u(y)\,dS\right)dr \qquad \text{(using the cor. of coarea formula)}$$

$$= \frac{1}{\epsilon^n}\int_0^{\epsilon} \eta\left(\frac{r}{\epsilon}\right) n\alpha(n) r^{n-1} u(x)\,dr \qquad \text{(by 1)}$$

$$= \frac{n\alpha(n)u(x)}{\epsilon^n}\int_0^{\epsilon} \eta\left(\frac{r}{\epsilon}\right) r^{n-1}\,dr$$

$$= \frac{u(x)}{\epsilon^n}\int_{B(0,\epsilon)} \eta\left(\frac{y}{\epsilon}\right) dy$$

$$= u(x)\int_{B(0,\epsilon)} \eta_{\epsilon}(y)\,dy \qquad \text{(by definition)}$$

$$= u(x)$$

So $u^{\epsilon} \equiv u$ in U_{ϵ} and so $u \in C^{\infty}(U_{\epsilon})$ for each $\epsilon > 0$.

Remark. The above property holds for each $\epsilon > 0$. It may happen u may not be smooth or even continuous upto to ∂U.

2.5.3 Estimates of derivatives

Theorem. Assume that u is harmonic in U then

$$\left|D^{\alpha}u(x_0)\right| \le \frac{C_k}{r^{n+k}}\|u\|_{L^1(B(x_0,r))} \tag{1}$$

for each ball $B(x_0,r)\subset U$ and each multiindex α of order k where

$$C_0=\frac{1}{\alpha(n)}$$

$$C_k=\frac{\left(2^{n+1}nk\right)^k}{\alpha(n)} \qquad k = 1, 2, \ldots \tag{2}$$

Proof. We prove it by induction .For $k = 0$, $\alpha = 0$.

To show $\left|u(x_0)\right| \le \frac{1}{r^n\alpha(n)}\|u\|_{L^1(B(x,r))}$

By mean value theorem

$$u(x_0)=\fint_{B(x_0,r)} u(y)\,dy \text{ for each ball } B(x_0,r)\subset U$$

$$u(x_0)=\frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} u(y)\,dy$$

$$\left|u(x_0)\right| \le \frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,r))} \tag{3}$$

$$\left|D^0u(x_0)\right| \le \frac{C_0}{r^n}\|u\|_{L^1(B(x_0,r))}$$

Hence the result.

$k = 1$

To show

$$\left|D\,u(x_0)\right| \le \frac{C_1}{r^{n+1}}\|u\|_{L^1(B(x_0,r))}$$

where $\qquad C_1=\dfrac{2^{n+1}\,n}{\alpha(n)}$

Consider

$$\Delta u_{x_i} = \frac{\partial^2}{\partial x_1^2}\left(u_{x_i}\right) + \ldots.. + \frac{\partial^2}{\partial x_n^2}\left(u_{x_i}\right)$$

$$= \frac{\partial}{\partial x_i}\left(\Delta u\right) = 0$$

So u_{x_i} is harmonic. By Mean Value theorem

$$\left|u_{x_i}\left(x_0\right)\right| = \left| \fint_{B(x_0,\frac{r}{2})} u_{x_i}\ dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0,\frac{r}{2}\right)} u_{x_i}\ dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0,\frac{r}{2}\right)} u\ v_i\ dS \right| \qquad \text{(By Guass -Green Theorem)}$$

$$= \left|\frac{2n}{r}\right| \left| \fint_{\partial B\left(x_0,\frac{r}{2}\right)} u\ v_i\ dS \right|$$

$$\leq \frac{2n}{r} \left\|u\right\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)} \qquad (4)$$

If $x \in \partial B\left(x_0, \frac{r}{2}\right)$ then $B\left(x, \frac{r}{2}\right) \subset B\left(x_0, r\right) \subset U$

By eq. (3)

$$\left|u(x)\right| \leq \frac{2^n}{\alpha(n) r^n} \left\|u\right\|_{L^1\left(B\left(x,\frac{r}{2}\right)\right)}$$

$$\leq \frac{2^n}{\alpha(n) r^n} \left\|u\right\|_{L^1(B(x_0,r))}$$

Hence

$$\|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)} \le \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))} \tag{5}$$

Combining (4) and (5)

$$\left|u_{x_i}(x_0)\right| \le \frac{2^{n+1}\, n}{\alpha(n)\, r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

or

$$\left|D^\alpha u(x_0)\right| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

Hence result is true for $k = 1$.

Assume that result is true for each multiindex of order less than or equal to $k - 1$ for all balls in U.

Fix $B(x_0, r) \subset U$ and α be multiindex with $|\alpha| = k$

$$D^\alpha u = \left(D^\beta u\right)_{x_i} \text{ for some } i = \{1, 2, \ldots, n)$$

where $|\beta| = k - 1$

Consider the ball $B\left(x_0, \frac{r}{k}\right)$,

$$\left|D^\alpha u(x_0)\right| = \left|\left(D^\beta u\right)_{x_i}\right|$$

Proceeding as in eq. (4)

$$\le \frac{kn}{r} \left\|D^\beta u\right\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{k}\right)\right)} \tag{6}$$

If $x \in \partial B\left(x_0, \frac{r}{k}\right)$ then

$$B\left(x, \frac{k-1}{k} r\right) \subset B(x_0, r) \subset U$$

Also by assumption, in the ball $B\left(x, \frac{k-1}{k} r\right)$

$$\left|D^{\beta}u(x_0)\right| \le \frac{\left[2^{n+1}\ n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}} \|u\|_{L^1\left(B\left(x,\frac{k-1}{k}r\right)\right)}$$

$$\le \frac{\left[2^{n+1}\ n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}} \|u\|_{L^1(B(x_0,r))} \tag{7}$$

Combining eq. (6) and (7)

$$\left|D^{\alpha}u(x_0)\right| \le \frac{kn}{r}\frac{\left[2^{n+1}\ n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}} \|u\|_{L^1(B(x_0,r))}$$

$$= \frac{\left(2^{n+1}\ nk\right)^{k}}{\alpha(n)r^{n+k}}\left(\frac{k}{k-1}\right)^{n}\frac{1}{2^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

$$\le \frac{\left(2^{n+1}\ nk\right)^{k}}{\alpha(n)r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

Since $\frac{1}{2}\left[\frac{k}{2(k-1)}\right]^{n} < 1 \quad \text{for } \forall k \ge 2$

Hence result holds for $|\alpha| = k$.

2.5.4 Liouville's theorem

Suppose $u : R^n \to R$ is harmonic and bounded. Then u is constant.

Proof. Fix $x_0 \in R^n$, $r > 0$ then by mean value theorem

$$\left|Du(x_0)\right| = \left|u_{x_i}(x_0)\right| = \left| ⨍_{B(x_0,\frac{r}{2})} u_{x_i}\ dx\right|$$

$$= \left|\frac{2^n}{\alpha(n)r^n}\int_{\partial B\left(x_0,\frac{r}{2}\right)} u\,\nu\, dS\right| \qquad \text{(By Guass Green's theorem)}$$

$$\leq \frac{2n}{r} \|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)}$$

If $x \in \partial B\left(x_0, \frac{r}{2}\right)$ then $B\left(x, \frac{r}{2}\right) \subset B(x_0, r)$

$$|u(x)| \leq \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))}$$

Hence $|u_{x_i}(x_0)| \leq \frac{2n}{r}\left(\frac{2}{r}\right)^n \frac{1}{\alpha(n)} \|u\|_{L^1(B(n_0,r))}$

$$= \frac{2^{n+1}\, n}{r^{n+1}\alpha(n)} \|u\|_{L^1(B(x_0,r))}$$

$$\leq \frac{n\ 2^{n+1}}{r} \|u\|_{L^\infty(R^n)}$$

$$\to 0 \ \text{ as } \ r \to \infty$$

Hence $Du = 0$ so u is constant.

Representation formula. For $n \geq 3$

Let $f \in C_C^2(R^n)$, then every bounded solution of Poisson's equation

$$-\Delta u = f \quad \text{in} \quad R^n \tag{1}$$

has the form

$$u(x) = \int_{R^n} \Phi(x-y) f(y)\, dy + C \qquad (x \in R^n)$$

where C is a constant and $\Phi(x)$ is the solution of Laplace equation.

Proof. For $n \geq 3$

$$\Phi(x) = \frac{1}{n\alpha(n)\,(n-2)\,|x|^{n-2}} \to 0 \ \text{ as } \ |x| \to \infty$$

$\Rightarrow \qquad \Phi(x)$ is bounded.

Let u be a solution of eq. (1) so

$$u = \int_{R^n} \Phi(x-y) f(y)\, dy$$

is bounded. Since $f \in C^2(R^n)$ and $\Phi(x)$ is bounded for $n \geq 3$.

Let $\bar{u}$ be bounded solution of eq. (1)

Define $w = u - \bar{u}$

$$\Delta w = 0$$

and w is bounded. (Difference of two bounded functions)

By Liouville's theorem

w = constant

or $u - \bar{u} = -C$

$$\bar{u} = u + C$$

Hence the result.

Remark. The above results does not hold for n = 2. Since $\Phi(x) = -\frac{1}{2\pi}\log|x|$ is not bounded as $|x| \to \infty$.

2.5.5 Analyticity property

Theorem. Let u is harmonic in $U \subset R^n$ then u is analytic.

Proof. Let x_0 be any point in U. We now show that u can be represented by a convergent power series in the neighbourhood of x_0 . The Taylor series of u about x_0

$$\sum_{\alpha} \frac{D^\alpha u(x_0)\,(x - x_0)^\alpha}{\alpha!} \tag{1}$$

converges.

Let $r := \frac{1}{4} dist(x_0, \partial U)$

$$\text{Let } M := \frac{1}{\alpha(n)\, r^n} \|u\|_{L^1(B(x_0,2r))} \tag{2}$$

$$< \infty$$

For each $x \in B(x_0, r)$, $B(x,r) \subset B(x_0, 2r) \subset U$

By estimates of derivatives

$$\left|D^\alpha u(x_0)\right| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

where $C_k = \dfrac{\left(2^{n+1}\, nk\right)^k}{\alpha(n)}$ for each $|\alpha| = k$

so

$$\left\|D^\alpha\, u(x_0)\right\|_{L^\infty(B(x_0,r))} \le \frac{\left(2^{n+1}\;\, nk\right)^k}{\alpha(n)\, r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

$$\le M\left(\frac{2^{n+1}\;\, n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|} \qquad \text{(using 1)} \qquad (3)$$

By Sterling's formula

$$\underset{k\to 0}{lt}\, \frac{k^{k+\frac{1}{2}}}{k!\; e^k} = \frac{1}{\sqrt{2}\;\pi}$$

$\Rightarrow \qquad k^k \le C\; k!\; e^k$ where C is a constant

or $\qquad |\alpha|^{|\alpha|} \le C|\alpha|!\; e^{|\alpha|}$ (4)

Also by Multinomial theorem

$$n^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}$$

$$|\alpha|! \le n^{|\alpha|}\;\; \alpha! \qquad (5)$$

Using (4) and (5) in equation (3)

$$\left\|D^\alpha u(x_0)\right\|_{L^\infty(B(x_0,r))} \le M\left(\frac{2^{n+1}\;\, n}{r}\right)^{|\alpha|} C\; e^{|\alpha|}\; n^{|\alpha|}\;\; \alpha!$$

$$\le MC\left(\frac{2^{n+1}\;\, n^2 e}{r}\right)^{|\alpha|}\; \alpha!$$

We claim that the power series in (1) converges provided

$$|x - x_0| < \frac{r}{2^{n+2}\ n^3 e}$$

The remainder term after N term is

$$R_N(x) = \sum_{|\alpha|=N}^{\infty} \frac{D^\alpha u\left(x_0 + t(x - x_0)\right)\ (x - x_0)^\alpha}{\alpha!}$$

for some $0 < t < 1$.

$$|R_n(x)| \le CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}\ n^2\ e}{r}\right)^N \left(\frac{r}{2^{n+2}\ n^3\ e}\right)^N$$

$$\le C\ M \sum_{|\alpha|=N} \left(\frac{1}{2n}\right)^N$$

$$\le \frac{C\ M}{2^N} \to 0 \quad \text{as} \quad N \to \infty$$

Hence series is convergent. So u (x) is analytic in neighbourhood of x_0. But x_0 is arbitrary point of U. So u is analytic in U.

2.5.6 Harnack's Inequality

Harnack's inequality shows that the values of non-negative harmonic functions within open connected subset of U, are comparable.

Statement For each connected open set $V \subset\subset U$, there exists a positive constant C, depending upon V, such that

$$\sup_V u \le C \inf_V u \qquad (1)$$

for all non-negative harmonic function u in U.

The equation (1) is equivalent to

$$\frac{1}{C}\ u(y) \le u(x) \le C\, u(y) \qquad \forall x, y \in V$$

Proof. Let $r := \frac{1}{4}\ dist\ (V, \partial U)$

Choose $x, y \in V$, $|x-y| \le r$

Then

$$u(x) = \fint_{B(x,2r)} u\, dz = \frac{1}{\alpha(n)\,(2r)^n} \int_{B(x,2r)} u\, dz$$

$$\ge \frac{1}{\alpha(n)\,(2r)^n} \int_{B(y,r)} u\, dz \qquad (\because B(x,2r) \supset B(x,r))$$

$$= \frac{1}{2^n} \fint_{B(y,r)} u\, dz$$

$$= \frac{1}{2^n}\, u(y)$$

or

$$2^n\, u(x) \ge u(y) \tag{2}$$

Interchanging the role of x and y

$$2^n u(y) \ge u(x) \tag{3}$$

Combining (2) and (3)

$$2^n\, u(y) \ge u(x) \ge \frac{1}{2^n}\, u(y) \qquad x, y \in V$$

Since V is connected, $\bar{V}$ is compact, so $\bar{V}$ can be covered by a chain of finite number of balls $\{B_i\}_{i=1}^{N}$ such that $B_i \cap B_j \ne \phi$ for $i \ne j$ each of radius $\frac{r}{2}$. So

$$u(x) \ge \left(\frac{1}{2^n}\right)^N u(y) \qquad (\text{Since } x, y \in V \text{ so } x, y \in \text{ ball } B_i)$$

$$u(x) \ge \frac{1}{C}\, u(y)$$

Similarly, $C\, u(y) \ge u(x)$

or

$$\frac{1}{C}u(y) \le u(x) \le C\,u(y)$$

for all $x, y \in V$.

Consider the inequality

$$u(x) \le C\ u(y)$$

Let $\{u(x_i)\}_{i=1}^{k} \in U$ be a sequence in n.b.d. of u (x).

