M.Sc. MATHEMATICS

MAL - 522

MEASURE AND INTEGTRATION THEORY

DIRECTORATE OF DISTANCE EDUCATION
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CONTENTS

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Lesson No. 1 **Written by Dr. Vizender Singh**

Lesson: Measurable Function and Properties

Structure:

1.1. Introduction

With the class of measurable sets in mind, we introduce a rich class of functions; namely, the class of measurable functions which includes the class of continuous functions as a proper subclass. The class of measurable functions plays a role of central importance in Lebesgue theory of integration. It will assume a place comparable to that of the class of functions which are bounded and continuous almost everywhere in the Riemann theory of integration and of functions of bounded variation in the instance of Stieltjes integrals. Roughly speaking, a function is integrable if its behaviour is not too irregular, and if the values it takes are not too large too often. The second requirement is equivalent to the existence of the equality of the upper and lower integrals. We now introduce the notion of measurability which gives precisely the conditions required for the integrability, given that the function is not too large. In many cases, it is easier to examine the measurability of a function than to investigate its upper and lower integrals directly.

1.2. Measurable Function

1.2.1 Definition. Let E be a measurable set and f a function defined on E. Then f is said to be measurable (Lebesgue function) if for any realm any one of the following four conditions is satisfied.

(a) $\{x|\, f(x) > \alpha\}$ is measurable

(b) $\{x|\, f(x) \geq \alpha\}$ is measurable

(c) $\{x|\, f(x) < \alpha\}$ is measurable

(d) $\{x|\, f(x) \leq \alpha\}$ is measurable.

We show first that these four conditions are equivalent. First of all we show that (a) and (b) are equivalent.

Since

$$\{x|\, f(x) > \alpha\} = \{x|\, f(x) \leq \alpha\}^c$$

and also we know that complement of a measurable set is measurable, therefore (a) $\Rightarrow$ (d) and conversely.

SimiIarIy since (b) and (c) are complement of each other, (c) is measurable if (b) is measurable and conversely.

Therefore, it is sufficient to prove that (a) $\Rightarrow$ (b) and conversely. Firstly we show that (b) $\Rightarrow$ (a) . The set $\{x|\, f(x) \geq \alpha\}$ is given to be measurable.

Now

$$\{x|\, f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x|\, f(x) \geq \alpha + \frac{1}{n}\right\}$$

But by (b), $\{x|\, f(x) \geq \alpha + 1\}$ is measurable. Also we know that countable union of measurable sets is measurable.

Hence $\{x|\, f(x) > \infty\}$ is measurable which implies that (b) $\Rightarrow$ (a) . Conversely, let (a) holds. We have

$$\{x \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > \alpha - \frac{1}{n}\}$$

The set $\{x | f(x) > \alpha - 1\}$ is measurable by (a). Moreover, intersection of measurable sets is also measurable. Hence $\{x | f(x) \geq \alpha\}$ is also measurable. Thus (a) ⇒ (b) . Hence the four conditions are equivalent.

Lemma.1.2.2. If m is an extended real number then these four conditions imply that $\{x | f(x) = \alpha\}$ is also measurable.

Proof. Let m be a real number, then

$$\{x | f(x) = \alpha\} = \{x | f(x) \geq \alpha\} \cap \{x | f(x) \leq \alpha\}.$$

Since $\{x | f(x) \geq \alpha\}$ and $\{x | f(x) \leq \alpha\}$ are measurable by conditions (b) and (d), the set $\{x | f(x) = \alpha\}$ is measurable being the intersection of measurable sets. Suppose $\alpha = +\infty$. Then

$$\{x | f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x | f(x) \geq n\}$$

which is measurable by the condition (b) and the fact that intersection of measurable sets is measurable.

Similarity when $\alpha = -\infty$, then

$$\{x | f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x | f(x) \leq -n\}$$

which is again measurable by condition (d). Hence the result follows.

Second definition of Measurable functions

We see that

$$\{x|\ f(x) > \alpha\}$$

is inverse image of $(\alpha, \infty]$. Similarly the sets $\{x|\ f(x) > \alpha\}$, $\{x|\ f(x) > \alpha\}$, $\{x|\ f(x) > \alpha\}$ are inverse images of $[\alpha, \infty]$, $[-\infty, \alpha)$ and $[-\infty, \alpha]$ respectively. Hence we can also define a measurable function as follows.

A function f defined on a measurable set E is said to be measurable if for any real α any one of the four conditions is satisfied :

(a) The inverse image $f^{-1}(\alpha, \infty]$ of the half- open interval $(\alpha, \infty]$ is measurable.
(b) For every real m, the inverse image $f^{-1}[\alpha, \infty]$ of the closed interval $[\alpha, \infty]$ is measurable.
(c) The inverse image $f^{-1}[-\infty, \alpha)$ of the half open interval $[-\infty, \alpha)$ is measurable.
(d) The inverse image $f^{-1}[-\infty, \alpha]$ of the closed interval $[-\infty, \alpha]$ is measurable.

Remark 1.2.3. It is immediate that a necessary and sufficient condition for measurability is that $\{x|\ a \leq f(x) \leq b\}$ should be measurable for all a, b [including the case $a = -\infty$, $b = +\infty$], for any set of this form can be written as the intersection of two sets

$$\{x \mid f(x) \geq a\} \cap \{x \mid f(x) \leq b\},$$

if f is measurable, each of these is measurable and so is $\{x \mid a \leq f(x) \leq b\}$. Conversely any set of the form occurring in the definition can easily be expressed in terms of the sets of the form

$$\{x \mid a \le f(x) \le b\}.$$

Remark 1.2.4. Since (α, ∞) is an open set, we may define a measurable function as "A function f defined on a measurable set E is said to be measurable if for every open set G in the real number system, $f^{-1}(\mathrm{G})$ is a measurable set.

Definition.1.2.5. Characteristic function of a set E is defined by

$$\chi_E(x) = \begin{cases} 1 & if \quad x \in E \\ 0 & if \quad x \notin E \end{cases}$$

This is also known as indicator function.

Example of a Measurable function

Example.1.2.6. A constant function with a measurable domain is measurable.

Solution. Let E be a measurable set and let : E → R* be a constant function definition by $f(x) = K$(constant) . Then for any real *m*, we have

$$\{x : f(x) > \alpha\} = \begin{cases} E & if \quad \alpha < k \\ \varphi & if \quad \alpha \ge k \end{cases}$$

Since both E and φ are measurable, it follows that the set $\{x : f(x) > \alpha\}$ and hence f is measurable.

Theorem. 1.2.7. For any real c and two measurable real- valued functions f, g the four functions *cf*, $f + g$, *fg* are measurable.

Proof. We consider the function cf. In case $c = 0$, cf is the constant function 0 and hence is measurable since every constant function is continuous and so measurable. In case $c > 0$ we have

$$\{x|\ cf(x) > \alpha\} = \{x|f(x) > \frac{\alpha}{c}\} = f^{-1}(\frac{\alpha}{c}, \infty],$$

and so measurable.

In case c < 0, we have

$$\{x|\ cf(x) < \frac{r}{c} > r\} = \{x|\ f(x) < \frac{r}{c}\}$$

and so measurable.

Now if f and g are two measurable real valued functions defined on the same domain, we shall show that $f + g$ is measurable. To show iit, it is sufficient to show that the set $\{x|\ f(x) + g(x) > \alpha\}$ is measurable. if $f(x) + g(x) > \alpha$, then $f(x) < \alpha$ - *g(x)* and by the Cor. of the axiom of Archimedes there is a rational number r such that

$$\alpha - g(x) < r < f(x)$$

Since the functions f and g are measurable, the sets

$$\{x|\ f(x) > r\} \text{ and } \{x|\ g(x) > \alpha - r\}$$

are measurable. Therefore, there intersection

$$S_r = \{x|\ f(x) > r\} \cap \{x|\ g(x) > \alpha - r\}$$

is a1so measurab1e.

It can be shown that

$$\{x|\ f(x) + g(x) > \alpha\} = U\{\ S_r|\ \ r \text{ is a rational}\}$$

Since the set of rational is countable and countable union of measurable sets is measurable, the set $U\{S_r|\ r$ is a rational$\}$ and hence $\{x|\ f(x) + g(x) > \alpha\}$ is measurable which proves that $f(x) + g(x)$ is measurable. From this part it follows that f - $g = f + (-g)$ is also measurable, since when g is measurable $(-g)$ is also measurable.

Next we consider fg.

