Or

Let f and g be bounded measurable functions defined on a measurable set E of finite measure. Then prove that :

(i) $\int_E (f+g) = \int_E f + \int_E g$

(ii) If $f \le g$ a.e, then prove that $\int_E f \le \int_E g$.

Hence $\left|\int_E f\right| \le \int_E |f|$.

(iii) If $\alpha \le f(x) \le \beta$, then prove that :

$$\alpha \; m(E) \le \int_E f(x)dx \le \beta m(E)$$

(iv) If E_1 and E_2 are disjoint measurable subsets of E, then prove that : **12**

$$\int_{E_1 \cup E_2} f \quad \int_{E_1} f + \int_{E_2} f$$

12. State and prove Vitali's covering lemma by giving full details. **12**

Roll No. .... **Exam Code : J-19**

Subject Code—0362

M. Sc. EXAMINATION

(For Batch 2011 Onwards Main & Reappear)

(Second Semester)

MATHEMATICS

MAL-522

Measure and Integration Theory

Time : 3 Hours *Maximum Marks : 70*

Section A

Note : Attempt any *Seven* questions. **7×5=35**

1. Let f be a continuous function and g be a measurable function, then prove that the composite function $f \circ g$ is measurable.

2. Prove that a continuous function defined on a measurable set is measurable. Is the converse true ? If not, justify by giving a suitable example.

3. Show by giving a suitable example (with solution) that there are sequences of measurable functions that converge in measure but fail to converge at any point.

4. Prove that a bounded measurable function f defined on a measurable set E of finite measure is Lebesgue integrable.

5. If $\int_E f = 0$ and $f(x) \geq 0$ on E, then prove that $f = 0$ *a.e.*

6. State bounded convergence theorem. Verify this theorem for the sequence of functions

$$f_n(x) = \frac{1}{\left(1+\frac{x}{n}\right)^n}, \ 0 \leq x \leq 1, \ n \in \mathrm{N}.$$

7. Prove that a measurable function f defined on E is integrable over E iff $|f|$ is integrable over E. Also, using this result, prove that if f is an integrable function, then it is finite-valued *a.e.*

8. State four properties of Dini's derivatives. Also give an example of the functions f and g for which $D^+(f+g) \neq D^+ f + D^+ g$.

9. If f is a function of bounded variation, then prove that it is measurable.

10. State and prove Fundamental theorem of integral calculus.

Section B

Note : Attempt all the questions.

11. State and prove Lebesgue dominated convergence theorem. Use this theorem to evaluate the following integral :

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx,$$

where $f_n(x) = \dfrac{n^{3/2} x}{1 + n^2 x^2}, \ n \in \mathrm{N}, \ x \in [0, 1]$

Also by giving a suitable example, show that the existence of an integrable 'dominant function' in this theorem is sufficient but not necessary for the interchange of the limit and integral operations. **12**

Or

If f and g are absolutely continuous functions on $[a, b]$, then prove that functions :

(i) $f + g$

(ii) $f\,g$

(iii) f/g $(g \neq 0)$

are also absolutely continuous functions on $[a, b]$. **12**

13. Define characteristic function and simple function. Give an example of simple function. Also show that a function is simple if and only if it is measurable and assumes only a finite number of values. **11**

Or

State and prove Lusin theorem. **11**

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