So

$$u(x_1) \le C\ u(y)$$

$$\ldots$$

$$u(x_n) \le C\ u(y)$$

$$l.u.b.\{u(x_i)\} \le u.b.\ \{u(x_i)\}$$

$$\sup\ \{u(x_i)\} \le C\ u(y)$$

or $$\sup_{x \in V}\ u(x) \le C\ u(y)$$

$$\Rightarrow \qquad C\ u(y) \ge \sup_{x \in V}\ u(x)$$

Let $\{u\ (y_j)\}_{j=1}^{k} \in U$ be sequence in the n.b.d. u (y)

$$u(y_1) \ge \frac{1}{C}\ \sup\ u(x)$$

$$u(y_2) \ge \frac{1}{C}\ \sup\ u(x)$$

$$u(y_m) \ge \frac{1}{C}\ \sup\ u(x)$$

$$\text{g.l.b.}\ \{u(y_j)\} \ge l.b.\ \{u(y_j)\}$$

$$\inf_{y \in V}\{u(y)\} \ge \frac{1}{C}\ \sup_{x \in V}\ u(x)$$

or

$$C \inf_{y \in V} \{u(y)\} \geq \sup_{x \in V} u(x)$$

Hence the result.

2.6 Suggested References

1. L.C. Evans, "Partial Differential Equations," American Mathematical Society, Rhade.
2. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum Outline Series, McGraw Hill Series.

Lesson 3

Green's Function

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Structure

3.1 Introduction

We now consider an important tool to solve the boundary value problem

$-\Delta u = f$ in U

u = g on ∂U

where $U \subset R^n$ is open and bounded and ∂U is C^1, which is; Green's function. We obtain the derivation of Green's function and discuss its characteristics .Later on , we find the solution of heat equation.

3.2 Derivation of Green's function

Consider the boundary value problem

$-\Delta u = f \quad$ in U (1)

$u = g \quad$ on ∂U (2)

Solution Let $u \in C^2(U)$ and fix $x \in U$, choose $\in > 0$ such that $B(x,\in) \subset U$.

In the region $V_\in = U - B(x,\in)$, applying Green's formula to $u(y)$ and $\Phi(y-x)$

$$\int_{V_\in} \left[u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\ dy \right]$$

$$= \int_{\partial V\in} \left[u(y)\frac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\frac{\partial u(y)}{\partial\nu} \right] ds(y)$$

where ν is the outward unit normal to $\partial V_\in$.Hence

$$-\int_{V_\in} \Phi(y-x)\ \Delta u(y)\ dy =$$

$$\int_{\partial U + \partial B(x,\in)} \left[u(y)\frac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\frac{\partial u(y)}{\partial\nu} \right] ds(y) \tag{3}$$

$\left(\because\ \Delta\Phi(y-x) = 0 \text{ for } x \neq y\right)$

Now

$$\left| \int_{\partial B(x,\in)} \Phi(y-x)\frac{\partial u(y)}{\partial\nu}\ ds(y) \right| \le \|Du\|_{L^\infty(\partial B(x,\in))} \left|\Phi(y-x)\right| \left|\int ds(y)\right|$$

$$\le C\ \frac{1}{\in^{n-2}} n\alpha(n) \in^{n-1}$$

$$\to 0 \quad \text{as} \quad \in \to 0. \tag{4}$$

Again

$$\int_{\partial B(x,\in)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,ds(y) = \int_{\partial B(0,\in)} u(y+x)\frac{\partial \Phi}{\partial \nu}(y)\,ds(y)$$

Using

$$D\Phi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} \qquad y \neq 0$$

$$\nu = \frac{-y}{|y|}$$

$$\nu = \int_{\partial B(0,\in)} u(y+x)\frac{1}{n\alpha(n)\in^{n-1}}\,ds(y)$$

$$= \frac{1}{n\alpha(n)\in^{n-1}}\int_{\partial B(x,\in)} u(y)\,ds(y)$$

$$= ⨍_{\partial B(x,\in)} u(y)\,ds(y)$$

$$\to u(x) \qquad \text{as} \quad \in \to 0 \tag{5}$$

Using (4) and (5) in equation (3) and making $\in \to 0$

$$-\int_U \Phi(y-x)\Delta(y)\,dy$$

$$= \int_{\partial U}\left[u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}\right] ds(y) + u(x)$$

$$(y \neq x)$$

Hence

$$u(x) = \int_{\partial U}\left[\Phi(y-x)\frac{\partial u}{\partial \nu} - u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\right] ds(y)$$

$$-\int_U \Phi(y-x)\Delta u(y)\,dy \tag{6}$$

Equation (6) is valid for any point $x \in U$ and any function $u \in C^2(u)$. Equ. (6) gives the solution of problem defined by equ. (1) and (2) provided that $u(y), \frac{\partial u}{\partial \nu}$ are known on the boundary ∂U and the value of Δu in *U*. But $\frac{\partial u}{\partial \nu}$ is unknown to us along the boundary. For it, we define a correction term formula. $\phi^x(y)$ given by the solution of

$$\Delta \phi^x = 0 \qquad \text{in } U$$

$$\phi^x = \Phi(y-x) \text{ on } \partial U \tag{7}$$

Applying Green's theorem to $\phi^x(y)$

$$\int_U \left[u(y)\Delta\phi^x - \phi^x \Delta u(y)\right] dy$$

$$= \int_{\partial U} \left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu}\right] ds$$

Thus

$$-\int_U \phi^x \Delta u(y)\, dy = \int_{\partial U} \left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu}\right] dx \qquad \text{(by equ. 7)} \tag{8}$$

Adding equ. (6) and (8)

$$u(x) = -\int_U \left[\Phi(y-x) - \phi^x(y)\right] \Delta u(y)\, dy$$

$$-\int_{\partial U} \frac{\partial}{\partial \nu}\left[\Phi(y-x) - \phi^x(y)\right] u(y)\, dy \tag{9}$$

We define the Green's function $G(x,y)$

$$G(x,y) := \Phi(y-x) - \phi^x(y) \qquad x, y \in U,\ x \neq y \tag{10}$$

for the region *U*.

From equ. (9) and (10)

$$u(x) = -\int_U G(x,y)\, \Delta u(y)\, dy - \int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) \tag{11}$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y).\nu(y)$$

is normal derivative of G w.r.t. y.

Equ. (11) is independent of $\frac{\partial u}{\partial \nu}$.

Hence the boundary value problem given by equ. (1) and (2) can be solved in term of Green's function and the solution is given by equ. (11).

Equ. (11) is known as **Representation Formula** for Green's function.

Remark: (i) By definition of Green's function $G(x,y)$, for given x,

$-\Delta G(x,y) = \delta(x)$ in U

$G = 0$ on ∂U

where $\delta(x)$ is Dirac Delta function.

(ii) It is not so easy to construct G (x, y) for arbitrary region. We can construct for simple geometries.

3.3 Characteristics of Green's function

Theorem. Show that for all $x, y \in U$, $x \neq y$ $G(x,y)$ is symmetric i.e. $G(x,y) = G(y,x)$

Proof. Fix x, $y \in U$ $(x \neq y)$.

Define $v(z) := G(x,z)$

$$= \Phi(z-x) - \phi^x(z) \qquad z \in U,\ z \neq x$$

$w(z) := G(y,z)$

$$= \Phi(z-y) - \phi^y(z) \qquad z \in U,\ z \neq y \tag{1}$$

So $\Delta v(z) = 0$ in U

Similarly $\Delta w(z) = 0$ in U $\quad (z \neq x, y)$. (2)

On ∂U

$$v(z) = 0$$

$$w(z) = 0 \tag{3}$$

Applying Green's formula on the region

$$V = U - \left[B(x,\in) \cup B(y,\in)\right] \text{ for sufficiently small } \in > 0$$

$$\int_V (w\Delta v - v\Delta w)\, dz = \int_{\partial V} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(z)$$

(ν is outward drawn normal.)

$$= \int_{\partial U + B(x,\in) + \partial B(y,\in)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(z) \quad x, y \in U,\ x \neq y$$

$$\Rightarrow \quad \int_{\partial B(x,\in)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) dz + \int_{\partial B(y,\in)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(z) = 0$$

or

$$\int_{\partial B(y,\in)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) ds(z) = \int_{\partial B(x,\in)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(y) \tag{4}$$

Let us compute

$$\int_{\partial B(x,\in)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds$$

As w is smooth near x, so

$$\left| \int_{\partial B(x,\in)} \frac{\partial w}{\partial \nu} ds(z) \right| \leq \|Dw\|_{\partial B(x,\in)} \left| \int ds \right|$$

$$\leq C \in^{n-1}$$

$$\to 0 \text{ as } \in \to 0 \tag{5}$$

$$\int_{\partial B(x,\in)} w \frac{\partial v}{\partial \nu} ds(z)$$

$$= \int_{\partial B(x,\in)} w(z) \frac{\partial}{\partial \nu}\left[\Phi(z-x) - \phi^x(z)\right] ds(z)$$

$$= \int_{\partial B(x,\in)} w(z) \frac{\partial \Phi(z-x)}{\partial \nu} ds(z) - \int_{\partial B(x,\in)} w(z) \frac{\partial \phi^x}{\partial \nu}(z)\, ds(z)$$

$$= \int_{\partial B(x,\in)} w(z) \frac{\partial \Phi(z-x)}{\partial \nu} ds(z) - \int_{B(x,\in)} \Delta\phi^x w(z)\, dz$$

$$= \int_{\partial B(x,\in)} w(z) \frac{\partial \Phi(z-x)}{\partial \nu} ds(z) - 0 \qquad \left(\because \phi^x \text{ is smooth in } U\right)$$

(6)

Now

$$\int_{\partial B(x,\in)} w(z) \frac{\partial \Phi(z-x)}{\partial \nu}\, ds(z) = \int_{\partial B(0,\in)} w(z+x) \frac{\partial \Phi(z)}{\partial \nu}\, ds(z)$$

$$= \frac{1}{n\alpha(n) \in^{n-1}} \int_{\partial B(x,\in)} w(z)\, ds(z)$$

$$= \int_{\partial B(x,\in)} w(z) ds(z)$$

$$\to w(x) \qquad \text{as} \quad \in \to 0 \tag{7}$$

Combining equ. (5), (6) and (7) and taking limit $\in \to 0$

$$\underset{\in\to 0}{lt} \int_{\partial B(x_1 \in 1)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(z) \to w(x)$$

Similarly, $\underset{\in\to 0}{lt} \int_{\partial B(y,\in)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) ds(z) \to v(y)$

Hence equation (4) gives

$w(x) = v(y)$

$\Rightarrow \qquad G(x,y) = G(y,x)$.

Hence the result.

Def. Let $x \in R_+^n$, then the reflection of a point x w.r.t. ∂R_+^n is the point

$$\bar{x} = (x_1, x_2, \ldots, x_{n-1}, -x_n)$$

Example. Solve the boundary value problem

$$\Delta u = 0 \quad \text{in } R_+^n$$

$$u = g \quad \text{on } \partial \mathrm{R}_+^n$$

with the help of Green's function.

Sol. Let $x, y \in R_+^n, \; x \neq y$.

By definition, $G(x,y) = \Phi(y-x) - \phi^x(y)$

We choose the corrector term

$$\phi^x(y) = \Phi(y - \bar{x}) \tag{1}$$

where $\bar{x}$ is reflection of x w.r.t. ∂R_+^n.

Clearly $\Delta \phi^x = 0 \quad$ in R_+^n

Now, $\Phi(y-\bar{x}) = \dfrac{1}{n(n-2)\alpha(n)\,|y-\bar{x}|^{n-2}} \qquad n \geq 3$

$$\frac{\partial \Phi}{\partial y_1}(y-\bar{x}) = -\frac{y_1 - x_1}{n\alpha(n)\,|y-\bar{x}|^n}$$

$$\frac{\partial^2 \Phi}{\partial y_1^2} = -\frac{1}{n\alpha(n)\,|y-\bar{x}|^n} + \frac{(y_1 - x_1)^2}{\alpha(n)\,|y-\bar{x}|^{n+2}}$$

$$\frac{\partial^2 \Phi}{\partial y_n^2} = -\frac{1}{n\alpha(n)\,|y-\bar{x}|^n} + \frac{(y_n + x_n)^2}{\alpha(n)\,|y-\bar{x}|^{n-2}}$$

Adding

$\Delta\Phi\left(y-\overline{x}\right)=0$.

On ∂R_{+}^{n} $\quad |y-x|=\left(y-\overline{x}\right)$

So $\quad \Phi\left(y-\overline{x}\right)=\Phi\left(y-x\right)$

Hence both conditions are satisfied.

So, Green's function

$$G\left(x,y\right)=\Phi\left(y-x\right)-\Phi\left(y-\overline{x}\right)$$

is well defined.

So using the representation formula

$$u\left(x\right)=0-\int_{\partial R_{+}^{n}} g\left(y\right)\frac{\partial G}{\partial \nu}\left(x,y\right)\,ds\left(y\right)$$

$$\frac{\partial G}{\partial \nu}\left(x,y\right)=DG.\hat{\nu}=-\frac{\partial G}{\partial y_n}\left(x,y\right)$$

$$\frac{\partial G}{\partial y_n}=\frac{\partial \Phi\left(y-x\right)}{\partial y_n}-\frac{\partial \Phi}{\partial y_n}\left(y-\overline{x}\right)$$

$$=-\left[\frac{y_n-x_n}{n\alpha\left(n\right)\,|y-x|^n}-\frac{y_n+x_n}{n\alpha\left(n\right)\,|y-x|^n}\right]$$

$$=\frac{2x_n}{n\alpha\left(n\right)\,|x-y|^n}\qquad\left(\text{on } \partial \mathrm{R}_{+}^{n},\ |y-x|=|y-\overline{x}|\right)$$

$$u\left(x\right)=\frac{2x_n}{n\alpha\left(n\right)}\int_{\partial R_{+}^{n}}\frac{g\left(y\right)}{|x-y|^n}\,ds\left(y\right)\qquad x\in R_{+}^{n}$$

This is the required solution and is known as Poisson's formula.

Def. Let $x\in R^n-\{0\}$

The point $\overline{x}=\dfrac{x}{|x|^2}$

is called the dual point of x w.r.t. $\partial B(0,1)$.