The measurability of $f\,g$ follows from the identity

$$f\,g = \frac{1}{2}[(f+g)^2 - f^2 - g^2],$$

if we prove that f^2 is measurable when f is measurable. For this it is sufficient to prove that $\{x \in E|\ f^2(x) > \alpha\}$, m is a real number, is measurable. Let α be a negative real number. Then it is clear that the set $\{x|\ f^2(x) > \alpha\} = E(\text{domain of the measurable function } f)$. But E is measurable by the definition of f . Hence $\{x|\ f^2(x) > \alpha\}$ is measurable when $m < 0$.

Now let $\alpha \geq 0$, then

$$\{x|\ f^2(x) > \alpha\} = \left\{x\middle|\ f(x) > \sqrt{\alpha}\right\} \cup \{x|\ f(x) < -\sqrt{\alpha}\}$$

Since f is measurable, it follows from this equality that

$$\{x|\ f^2(x) > \alpha\}$$

is measurable for $\alpha \geq 0$.

Hence f^2 is also measurable when f is measurable. Therefore, the theorem follows from the above identity, since measurability of f and g imply the measurability of $+g$. From this we may also conclude that $f/g(g \neq 0)$ is also measurable.

Theorem 1.2.8. If f is measurable, then $|f|$ is also measurable.

Proof. It suffices to prove the measurability of the set

$$\{x \mid \; |f(x)| > \alpha\}$$, where α is any real number.

If $\alpha < 0$, then

$$\{x| \; |f(x)| > \alpha\} = E \text{ (domain of } f\text{)}$$

But E is assumed to be measurable. Hence $\{x| \; |f(x)| > \alpha\}$ is measurable for $\alpha < 0$.

If $\alpha \geq 0$ then

$$\{x|\, |f(x)| > \alpha\} = \{x|\, f(x) > \alpha\} \cup \{x|\, f(x) < -\alpha\}$$

The right hand side of the equality is measurable since f is measurable. Hence $\{x|\, |f(x)| > \alpha\}$ is also measurable. Hence the theorem is proved.

Theorem 1.2.9. Let $\{f_n\}_{n=1}^{m}$ be a sequence of measurable functions. Then

$$sup\{f_1, f_2, \dots, f_n\}, \; inf\{f_1, f_2, \dots, f_n\},$$

$sup\ f_n$, f_n , $lim \quad f_n$ and $lim \quad f_n$ are measurable.

Proof. Define a function

$$M(x) = \; sup\,\{f_1, f_2, \dots, f_n\}$$

We shall show that $\{x|\, M(x) > \alpha\}$ is measurable.

In fact

$$\{x|\, M(x) > \alpha\} = \bigcup_{i=1}^{n} \{\, x|\, f_i(x) > \alpha\}$$

Since each f_i is measurable, each of the set $\{x|\, f_i(x) > \alpha\}$ is measurable and therefore their union is also measurable. Hence $\{x|\, M(x) > \alpha\}$ and so $M(x)$ is measurable.

Similarly we define the function

$$m(x) = \ inf\ \{f_1, f_2, \dots, f_n\}$$

Since $m(x) < \alpha$ iff $f_i(x) < m$ for some i we have

$$\{x|\ m(x) < \alpha\} = \bigcup_{i=1}^{n} \{\ x|\ f_i(x) < \alpha\}$$

and since $\{x|\ f_i(x) < \alpha\}$ is measurable on account of the measurability of f_i, it follows that $\{x|\ m(x) < \alpha\}$ and so $m(x)$ is measurable.

Define a function

$$M'(x) = \ sup f_n(x) = \ sup(f_1, f_2, \dots, f_n)$$

We shall show that the set

$$\{x|\ M'(x) > \alpha\}$$

is measurable for any real m.

Now

$$\{x|\ M'(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{\ x|\ f_n(x) > \alpha\}$$

is measurable, since each f_n is measurable.

Similarly if we define

$$m'(x) = inf\ f_n(x),$$

then

$$\{x|\ m'(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{\ x|\ \ f_n(x) < \alpha\}$$

and therefore measurability of f_n implies that of $m'(x)$. Now since

$$\overline{lim} \quad f_n = lim \quad sup \ f_n = inf\{ \ sup \ f_n \ \}$$

and $$\underline{lim} \quad f_n = \ sup \ \{ \ inf \ f_n\},$$

the upper and lower limits are measurable.

Finally if the sequence is convergent, its limit is the common value of $lim \quad f_n$ and $lim \quad f_n$ and hence is measurable.

Definition.1.2.10. Let f and g be measurable functions. Then we define

$$f^+ = \ Max \ (f, 0)$$

$$\mathrm{f}^- = \ \mathrm{Max} \ (-\mathrm{f}, 0)$$

$$f \vee g = \frac{f+g+|f-g|}{2} \ i.e. \ \ Max \ (f, g)$$

and

$$f \wedge g = \frac{f+g-|f-g|}{2} \ i.e. \ \ min \ (f, g)$$

Theorem 1.2.11. Let f be a measurable function. Then $f^+$and f^- are both measurable.

Proof. Let us suppose that > 0 . Then we have

$$f^+ = \mathrm{f} \ \text{ and } \ f^- \ = 0 \qquad \text{(i)}$$

So in this case we have

$$f = f^+ - f^-$$

Now let us take f to be negative. Then

$$f^{+} = Max\,(-f, 0) = 0$$

$$f^{-} = Max\,(f, 0) = 0$$

(ii)

Therefore on subtraction

$$f = f^{+} - f^{-}$$

In case $= 0$, then

$$f^{+} = 0, f^{-} = 0$$

(iii)

Thus for all f we have

$$f = f^{+} - f^{-}$$

(iv)

Also adding the components of (i) we have

$$f = |f| = f^{+} + f^{-}$$

(v)

Since f is positive.

And from (ii) when f is negative we have

$$f^{+} + f^{-} = 0 - f = -f = |f|$$

(vi)

In case f is zero, then

$$f^{+} + f^{-} = 0 - 0 = -0 = |f| \qquad \text{(vii)}$$

That is for all f, we have

$$|f| = f^{+} + f^{-} \qquad \text{(viii)}$$

Adding (iv) and (viii) we have

$$f + |f| = 2f^{+}$$

$$\Rightarrow f^{+} = \frac{1}{2}(f + |f|) \qquad \text{(ix)}$$

Similarly on subtracting we obtain

$$f^{-} = \frac{1}{2}(|f| - f) \qquad \text{(x)}$$

Since measurability of f implies the measurability of $|f|$ it is obvious from (ix) and (x) that f^{+} and f^{-} are measurable.

Theorem 1.2.12. If f and g are two measurable functions, then $f \vee g$ and $f \wedge g$ are measurable.

Proof. We know that

$$f \vee g = \frac{f + g + |f - g|}{2}$$

$$f \wedge g = \frac{f + g - |f - g|}{2}$$

Now measurability of $f \Rightarrow$ measurability of $|f|$. Also if f and g are measurable, then $f + g$, f - g are measurable. Hence $f \vee g$ and $f \wedge g$ are measurable.

We now introduce the terminology "almost everywhere" which will be frequently used in the sequel.

Definition 1.2.13. A statement is said to hold almost everywhere in E if and only if it holds everywhere in E except possibly at a subset D of measure zero.

Examples (a) Two functions f and g defined on E are said to be equal almost everywhere in E iff $f(x) = g(x)$ everywhere except a subset D of E of measure zero.

(b) A function defined on E is said to be continuous almost everywhere in E if and only if there exists a subset D of E of measure zero such that f is continuous at every point of $E - D$.

Theorem 1.2.14. (a) If f is a measurable function on the set E and $E_1 \subset E$ is measured set, then f is a measurable function on E_1.

(b) If f is a measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets, then f is measurable.

Proof. (a) For any real m, we have $\{x \in E_1, f(x) > \alpha\}$ right- hand side is measurable. $\{x \in E;\ f(x) > \alpha\} \cap E_1$. The result follows as the set on the right-hand side is measurable.

(b) Write $E = \bigcup_{i=1}^{\infty} E_i$ Clearly, E, being the union of measurable set is measurable. The result now follows, since for each real m, we have $E = \{\, x \in E, f(x) > \alpha\} = \{\bigcup_{i=1}^{\infty} E_i \mid f(x) > \alpha\}$.

Corollary 1.2.15. Let $\{f_n\}$ be a sequence of measurable functions such that $lim_{n\to\infty} f_n = f$ almost everywhere. Then f is a measurable function.

