12. Define dual space. Prove that $\left(l_p^n\right)^* = l_q^n$ and $\left(l_1^n\right)^* = l_\infty^n$ where :

$$l_p^n = \left\{ x = \left(x_1, x_2, \ldots\ldots, x_n\right) : \|x\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\}$$

$$l_1^n = \left\{ x = \left(x_1, x_2, \ldots\ldots, x_n\right) : \|x\| = \sum_{i=1}^{n} |x_i| \right\}$$

$$l_\infty^n = \left\{ x = \left(x_1, x_2, \ldots\ldots, x_n\right) : \|x\| = \max_{1 \le i \le n} |x_i| \right\}$$

Or

If P is a projection on a Banach space B and if M and N are its range and null space, then prove that M and N are closed linear subspace of B such that $B = M \oplus N$. Also state and prove its converse part.

Subject Code—0366-X

M. Sc. EXAMINATION

(Fourth Semester)

(Prior to 2011 Re-appear)

MATHEMATICS

MAL-641

Functional Analysis

Time : 3 Hours *Maximum Marks :* 100

Section A

Note : Attempt any *Seven* questions. **7×7=49**

1. Prove that a normed linear space X is complete iff every absolutely summable series in X is convergent (summable).

2. State and prove Minkowski's inequality for L^p space.

3. Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M, then there exists a functional F in N* such that F(M) = {0} and $F(x_0) \neq 0$

4. Let X and Y be normed spaces over the field **K** and $T : X \xrightarrow{\text{onto}} Y$ be a linear operator. Then, T^{-1} exists and is a bounded linear operator iff ∃ a constant K > 0 such that $\|Tx\|_y \geq K\|x\|_X, \forall x \in X$.

5. Prove that a linear transformation is closed iff its graph is a closed subspace.

6. Let X and Y be normed spaces and T : X → Y be linear operator. Then prove that T is compact iff it maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in y which has a convergence.

7. State and prove Schwarz's inequality in an Inner product space.

8. Show that the linear space C[a, b] equipped with the norm given by $\|x\|_\infty = \sup_{t \in [a, b]} |x(t)|, x \in C[a, b]$, is not an inner product space and hence not a Hilbert space.

9. A subspace M of a Hilbert space H is closed in H iff $M = M^{\perp\perp}$.

10. If N_1 and N_2 are normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and N_1N_2 are normal.

Section B

Note : Attempt all the questions. **3×17=17**

11. State and prove Riesz-Fisher theorem for the completeness of L^p space.

Or

State and prove Riesz-representation theorem for bounded linear functional on C[a, b].

13. Let H be a Hilbert space and let $<e_i>$ be an orthonormal set in H. Then the following conditions are all equivalent to each other :

(i) $<e_i>$ is complete

(ii) $x \perp < e_i > \Rightarrow x = 0$

(iii) If x is any arbitrary vector in H, then $x = \sum (x, e_i) e_i$.

(iv) If x is any arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

Or

Let H be a Hilbert space and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

13. Let H be a Hilbert space and let $<e_i>$ be an orthonormal set in H. Then the following conditions are all equivalent to each other :

(i) $<e_i>$ is complete

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Or

Let H be a Hilbert space and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.