Example. Solve the boundary value problem

$$\Delta u = 0 \quad \text{in } B(0,1)$$

$$u = g \quad \text{on } \partial \mathrm{B}(0,1) \qquad (*)$$

Sol. Fix any point $x \in B^0(0,1)$ and $y \neq x$

The Green's function is given by

$$G(x,y) = \Phi(y-x) - \phi^x(y) \qquad (1)$$

We choose $\phi^x(y) = \Phi(|x|\,(y-\bar{x}))$. (2)

where $\bar{x}$ dual of x w.r.t. $\partial B(0,1)$

As we know $\Phi(y-x)$ is harmonic. So is $\Phi(y-\bar{x})$ for $y \neq x$

Similarly $|x|^{2-n}\,\Phi(y-\bar{x})$ is harmonic for $y \neq x$

or $\Phi(|x|\,(y-\bar{x}))$ is harmonic for $y \neq x$

So, $\Delta\phi^x = 0$ in B (0, 1)

On $\partial B(0,1)$:

$$\phi(x) = \Phi(|x|\,(y-\bar{x}))$$

But

$$|x|^2\,|y-\bar{x}|^2 = |x|^2\left\{\left(y_1 - \frac{x_1}{|x|^2}\right)^2 + \ldots + \left(y_n - \frac{x_n}{|x|^2}\right)^2\right\}$$

$$= |x|^2\left\{|y|^2 + \frac{1}{|x|^2} - \frac{2x.y}{|x|^2}\right\}$$

$$= \left|x^2 + 1 - 2x.y\right| \qquad (\because |y| = 1)$$

$$= |x|^2 + |y|^2 - 2x\,y$$

$$= |x-y|^2$$

So $\phi(x) = \Phi\left(|x|(y-\bar{x})\right) = \Phi(y-x)$

Hence both conditions of $\phi^x(y)$ are satisfied.

So

$$G(x,y) = \Phi(y-x) - \Phi\left(|x|(y-\bar{x})\right) \tag{3}$$

is well defined.

Hence solution of problem (*) is given by

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G}{\partial \nu}ds(y) \tag{4}$$

Now on $\partial B(0,1)$

$$\frac{\partial G}{\partial \nu} = \frac{\partial G}{\partial y}.\nu,$$

ν being the unit normal.

$$= \frac{\partial G}{\partial y}\frac{y}{|y|}$$

$$= \sum_i \frac{\partial G}{\partial y_i} y_i \qquad (\because |y| = 1)$$

$$\frac{\partial G}{\partial y_i} = \frac{x_i - y_i}{n\alpha(n)\,|x-y|^n} + \frac{y_i\,|x|^2 - x_i}{n\alpha(n)\,|x-y|^n}$$

$$= \frac{y_i\,|x|^2 - y_i}{n\alpha(n)\,|x-y|^n}$$

$$\frac{\partial G}{\partial \nu} = -\frac{\left(1-|x|^2\right)}{n\alpha(n)|x-y|^n}$$

So equation (4) gives

$$u(x) = \int_{\partial B(0,1)} g(y) \frac{1-|x|^2}{n\alpha(n)|x-y|^n} \, ds(y)$$

This is the required solution.

3.4 Energy methods

Uniqueness of solution

There exists at most one solution $u \in C^2(U)$ of

$-\Delta u = f$ in U

$u = g$ on ∂U (*)

where U is a open, bounded set.

Proof. Let $\bar{u}$ be another solution of problem (*)

Let $w = u - \bar{u}$

then $\Delta w = 0$ in U

$w = 0$ on ∂U

Consider

$$\int_U w\Delta w \, dx = \int_U w\left(w_{x_i}\right)_{x_i} \, dx$$

Integrating by parts

$$= -\int_U w_{x_i} w_{x_i} \, dx + \int_{\partial U} w_{x_i} \, w \, \nu \, dS,$$

(ν being the unit normal).

$$= -\int_U |Dw|^2 \, dx + 0$$

$$\Rightarrow |Dw|^2 = 0 \quad \text{in } U$$

$$\Rightarrow Dw = 0 \quad \text{in } U$$

$$\Rightarrow w = \text{Constant} \quad \text{in } U$$

But $w = 0$ on ∂U

Hence $w = 0$ in U

$$\Rightarrow \quad u = \bar{u}$$

Hence uniqueness of solution.

Def. We define the energy functional for Poisson's equation $\Delta u = -f$ by the expression

$$I[w] = \int_U \left[\frac{1}{2}|Dw|^2 - wf \right] dx$$

where $w \in A$ and A is the admissible set

$$A = \left\{ w \in C^2\left(\bar{U}\right) \middle| w = g \text{ on } \partial U \right\}$$

Theorem. Let $u \in C^2\left(\bar{U}\right)$ be a solution of Poisson's equation. Then

$$I[u] = \min_{w \in A} I[w] \tag{1}$$

Conversely if $u \in A$ satisfies (1) then u is a solution of boundary value problem.

$$\begin{aligned} -\Delta u &= f && \text{in } U \\ u &= g && \text{on } \partial U \end{aligned} \tag{2}$$

Proof. Let $w \in A$ and u be a solution of Poisson's equation. So

$-\Delta u = f$ in U

$$0 = \int_U (-\Delta u - f)\,(u - w)\,dx$$

$$= -\int_U \Delta u(u - w)\,dx - \int_U f\,(u - w)\,dx$$

Integrating by parts

$$0 = \int_U Du.\,D(u - w)\,dx - \int_{\partial U} (u - w) D u.\nu\,dS - \int_U f\,(u - w)\,dx$$

$$= \int_U (Du.Du - fu)\, dx - 0 - \int_U (Du.Dw - f\ w)\, dx$$

$$\Rightarrow \int_U \left(|Du|^2 - f\ u\right) dx = \int_U (Du.Dw - fw)\, dx$$

$$\Rightarrow \int_U \left(|Du|^2 - fu\right) dx \le \int \left[\frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2 - fw\right] dx$$

(By Cauchy- Schwatz's inequality)

i.e. $\left[\frac{1}{2}|Du|^2 - fu\right] dx \le \int \left[\frac{1}{2}|Dw|^2 - fw\right] dx$

$$I[u] \le I[w]$$

Since $u \in A$, so

$$I[u] = \min_{w \in A} I[w]$$

Conversely

Suppose $I[u] = \min_{w \in A} I[w]$

For any $v \in C_c^\infty(U)$

define $i(\tau) = I[u + \tau v] \qquad \tau \in R$

So $i(\tau)$ attains minimum for $\tau = 0$

$$i'(\tau) = 0 \text{ for } \tau = 0$$

$$i(\tau) = \int_U \left[\frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v) f\right] dx$$

$$= \int_U \left[\frac{1}{2}\left(|Du|^2 + \tau^2 |Dv|^2\right) + \tau Du\ Dv - (u + \tau v) f\right] dx$$

$$i'(0) = \int_U [Du.Dv - vf]\, dx$$

Integrating by parts

$$0 = -\int_U v\,\Delta u \;\; dx + \int_{\partial U} D u.\nu \;\; dS - \int_U vf \;\; dx$$

$$0 = \int_U [-\Delta u - f] v \;\; dx + 0 \qquad \left(\because v \in C_C^{\infty}(U)\right)$$

This is true for each function $v \in C_c^{\infty}(U)$

So $\quad \Delta u = -f \quad \text{in} \quad U$

So u is a solution of Poisson's equation.

3.5 Heat Equation

The linear partial differential equation

$u_t - \Delta u = 0$

where

$x \in U \subset R^n$

$u : \bar{U} \times [-,\infty) \to R$

is known as homogeneous Heat equation or Diffusion equation.

The equation

$$u_t - \Delta u = f(x,t)$$

where $f : U \times [0,\infty) \to R$

is known as non-homogeneous heat equation.

3.5.1 Fundamental solution

Consider the homogeneous heat equation

$$u_t - \Delta u = 0 \qquad (1)$$

We seek a solution of equation (1) of the form

$$u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right)$$

where $\alpha,\ \beta$ are to be determined.

$$u(x,t) = \frac{1}{t^{\alpha}} v(y) \tag{2}$$

where $y = x / t^{\beta}$ (3)

Differentiating w.r.t. t, x

$$u_t = -\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta y \; Dv}{t^{\alpha+1}}$$

$$\Delta u = \frac{1}{t^{\alpha+2\beta}} \; \Delta v$$

Using in equation (1) and simplifying

$$\alpha \, v(y) + \beta y \; Dv + \frac{1}{t^{2\beta-1}} \Delta v = 0 \tag{4}$$

To make equation (4) independent of t, we put $\beta = 1/2$, so that

$$\alpha \, v(y) + \beta y \,.Dv + \Delta v = 0 \tag{5}$$

We seek a radial solution of equation (5) as

$$v(y) := w(r) \text{ where } r = |y| \tag{6}$$

From equations (5) and (6)

$$\Delta v(y) = w'' + \omega'(r)\left(\frac{n-1}{r}\right)$$

Using in equation (6),

$$w'' + \left(\frac{r}{2} + \frac{n-1}{r}\right) w' + \alpha w = 0$$

To make it exact differential, we put $\alpha = n/2$ and multiply by r^{n-1}.This gives

$$\left(r^{n-1} w'\right)' + \frac{\left(r^n w\right)'}{2} = 0$$

Integrating

$$r^{n-1} w' + \frac{r^n w}{2} = c,$$

where c is a constant.

Assuming $r \to 0,\ w,\ w' \to 0,\ c \to 0$

Hence $$w' + \frac{wr}{2} = 0$$

$$\Rightarrow \quad \frac{w'}{w} = -\frac{r}{2}$$

Integrating

$$w = b\ e^{-r^2/4}$$

where b is a constant.

So $v(y) = be^{-|y|^2/4}$

Hence $u(x,t) = \dfrac{b}{t^{n/2}} e^{-|x|^2/4t}$

is solution of equation (1).

To find b, we normalize the solution.

$$\int_{R^n} u(x,t)\ dx = 1$$

$$\frac{b}{t^{n/2}} \int_{R^4} e^{-|x|^2/4t}\ dx = 1$$

$$\frac{b}{t^{n/2}} \left(2\sqrt{\pi t}\right)^n = 1$$

or $$b = \frac{1}{(4\pi)^{n/2}}$$

Hence fundamental solution is

$$u(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}}\ e^{-|x|^2/4t} & t > 0 \qquad x \in R^n \\ 0 & t < 0 \end{cases}$$

Note: The fundamental solution of equation (1) is singular at (0, 0).

3.5.2 Solution of Initial Value Problem

Assume that $g \in C\left(R^n\right) \cap L^\infty\left(R^n\right)$

and define

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4t}} g(y)\ dy \tag{1}$$

$$= \int_{R^n} \Phi(x-y) g(y)\ dy$$

where $\Phi(x)$ is fundamental solution of heat equation.

Then

(i) $u \in C^\infty\left(R^n \times (0,\infty)\right)$

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \qquad x \in R^n,\ t > 0$

(iii) $\underset{(x,t)\to\left(x^0,0\right)}{lt} u(x,t) = g\left(x^0\right)$ for each $x^0 \in R^n$

$x \in R^n,\ t > 0$

Proof.

(i) Since the function $\frac{1}{t^{n/2}} e^{\frac{-|x|^2}{4t}}$ is infinitely differentiable with uniform bounded derivatives of all order on $R^n \times [\delta,\infty)$ for $\delta > 0$

So $u \in C^\infty\left(R^n \times (0,\infty)\right)$

(ii) $u_t = \int_{R^n} \Phi_t(x-y,t)\ g(y)\ dy$

$\Delta u = \int_{R^n} \Delta\Phi(x-y,t)\ g(y)\ dy$

Therefore, $u_t - \Delta u = 0$ since $\Phi(x-y)$ is a solution of heat equation.

(iii) *Fix* $x^0 \in R^n$. Since g is continuous ,given $\in > 0$, there exists a $\delta > 0$ such that

$$\left|g(y)-g\left(x^0\right)\right| < \in \quad \text{whenever} \quad \underset{y\in R^n}{\left|y-x^0\right|} < \delta \tag{2}$$

Then if $\left|x-x^0\right| < \dfrac{\delta}{2}$

$$\left|u(x,t)-g\left(x^0\right)\right| = \left|\int_{R^n} \Phi(x-y,t)\left[g(y)-g\left(x^0\right)\right] dy\right|$$

$$\left(\because \int_{R^n} \Phi(x-y,t)\, dx = 1\right)$$

$$\leq \left|\int_{B(x_0,\delta)} \Phi(x-y,t)\left[g(y)-g\left(x^0\right)\right] dy\right| +$$

$$\left|\int_{R^n - B(x^0,\delta)} \Phi(x-y,t)\left[g(y)-g\left(x^0\right)\right] dy\right|$$

$$\leq \int_{B(x_0,\delta)} \Phi(x-y,t)\left|g(y)-g\left(x^0\right)\right| dy$$

$$+ \int_{R^n - B(x_0,\delta)} \Phi(x-y,t)\left|g(y)-g\left(x^0\right)\right| dy$$

$$\left|u(x,t)-g\left(x^0\right)\right| \leq I + J \tag{3}$$

where $I = \displaystyle\int_{B\left(x^0,\delta\right)} \Phi(x-y,t)\left|g(y)-g\left(x^0\right)\right| dy$

$$J = \int_{B\left(x^0,\delta\right)} \Phi(x-y,t)\left|g(y)-g\left(x^0\right)\right| dy \tag{4}$$

$$I \leq \in \int_{R^n - B(x^0, \delta)} \Phi(x-y,t)\, dy$$

$$I \leq \in \tag{5}$$

Further if $\left|x - x^0\right| < \dfrac{\delta}{2}$

and $\delta \leq \left|y - x^0\right|$

then $\left|y - x^0\right| \leq \left|y - x + x - x^0\right|$

$$\leq |y-x| + \left|x - x^0\right|$$

$$\leq |y-x| + \delta / 2$$

$$\leq |y-x| + \frac{\left|y - x^0\right|}{2}$$

or $|y-x| \geq \dfrac{1}{2}\left|y - x^0\right|$

Hence

$$J = \int_{R^n - B(x^0, \delta)} \Phi(x-y,t) \left|g(y) - g\left(x^0\right)\right| dy$$

$$\leq 2\|g\|_{L^\infty\left(R^n - B\left(x^0, \delta\right)\right)} \int_{R^n - B(x^0, \delta)} \Phi(x-y,t)\, dy$$

$$\leq \frac{c}{t^{n/2}} \int_{R^n - B(x^0, \delta)} e^{\frac{-|x-y|^2}{4t}}\, dy$$

$$J \leq \frac{c}{t^{n/2}} \int_{R^n - B(x^0, \delta)} e^{-\frac{\left|y - x^0\right|^2}{16t}} dy$$

$$\leq \frac{c}{t^{n/2}} \int_{\delta}^{\infty} \left\{ e^{\frac{-r^2}{16t}} \, r^{n-1} \right\} dr \qquad \text{(By Cor. of coarea formula)}$$

$$\to 0 \quad \text{as} \quad t \to 0 \tag{6}$$

Using (5) and (6) in (3)

$$\left| u(x,t) - g\left(x^0\right) \right| < \in$$

So $\quad u(x,t) \to g\left(x^0\right) \quad$ as $\quad (x,t) \to \left(x^0, 0\right)$

Hence the result.