Proof. We have already proved that if $\{f_n\}$ is a sequence of measurable functions then $lim_{n\to\infty} f_n$ is measurable. Also it is given that $lim_{n\to\infty} f_n = f$ a.e. Therefore using the above theorem it follows that f is measurable.

Theorem 1.2.16. Characteristic function χ_A is measurable if and only if A is measurable.

Proof. Let A be measurable. Then

$$\chi_A(x) = \begin{cases} 1 \; if & x \in A \\ 0 \; if & x \notin A \; i.e. x \in A^c \end{cases}$$

Hence it is clear from the definition that domain of χ_A is A ∪ A^c which is measurable due to the measurability of A. Therefore, if we prove that the set $\{x| \, \chi_A > \alpha\}$ is measurable for any real m, we are through.

Let α ≥ 0 . Then

$$\{x \mid \chi_A > \alpha\} = \{x \mid \chi_A = 1\}$$

$$= \text{A} \;\; \text{(by the definition of Ch. fn.)}$$

But A is given to be measurable. Hence for α ≥ 0. The set $\{x \mid \chi_A > \alpha\}$ is measurable.

Now let us take α < 0 . Then

$$\{x \mid \chi_A > \alpha\} = A \cup A^c$$

Hence $\{x \mid \chi_A > \alpha\}$ is measurable for α < 0 also, since A ∪ A^c has been proved to be measurable. Hence if A is measurable, then χ_A is also measurable.

Conversely, let us suppose that $\chi_A(x)$ is measurable. That is, the set $\{x \mid \chi_A > \alpha\}$ is measurable for any real m.

Let m ≥ 0 . Then

$$\{x \mid \chi_A > \alpha\} = \{x \mid \chi_A = 1\} = A$$

Therefore, measurability of $\{x \mid \chi_A > \alpha\}$ implies that of the set A for $\alpha \geq 0$.

Now consider < 0. Then

$$\{x \mid \chi_A > \alpha\} = A \cup A^c$$

Thus measurability of $\chi_A(x)$ implies measurability of the set A ∪ A^c which imply A is measurable.

Remark 1.2.27. With the help of above result, the existence of non- measurable function can be demonstrated. In fact, if A is non- measurable set then χ_A cannot be measurable.

Definition 1.2.18. A function φ, defined on a measurable set E, is called **simple** if there is a finite disjoint class $\{E_1, E_2, \ldots, E_n\}$ of measurable sets and a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of real numbers such that

$$f(x) = \begin{cases} \alpha_i \ \ if \ \ x \in E_i, i = 1,2, \ldots, n \\ 0 \ \ if \ \ x \notin E_1 \cup E_2 \cup \ldots \cup E_n \end{cases}$$

Thus, a function is simple if it is measurable and takes only a finite number of different values.

The simplest example of a simple function is the characteristic function χ_E of a measurable set E.

Definition 1.2.19. A function f is said to be a step function if

$$f(x) = C_i, \quad \xi_{i-1} < x < \xi_i$$

for some subdivision of $[a, b]$ and some constants C_i. Clearly, a step function is a simple function.

Theorem 1.2.20. Every simple function ϕ on E is a linear combination of characteristic functions of measurable subsets of E.

Proof. Let ϕ be a simple function and c_1, c_2,…,c_n denote the non- zero real numbers in its image (E) . For each $= 1,2, \ldots, n$, let

$$A_i = \{x \in E: \phi(x) = C_i\}$$

Then we have

$$\phi = \sum_{i=1}^{n} C_i \, \chi_{A_i}$$

On the other hand, if ϕ(E) contains no non- zero real number, then $\phi = 0$ and is the characteristic function χ_ϕ ofthe empty subset of E.

It follows from Theorem 10 that simple functions, being the sum of measurable functions, is measurable. Also, by the definition, if f and g are simple functions and c is a constant, then $f + c$, cf, $f + g$ and fg are simple.

Approximation Theorem

Theorem 1.2.21. For every non- negative measurable function f, there exists a non-negative non-decreasing sequence $\{f_n\}$ of simple functions such that

$$\lim_{n\to\infty} f_n\,(x) = f(x), x \in E$$

In the general case if we do not assume non-negativeness of , then we say. For every measurable function f, there exists a sequence $\{f_n\}$, $n \in N$ of simple function which converges (pointwise) to f .

i.e. “Every measurable function can be approximated by a sequence of simple functions.”

Proof. Let us assume that $f(x) \geq 0$ and $\in E$. Construct a sequence

$$f_n(x) = \begin{cases} \dfrac{i-1}{2^n} & \text{for } \dfrac{i-1}{2^n} \leq f(x) < \dfrac{i}{2^n}, i = 1,2,\ldots,n2^n \\ n & \text{for } f(x) \geq n \end{cases}$$

for every $n \in N$. If we take $n = 1$, then

$$f_1(x) = \begin{cases} \dfrac{i-1}{2} & for \ \dfrac{i-1}{2} \leq f(x) < \dfrac{i}{2}, i = 1,2, \\ 1 & for \ f(x) \geq 1 \end{cases}$$

That is,

$$f_1(x) = \begin{cases} 0 & for \ \ 0 \leq f(x) < \frac{1}{2} \\ \frac{1}{2} & for \ \ \frac{1}{2} \leq f(x) < 1 \\ 1 & for \ \ f(x) \geq 1 \end{cases}$$

Similarly by taking $n = 2$, we obtain

$$f_2(x) = \begin{cases} \frac{i-1}{4} & for \ \ \frac{i-1}{4} \leq f(x) < \frac{i}{4}, i = 1,2, \dots ,8 \\ 2 & for \ \ f(x) \geq 2 \end{cases}$$

That is,

$$f_2(x) = \begin{cases} 0 & for \ \ 0 \leq f(x) < \frac{1}{4} \\ \frac{1}{4} & for \ \ \frac{1}{4} \leq f(x) < \frac{1}{2} \\ \frac{7}{4} & for \ \ \frac{7}{4} \leq f(x) < 2 \\ 2 & for \ \ f(x) \geq 2 \end{cases}$$

Similarly we can write $f_3(x)$ and so on. Clearly all f_n are positive whenever f is positive and also it is clear that $f_n \leq f_{n+1}$. Moreover f_n takes only a finite number of values. Therefore $\{f_n\}$ is a sequence of non- negative, non-decreasing functions which assume only a finite number of values.

Let us denote

$$E_{ni} = f^{-1}\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right] = \{x \in E| \ \ \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\}$$

and

$$E_n = f^{-1}[n, \infty] = \{x \in E | f(x) \geq n\}$$

Both of them are measurable. Let

$$f_n = \sum_{i=l}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n_i}} + n\chi_{E_n}$$

for every $n \in N$.

Now $\sum_{i=l}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n_i}}$ is measurable, since E_{n_i} has been shown to be measurable and characteristic function of a measurable set is measurable. Similarly χ_{E_n} is also measurable since E_n is measurable. Hence each f_n is measurable. Now we prove the convergence of this sequence.

Let $f(x) < \infty$. That is f is bounded. Then for some n we have

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$$

$$\Rightarrow \frac{i-1}{2^n} - \frac{i-1}{2^n} \leq f(x) - \frac{i-1}{2^n} < \frac{i}{2^n} - \frac{i-1}{2^n},$$

$$\Rightarrow 0 \leq f(x) - \frac{i-1}{2^n} < \frac{l}{2^n}$$

$$\Rightarrow 0 \leq f(x) - f_n(x) < \frac{l}{2^n} \qquad \text{(by the def of } \mathrm{f_n}(\mathrm{x}))$$

$$\Rightarrow f(x) - f_n(x) < \in$$

$$\Rightarrow lim_{n\to\infty} f_n\,(x) = f(x)$$

and this convergence is uniform. Let us suppose now that f is not bounded. Then since

$$f_n(x) = n \text{ for } f(x) \geq n$$

$$\mathrm{lim}_{\mathrm{n}\to\infty}\, \mathrm{f_n}\,(\mathrm{x}) = \infty = \mathrm{f(x)}$$

When **we do not assume non-negativenss of the function** then since we know that f^+ and f^- are both non-negative, we have by what we have proved above

$$f^{+} = lim_{n\to\infty}\, \phi_n'(x) \qquad \text{(i)}$$

$$f^{-} = lim_{n\to\infty}\, \phi_n''(x) \qquad \text{(ii)}$$

where $\phi_n'(x)$ and $\phi_n''(x)$ are simple functions. Also we have proved already that

$$f = f^{+} - f^{-}$$

Now from (i) and (ii) we have

$$f^{+} - f^{-} = \lim_{n\to\infty} \phi_n'(x) - \lim_{n\to\infty} \phi_n''(x)$$

$$= lim_{n\to\infty}\big(\phi_n'(x) - \phi_n''(x)\big)$$

$$= lim_{n\to\infty}\, \phi_n(x)$$

(since the difference of two simple functions is again a simple function). Hence the theorem.

Littlewood's three principles of measurability

The following three principles concerning measure are due to Littlewood.

First Principle: Every measurable set is a finite union of intervals.

Second Principle: Every measurable function is almost a continuous function.

Third Principle. If $\{f_n\}$ is a sequence of measurable function defined on a set E of finite measure and if $f_n(x) \to f(x)$ on E, then $f_n(x)$ converges almost uniformly on E.

First of all we consider third principle. We shall prove Egoroff‟s theorem which is a slight modification of third principle of Littlewood's.

Egoroff's Theorem

Theorem 1.2.22. Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure such that $f_n(x) \to f(x)$ almost everywhere. Then to each $\in > 0$ there corresponds a measurable subset E_0 of E such thatm $E_0^c < \in$ and $f_n(x)$ converges to $f(x)$ uniformly on E_0.

Proof. Since $f_n(x) \to f(x)$ almost everywhere and $\{f_n\}$ is a sequence of measurable functions, therefore $f(x)$ is also a measurable function. Let

$$H = \{x \mid lim_{n\to\infty} f_n(x) = f(x)\}$$

Clearly measure of E- H is zero.

For each pair (k, n) of positive integers, let us define the set

$$E_{kn} = \bigcap_{m=n}^{\infty} \{x| \ |f_m(x) - f(x)| < \frac{1}{k}\}$$

(Since each $f_m - f$ is a measure function, the sets E_{kn} are measurable).

Then for each k, if we put

$$E' = \bigcup_{n=1}^{\infty} E_{kn}$$

Then it is clear that

$$E' = \bigcup_{n=1}^{\infty} E_{kn} \supset H$$

In fact, if $x \in H$ then $x \in E' \Rightarrow H \subset E'$.

We have also

$$E_{k(n+1)} = \bigcap_{m=n+1}^{\infty} \{x| \, |f_m(x) - f(x)| < \frac{1}{k}\}$$

Clearly

$$E_{kn} = E_{k(n+1)} \cap \{x| \, |f_n(x) - f(x)| < \frac{1}{k}\}$$

Hence $E_{k(n+1)}$ cannot be a proper subset of E_{kn}. That is,

$$E_{kn} \subset E_{k(n+1)}$$

Thus for each k the sequence $[E_{kn}]$ is an expanding sequence of measurable sets. Therefore

$$\lim_{n\to\infty} m\,(E_{kn}) = m\left(\bigcup_{n=1}^{\infty} E_{kn}\right)$$

$$\geq \mathrm{m(H)} = \mathrm{m(E)}$$

Whence

$$lim_{n\to\infty}\, m\,(E_{kn}^{c}) = 0. \qquad \text{(i)}$$

Thus, given $\in > 0$, we have that for each k there is a positive integer n_k such that

$$|mE_{kn}^{c} - 0| < \frac{\in}{2^k}, n \geq n_k$$

$$i.e.\ |mE_{kn}^{c}| < \tfrac{\in}{2^k}, \qquad n \geq n_k \qquad \text{(ii)}$$

Let

$$E_0 = \bigcap_{k=1}^{\infty} E_k\, n_k,$$

then E_0 is measurable and

$$mE_0^{c} = m\left(\bigcap_{k=1}^{\infty} E_{kn_k}\right)^{c}$$

$$= m\left(\cup_{k=1}^{\infty} E_{kn_k}^{c}\right)$$

$$\leq \sum_{k=1}^{\infty} m\, E_{kn_k}^{c}$$

$$= \textstyle\sum_{k=1}^{\infty} \frac{\in}{2^k} \qquad \text{(using (ii))}$$

$$= \in \sum_{k=1}^{\infty} \frac{1}{2^k} = \in.$$

It follows from the definition of E_{kn} that for all $m \geq n_k$,

$$|f_m(x) - f(x)| < \frac{1}{k} \qquad \text{(iii)}$$

for every $x \in E_{kn_k}$. Since $E_0 \subset E_{kn_k}$ for every k, the condition $m \geq n_k$ yields (iii) for every $x \in E_0$. Hence $f_n(x) \to f(x)$ uniformly on E_0. This completes the proof of the theorem.

Now we pass to the **second principle of Littlewood.** This is nothing but approximation of measurable functions by continuous functions. In this connection we shall prove the following theorem known as Lusin Theorem after the name of a Russian Mathematician Lusin, N.N.

Lusin's Theorem

Theorem 1.2.23.. Let f be a measurable function defined on $[a, b]$. Then to each $\in > 0$, there corresponds a measurable subset E_0 of $[a, b]$ such that $mE_0^c < \in$ and f is continuous on E_0.

Proof. Let f be a measurable function defined on $[a, b]$. We know that every measurable function is the limit of a sequence $\{\phi_n(x)\}$ of simple functions whose points of discontinuity form a set of measure zero. Thus we have

$$\lim_{n\to\infty} \phi_n(x) = f, \qquad x \in [a, b]$$

By Egoroff's theorem, to each $\in > 0$ there exists a subset E_0 of $[a, b]$ such that $mE_0^c < \in$ and $\phi_n(x)$ converges to $f(x)$ uniformly on E_0. But we know that if $\{\phi_n(x)\}$ is a sequence of continuous function converging uniformly to a function (x) , then $f(x)$ is continuous. Therefore $f(x)$ is continuous on E_0. This completes the proof of the theorem.

Theorem 1.2.24. Let f be a measurable function defined on $[a, b]$ and assume that f takes values $\pm\infty$ on a set of measure zero. Then given $\in > 0$ we can find a continuous function g and a step function h such that

$$|f - g| < \in, \qquad (f - h) < \in,$$

except on a set of measure less than $\in$.

Proof. Let H be a subset of $[\mathrm{a}, \mathrm{b}]$ where $f(x)$ is not $\pm\infty$. Then by the hypothesis of the theorem $= m([a, b])$. We know that every measurable function can be expressed as a almost everywhere limit of a sequence of step functions which are continuous on a set of measure zero. That is, we can find a sequence of step functions such that

$$lim_{n\to\infty} \phi_n(x) = f(x) \text{ a.e. on H.}$$

Let $F \subset H$ such that $\phi_n(x) \to f(x)$ and is continuous everywhere on F.

By Egoroff's theorem for a given $\in > 0$ we can find a subset $F' \subset H$ such that $\phi_n(x) \to f(x)$ uniformly on F' and

$$M(F - F') < \in$$

But we know that if $\{f_n\}$ is a sequence of continuous function converging uniformly to a function (x) , then $f(x)$ is continuous. Therefore $f(x)$ is continuous on F'.

Define a continuous function $g(x)$ on $[\mathrm{a}, \mathrm{b}]$ such that

$$g(x) = \begin{cases} 0 & if \quad x \notin F' \\ f(x) & if \quad x \in F' \end{cases}$$

Therefore on F' we have

$$|f - g| < \in$$

We have already shown that

$$m([a, b] - F') < \in.$$

Also we have shown that $\phi_n(x) \to f(x)$ where $\phi_n(x)$ is a sequence of step function, so $f(x)$ is also a step function. Hence the theorem.

In order to prove the **first principle of Littlewood** we prove two theorems on approximations of measurable sets.

Theorem 1.2.25. A set E in R is measurable if and only if to each $\in > 0$, there corresponds a pair of sets F, G such that $F \subset E \subset G$, F is closed, G is open and $m(G- F) <\in$.

Proof. Sufficiency: Taking $\in= 1/n$, let the corresponding pair of sets be F_n, G_n with

$$m(G_n - F_n) < 1/n$$

Let

$$X = \bigcup_n F_n\,,\;\; Y = \bigcap_n G_n$$

It follows that $Y - X \subset G_n - F_n$ and

$$m(Y- X) \leq m(G_n - F_n) < 1/n$$

So that

$$m(Y - X) = 0\ .$$

Since

$$E - X \subset Y - X,$$

so

$$m(E - X) = 0.$$

Therefore, $E—X$ is measurable.

But $= (E - X) \cup X$. Therefore E is measurable, since X is measurable and $E-\ X$ is measurable.