3.5.3 Non-homogeneous heat equation

Consider the non-homogeneous heat equation

$$u_t - \Delta u = f(x,t) \qquad \text{on } R^n \times (0,\infty)$$

$$u = 0 \qquad \text{on } R^n \times \{t = 0\}$$

then

$$u(x,t) = \int_0^t \frac{1}{\left[4\pi(t-s)\right]^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s) \, dy \, ds \tag{1}$$

$$x \in R^n, \; t > 0$$

$$= \int_0^t \int_{R^n} \Phi(x-y, t-s) f(y,s) dy \, ds \tag{2}$$

where $f \in C_1^2\left(R^n \times [0,\infty)\right)$ and has compact support then

(i) $\quad u \in C_1^2\left(R^n \times (0,\infty)\right)$

(ii) $\quad u_t(x,t) - \Delta u(x,t) = f(x,t)$

(iii) $\underset{\substack{(x,t)\to(x^0,0) \\ x\in R^n,\ t>0}}{lt} u(x,t)=0$ for each point $x^0 \in R^n$

Proof. (i) Since Φ has a singularity at (0, 0) we cannot differentiate under the integral sign. Substituting the variable $x - y = 0$, $t - s = \phi$ and again converting to original variable

$$u(x,t)=\int_0^t \int_{R^n} \Phi(y,s) f(x-y,t-s)\,dy\; ds$$

Since $f \in C^2\left(R^n \times [0,\infty)\right)$ and $\Phi(y,s)$ is smooth near $s = t > 0$, we compute

$$u_t(x,t)=\int_0^t \int_{R^n} \Phi(y,s) f_t(x-y,t-s)\,dy\; ds$$

$$+\int_{R^n} \Phi(y,t) f(x-y,0)\; dy$$

(By Leibnitz's Rule)

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t)=\int_0^t \int_{R^n} \Phi(y,s) \frac{\partial^2}{\partial x_i \partial x_j} f(x-y,t-s)\,dy\; ds$$

Thus

$$u_t,\ D^2u \in C^2\left(R^n \times (0,\infty)\right)$$

(ii) $u_t(x,t) - \Delta u(x,t)$

$$=\int_0^t \int_{R^n} \Phi(y,s)\left(\frac{\partial}{\partial t} - \Delta_x\right) f(x-y,\ t-s)\,dy\; ds$$

$$+\int_{R^n} \Phi(y,t) f(x-y,0)\; dy$$

$$= \int_0^t \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds + K$$

where

$$K = \int_{R^n} \Phi(y,t)\, f(x-y,0)\, dy \tag{3}$$

$$u_t - \Delta u(x,t) = \int_0^{\in} \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds$$

$$+ \int_{\in}^{t} \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds$$

$$= I_{\in} + J_{\in} + K \tag{4}$$

where

$$I_{\in} = \int_0^{\in} \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds$$

$$J_{\in} = \int_{\in}^{t} \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds$$

Now,

$$I_{\in} = \int_0^{\in} \int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\, ds$$

$$\leq \left(\|f_t\|_{L^\infty(R^n)} + \|D^2 f\|_{L^\infty(R^n)}\right) \int_0^{\in} \int_{R^n} \Phi(y,s)\,dy\, ds$$

$$\leq C\int_0^{\in} ds$$

$$\leq C \in \tag{5}$$

$$J_{\in} = \int_{\in}^{t}\int_{R^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,\, t-s)\,dy\ \ ds$$

Integrating by parts

$$= \int_{\in}^{t}\int_{R^n} \left[\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)\right] f(x-y,\, t-s)\,dy\ ds - \left|\int_{R^n} \Phi(y,s) f(x-y,\, t-s)\,dy\right|_{\in}^{t}$$

(surface integrals are zero since f has compact support)

$$= -\int_{R^n} \Phi(y,t) f(x-y,\, 0) + \int_{R^n} \Phi(y,\in) f(x-y,\, t-\in)\,dy$$

$$= \int_{R^n} \Phi(y,\in) f(x-y,\, t-\in)\,dy - K \tag{6}$$

Using (5) and (6) in (4)

$$u_t - \Delta u(x,t) = \int_{R^n} \Phi(y,\in) f(x-y,\, t-\in)\ dy - K + C\in + K$$

Taking limit as $\in \to 0$

$$u_t - \Delta u(x,t) = \underset{\in\to 0}{lt} \int_{R^n} \Phi(y,\in) f(x-y,\, t-\in)\,dy$$

$$= \underset{\in\to 0}{lt} \int_{R^n} \Phi(x-y,\, \in) f(y,\, t-\in)\,dy$$

$$= f(x,t)$$

(using the result $\underset{(x,t)\to(x_0,0)}{Lt}$ $then \int_{R^n} \Phi(x-y,\, t) g(y)\,dy \ \to g(x_0)$)

(iii) $u(x,t)=\int_0^t \int_{R^n} \Phi(y,s) f(x-y,\, t-s)\, dy\, ds$

$$\|u\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)} \int_0^t \int_{R^n} \Phi(y,s)\, dy\, ds$$

$$= \|f\| \int_0^t ds$$

$$= \|f\|\, t$$

Taking limit as $t \to 0$

$\underset{t\to 0}{lt}\; u(x,\, t) = 0$ for each $x \in R^n$.

Def. We define the parabolic cylinder

$$U_T := U \times (0,T]$$

where $U \subset R^n$ is open and bounded.

The parabolic boundary of U_T

$$\Gamma_T := \bar{U}_T - U_T$$

Remark. U_T is interpreted as the parabolic interior of $\bar{U} \times (0,T]$ including the top $U \times \{t = T\}$. But Γ_T includes bottom and vertical sides of $U \times [0,T)$

3.6 Self –Assessment Questions

Q. Find the solution of boundary value problem

$\Delta u = 0$ in $B^0(0,r)$

$u = g$ on $\partial B(0,r)$

Ans. $u(x) = \dfrac{r^2 - |x|^2}{n\alpha(n) r} \displaystyle\int_{\partial B(0,r)} \dfrac{g(y)}{|x-y|^n}\, ds(y)$

3.7 Suggested References

1. L.C. Evans, "Partial Differential Equations," American Mathematical Society, Rhade.
2. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum Outline Series, McGraw Hill Series.

Lesson 4

Solution of Wave Equation

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Structure

4.1 Introduction

In this lesson, we seek the solution of wave equation. The homogeneous wave equation

$$u_{tt} - \Delta u = 0$$

where $t > 0$, $x \in U \subset R^n$ is open and

$$u : \bar{U} \times [0, \infty) \to R.$$

The non-homogeneous wave equation.

$$u_{tt} - \Delta u = f(n,t)$$

where $f : U \times [0,\infty) \to R$ is a prescribed function.

4.2 Solution of 1-D wave equation

First we find the solution of wave equation in the one dimensional case. Consider the initial value problem

$$u_{tt} - u_{xx} = 0 \qquad \text{in } \mathrm{R} \times (0,\infty) \tag{1}$$

$$u = g \qquad u_t = h \qquad \text{on } R \times \{t = 0\} \tag{2}$$

where g and h are prescribed functions.

Factorizing equation (1)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0 \tag{3}$$

Let

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) \tag{4}$$

From (3) and (4)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v\,(x,t) = 0 \qquad x \in R,\ t > 0$$

$$v_t + v_x = 0 \quad (5)$$

which is a transport equation with constant coefficients whose solution is

$$v(x,t) = a(x-t) \tag{6}$$

where

$$a(x) := v(x,0) \tag{7}$$

Using equation (6) in equation (4)

$$u_t - u_x = a(x-t) \quad \text{in } R \times (0,\infty)$$

$$= f(x,\ t).$$

This is a non-homogeneous transport equation whose solution is

$$u(x,t) = \int_0^t f(x+t-s,\ s)\ ds + b(x+t)$$

where

$$b(x) = u(x,0)$$

or

$$u(x,t) = \int_0^t a(x+t-2s)\,ds + b(x+t)$$

Changing the variable $x+t-2s = y$

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x+t)$$

Using equation (2)

$$g(x) = b(x)$$

so

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\ dy$$

To find $a\,(x)$: We have

$$u_t(x,0) - u_x(x,0) = v(x,0)$$

$$h(x) - g'(x) = a(x) \qquad \text{(using 2)}$$

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t}\left[h(y) - g'(y)\right]\,dy$$

or

$$u(x,t) = \frac{1}{2}\left[g(x+t) - g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy \qquad x \in R,\ t > 0 \qquad (8)$$

This is required solution of wave equation.Equation (8) is known as ***D Alembert's formula***.

Note. The general solution of 1-D wave equation

$$(u_t + u_x)\,(u_t - u_x) = 0$$

is the sum of general solution of $u_t + u_x = 0$ and $u_t - u_x = 0$

i.e. $u(x,t) = F(x+t) + G(x-t)$

To find the solution of wave equation over $R^n\,(n \geq 2)$, we first prove a lemma.

Def. We define

$$U(x;\, r,t) = \oint\!\!\!\!\oint_{\partial B(x,r)} u(y,t)\; ds(y)$$

$$G(x;\, r,t) = \oint\!\!\!\!\oint_{\partial B(x,r)} g(y)\; ds(y)$$

$$H(x;\, r,t) = \oint\!\!\!\!\oint_{\partial B(x,r)} h(y)\; ds(y)$$

Lemma. Fix $x \in R^n$, satisfying

$$u_{tt} - \Delta u = 0 \qquad \text{in } R^n \times (0,\infty) \tag{1}$$

$$u = g,\ u_t = h \qquad \text{on } R^n \times \{t = 0\} \tag{2}$$

then

$$U \in C^m\left(\bar{R}_+ \times [0,\infty)\right) \text{ and}$$

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \qquad \text{in } R_+ \times (0,\infty) \tag{3}$$

$$U = G\,, \qquad U_t = H \qquad \text{on } R_+ \times \{t = 0\} \tag{4}$$

Equation (3) is known as ***Euler Poisson Darboux Equation***.

Proof. We know

$$U(x;r,t) = \oint\!\!\!\!\oint_{\partial B(x,r)} u(y,t)\,ds(y)$$

Shifting to unit Ball $B(0,1)$

$$U(x;r,t) \qquad = \fint_{\partial B(0,1)} u(x+rz)\,ds(z)$$

Differentiating w.r.t. r

$$U_r = \fint_{\partial B(0,1)} Du(x+rz).z\,ds(z)$$

$$= \fint_{\partial B(x,r)} Du(y).\frac{y-x}{r}\,ds(y)$$

$$= \fint_{\partial B(x,r)} Du(y).\upsilon\;ds(y)$$

(where υ is unit outward normal).

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \upsilon}\,ds(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} \frac{\partial u}{\partial \upsilon}\,ds(y)$$

$$= \frac{1}{n\alpha(x)r^{n-1}}\int_{B(x,r)} \Delta u\;dy \qquad \text{(By Green 's formula)}$$

$$= \frac{r}{n}\fint_{B(x,r)} \Delta u\;dy$$

Hence

$$U_r(x;\,r,t) = \frac{r}{n}\fint_{B(x,r)} \Delta u\;dy \tag{5}$$

Again differentiating w.r.t. r

$$U_{rr}(x;\,r,t) = \frac{1}{n\alpha(n)}\frac{\partial}{\partial r}\left[\frac{1}{r^{n-1}}\int_{B(x,r)} \Delta u\;dy\right]$$

$$= \frac{1-n}{n\alpha(n)r^{n}}\int_{B(x,r)} \Delta u\;dy + \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} \Delta u\;ds$$

$$= \left(\frac{1}{n} - 1 \right) \fint_{B(x,r)} \Delta u \; dy + \fint_{\partial B(x,r)} \Delta u \; ds \tag{6}$$

From equation (5) and (6), we observe that

$$\underset{r \to 0}{lt} \, U_r (x; r, t) = 0$$

$$\underset{r \to 0}{lt} \, U_{rr} = \Delta u + \left(\frac{1}{n} - 1 \right) \Delta u$$

$$= \frac{1}{n} \Delta u (x, t)$$

So

$$U \in C^m \left(\bar{R}_+ \times [0, \infty) \right)$$

By equation (5)

$$U_r (x; r, t) = \frac{r}{n} \fint_{B(x,r)} \Delta u \; dy$$

$$= \frac{r}{n} \fint_{B(x,r)} u_{tt} \; dy$$

$$= \frac{1}{n \alpha (n) r^{n-1}} \int_{B(x,r)} u_{tt} \; dy$$

$$r^{n-1} U_r =: \frac{1}{n \alpha (n)} \int_{B(x,r)} u_{tt} \; dy$$

Differentiating w.r.t. r

$$r^{n-1} U_{rr} + (n-1) r^{n-2} U_r = \frac{1}{n \alpha (n)} \frac{d}{dr} \left[\int_{B(x,r)} u_{tt} \; dy \right]$$

$$= \frac{1}{n \alpha (n)} \int_{\partial B(x,r)} u_{tt} \; ds$$

$$= r^{n-1} \fint_{\partial B(x,r)} u_{tt} \; ds$$

or

$$U_{rr} + \frac{(n-1)}{r} U_r = U_{tt} \tag{7}$$

which is required equation.

Also $u = g$ on $R^n \times \{t = 0\}$

$$\int_{\partial B(x,r)} u(y,0)\, dS(y) = \int_{\partial B(x,r)} g(y)\, dS(y)$$

Dividing by $n\alpha(n)r^{n-1}$

$$U(x,0) = G(x)$$

Similarly we can show

$$U_t(x,0) = H(x) \qquad \text{for } R_+ \times \{t = 0\}$$

4.3 Kirchoff's formula

Consider the initial value problem

$$u_{tt} - \Delta u = 0 \qquad \text{in } R^3 \times (0,\infty) \tag{1}$$

$$u(x) = g(x)\ , u_t = h \quad \text{on } R^3 \times \{t = 0\} \tag{2}$$

Sol. First we prove that

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \ \text{ in } R_+ \times (0,\infty) \tag{3}$$

$$\tilde{U} = \tilde{G} \ , \ \tilde{U}_t = \tilde{H} \quad \text{on } R_+ \times \{t = 0\}$$

$$\tilde{U} = 0 \qquad \text{on } \{r = 0\} \times (0,\infty) \tag{4}$$

where

$$\tilde{U} := rU$$

$$\tilde{G} := rG$$

$$\tilde{H} := rH$$

We know Euler Poisson Darboux Equation for n=3 is

$$U_{tt} - U_{rr} - \frac{2}{r} U_r = 0 \text{ in } R_+ \times (0, \infty) \tag{5}$$

$$U = G, \qquad U_t = H \text{ on } R_+ \times \{t = 0\} \tag{6}$$

$$\bar{U}_{tt} := rU_{tt}$$

$$= r\left\{U_{rr} + \frac{2}{r} U_r\right\} \qquad \text{(using 5)}$$

$$= rU_{rr} + 2rU_r$$

$$= (rU_r + U)_r$$

$$= (\tilde{U}_r)_r$$

$$= \tilde{U}_{rr} \tag{7}$$

So $\tilde{U}$ satisfies the 1-D wave equation.