Necessity: We now assume that E is measurable. We first prove this part under the assumption that E is bounded. Since E is measurable and bounded, we can choose an open set $G \supset E$ such that

$$m(G) < m(E) + \frac{\in}{2} \qquad \text{(i)}$$

Choose a compact (closed and bounded) set $\supset E$, and then choose an open set V such that $S - E \subset V$ and

$$m(V) < m\,(S\text{-}\ E) + \frac{\epsilon}{2}$$

(ii)

Let $F = S - V$. Then F is closed (since S- $V = S \cap V^c$ which is closed being the intersection of closed sets) and $\subset E$. We have

$$m(F) = m(S) - m(S \cap V)$$

$$\geq m(S) - m(V)$$

$$> m(S) - m(S - E) - \frac{\epsilon}{2}$$

(Using (ii))

$$= m(E) - \frac{\epsilon}{2}$$

(iii)

Then

$$m(G\text{-}\ F) = m(G) - m(F)$$

$$= m(G) - m(E) + m(E) - m(F)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$ (using (i) and (iii))

This finishes the proof for the case in which E is bounded.

Now, let E be the measurable but unbounded. Let

$$S_n = \{x \mid |x| \leq n\}, \qquad n \in Z$$

$$E_1 = E \cap S_1$$

$$E_n = E \cap (S_n - S_{n-1}), n \geq 2\ .$$

Then

$$E = \bigcup_n E_n,$$

where each E_n is bounded and measurable.

Using what has already been established, let F_n, G_n be a pair of sets such that $F_n \subset E_n \subset G_n$, F_n is closed, G_n is open, and $m(G_n - F_n) < \frac{E}{2^n}$.

Let $= \bigcup_n F_n$, $G = \bigcup_n \in_n$. Then $G - F \subset \bigcup_n (G_n - F_n)$ and so

$$m(G - F) \leq m\{\bigcup_n (G_n - F_n)\}$$

$$\leq \sum m\,(G_n - F_n)$$

$$= \sum_n \frac{\in}{2^n}$$

$$= \in \sum_n \frac{l}{2^n}$$

$$= \in.$$

We see that G is open and that $F \subset E \subset G$, so all that remains to prove is that F is closed. Suppose $\{x_i\}$ is a convergent sequence (say $x_i \to x$) with $x_i \in F$ for each i. Then $\{x_i\}$ is bounded and so is contained in S_N for certain . Now

$$F_n \subset S_n - S_N \text{ if } n > N.$$

Therefore,

$$x_i \in \bigcup_{n=1}^{N} F_n \text{ for each } i.$$

But then the limit x is in $\bigcup_{n=1}^{N} F_n$, for this last set is closed. Therefore F is closed. This finishes the proof.

Definition 1.2.26. If A and B are two sets, then

$$A \,\Delta\, B = (A - B) \cup (B - A)\ .$$

Theorem 1.2.27. If E is a measurable set of finite measure in R and if $\in > 0$, there is a set G of the form $G = \cup_{n=1}^{N} I_n$ where $I_1, I_2, ..., I_N$ are open intervals, such that $m(E\Delta G) < \in$.

Proof. Let us assume at first that E is bounded. Let X be an open interval such that $E \subset X$. There exist Lebesgue covering $\{I_n\}$ and $\{J_n\}$ of E and X E respectively such that

$$\sum_n |\, I_n| < m(\in) + \frac{\in}{3},$$

$$\sum |J_n| < m(X - E) + \frac{\in}{3},$$

and such that each I_n and J_n is contained in X. Choose N so that

$$\sum_{n>N} |\, I_n| < \frac{\in}{3}$$

and define sets G, H, K as follows

$$G = \cup_{n=1}^{N} I_n\ ,\ H \cup_{n>N} I_n\ ,\ K = G \cap\ \ \cup_n J_n$$

Observe that $E - G \subset H$ and $G - E \subset K$ so that $E\Delta G \subset H \cup K$ and therefore

$$m(E\Delta G) \leq m(H \cup K) \leq m(H) + m(K)$$

We know that

$$m(H) \leq \sum_{n>N} m\,(I_n)$$

$$= \textstyle\sum_{n>N} |\, I_n| < \frac{\in}{3} \qquad \text{(by our choice)}$$

Hence it suffices to prove that $m(K) < \frac{2\in}{3}$.

Since

$$K = G \cap \bigcup_n J_n$$

$$= \cup\, G \bigcap J_n$$

therefore $m(K) = \sum_n m\,(G \cap J_n)$. So we seek an estimate of $\sum_n m\,(G \cap J_n)$. Now we can see that

$$\mathrm{X} = [\cup_\mathrm{n}\ \mathrm{I_n}] \cup [\cup_\mathrm{n}\,(\mathrm{J_n} - \mathrm{G})],$$

whence

$$m(X) = m\left[\bigcup_n I_n\right] + m\left[\bigcup_n (J_n - G)\right]$$

$$\leq \sum_n |\,I_n| + \sum_n m\,(J_n - G)$$

We also have

$$\sum_n |\,I_n| + \sum_n |J_n| < m(E) + m(X - E) + \frac{2 \in}{3}$$

$$= m(X) + \frac{2 \in}{3},$$

whence

$$\sum_n |\,I_n| + \sum_n |J_n| < \sum_n |\,I_n| + \sum_n m\,(J_n - G) + \frac{2 \in}{3}$$

and therefore, since $J_n = (J_n - G) \cup (J_n \cap G)$,

$$m(K) \leq \sum_n m\,(G \cap J_n) = \sum_n m\,(J_n) - \sum_n m\,(J_n - G)$$

$$< \frac{2 \in}{3}$$

Hence when E is bounded

$$m(E\Delta G) < \frac{\in}{3} + \frac{2\in}{3} = \in$$

For the general case, let

$$S_n = \{x \mid |x| \le n\},$$

$$T_1 = S_1$$

$$T_1 = S_n - S_{n-1}, n \ge 2$$

Let $E_n = E \cap S_n$. Then

$$E = \bigcup_{i=1}^{\infty} (E \cap T_i)$$

$$E - E_n = \bigcup_{i=n+1}^{\infty} (E \cap T_i)$$

Because $m(E) < +\infty$, we have

$$m\,(E - E_n) = \sum_{i=n+1}^{\infty} m\,(E \cap T_i) \to 0 \text{ as } n \to \infty.$$

But $E\Delta E_n = E - E_n$(since $E\Delta E_n = (E - E_n) \cup (E_n - E)$ and $E_n - E$ is empty} and so $m(E\Delta E_n) \to 0$. Using what has already been proved we can find a sequence G_n which is finite union of open intervals such that $m(E_n\Delta G_n) < 1/n$. Now the following inequality is true.

$$m(E\Delta G_n) \le m(E\Delta E_n) + m(E_n\Delta G_n)\ ,$$

Since $\Delta G_n = (E\Delta E_n) \cup (E_n\Delta G_n)$. We see therefore that $m(E\Delta G_n) \to 0$. If $\in > 0$, we shall have $m(E\Delta G_n) < \in$ for a suitable value of n, and then G_n will serve our purpose. This completes the proof of the theorem.

Theorem 1.2.28. Let E be a set with $m^*E < \infty$. Then E is measurable iff given $\in > 0$, there is a finite union B of open intervals such that

$$m^*(E\Delta B) < \in$$

Proof. Suppose E is measurable and let $\in > 0$ be given. The (as already shown} there exists an open set $O \supset E$ such that $m^*(E) < \frac{\in}{2}$. As m^*E is finite, so is m^*O. Since the open set O can be written as the union of countable {disjoint) open intervals $\{I_i\}$, there exists an $n \in N$ such that

$$\sum_{i=n+1}^{\infty} l\,(I_i) < \frac{\in}{2}$$

(In fact $m^*O = \sum_{i=n+1}^{\infty} l\,(I_i) < \infty \Rightarrow \sum_{i=n+1}^{\infty} l\,(I_i) < \frac{\in}{2}$ because $m * O < \alpha$)

Set $B = \bigcup_{i=1}^{\infty} I_i$. Then

$$E\Delta B = (E - B) \cup (B \backslash E) \subset (O \backslash B) \cup (O \backslash E)$$

Hence

$$m^*(E\Delta B) \leq m^*\left(\bigcup_{i=n}^{\infty} I_i\right) + m^*(O \backslash E) < \frac{\in}{2} + \frac{\in}{2} = \in.$$

Conversely, assume that for a given $\in > 0$, there exists a finite union $B = \bigcup_{i=n}^{\infty} I_i$ if open intervals with $m * (E\Delta B) < \in$. Then using "Let∈ be any set. The given $\in > 0$ there exists an open set $O \supset E$ such that $m^*O < m^*E + \in$ there is an open set $O \supset E$ such that

$$m^*O < m^*E + \in \qquad \text{(i)}$$

If we can show that $m^*(O - E)$ is arbitrary small, then the result will follow from "Let E be set. Then the following are equivalent

(i) E is measurable and

(ii) (ii) given $\in > 0$ there is an open set $O \supset E$ such that $m^*(O - E) < \in$ ".