Also $\tilde{U}(r,0) = rU(r,0)$

$$= rG(r)$$

$$= \tilde{G}$$

Similarly $\tilde{U}_t(r,0) = \tilde{H}(r)$

Hence, by D Alembert's formula, we have $0 \le r \le t$

$$\tilde{U}(x,r;\, t) = \frac{1}{2}\left[\tilde{G}(t+r) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\, dy \tag{8}$$

Now

$$u(x,t) = \underset{r \to 0}{lt}\, U(x,r;t) \qquad \text{(by def.)}$$

$$= \underset{r \to 0}{lt} \frac{\tilde{U}(x,r;t)}{r}$$

$$= \underset{r \to 0}{lt} \left\{\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r}\right] + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\, dy\right\}$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$= \frac{\partial}{\partial t}\left[t \oint_{\partial B(x,t)} g\, ds \right] + t \oint_{\partial B(x,t)} h\, ds \tag{9}$$

But

$$\frac{\partial}{\partial t}\left[\oint_{\partial B(x,t)} g ds \right] = \frac{\partial}{\partial t}\left[\oint_{\partial B(0,1)} g(x+tz)\, ds(z) \right]$$

$$= \oint_{\partial B(0,1)} Dg(x+tz).z\, ds(z)$$

$$= \oint_{\partial B(x,t)} Dg(y).\left(\frac{y-x}{t} \right) ds(y)$$

So

$$u(x,t) = \oint_{\partial B(x,t)} g ds + \oint_{\partial B(x,t)} Dg(y)\,(y-x) ds(y) + \oint_{\partial B(x,t)} t\, h(y)\, ds(y)$$

$$= \oint_{\partial B(x,t)} \left[g + t\, h(y) + D\, g(y).(y-x) \right] ds(y) \tag{10}$$

This is required solution.Equation (10) is known as **Kirchoff's formula.**

4.4 Solution of 2-D Wave Equation

Now we find the solution of wave equation by the method of descent. Consider initial value problem

$u_{tt} - \Delta u = 0$ in $R^2 \times (0,\infty)$ (1)

$u = g$, $u_t = h$ on $R^2 \times \{t=0\}$ (2)

Sol. We regard it as a problem for n = 3 in which the third spatial variable x_3 does not appear.Let us write

$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$ (3)

So equation (1) and (2) are modified to

$$\bar{u}_{tt} - \Delta\bar{u} = 0 \quad \text{in} \quad R^3 \times (0,\infty) \tag{4}$$

$$\bar{u} = \bar{g},\ \bar{u}_t = \bar{h} \quad \text{on} \quad R^3 \times \{t = 0\} \tag{5}$$

where

$$\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$$

$$\bar{h}(x_1, x_2, x_3) = g(x_1, x_2)$$

If $x = (x_1, x_2) \in R^2$ then $\bar{x} \in R^3$

The solution of initial value problem defined in equation (4) and (5) is given by Kirchoff's formula i.e.

$$\bar{u}(\bar{x}, t) = \frac{\partial}{\partial t}\left(t \oiint_{\partial\bar{B}(\bar{x},t)} \bar{g}\ d\bar{s} \right) + t \oiint_{\partial\bar{B}(\bar{x},t)} \bar{h}\ d\bar{s} \tag{6}$$

where $\bar{B}(\bar{x}, t)$ denotes the ball in R^3 with centre $\bar{x}$ and radius $t > 0$ and $d\ \bar{s}$ denotes the two-dimensional surface measure on $\partial\bar{B}(\bar{x}, t)$.

Now

$$\oiint_{\partial\bar{B}(\bar{x},t)} \bar{g}\ d\bar{S} = \frac{1}{4\pi t^2} \int_{\partial\bar{B}(\bar{x},t)} \bar{g}\ d\bar{S}$$

$$= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y) \left[1 + \left(\frac{d\gamma}{dy} \right)^2 \right]^{1/2} dy$$

where factor '2' is taken as $B(\bar{x}, t)$ consists of two hemisphere and

$\gamma(y) = \sqrt{t^2 - |y - x|^2}$ is the parametric equation of any $y \in B(x, t)$

$$\oiint_{\partial\bar{B}(\bar{x},t)} \bar{g}\ d\bar{s} = \frac{1}{2\pi t^2} \int_{B(x,t)} g(y) \frac{t}{\sqrt{t^2 - |y - x|^2}} dy$$

$$= \frac{t}{2}\left[\frac{1}{\pi t^2}\int_{B(x,t)} g(y)\frac{t}{\sqrt{t^2-|y-x|^2}}\,dy\right]$$

$$= \frac{t}{2}\,⨍_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy \qquad (7)$$

Similarly,

$$\int_{\partial\bar{B}(\bar{x},t)} \bar{h}\,d\bar{S} = \frac{t}{2}\,⨍_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy \qquad (8)$$

Using (7) and (8) in equation (6)

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right)$$

$$+\frac{t^2}{2}\,⨍_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy \qquad (9)$$

$$\frac{\partial}{\partial t}\left(t^2\,⨍_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right) = \frac{\partial}{\partial t}\left(t\,⨍_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz\right)$$

$$= ⨍_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz + t\,⨍_{B(0,1)} \frac{D.g(x+tz)z}{\sqrt{1-|z|^2}}\,dz$$

$$= t\,⨍_{B(x,t)} \frac{g(y)dy}{\sqrt{t^2-|y-x|^2}} + t\,⨍_{B(x,t)} \frac{Dg(y).(y-x)dy}{\sqrt{t^2-|y-x|^2}}$$

Hence equation (9) gives

$$u(x,t) = \frac{t}{2}\,⨍_{B(x,t)} \frac{g(y)dy}{\sqrt{t^2-|y-x|^2}} + \frac{t}{2}\,⨍_{B(x,t)} \frac{Dg(y)(y-x)dy}{\sqrt{t^2-|y-x|^2}}$$

$$+\frac{t^2}{2}\iint_{B(x,t)}\frac{h(y)\,dy}{\sqrt{t^2-|y-x|^2}}$$

$$u(x,t)=\frac{1}{2}\iint_{B(x,t)}\frac{t\,g(y)+t^2h(y)+Dy(y).(y-x)}{\sqrt{t^2-|y-x|^2}}dy \qquad (10)$$

where $x \in R^2$.Eq. (10) is required solution Equation (10) is known as **Poisson's Formula.**

4.5 Solution of wave equation for n ≥3

To find the solution of wave equation for n > 3 we derive some identities.

Suppose $\phi : R \to R$ be C^{k+1} for $k = 1,2,...$

I. $$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}\right)$$

II. $$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\sum_{j=0}^{k-1}\beta_j^k\, r^{j+1}\frac{d^j\phi(r)}{dr^j}$$

where

$\beta_j^k\,(j=0,1,2,...,k-1)$ are independent of *r*.

III. $\beta_0^k = 1.3.5....(2k-1)$

Proof

I. We prove it by induction. For k = 1 .We have to show

$$\frac{d^2}{dr^2}\left(r\phi(r)\right)=\left(\frac{1}{r}\,\frac{d}{dr}\right)\left(r^2\frac{d\phi}{dr}\right)$$

$$= \frac{d}{dr}\left[r\phi'(r)+\phi(r)\right]$$

$$= r\phi''(r)+2\phi'(r)$$

$$=\frac{1}{r}\left[r^2\phi''(r)+2r\phi'(r)\right]$$

$$=\frac{1}{r}\,\frac{d}{dr}\left[r^2\phi'(r)\right] \text{ = R.H.S.}$$

Suppose result holds for k. So

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}\right) \qquad \text{(a)}$$

We have to prove for $k+1$ i.e.

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k+1}\phi(r)\right)=\left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\left(r^{2k+2}\frac{d\phi}{dr}\right)$$

Now

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k+1}\phi(r)\right)$$

$$=\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[\frac{1}{r}\frac{d}{dr}\left\{r^{2k+1}\phi(r)\right\}\right]$$

$$=\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[\frac{1}{r}(2k+1)r^{2k}\phi(r)+r^{2k}\phi'(r)\right]$$

$$=\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[(2k+1)r^{2k-1}\phi(r)+r^{2k-1}\left\{\phi'(r)r\right\}\right]\text{(Using (a))}$$

$$=(2k+1)\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\phi'(r)\right]+\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\frac{d}{dr}\left\{r\phi'\right\}\right]$$

$$=(2k+1)\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\phi'(r)\right]+\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\phi'+r^{2k+1}\phi''\right]$$

$$=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[(2k+2)r^{2k}\phi'+r^{2k+1}\phi''(r)\right]$$

$$=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\frac{1}{r}\left[(2k+2)r^{2k+1}\phi'+r^{2k+2}\phi''(r)\right]$$

$$=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\frac{1}{r}\frac{d}{dr}\left[r^{2k+2}\phi'\right]$$

$$=\left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\left[r^{2k+2}\phi'\right]$$

Hence the result holds for $k+1$.

So result is true for all k = 1, 2, …

II. Try yourself.

III. Try yourself.

Def. Assume n is odd ,say n = 2 k + 1 ,$(k \geq 1)$. We define

$$\tilde{U}(r,t) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$$

$$\tilde{G}(r,t) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}G(x;r)\right)$$

$$\tilde{H}(r,t) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}H(x;r)\right)$$

$$\tilde{U}(r,0) = \tilde{G}(r), \tilde{U}_t(r,0) = \tilde{H}(r)$$

Lemma. $\tilde{U}$ satisfies the 1-D wave equation.

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \qquad \text{in } R_+ \times (0,\infty) \tag{1}$$

$$\tilde{U} = \tilde{G};\ \tilde{U}_t = \tilde{H} \qquad \text{on } R_+ \times \{t=0\}$$

$$\tilde{U} = 0 \qquad \text{on } \{r=0\} \times (0,\infty) \tag{2}$$

Proof. $\tilde{U}_{rr} = \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U\right)$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial U}{\partial r}\right) \qquad \text{(by identity I)}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^{2k}\frac{\partial U}{\partial r}\right)\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[\frac{2k}{r}r^{2k-1}U_r + r^{2k-1}U_{rr}\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U_{rr} + 2k\ r^{2k-2}U_r\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left\{U_{rr} + \frac{(n-1)}{r}U_r\right\}\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U_{tt}\right]$$

$$= \tilde{U}_{tt}\,.$$

Also

$$\tilde{U} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U\right)$$

$$= \sum_j \beta_j^k r^{j+1}\frac{d^jU}{dr^j} \qquad \text{(by identity II)}$$

$$\tilde{U}(0,t) = 0\,.$$

By definition $\tilde{U}(r,0) = \tilde{G}$

$$\tilde{U}(r,0) = \tilde{H} \qquad \text{on } R_+ \times \{t=0\}$$

Hence the lemma.

4.5.1 Solution for odd *n* $(n \geq 3)$

Consider the initial value problem

$$u_{tt} - \Delta u = 0 \text{ in } R^n \times (0,\infty) \tag{1}$$

$$u = g \;\;,\;\; u_t = h \quad \text{on } R^n \times \{t=0\} \tag{2}$$

Solution. By lemma, $\tilde{U}$ satisfies the 1-D wave equation and the initial condition. Therefore, by D Alembert's formula, on half-line $0 \leq r < t$

$$\tilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(t+r) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r}\tilde{H}(y)\,dy \tag{3}$$

for all $r \in R,\; t \geq 0$

$$\tilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\,U(x;r,t)\right)$$

$$= \beta_0^k \, r\, U + \beta_1^k \, r^2 \frac{\partial U}{\partial r} + \ldots$$

$$\Rightarrow \frac{\tilde{U}(r,t)}{\beta_0^k r} = U + O(r)$$

Taking limit as $r \to 0$

$$\underset{r\to 0}{lt} \frac{\tilde{U}(r,t)}{\beta_0^k r} = \underset{r\to 0}{lt}\, U(x,r;t) = \underset{r\to 0}{lt} \oint\!\!\!\!\!\!\iint_{\partial B(x,r)} u(y)\, dS(y)$$

$$= u(x,t)$$

So

$$u(x,t) = \frac{1}{\beta_0^k} \underset{r\to 0}{lt} \left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{H}(y)\, dy \right] \qquad \text{(using 3)}$$

$$= \frac{1}{\beta_0^k} \left[\tilde{G}'(t) + \tilde{H}(t) \right]$$

Since $n = 2k + 1$

$\beta_0^k = 1.3....2k - 1$

$= 1.3. \ldots (n-2) \qquad (\because n = 2k+1)$

$= \gamma_n$ (say)

Hence,

$$u(x,t) = \frac{1}{\gamma_n} \left[\frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \oint\!\!\!\!\!\!\iint_{\partial B(x,t)} g ds \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \oint\!\!\!\!\!\!\iint_{\partial B(x,t)} h \, dS \right) \right] \qquad (4)$$

is required solution for odd *n*.

Note. Putting *n* = 3, we obtain Kirchoff's formula.

4.5.2 Solution for even *n*

Suppose that n is even i.e. $n \geq 2 . 2m = n + 2$ (say),so $m \geq 2$.

We again use the method of Descent.

Consider the initial value problem

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad R^n \times (0, \infty) \tag{1}$$

$$u = g \quad , \quad u_t = h \quad \text{on} \quad R^n \times \{t = 0\} \tag{2}$$

Sol. Since n is even, $n + 1$ is odd.

Suppose

$$\bar{u}(x_1, x_2, \ldots, x_{n+1}, t) := u(x_1, x_2, \ldots, x_n, t) \tag{3}$$

is the solution of wave equation in $R^{n+1} \times (0, \infty)$ i.e.

$$\bar{u}_{tt} - \Delta \bar{u} = 0 \quad \text{in } R^{n+1} \times (0, \infty) \tag{4}$$

with initial condition.

$$\bar{u} = \bar{g} \quad , \quad \bar{u}_t = \bar{h} \quad \text{on} \quad R^{n+1} \times \{t = 0\} \tag{5}$$

where

$$\bar{g}(x_1, x_2, \ldots, x_{n+1}) := g(x_1, x_2, \ldots, x_n)$$

$$\bar{h}(x_1, x_2, \ldots, x_{n+1}) := h(x_1, x_2, \ldots, x_n).$$

The solution of equation (4) subject to (5) is

$$\bar{u}(\bar{x}, t) = \frac{1}{\gamma_{n+1}} \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n+1} \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{s} \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \oint_{\partial \bar{B}(\bar{x}, t)} \bar{h} d\bar{s} \right) \right\} \tag{6}$$

where $\bar{B}(\bar{x}, t)$ denotes the ball in R^{n+1} with centre $\bar{x}$ and radius t and $d\bar{s}$ denotes the n-dimensional surface measure on $\partial \bar{B}(\bar{x}, t)$

Now

$$\oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{s} = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} g(y) \left[1 + |D\gamma(y)|^2 \right]^{1/2} dy$$

where the factor '2 'is due to the fact that the surface area consists of two hemispheres and $\partial\bar{B}(\bar{x},t)\cap(y_{n+1}\geq 0)$ has the equation $\gamma(y)=\sqrt{t^2-|y-x|^2}$ $,y\in B(x,t)$ and $\partial\bar{B}(\bar{x},t)\cap(y_{n+1}\leq 0)$ is the graph of -$\gamma(y)$.

$$\iint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{s} = \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} g(y)\left[\frac{t}{\sqrt{t^2-|y-x|^2}}\right]dy$$

$$= \frac{2\alpha(n)t}{(n+1)\alpha(n+1)}\iint_{B(x,t)}\frac{g(y)dy}{\sqrt{t^2-|y-x|^2}} \tag{7}$$

Similarly

$$\iint_{\partial\bar{B}(\bar{x},t)} \bar{h}d\bar{s} = \frac{2\alpha(n)t}{(n+1)\ \alpha(n+1)}\iint_{B(x,t)}\frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy \tag{8}$$

Using equation (7) and (8) in equation (6)

$$u(x,t)=\frac{2}{\gamma_{n+1}}\frac{\alpha(n)}{(n+1)\alpha(n+1)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\iint_{B(x,t)}\frac{gdy}{\sqrt{t^2-|y-x|^2}}\right)\right.$$

$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\iint_{B(x,t)}\frac{h\ dy}{t^2-|y-x|^2}\int\right)\right]$$

But

$$\frac{2\alpha(n)}{\gamma_{n+1}(n+1)\alpha(n+1)}=\frac{1}{2.4....(n-2)n}$$

$$=\frac{1}{\gamma_n}\ \text{(say)} \tag{9}$$

Hence

$$u(x,t)=\frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \int_{B(x,t)} \frac{g\,dy}{\sqrt{t^2-|y-x|^2}}\right)\right.$$

$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \int_{B(x,t)} \frac{h\ dy}{t^2-|y-x|^2}\int\right)\right]$$

is required solution, where n is even.

Note. For n = 2, $\gamma_2 = 2$, we get the Poisson's formula.

4.6 Solution of Non-homogeneous wave equation

Consider the initial value problem

$$u_{tt}-\Delta u = f \qquad \text{in } R^n \times (0,\infty) \tag{1}$$

$$u = 0 \quad , u_t = 0 \quad \text{on } R^n \times \{t=0\} \tag{2}$$

where $f \in C^{[n/2]+1}\left(R^n \times (0,\infty)\right)$; $[n/2]$ denotes the greatest integer function, then solution of equation (1) subject to (2) is

$$u(x,t)=\int_0^t u(x,t;s)\ ds \qquad x\in R^n \ \ t\geq 0 \tag{3}$$

where $u(x,t;s)$ is a solution of

$$u_{tt}(x,t;s)-\Delta u(x,t;s)=0 \qquad \text{in } R^n \times (s,\infty)$$

$$u(x,t;s)=0;\quad u_t(x,t;s)=f(x,t;s) \text{ on } R^n\times\{t=s\} \tag{4}$$

Sol. To show that equation (3) is a solution of equation (1) subject to (2) we need to show

(i) $u \in C^2\left(R^n \times [0,\infty)\right)$

(ii) $u_{tt}-\Delta u = f(x,t)$ in $R^n \times (0,\infty)$

(iii) $\underset{(x,t)\to(x^0,0)}{lt} u(x,t)=0$

$$\lim_{(n,t)\to\left(x^0,0\right)} u_t(x,t) = 0$$

for each point $x^0 \in R^n$.

Proof. (i) $\left[\frac{n}{2}\right]$ denotes the greatest integer function.

If n is even $\left[\frac{n}{2}\right]+1 = \frac{n-1}{2}+1 = \frac{n+1}{2}$

If n is odd $\left[\frac{n}{2}\right]+1 = \frac{n}{2}+1$

From previous article,

$$u(x,t;s) \in C^2\left(R^n \times (\delta,\infty)\right) \text{ for each } \delta \ge 0$$

so $\quad u \in C^2\left(R^n \times [0,\infty)\right)$

(ii) $\quad u(x,t): = \int_0^t u(x,t;s)\, ds$

Differentiating w.r.t. t

$$u_t(x,t): = \int_0^t u_t(x,t;s)\, ds + u(x,t;t)$$

$$= \int_0^t u_t(x,t;s)\, ds \qquad \text{(by 4)}$$

Again differentiating w.r.t. t

$$u_{tt}(x,t): = \int_0^t u_{tt}(x,t;s)\,ds + u_t(x,t;t)$$

$$= \int_0^t u_{tt}(x,t;s)\,ds + f(x,t) \qquad \text{(by 4)} \qquad (5)$$

$$\Delta u(x,t) := \int_0^t \Delta u(x,t;s)\, ds$$

$$= \int_0^t u_{tt}(x,t;s)\, ds \qquad \text{(by 3)} \qquad (6)$$

$$u_{tt}(x,t) - \Delta u(x,t) = \int_0^t \left[u_{tt}(x,t;s) - \Delta u(x,t;s) \right] ds + f(x,t)$$

$$= f(x,t)$$

(iii) Also $u(x,0) = 0$

$$u_t(x,0) = 0$$

The solution of non-homogeneous wave equation is given by equation (3)

Q. Find the solution of

$$u_{tt} - u_{xx} = f(x,t) \quad \text{in } R \times (0,\infty)$$

$$u = 0 \ , \ u_t = 0 \quad \text{on } R \times \{t = 0\}$$

Sol. The solution of homogeneous wave equation in

$$u_{tt} - u_{xx} = 0 \quad \text{in } R \times (0,\infty)$$

$$u = g \ , \ u_t = h \quad \text{on } R \times \{t = 0\}$$

is given by D Alembert's formula.

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t) + \frac{1}{2}\int_{n-t}^{n+t} h(y)\, dy \right]$$

Hence

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\, dy$$

(Replacing t by $t - s$)

Hence

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\, dy\, ds$$

Replacing $t - s$ *by* s ,We find

$$u(x,t)=\frac{1}{2}\int_0^t\int_{x-s}^{x+s} f(y,t-s)\,dy\,ds$$

is the required solution.

4.7 Energy Methods

Uniqueness of solution

Let $U \subset R^n$ be a bounded, open set with a smooth boundary ∂U and

$U_T = U \times (0,T]$

$\Gamma_T = \bar{U}_T - U_T$ where $T > 0$

there exists at most one function $u \in C^2(\bar{U}_T)$ of the initial value problem.

$$u_{tt} - \Delta u = f \qquad \text{in } U_T \tag{1}$$

$$u = g \text{ on } \Gamma_T; \quad U_t = h \text{ on } U \times \{t=0\} \tag{2}$$

Proof. Let $\bar{u}$ be another solution of equation (1). We take

$w(x,t) = u - \bar{u}$,

So

$$w_{tt} - \Delta w = 0 \qquad \text{in } U_T \tag{3}$$

$$w = 0 \text{ on } \Gamma_t; \quad w_t = 0 \quad \text{on } U \times \{t=0\} \tag{4}$$

Define

$$e(t) := \frac{1}{2}\int_U \left[w_t^2 + |Dw|^2 \right] dx \qquad 0 \le t \le T$$

Differentiating w.r.t. t

$$\dot{e}(t) = \int_U [w_t w_{tt} + Dw\, Dw_t]\, dx$$

(Integrating the 2nd integral by parts)

$$= \int_U \left[w_t w_{tt} - D^2 w\, w_t \right] dx + \int_{\partial U} w_t Dw\, \hat{v}\, dS$$

$$= \int_U w_t \left[w_{tt} - \Delta w \right] dx + 0 \qquad \text{(using 4)}$$

$$= 0 \qquad \text{(by 3)}$$

So $e(t)$ = Constant for all t.

But $e(0) = \frac{1}{2} \int_U \left[w_t^2(x,0) + \left| D w(x,0) \right|^2 \right] dx$

$= 0.$

So $e(t)$ is zero for all t.

i.e. $Dw \equiv w_t = 0$ within U_T

Since $w = 0$ on $U \times \{t = 0\}$

$w = u - \bar{u} = 0$ in U_T

$u = \bar{u}$ in U_T

Def. Let $u \in C^2$ be a solution of

$u_{tt} - \Delta u = 0$ in $R^n \times (0, \infty)$

Fix $x_0 \in R^n$, $t_0 > 0$.

Consider the set

$C = \left\{ (x,t) \middle| \ 0 \le t \le t_0; \ |x - x_0| \le t_0 - t \right\}$

which defines a cone.

Theorem. If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$ then $u = 0$ within cone C.

Proof. We define

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} \left[u_t^2(x,t) + \left| Du(x,t) \right|^2 \right] dx \qquad 0 \le t \le t_0$$

Differentiating w.r.t. t.

$$\dot{e}(t) = \int_{B(x_0, t_0 - t)} \left[u_t u_{tt}(x,t) + \left(Du \ Du_t \right) \right] dx$$

$$-\frac{1}{2}\int_{\partial B(x_0,t_0-t)}\left[u_t^2+|Du|^2\right]ds \qquad \text{(By Cor of coarea formula)}$$

Integrating by parts (2nd term of 1st integral)

$$=\int_{B(x_0,t_0-t)}\left[u_t u_{tt}-u_t\ \Delta u\right]dx$$

$$+\int_{\partial B(x_0,t_0-t)}\frac{\partial u}{\partial v}u_t\ ds-\frac{1}{2}\int_{\partial B_0(x_0,t_0-t)}\left[u_t^2+|Du|^2\right]dS$$

$$=0+\int_{\partial B(x_0,t_0-t)}\left[u_t\frac{\partial u}{\partial v}-\frac{1}{2}u_t^{\ 2}-\frac{1}{2}|Du|^2\right]dx$$

$$\leq\int_{\partial B(x_0,t_0-t)}\left[u_t^2+|Du|^2-\frac{1}{2}u_t^2-\frac{1}{2}|Du|^2\right]dx$$

(by Cauding Schwartz Inequality)

≤ 0.

So $e\ (t)$ is a decreasing function of t

$e(t)\leq e(0)$

But e (0) = 0

$e(t)\leq 0$

hence e(t)=0 $\left(\because\ e(t)\right.$ is a sum of square quantities$\left.\right)$

$u_t=Du=0$ within C

$\Rightarrow u$ is constant within C

Hence u = 0 within C $\left(\because\ u=0 \text{ for } t=0\right)$

4.8 Self Assessment Questions

Q. Find the solution of

$u_{tt}-\Delta u=f(x,t)$ in $R^3\times(0,\infty)$

$u=0$, $u_t=0$ on $R^3\times\{t=0\}$

Ans. $u(x,t) = \frac{1}{4\pi} \int\limits_{B(x,t)} \frac{f(y, t - |y - x|)}{|y - x|} dy$

4.9 Suggested Readings

1. L.C. Evans, "Partial Differential Equations," American Mathematical Society, Rhade.
2. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum Outline Series, McGraw Hill Series.

Lesson 5
Other Techniques to Represent Solution

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Structure

5.1 Introduction

There are several other techniques to solve the linear and non-linear partial differential equations. e.g. Separation of variables ,Similarity solutions, Connecting non-linear partial differential equations to linear partial differential equations, Transform methods. Here we will discuss them.

5.2 Separation of variables

In this method, we assume a solution given by sum or product of undetermined functions and form ordinary differential equations, which are solved. This technique is well understood by examples.

Exp. Consider the boundary value problem in heat equation

$$u_t - \Delta u = 0 \text{ in } U \times (0, \infty) \tag{1}$$

$$u = 0 \text{ on } \partial U \times [0, \infty) \tag{2}$$

$$u = g \text{ on } U \times \{t = 0\}$$

where $g : U \to R$ is given.

Sol. Let the solution of equ. (1) be given by

$$u(x,t) = v(t)\, w(x) \qquad x \in U,\ t \geq 0 \tag{3}$$

From (1) and (3)

$$v' w(x) - \Delta w\, v(t) = 0$$

Dividing by $w(x)\, v(t)$

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} \tag{4}$$

L.H.S. of equ. (4) is a function of t only and R.H.S. is a function of x only. Equ. (4) is true if each side is equal to some constant, say, μ.

$$\frac{v'(t)}{v(t)} = \mu = \frac{\Delta w(x)}{w(x)}$$

$$\Rightarrow \qquad v'(t) - \mu v(t) = 0 \tag{5}$$

and

$$\Delta w(x) - \mu w(x) = 0 \tag{6}$$

Considering equ. (5) and integrating

$$v = C\, e^{\mu t} \tag{7}$$

where C is a constant.

Taking equ. (6), comparing with the

$$\left.\begin{aligned} -\Delta w = \lambda w \quad & \text{in } U \\ w = 0 \quad & \text{on } \partial U \end{aligned}\right\} \tag{8}$$

then λ is eigen value and $w(\neq 0)$ is the corresponding eigen function.So

$\mu = -\lambda$ is eigen value of equ. (6) and w is corresponding eigen function.

Hence solution of problem defined by equ. (1) and (2) is

$$u(x,t) = Ce^{-\lambda t} w(x) \tag{9}$$

where C is a constant to be determined from the initial condition at $t = 0$, which gives

$$g = Cw$$

so $\quad u = Ce^{-\lambda t} w$

where $g = Cw$

is required solution.

Particular case:

(a) If $\lambda_1, \lambda_2, ..., \lambda_m$ are eigen values of problem (8) and $w_1, w_2, ..., w_m$ are the corresponding eigen functions and $c_1, c_2, ..., c_m$ are constants then solution of equ. (1)

$$u(x,t) = \sum_{i=1}^{m} c_i e^{-\lambda_i t} w_i(x)$$

provided $\sum_{i=1}^{m} d_i w_i = g$.

(b) Let $\lambda_1, \lambda_2, ...$ be a countable set of eigen values with corresponding eigen function $w_1, w_2, ...$ so that

$$u = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t} w_i(x)$$

provided that $\sum_{i=1}^{\infty} c_i w_i(x) = g$ in *U*.

Exp. Find the solution of the non-linear porous medium equation

$$u_t - \Delta(u^\gamma) = 0 \text{ in } R^n \times (0,\infty) \tag{1}$$

where $u \geq 0$ and $\gamma > 1$ is a constant.