Write

$$S = \bigcup_{i=1}^{n} (I_i \cap O)$$

Then $S \subset B$ and so

$$S\Delta E = (E \backslash S) \cup (S \backslash E) \subset (E - S) \cup (B - E).$$

However,

$$E \backslash S = (E \cap O^c) \cup (E \cap B^c) = E - B \qquad \text{(because } E \subset O\text{)}.$$

Therefore

$$S\Delta E \subset (E - B) \cup (B - E) = E\Delta B,$$

and as such $m^*(S\Delta E) <\in$. However,

$$E \subset S \cup (S\Delta E)$$

and so

$$m^*E < m^*S + m^*(S\Delta E)$$
$$< m^*S + \in \qquad \text{(ii)}$$

Also

$$O - E = (O - S) \cup (S\Delta E)$$

Therefore

$$m^*(O\backslash E) < m^*O - m^*S + \in$$

$$< m^*E + \in - m^*S + \in \qquad \text{(using(i))}$$

$$< m^*S + E + E - m^*S + E \qquad \text{(using(ii))}$$

$$< m^*S + E + E - m^*S + E$$

$$= 3 \in.$$

Hence E is measurable.

Convergence in Measure

Definition 1.2.29. A sequence $\langle f_n \rangle$ of measurable functions is said to converge to f in measure if, given $\in > 0$, there is an N such that for all $n \geq N$ we have

$$m\{x |\, f(x) - f_n(x)| \geq \in\} <\in.$$

F. Riesz Theorem

Theorem 1.2.30 "Let $\langle f_n \rangle$ be a sequence of measurable functions which converges in measure to f. Then there is a subsequence $\langle f_{nk} \rangle$ which converges to f **almost everywhere."**

Proof. Since $\langle f_n \rangle$ is a sequence of measurable functions which converges in measure to f, for any positive integer k there is an integer n_k such that for $n \geq n_k$ we have

$$m\left\{x \ \mid f_n(x) - f(x)| \geq \frac{1}{2^k}\right\} < \frac{1}{2^k}$$

Let

$$E_k = \left\{x \ \mid \ |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\right\}$$

Then if $x \notin \bigcup_{k=i}^{\infty} E_k,$

we have

$$|f_{n_k}(x) - f(x)| < \tfrac{1}{2^k} \ \ for \ \ k \geq i \text{ and } so \ f_{n_k}(x) \to f(x)$$

Hence $f_{n_k}(x) \to f(x)$ for any $x \notin A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} E_k$

But

$$mA \leq m\left[\bigcup_{k=i}^{\infty} E_k\right]$$

$$= \sum_{k=i}^{\infty} m\,E_k = \frac{1}{2^{k-1}}$$

Hence measure of A is zero.

Example 1.2.31. An example of a sequence $\langle f_n \rangle$ which converges to zero in measure on $[0,1]$ but such that $\langle f_n(x) \rangle$ does not converge for any x in $[0,1]$ can be constructed as follows :

Let $= k + 2^v$, $0 \leq k < 2^v$, and set $f_n(x) = 1 \ if \ x \in [k2^{-v}, (k+1)2^{-v}]$ and $f_n(x) = 0$ otherwise. Then

$$m\{\,x|\ |f_n(x)| > \in \} \leq 2/n$$

and so $f_n \to 0$ in measure, although for any $x \in [0,1]$, the sequence $\langle f_n(x) \rangle$ has the value for arbitrarily large values of n and so does not converge.

Definition 1.2.32. A sequence $\{f_n\}$ of a.e. finite valued measurable functions is said to be fundamental in measure, if for every $\in > 0$,

$$m\{x : |f_n(x) - f_m(x)| \geq \in\} \to 0 \text{ as } n \text{ and } m \to \infty.$$

Definition 1.2.33. A sequence $\{f_n\}$ of real valued functions is said to be fundamental a.e. if there exists a set E_0 of measure zero such that, if $x \notin E_0$and $\in > 0$, then an integer $n_0 = (x, \in)$ can be found with the property that

$$|f_n(x) - f_m(x)| < \in, \text{ whenever } n \geq n_0 \text{ and } \geq n_0 .$$

Definition 1.2.34. A sequence $\{f_n\}$ of a.e. finite valued measurable functions will be said to converge to the measurable function f almost uniformly if, for every $\in > 0$, there exists a measurable set F such that $m(F) < \in$ and such that the sequence $\{f_n\}$ converges to f uniformly on F^c.

In this Language, Egoroff"s Theorem asserts that on a set of finite measure convergence a.e. implies almost uniform convergence.

The following result goes in the converse direction.

Theorem 1.2.35. If $\{f_n\}$ is a sequence of measurable functions which converges to f almost uniformly, then $\{f_n\}$ converges to f a.e.

Proof. Let F_n beameasurable set such that $m(F_n) < 1/n$ and such that the sequence $\{f_n\}$ converges to f uniformly on F_n^c, $n = 1,2, ...$

If $$F = \cap_{n=1}^{\infty} F_n,$$

then

$$m(F) \leq \mu(F_n) < 1/n$$

so that $m(F) = 0$, and it is clear that, for $x \in F^c$, $\{f_n(x)\}$ converges to $f(x)$.

Theorem 1.2.36. Almost uniform convergence implies convergence in measure.

Proof. If $\{f_n\}$ converges to f almost uniformly, then for any two positive numbers $\in$ and δ there exists a measurable set F such that $m(F) < \delta$ and such that $|f_n(x) - f(x)| <\in$, whenever x belongs to F^c and n is sufficiently large.

Theorem 1.2.37. If $\{f_n\}$ converges in measure to f, then $\{f_n\}$ is fundamental in measure. If also $\{f_n\}$ converges in measure to g, then $f = g$ .a.e.

Proof. The first assertion of the theorem follows from the relation

$$\{x: |f_n(x) - f_m(x)| \geq \in\}$$
$$\subset \left\{x: |f_n(x) - f(x)| \geq \frac{\in}{2}\right\} \cup \left\{x: |f_m(x) - f(x)| \geq \frac{\in}{2}\right\}$$

To prove the second assertion, we have

$$\{x: |f(x) - g(x)| \geq \in\} \subset \left\{x: f_n(x) - f(x)| \geq \frac{\in}{2}\right\} \cup \left\{x: |f_n(x) - g(x)| \geq \frac{\in}{2}\right\}$$

Since by proper choice of n, the measure of both sets on the right can be made arbitrarily small, we have

$$m(\{x: |f(x) - g(x)| \geq \in\}) = 0$$

for every $\in > 0$ which implies that $f = g$ a.e.

Theorem 1.2.38. If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then some subsequence $\{f_{n_k}\}$ is almost uniformly fundamental.

Proof. For any positive integer k we may find an integer $\bar{n}(k)$ such that if $n \geq \bar{n}(k)$ and $\geq \bar{n}(k)$, then

$$m\left(\left\{x: |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}.$$

We write

$n_1 = \bar{n}(1),\ \ n_2 = (n_1 + 1) \cup \bar{n}(2),\ \ n_3 = (n_2 + 1) \cup \bar{n}(3)$ then $n_1 < n_2 < n_3 < \ldots.,$

So that the sequence $\{f_{n_k}\}$ is indeed on subsequence of $\{k_n\}$. If

$$E_k = \left\{x: |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right\}$$

And $k \leq i \leq j$, then, or every x which does not belong to $E_k \cup E_{k+1} \cup E_{k+2} \cup$

…., we have

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{m=i}^{\infty} |\, f_{n_m}(x) - f_{n_{m+1}}(x)| < \sum_{m=i}^{\infty} \frac{1}{2^m} = \frac{1}{2^{i-1}},$$

so that, in other words, the sequence $\{f_{n_i}\}$ is uniformly fundamental on $E\backslash(E_k \cup E_{k+1} \cup \ldots.)$.

Since

$$m(E_k \cup E_{k+1} \cup \ldots.) \leq \sum_{m=k}^{\infty} m\,(E_m) < \frac{1}{2^{k-1}}$$

This completes the proof of the theorem.

Theorem 1.2.39. If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that $\{f_n\}$ converges in measure to f.

Proof. By the above theorem we can find a subsequence $\{f_{n_k}\}$ which is almost uniformly fundamental and therefore fundamental a.e. We write $f(x) = lim_{k\to\infty} f_{n_k}(x)$ for every x for which the limit exists. We observe that, for every $\in > 0$,

$$\{x: |f_n(x) - f(x)| \geq \in] \subset \left\{x: |f_n(x) - f_{n_k}(x)| \geq \frac{\in}{2}\right\} \cup \left\{x: |f_{n_k}(x) - f(x)| \geq \frac{\in}{2}\right\}.$$

The measure of the first term on the right is by hypothesis arbitrarily small if n and n_k are sufficiently large, and the measure of the second term also approaches 0 (as $k \to \infty$), since almost uniform convergence implies convergence in measure. Hence the theorem follows.

Remark. Convergence in measure does not necessarily imply convergence pointwise at any point. Let

$$E_{r,k} = \left[\frac{r-1}{2^k}, \frac{r}{2^k}\right] \;(r = 1,2, \ldots, 2^k,\; k = 1,2, \ldots\},$$

and arrange these intervals as a single sequence of sets $\{F_n\}$ by taking first those for which $k = 1$, then those with $k = 2$, etc. If m denotes Lebesgue measure on [0,1], and $f_n(x)$ is the indicator function of $\mathrm{F_n}$, then for $0 < \in < 1$,

$$\{x: |f_n(x)| \geq \in\} = F_n$$

so that, for any $\in > 0$, $m\{x : |f_n(x)| \geq \in\} \leq m(F_n) \to 0$. This means that $f_n \to 0$ in measure in [0,1]. However, at no point x ∈ [0,1] does $f_n(x) \to 0$; in fact, since every x is in infinitely many of the sets $\mathrm{F_n}$ and infinitely many of the sets $(\Omega - F_n)$ we have

$$lim \quad inf\ f_n(x) = 0\ ,\ lim \quad sup\ f_n(x) = 1 \text{ for all } x \in [0,1].$$

1.3 Check Your Progress

Q.1. Let E be a set of rationals in [0,1]. Then the characteristic function $\chi_E(x)$ is measurable.

Q.2. A continuous function is measurable function.

Q.3. For any real c and two measurable real- valued functions , g, the function $f + c$ is measurable.

Q.4. Fill in the blanks.

Let f and g be any two functions which are --------------- in E. If f is measurable so is g.

Proof. Since f is measurable, for any real m the set $\{x|\, f(x) > \alpha\}$ is measurable. We shall show that the set $\{x|\ g(x) > \alpha\}$ is measurable. To do so we put

$$E_1 = \{x|\, f(x) > \alpha\}$$

and

$$E_2 = \{x|\ g(x) > \alpha\}$$

Consider the sets

$$E_1 - E_2 \text{ and } E_2 - E_1$$

Since $f = g$ almost everywhere, measures of these sets are zero. That is, both of these sets are measurable. Now

$$E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$$

= -----------------------------------

Since E_1, $E_2 - E_1$ and $(E_1 - E_2)^c$ are measurable therefore it follows that E_2 is measurable. Hence the theorem is proved.

Q.5. If a function f is continuous almost everywhere in E, then f is measurable.

Proof. Since f is continuous almost everywhere in E, there exists a subset D of E with $\mathrm{m}^*\mathrm{D} = 0$ such that f is continuous at every point of the set $\mathrm{C} = \mathrm{E} - \mathrm{D}$. To prove that f is measurable, let m denote any given real number. It suffices to prove that the inverse image

$B =$ -----------------------

of the interval (α, ∞) is measurable.

For this purpose, let x denote an arbitrary point in $B \cap C$. Then $f(x) > \alpha$ and f is continuous at x. Hence there exists an open interval U_x containing x such that $f(y) > \alpha$ hold for every point y of $E \cap U_x$. Let

$U =$ ------------------------

Since $x \in E \cap U_x \subset B$ holds for every $x \in B \cap C$, we have

$$B \cap C \subset E \cap \cup \subset B$$

This implies

$$B = (E \cap U) \cup (B \cap D)$$

As an open subset of R, U is measurable. Hence $E \cup U$ is measurable. On the other hand, since

$$m^*(B \cap D) \leq m^*D = 0$$

$B \cap D$ is also measurable. This implies that B is measurable. This completes the proof of the theorem.

1.4 <u>Summary</u>

This chapter presents the definition and the theorem related to measurable functions. Various properties of measurable functions are also reviewed. The principal results of these properties are summarized. The Lusin's theorem is also reviewed in the chapter. This theorem asserts that a measurable function is almost continuous. It has been used as a basis for the definition of measurable functions.

1.5 Keywords

Measurable Functions, Continuous Function, Sequence of Function and Convergence Behavior of Measurable Functions, Almost Everywhere, Characteristics Function, Simple Function, Step Function.

1.6 Self-Assessment Test

Q.1. Show that the function $f(x) = [x]$, where $[.]$ is the greatest integer function, is measurable?

Q.2. Show that the cardinality of the class of measurable function is 2^c.

Q.3. Let $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that

$$f(x) = \begin{cases} \frac{1}{x(x-1)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is measurable?

Q.4. Show that $f(x) = \begin{cases} 1 \; if \; x \in A \\ -1 \; if \; x \in A^c \end{cases}$

where A is a non-measureble subset of a measurable set E is not measurable while $f^2(x) \; \forall \; x \in E$ is measurable.

Q.5. Show that the function f defined on E = [0,1] is measurable where

$$f(x) = \begin{cases} 3 & if & x = 0 \\ \frac{1}{x} & if & 0 < x < 1 \\ 5 & if & x = 1 \end{cases}$$

1.7 Answers to check your progress

A.1 For the set of rationals in the given interval, we have

$$\chi_E(x) = \begin{cases} 1 & if \quad x \in E \\ 0 & if \quad x \notin E \end{cases}$$

It is sufficient to prove that the set

$$\{x \,|\chi_E(x) > \alpha\}$$

is measurable for any real α.

Let us suppose first that α ≥ 1. Then

$$\{x \mid \chi_E(x) > \alpha\} = \{x \mid \chi_E(x) > 1\}$$

Hence the set $\{x \mid \chi_E(x) > \alpha\}$ is empty in this very case. But outer measure of any empty set is zero. Hence for α ≥ 1, the set $\{x \mid \chi_E(x) > \alpha\}$ and so $\chi_E(x)$ is measurable.

Further let $0 \leq \mathrm{m} < 1$. Then

$$\{x \mid \chi_E(x) > \alpha\} = E$$

But E is countable and therefore measurable. Hence $\chi_E(x)$ is measurable.

Lastly, let $\alpha < 0$. Then

$$\{x \mid \chi_E(x) > \alpha\} = [0,1]$$

and therefore measurable. Hence the result.

A.2. If the function f is continuous, then $f^{-1}(\alpha, \mathrm{m})$ is also open. But every open set is measurable. Hence every continuous function is measurable. We may also argue as follows:

If f is continuous then

$$\{x \mid f(x) < \alpha, x \in (a,b)\}$$

is closed and hence

$$\{x \mid f(x) \geq m\} = \{x \mid f(x) < m\}^c$$

is open and so measurable.

All the ordinary functions of analysis may be obtained by limiting process from continuous function and so are measurable.

A.3. We are given that f is a measurable function and c is any real number. Then for any real number α

$$\{x \mid f(x) + c > \alpha\} = \{x \mid f(x) > \alpha - c\}$$

But $\{x \mid f(x) > \alpha - c\}$ is measurable by the condition (a) of the definition. Hence $\{x \mid f(x) + c > \alpha\}$ and so $f(x) + c$ is measurable.