Sol. We seek a solution of equ. (1) of the type

$$u(x,t) = v(t)w(x) \tag{2}$$

From (1) and (2)

$$w(x)v'(t) - (\Delta w^\gamma)v^\gamma = 0$$

Dividing by wv^γ

$$\frac{v'(t)}{v^\gamma} = \frac{\Delta w^\gamma}{w} \tag{3}$$

L.H.S. is a function of *t* only and R.H.S. is a function of *x* only. Equ. 3 is true if each side is equal to some constant say μ.

$$\frac{v'(t)}{v^\gamma} = \mu$$

$$\frac{v^{-\gamma+1}}{-\gamma+1} = \mu t + \lambda,$$

where λ is a constant.

$$v^{1-\gamma} = (1-\gamma)\mu t + \lambda,$$

$$v = \left[(1-\gamma)\ \mu t + \lambda\right]^{\frac{1}{1-\gamma}} \tag{4}$$

$$\Delta w^\gamma = \mu\, w \tag{5}$$

Suppose $w = |x|^\alpha$ is solution of equ. (5) where α is a constant to be determined.

$$\Delta w^\gamma = \Delta |x|^{\alpha\gamma}$$

$$= |x|^{\alpha y-2} \ \alpha\gamma\left[n+\alpha\gamma-2\right]$$

Using in equ. (5)

$$|x|^{\alpha\gamma-2}\alpha\gamma\left(n+\alpha\gamma-2\right)=\mu|x|^{\alpha} \tag{6}$$

In order to hold equ. (6) in R^n, we must have

$$\alpha=\alpha\gamma-2 \quad \Rightarrow \ \alpha=\frac{2}{\gamma-1} \text{ and}$$

$$\mu=\alpha\gamma\left(n+\alpha\gamma-2\right)>0 \tag{7}$$

So solution of equ. (1) is

$$u=\left[\left(1-\gamma\right)\ \mu t+\lambda\right]^{\frac{1}{1-\gamma}}\ |x|^{\alpha}. \tag{8}$$

where α, μ are given by equ. (7)

Remark. In equ. (8) u is singular when

$$\left(1-\gamma\right)\mu t+\lambda=0 \text{ or}$$

$$t=\frac{\lambda}{\left(\gamma-1\right)\mu}=t^{*} \text{ (say)}$$,t^* is called the critical time.

5.3 Similarity solution

Certain symmetries of partial differential equation help to convert them in ordinary differential equations.

Def. Plane Travelling Wave:

A solution $u\left(x,t\right)$ of the partial differential equations of two variables $x,t\in R$ of the form

$$u\left(x,t\right)=v\left(x-\sigma t\right) \qquad x\in R,\ t\in R$$

represents a travelling wave with speed σ and velocity profile v.

Generalization. A solution $u\left(x,t\right)$ of a partial differential equations in n + 1 variables $x=\left(x_1,\ x_2,\ \ldots,\ x_n\right)\in R^n$, $t\in R$ having the form

$$u(x,t) = v(y.x - \sigma t)$$

is called a plane wave with wave front normal to $y \in R^n$.

Exponential solution: The exponential solution of partial differential equations is

$$u(x,t) = e^{i(y\cdot x + wt)}$$

where $w \in \mathbb{C}$, $y = (y_1, y_2, ..., y_n) \in R^n$, w being frequency and $\{y_i\}_{i=1}^{n}$ the wave number.

Exp. The heat equation

$$u_t - \Delta u = 0$$

has the exponential solution.

$$u = e^{i\left(yx + i\,|y|^2 t\right)}$$

$$= e^{-|y|^2 t}\left[e^{iyx}\right]$$

$e^{-|y|^2 t}$cos yx and $e^{-|y|^2 t}$ sin yx are solutions of equ.(1).Here the term $e^{-|y|^2 t}$ corresponds the dissipation of energy.

Exp. The wave equation

$$u_{tt} - \Delta u = 0$$

has the exponential solution

$$u = e^{i(y\cdot x \pm |y| t)}$$

Since w is real, no dissipation effects occur.

Exp. The dispersive equation

$$u_t + u_{xxx} = 0 \qquad \text{in } R \times (0,\infty)$$

has the exponential solution

$$u = e^{i\left(y\cdot x + y^3 t\right)}$$

No dissipation of energy. Also the velocity of propagation depends on frequency. Hence dispersion takes place.

Exp. Barenblaltt's solution

Consider the porous medium equation

$$u_t - \Delta u^\gamma = 0 \quad \text{in} \quad R^n \times (0,\infty) \tag{1}$$

where $u \geq 0$ and $\gamma > 1$ is a constant.

Sol. We seek a solution of equ. (1) of the form

$$u(x,t) = \frac{1}{t^\alpha}\, v\left(\frac{x}{t^\beta}\right) \qquad x \in R^n,\ t > 0$$

where $\alpha,\ \beta$ are unknowns.

$$u(x,t) = \frac{1}{t^\alpha}\, v(y)$$

where $y = x / t^\beta$ (2)

From equ. (1) and (2)

$$\alpha v(y) + \beta y.Dv(y) + \frac{1}{t^{\alpha\gamma+2\beta-\alpha-1}}\, v^\gamma = 0 \tag{3}$$

To convert equ. (3) into an equation independent of *t*, we must have

$$\alpha\gamma + 2\beta - \alpha - 1 = 0 \quad \text{or}$$

$$\alpha = \frac{1-2\beta}{\gamma-1} \tag{4}$$

Hence equ. (3) gives

$$\alpha v + \beta y.Dv + \Delta v^\gamma = 0 \tag{5}$$

We seek a radial solution of equ. (5)

Let it be

$$v(y) = w(r) \text{ where } r = |y| \tag{6}$$

From equ. (5) and (6)

$$\alpha w + \beta w'(r) r + \left(w^\gamma\right)'' + (n-1)\left(w^\gamma\right)' r^{n-2} = 0$$

where dash denotes derivative w.r.t. r.

To make it exact differential, multiplying by r^{n-1} and taking $\alpha = n\beta$

$$\beta\left(r^n w\right)' + \left[r^{n-1}\left(w^\gamma\right)'\right]' = 0$$

Integrating and assuming that as $r \to 0 \;\; w,\, w' \to 0$

$$\beta r^n w + r^{n-1}\left(w^\gamma\right)' = 0$$

$$\left(w^\gamma\right)' = -\beta r\, w$$

$$\gamma w^{\gamma-1} = -\beta r w$$

or $w^{\gamma-2} = \dfrac{-\beta}{\gamma} r$

Again integrating

$$w^{\gamma-1} = \frac{-\beta r^2}{2\gamma(\gamma-1)} + b$$

where b is a constant.

$$\Rightarrow \qquad w(r) = \left[b - \frac{(\gamma-1)\beta}{2\gamma}\; r^2\right]^{\frac{1}{\gamma-1}}$$

Hence

$$v(y) = \left[b - \frac{(\gamma-1)\beta|x|^2}{2\gamma\; t^{2\beta}}\right]^{\frac{1}{\gamma-1}}$$

where $\qquad \alpha = n\beta = \dfrac{1-2\beta}{\gamma-1}$

i.e. $\qquad \beta = \dfrac{1}{2-n+n\gamma}$

$$\alpha = \frac{n}{2-n+n\gamma}$$

5.4 Connecting non-linear partial differential equations to linear partial differential equations

5.4.1 Cole-Hopf transformation

Consider the initial value problem for a quasi-linear parabolic equation

$$u_t - a\Delta u + b|Du|^2 = 0 \text{ in } R^n \times (0,\infty) \quad (1)$$

$$u = g \text{ on } R^n \times \{t = 0\} \quad (2)$$

where $a > 0$;a,b are constants..

Sol. Let $w = \phi(u)$ (3)

where u is a smooth solution of equ. (1) and $\phi : R \to R$ is a smooth function. We seek ϕ such that w solves the linear equation.

From (1) and (3)

$$Dw = \phi'(u)\; Du$$

$$\Delta w = \phi'(u)\Delta u + \phi''(u)|Du|^2$$

$$w_t = \phi'(u)\left[a\Delta u - b|Du|^2\right]$$

$$= a\left[\Delta w - \phi''(u)|Du|^2\right] - b\phi'(u)|Du|^2$$

Hence, $w_t - a\Delta w = -\left[a\phi''(u) + b\,\phi'(u)\right]\,|Du|^2$

We choose ϕ such that

$$a\phi''(u) + b\phi'(u) = 0 \quad (4)$$

So we have

$$w_t - a\Delta w = 0 \quad (5)$$

To find the solution of equ. (4)

Auxiliary equation is $am^2 + bm = 0$

roots with $m = 0, -b/a$

Hence

$\phi(u) = e^{-(b/a)u}$+C ,where C is a constant.

Neglecting the constant

$$w(x,t) = e^{-(b/a)u} \qquad (*)$$

$$w(x,0) = e^{-(b/a)g} \qquad (6)$$

Combining (5) and (6)

$$w_t - a\Delta w = 0 \quad \text{in } R^n \times (0,\infty)$$

$$w = e^{-\frac{b}{a}g} \quad \text{on } R^n \times \{t=0\}$$

which is heat equation having the solution

$$w(x,t) = \frac{1}{(4\pi at)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4at}} \; e^{\frac{-b}{a}g} \; dy \qquad x \in R^n$$

or $\quad u(x,t) = -\frac{a}{b}\log w$

$$u(x,t) = -\frac{a}{b}\log\left[\frac{1}{(4\pi at)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4at}} \; e^{\frac{-b}{a}g} \; dy\right] \qquad x \in R^n,\ t>0$$

This is the required solution. Equation (*) is known as Cole Hopf transformation.

Exp. Find the solution of Burger's equation with viscosity

$$u_t - a\ u_{xx} + u\ u_x = 0 \quad \text{in } \ R \times (0,\infty)$$

$$u = g \qquad \text{on } R \times \{t=0\} \qquad (1)$$

Sol. Let us take

$$w(x,t) := \int_{-\infty}^{x} u(y,t)\ dy$$

$$h(x) := \int_{-\infty}^{x} g(y)\ dy \qquad (2)$$

so that $w_x = u,\ w_x(x,0) = u(x,0) = g(x) = h'(x)$

From (1) and (3)

$$w_{xt} - a\ w_{xxx} + w_x\ w_{xx} = 0 \text{ in } R \times (0,\infty)$$

$$\frac{\partial}{\partial x}\left[w_t - a\ w_{xx} + \frac{1}{2} w_x^2\right] = 0 \text{ in } R \times (0,\infty)$$

Thus, problem is converted to

$$w_t - a\ w_{xx} + \frac{1}{2} w_x^2 = 0 \text{ in } R \times (0,\infty)$$

$$w(x,0) = h(x) \qquad \text{on } R \times \{t = 0\} \tag{4}$$

The equation (4) is a quasi-linear parabolic equation (previous example) with $b = \frac{1}{2}$.So solution of equ. (4) is

$$w(x,t) = -2a \log\left[\frac{1}{(4\pi at)^{n/2}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4at} - \frac{h(y)}{2a}}\ dy\right]$$

Differentiating w.r.t. x

$$u = w_x = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t}\ e^{\frac{-|x-y|^2}{4at} - \frac{h(y)}{2a}}\ dy}{\displaystyle\int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4at} - \frac{h(y)}{2a}} dy}$$

This is required solution.

5.4.2 Potential function

By use of potential function, non-linear partial differential equations can be converted to linear partial differential equations.

Exp. Consider the Euler's equation for inviscid, incompressible flow

$$\boldsymbol{u}_t + \boldsymbol{u}.D\boldsymbol{u} = -Dp + \boldsymbol{f} \text{ in } R^3 \times (0,\infty)$$

$$div\ \boldsymbol{u} = 0 \text{ in } R^3 \times (0,\infty)$$

$$\boldsymbol{u} = g \quad \text{on } R^3 \times \{t = 0\} \tag{1}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are prescribed functions, u and p are unknowns.

Sol. Let the external body force be derived from potential function *h*, such that

$$\boldsymbol{f} = Dh \tag{2}$$

Let the velocity $\underline{u}$ be derived from the potential v s.t.

$$\boldsymbol{u} = Dv \tag{3}$$

From equ. (1) and (3)

$$div\ \boldsymbol{u} = \Delta v = 0 \tag{4}$$

So from equ. (4) we can find v and thus $\underline{u}$.

From (1) and (3)

$$Dv_t + Dv\ D(Dv) = -Dp + Dh$$

or

$$D\left[v_t + \frac{1}{2}|Dv|^2 + p - h\right] = 0$$

Integrating

$$v_t + \frac{1}{2}|Dv|^2 + p = h$$

which is Bernoulli's equation to get *p*.

5.5 Transform Methods

5.5.1 Fourier Transforms

We now discuss the transform methods to solve linear and non-linear partial differentiation equations. First we define Fourier transform over L^1 and L^2 spaces, respectively.

Def. Let $u \in L^1(R^n)$, we define the Fourier transform of $u(x)$, denoted by $\hat{u}(y)$ as

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-i\,x.y} \; u(x)\; dx \qquad\qquad y \in R^n$$

and its inverse Fourier transform

$$\breve{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ix.\,y} \; u(x)\; dx \qquad y \in R^n$$

Since $\left|e^{\pm i\,xy}\right| = 1 \quad and \qquad u \in L^1\left(R^n\right)$

So integral converges for each y.

Plancherel's theorem

Assume that $u \in L^1\left(R^n\right) \cap L^2\left(R^n\right)$ then

$\hat{u},\ \breve{u} \in L^2\left(R^n\right)$ and

$$\|\hat{u}\|_{L^2\left(R^n\right)} = \|\breve{u}\|_{L^2\left(R^n\right)} = \|u\|_{L^2\left(R^n\right)} \tag{1}$$

Proof. To prove (1), we prove three results

(i) $$\int_{R^n} v(y)\,\hat{w}(y)\; dy = \int_{R^n} \hat{v}(x)\,w(x)\; dx$$

$$\text{L.H.S.} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} v(y) \int_{R^n} \left[w(x) e^{-ix.y}\; dx\right] dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} w(x) \int_{R^n} v(y)\; e^{-ixy}\; dy\; dx$$

$$= \int_{R^n} w\;(x)\;\hat{v}(x)\; dx$$

Hence the result.

(ii) If $u, v \in L^1\left(R^n\right) \cap L^2\left(R^n\right)$

then $(u * v) = (2\pi)^{n/2}\, \hat{u}\,\hat{v}$

where * denotes the convolution operator.

By def.

$$u * v = \int_{R^n} u(z)v(x-z)\,dz$$

$$(u*v)^{\wedge} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-i x y} \left\{ \int_{R^n} u(z)v(x-z) \right\} dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} v(x-z) e^{-i y(x-z)} dx \int_{R^n} e^{-izy} u(z)\,dz$$

$$= \hat{v}(y) \int_{R^n} e^{-i z y} u(z)\,dz$$

$$= \hat{v}(y)\left[\hat{u}(y)(2\pi)^{n/2} \right]$$

$$= (2\pi)^{n/2}\, \hat{u}(y)\hat{v}(y)$$

(iii) Consider

$$\int_{R^n} e^{-i x y - t|x|^2}\,dx = \prod_{i=1}^{n} \int_R e^{-ix_i y_i - t x_i^2}\,dx$$

But $\int_R e^{-ix_i y_i - t x_i^2}\,dx_i = \int_{-\infty}^{\infty} e^{-t\left[x_i^2 + \frac{ix_i y_i}{t}\right]}\,dx$

$$= \int_{-\infty}^{\infty} e^{-t\left(x_i + \frac{iy_i}{2t}\right)^2 + t\left(\frac{iy_i}{2t}\right)^2}\,dx_i$$

$$= \frac{e^{-y_i^2/4t}}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2}\,dz \quad \text{where} \quad z = \sqrt{t}\left(x_i + \frac{iy_i}{2t}\right)$$

$$= \frac{e^{-y_i^2/4t}}{\sqrt{t}}\sqrt{\pi}$$

Hence

$$\int_{R^n} e^{-ix.y-t|x|^2} \, dx = \left(\frac{\pi}{t}\right)^{n/2} e^{-|y|^2/4t}$$

Proof of theorem:

Choosing a function for $\in > 0$

$$v_\in(x) = e^{-\in|x|^2}$$

$$\hat{v}_\in(y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ixy} v_\in(x) \, dx \quad \text{(Using result (iii), putting } t = \in \text{)}$$

$$= \frac{1}{(2\in)^{n/2}} e^{-|y|^2/4\in} \qquad (2)$$

Hence

$$\int_{R^n} \hat{w}(y) e^{-\in|y|^2} \, dy = \frac{1}{(2\in)^{n/2}} \int_{R^n} w(x) e^{-|x|^2/4\in} \, dx \quad \text{(Using result (i))}$$

(3)

Taking limit as $\in \to 0$

$$\int_{R^n} \hat{w}(y) \, dy = \underset{\in\to 0}{lt} \frac{1}{(2\in)^{n/2}} \int_{R^n} w(x) \, e^{-|x|^2/4\in} \, dx$$

$$= (2\pi)^{n/2} \, w(0) \qquad \text{where} \quad \frac{x_i^2}{4\in} = z_i^2 \qquad (4)$$

Suppose $u \in L^1(R^n) \cap L^2(R^n)$

and set $v(x) := \bar{u}(-x)$, $\bar{u}$ is the conjugate of u.

$$w(x) := u * v$$

$$= \int_{R^n} u(z) \, v(x-z) \, dz$$

$$\hat{w} = (2\pi)^{n/2} \, \hat{u} \, \hat{v} \qquad \text{(by result II)}$$

But $\hat{v} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ixy} \bar{u}(-x)\, dx$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ixy}\ \bar{u}(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} \overline{e^{-ixy} u(x)}\, dx$$

$$= \overline{\hat{u}(y)}$$

$$\therefore\ \hat{w} = (2\pi)^{n/2}\ |\hat{u}|^2 \qquad (5)$$

From (4) and (5)

$$(2\pi)^{n/2} \int_{R^n} |\hat{u}|^2\ dy = (2\pi)^{n/2}\, w(0)$$

$$\text{or} \quad \int_{R^n} |\hat{u}|^2\ dy = \int_{R^n} u(z)\bar{u}(z)\, dz$$

$$= \int_{R^n} |u|^2\ dz \qquad \text{(by def.)}$$

$$\|\hat{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$$

Similarly $\|\breve{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$

(The result can be obtained by previously argument changing i to $-i$)

Hence

$$\|u\|_{L^2(R^n)} = \|\hat{u}\|_{L^2(R^n)} = \|\breve{u}\|_{L^2(R^n)}$$

Note

Since $u \in L^2(R^n)$ choose a sequence $\{u_k\}_{k=1}^{\infty} \subset L^1(R^n) \cap L^2(R^n)$ with

$u_k \to u$ in $L^2(R^n)$.

By (1)

$$\|\check{u}_k - u_j\|_{L^2(R^n)} = \|\hat{u}_k - \hat{u}_j\|_{L^2(R^n)} = \|u_k - u_j\|_{L^2(R^n)}$$

$\{\hat{u}_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(R^n)$ which converges to $\hat{u}$.

So $\hat{u}_k \to \hat{u}$ in $L^2(R^n)$

Def *Fourier Transform of u over* $L^2(R^n)$

Let $u \in L^2(R^n)$ then

$$\|\hat{u}\|_{L^2(R^n)} = \|\breve{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$$

So $\hat{u},\ \breve{u} \in L^2(R^n)$ (by above theorem)

hence $\hat{u},\ \breve{v}$ are well defined over $L^2(R^n)$.

Properties of Fourier transform:

Assume $u, v \in L^2(R^n)$

(i) $\int_{R^n} u\,\bar{v}\ dx = \int_{R^n} \hat{u}\,\bar{\hat{v}}\ dy$

(ii) $\widehat{D^\alpha u} = (iy)^\alpha\ \hat{u}$

for each multiindex α s.t. $D^\alpha u \in L^2(R^n)$

Proof. Let $u, v \in L^2(R^n)$ and $\alpha \in C^n$

then

$$\|u + \alpha v\|^2 = \|\hat{u} + \alpha\hat{v}\|^2 \text{ (Using Plancherel's theorem)}$$

i.e. $\int_{R^n} (u + \alpha v)(\bar{u} + \bar{\alpha}\bar{v})\ dx = \int_{R^n} (\hat{u} + \alpha\hat{v})(\bar{\hat{u}} + \bar{\alpha}\bar{\hat{v}})\ dy$

$$\Rightarrow \int_{R^n} \left[|u|^2 + |\alpha v|^2 + \bar{u}(\alpha v) + u(\bar{\alpha}\bar{v})\right] dx$$

$$= \int_{R^n} \left[|\hat{u}|^2 + |\alpha\hat{v}|^2 + \bar{\hat{u}}(\alpha \hat{v}) + \hat{u}\left(\bar{\alpha}\, \bar{\hat{v}}\right) \right] dy$$

or $$\int_{R^4} \left[\bar{u}(\alpha v) + u(\bar{\alpha}\bar{v}) \right] dx = \int_{R^n} \left(\alpha \bar{\hat{u}}\hat{v} + \bar{\alpha}\hat{u}\bar{\hat{v}} \right) dy \qquad (1)$$

Taking $\alpha = 1, i$ in (1) respectively and subtracting we obtain

$$\int_{R^n} u\bar{v}\ dx = \int_{R^n} \left(\hat{u}\bar{\hat{v}} \right) dy$$

(ii) If u is smooth and has compact support

$$\widehat{D^\alpha u} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ix\,y} D^\alpha u\ dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{R^n} D^\alpha\ e^{-ixy}\ u(x)\ dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{R^n} e^{-ixy} (-1)^\alpha (iy)^\alpha u(x)\ dx$$

$$= (iy)^\alpha\ \hat{u}(y)$$

Exp. Solve the partial differential equation

$$-\Delta u + u = f \text{ in } \mathrm{R}^n \qquad (1)$$

where $f \in C^2\left(R^n\right)$.

Sol. Taking Fourier transform of equation (1)

$$-(iy)^2\ \hat{u} + \hat{u} = \hat{f}, \qquad y \in R^n$$

$$\hat{u} = \frac{\hat{f}}{1+y^2} \qquad (2)$$

Taking inverse Fourier transform of (2)

$$u = \left(\frac{\hat{f}}{1+y^2} \right)^{v}$$

$u = f * B$ where

$$B = \frac{1}{\left(1+y^2\right)^{\vee}}$$

To find B, we know that

$$\frac{1}{a} = \int_0^{\infty} e^{-ta}\ dt$$

so $$\frac{1}{1+|y|^2} = \int_0^{\infty} e^{-t\left(1+\left|y^2\right|\right)} dt$$

$$\Rightarrow \left(\frac{1}{1+|y|^2}\right)^{\vee} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} \int_0^{\infty} e^{-t\left(1+|y|^2\right)} e^{ixy} dt\ dy$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} e^{-t} \left(\frac{\pi}{t}\right)^{n/2} e^{-|x|^2/4t}\ dt$$

$$= \frac{1}{2^{n/2}} \int_0^{\infty} \frac{e^{-t} - \frac{|x|^2}{4t}}{t^{n/2}}\ dt \qquad x \in R^n \tag{3}$$

So,

$$u(x,t) = \frac{1}{(4\pi)^{n/2}} \int_0^{\infty} \int_{R^n} \frac{f(y) e^{-t-\frac{|x-y|^2}{4t}}}{t^{n/2}}\ dy\ dt$$

Here ,B given in equation (3), is called the Bessel 's potential.

Exp. Find the solution of initial value problem of heat equation

$$u_t - \Delta u = 0 \text{ in } R^n \times (0,\infty) \tag{1}$$

$$u = g \quad \text{ on } R^n \times \{t = 0\} \tag{2}$$

Sol. Taking Fourier transform of equation (1) and (2) w.r.t. the spatial variable *x*.

$$\hat{u}_t - (iy)^2\,\hat{u} = 0 \qquad \text{for } t > 0 \tag{3}$$

$$\hat{u} = \hat{g} \qquad \text{for } t = 0 \tag{4}$$

or

$$\frac{\hat{u}_t}{\hat{u}} = -y^2$$

Integrating

$\hat{u} = Ce^{-y^2 t}$, where C is a constant.

Since $C = \hat{g}$ (Using (4)

$$\hat{u} = \hat{g}e^{-\left|y^2\right|t}$$

Taking inverse Fourier transform

$$u = \frac{g * F}{(2\pi)^{n/2}}$$

where,

$$F = \left(e^{-t|y|^2}\right)^{\vee}$$

$$= \frac{1}{(2t)^{n/2}}\,e^{-|x|^2/4t}$$

Hence solution is

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} g(y)\,e^{\frac{-|x-y|^2}{4t}}\,dy$$

Exp. Solve the Schroödinger's equation.

$$iu_t + \Delta u = 0 \qquad \text{in } R^n \times (0,\infty) \tag{1}$$

$$u = g \qquad \text{on } R^n \times \{t = 0\} \tag{2}$$

where u and g are complex valued functions .

Sol. Equ. (1) can be rewritten as

$$\frac{\partial u}{\partial(-it)} + \Delta u = 0$$

which is obtained from heat equation replacing t by it, hence we get

$$u(x,t) = \frac{1}{(4\pi it)^{n/2}} \int_{R^n} e^{\frac{i|x-y|^2}{4t}} g(y)\, dy \qquad t \neq 0 \tag{3}$$

which is required solution.

Remark. From equ. (3), we can obtain the fundamental solution of Schrodinger equation

$$\Psi(x,y) := \frac{1}{(4\pi it)^{n/2}} e^{\frac{i|x|^2}{4t}} \qquad x \in R^n,\ t \neq 0$$

Exp. Find the solution of initial value problem

$$u_{tt} - \Delta u = 0 \text{ in } R^n \times (0,\infty) \tag{1}$$

$$\left.\begin{aligned} u &= g \\ u_t &= 0 \end{aligned}\right\} \quad \text{on } R^n \times \{t = 0\} \tag{2}$$

Sol. Taking Fourier transform of equation (1) w.r.t. x

$$\hat{u}_{tt} + |y|^2 \hat{u} = 0 \qquad \text{for } t > 0 \tag{3}$$

$$\hat{u} = \hat{g} \quad \hat{u}_t = 0 \qquad \text{for } t = 0 \tag{4}$$

We seek an exponential solution of equ. (3). Let

$\hat{u} = \beta e^{t\gamma}$ where $\beta, \gamma \in$ □ are to be determined.

From (1) and (3)

$$|\gamma|^2 + |y|^2 = 0$$

$$\gamma = \pm i|y|$$

$$\hat{u} = \beta_1 e^{i|y|t} + \beta_2 e^{-i|y|t}$$

Using equation (2), we obtain

$$\beta_1 - \beta_2 = 0$$

$$\Rightarrow \beta_1 = \beta_2 = \beta \quad \text{(say)}$$

and $\quad \beta = \hat{g}/2$

Hence

$$\hat{u}(y,t) = \frac{\hat{g}}{2}\left(e^{i|y|t} + e^{-i|y|t}\right)$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{R^4} \frac{\hat{g}}{2}\left(e^{i|y|t} + e^{-i|y|t}\right) e^{ixy}\, dy \qquad x \in R^n,\ t \geq 0$$

is the required solution.

5.5.2 Laplace Transforms

Laplace transform method is useful for functions defined on R_+ i.e. $(0,\infty)$ if $u \in L^1(R_+)$, we define the Laplace transform of u,

$$\mathbf{L}\left(\bar{u}(s)\right) := \int_0^\infty e^{-st} u(t)\, dt \quad s \geq 0$$

We denoted by $\bar{u}$.

Exp. Solve the heat equation

$$v_t - \Delta v = 0 \text{ in } U \times (0,\infty) \tag{1}$$

$$v = f \quad \text{on } U \times \{t = 0\} \tag{2}$$

Sol. Taking Laplace transform of (1) w.r.t. t

$$\Delta \bar{v}(x,s) = \int_0^\infty e^{-st} \Delta v(x,t)\, dt$$

$$= \int_0^\infty e^{-st} v_t(x,t)\, dt$$

$$= e^{-st}v(x,t)\Big|_0^\infty + \int_0^\infty s\, e^{-st}v(x,t)\, dt$$

$$= e^{-st}v(x,t)\Big|_0^\infty + s\bar{v}(x,s)$$

$$= -f(x) + s\bar{v}(s)$$

Hence

$$-\Delta\bar{v}(s) + s\bar{v}(s) = f \tag{3}$$

Equation (3) is called Resolvent equation. The solution of resolvent equation is the Laplace transform of equation (1).

5.6 Self Assessment Questions

Exp. Solve the Hamilton Jacobi equation

$$u_t + H(Du) = 0 \text{ in } R^n \times (0,\infty)$$

where H is the Hamilton function.

Exp. Find the exponential solution of Schrodinger 's equation

$$i\,u_t + \Delta u = 0 \quad \text{in} \quad R^n$$

Q. Solve the telegraph equation.

$$u_{tt} + 2du_t - u_{xx} = 0 \text{ in } R \times (0,\infty)$$

$$u = g \qquad u_t = h \text{ on } R \times \{t = 0\}$$

for $d > 0$.

Q Prove that

(i) If $u, v \in L^1(R^n) \cap L^2(R^n)$ then

$$(u * v)^n = (2\pi)^{n/2}\,\hat{u}\,\hat{v}$$

(ii) $u = (\hat{u})^\vee$

5.7 Suggested Readings

1. L.C. Evans, "Partial Differential Equations," American Mathematical Society, Rhade.
2. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum Outline Series, McGraw Hill Series.