A.4. (i) equal almost everywhere

(ii) $= [E_1 \cup (E_2 - E_1)] \cap (E_1 - E_2)^c$

A.5. (i) $B = f^{-1}(\alpha, \infty) = \{x \in E \mid f(x) > \alpha\}$

(ii)

$$U = \bigcup_{x \in B \cap C} U_x$$

1.8 References/ Suggested Readings

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MAL-522: M. Sc. Mathematics (Measure and Integration theory)

Lesson No. 2 **Written by Dr. Vizender Singh**

Lesson: Lebesgue Integral of a Function

Structure:

2.1 Introduction

The theory of Riemann integration though very useful and adequate for solving many mathematical problems, both pure and applied, is not free from defects. It does not meet the needs of a number of important branches of mathematics and physics of comparatively recent development. First of all, the Riemann integral of a function is defined on a closed interval and cannot be defined on an arbitrary set. Investigations in probability theory, partial differential equations, hydromechanics and quantum mechanics often pose problems which require integration over sets. Second and more important is the fact that the Riemann integrability depends upon the continuity of the function. Of course, there are functions which are discontinuous and yet Riemann-integrable, but these functions are continuous almost everywhere. Again, given a sequence of Riemann integrable functions converging to some function in a domain, the limit of the sequence of integrated functions may not be the Riemann integral of the limit function. ·In fact, the Riemann integral of the limit function may not even exist. This is a major drawback of the Riemann theory of integration, apart from the fact that even relatively simple functions are not integrable in the sense *ot* Riemann integration. H. Lebesgue in his classical work, introduced the concept of an integral, known after his name the Lebesgue integral, based on the measure theory that generalizes the

Riemann integral. It has the advantage that it takes care of both bounded and unbounded functions and simultaneously allows their domains to be more general sets and thereby enlarges the class of functions for which the Lebesgue integral is defined. Also, it gives more powerful and useful convergence theorems relating to the interchange of the limit and integral valid under Jess restrictive conditions required for the Riemann integral.

The shortcomings of the Riemann integral suggested the further investigations in the theory of integration. We give a resume of the Riemann Integral first.

Let f be a bounded real- valued function defined on the interval [a,b] and let

$$a = \xi_0 < \xi_1 < < \xi_n = b$$

be a partition of [a,b]. Then for each partition we define the sums

$$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})\, M_i$$

and

$$s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1})\, m_i,$$

where

$$M_i = \sup_{\xi_i < x < \xi_{i-1}} f(x) \;,\;\; m_i = \inf_{\xi_{i-1} < x < \xi_i} f(x)$$

We then define the upper Riemann integral of f by

$$R \int_a^b f(x)dx = \inf S$$

With the infimum taken over all possible subdivisions of [a,b].

Similarly, we define the lower integral

$$R \int_{\underline{a}}^{b} f\ (x)dx = \ sup\ s\ .$$

The upper integral is always at least as large as the lower integral, and if the two are equal we say that f is Riemann integrable and call this common value the Riemann integral of f. We shall denote it by

$$R \int_a^b f\ (x)dx$$

To distinguish it from the Lebesgue integral, which we shall consider later.

By a **step function** we mean a function ψ which has the form

$$\psi(x) = c_i,\ \xi_{i-1} < x < \xi_i$$

for some subdivision of $[a, b]$ and some set of constants c_i .

The integral of $\psi(x)$ is defined by

$$\int_a^b \psi(x)dx = \sum_{i=1}^n c_i\,(\xi_i - \xi_{i-1}).$$

With this in mind we see that

$$R\,\overline{\int_a^b} f(x)dx\ =\ \inf \int_a^b \psi(x)dx$$

for all step function $\psi(x) \geq f(x)$.

Similarly,

$$R\,\underline{\int_a^b} f(x)dx =\ \sup \int_a^b \phi(x)dx$$

for all step functions $\varphi(x) \leq f(x)$.

Example 2.2.1. If

$$f(x) = \begin{cases} 1 & if\ \ x\ is\ rational \\ 0 & if\ \ x\ is\ irrational' \end{cases}$$

Then

$$R\,\overline{\int_a^b} f(x)dx\ = b - a$$

And $$R\,\underline{\int_a^b} f\,(x)dx = 0.$$

Thus we see that $f(x)$ is not integrable in the Riemann sense.

The Lebesgue Integral of a bounded function over a set of finite measure:

The example we have cited just now shows some of the shortcomings of the Riemann integral. In particular, we would like a function which is 1 on a measurable set and zero elsewhere to be integrable and have its integral the χ measure of the set.

The function χ_E defined by

$$\chi_E\,(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called the characteristic function on E. A linear combination

$$\varphi(x) = \sum_{i=1}^{n} a_i \ \chi_{E_i}(x)$$

is called a simple function if the sets E_i are measurable. This representation for φ is not unique. However, we note that a function φ is simple if and only if it is measurable and assumes only a finite number of values. If φ is a simple function and $[a_1 \ a_n]$ the set of non- zero values of , then

$$\varphi = \sum a_i \chi_{A_i},$$

where $A_i = \{x \mid \varphi(x) = a_i\}$. **This representation for φ is called the canonical representation and it** is characterised by the fact that the A_i are disjoint and the a_i distinct and nonzero.

If vanishes outside a set of finite measure, we define the integral of φ by

$$\int \varphi(x)\, dx = \sum_{i=1}^{n} a_i\, mA_i$$

when φ has the canonical representation $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$. We sometimes abbreviate the expression for this integral to $\int \varphi$. If E is any measurable set, we define

$$\int_E \ \varphi = \int \varphi \chi_E \ .$$

It is often convenient to use representations which are not canonical, and the following lemma is useful.

Lemma 2.2.2. If E_1, E_2,.., E_n are disjoint measurable subset of E then every linear combination

$$\varphi = \sum_{i=1}^{n} c_i \chi_{E_i}$$

with real coefficients $c_1, c_2, \dots, c_n$ is a simple function and

$$I \int \ \varphi = \sum_{i=1}^{n} c_i\, mE_i.$$

Proof. It is clear that φ is a simple function. Let a_1, $a_2 \ldots a_n$ denote the non- zero real number in (E) . For each j $= 1,2, ..., $ n .

Let

$$A_j = \bigcup_{c_i = a_j} E_i$$

Then we have

$$A_j = \phi^{-1}(a_j) = \{x| \ \phi(x) = a_j\}$$

and the canonical representation

$$\varphi = \sum_{j=1}^{n} a_j \chi_{A_j}$$

Consequently, we obtain

$$\int \ \varphi = \sum_{j=1}^{n} a_j \, mA_j$$

$$= \sum_{j=1}^{n} a_j \, m \left[\bigcup_{c_i = a_j} E_i \right]$$

$$= \textstyle\sum_{j=1}^{n} a_j \sum_{c_i = a_j}^{n} m \, E_i$$

(Since E_i are disjoint, additivity of measures applies)

$$= \sum_{j=1}^{n} c_j \, mE_i$$

This completes the proof of the theorem.

Theorem 2.2.3. Let φ and ψ be simple functions which vanish outside a set of finite measure. Then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi,$$

and, if φ $\geq \psi$ a.e, then

$$\int \varphi \geq \int \psi.$$

Proof. Let $\{A_i\}$ and $\{B_i\}$ be the sets which occur in the canonical representations of φ and ψ. Let A_0 and B_0 be the sets where φ and ψ are zero.

Then the sets E_k obtained by taking all the intersections $A_i \cap B_j$ form a finite disjoint collection of measurable sets, and we may write

$$\varphi = \sum_{k=1}^{N} a_k \, \chi_{E_k}$$

$$\psi = \sum_{k=1}^{N} b_k \, \chi_{E_k},$$

and so

$$a\varphi + b\psi = a \sum_{k=1}^{N} a_k \, \chi_{E_k} + b \sum_{k=1}^{N} b_k \, \chi_{E_k}$$

$$= \sum_{k=1}^{N} a \, a_k \chi_{E_k} + \sum_{k=1}^{N} b \, b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} (aa_k + bb_k) \, \chi_{E_k}$$

Therefore

$$(a\varphi + b\psi) = \sum_{k=1}^{N} (aa_k + bb_k) \, mE_k$$

$$= \sum_{k=1}^{N} (aa_k) \, m_{E_k} + \sum_{k=1}^{N} (bb_k) \, mE_k$$

$$= a \sum_{k=1}^{N} a_k \, mE_k + b \sum_{k=1}^{N} b_k \, mE_k$$

$$= a \int \varphi + b \int \psi$$

To prove the second statement, we note that S

$$\int \varphi - \int \psi = I(\varphi - \psi) \geq 0,$$

since the integral of a **simple function which is greater than or equal to zero almost** everywhere is non- negative by the definition of the integral.

Remark 2.2.4 We know that for any simple function φ we have

$$\varphi = \sum_{k=1}^{N} a_i \, \chi_{E_i}$$

Suppose that this representation is neither canonical nor the sets E_i's are disjoint. Then using the fact that **characteristic functions are always simple functions** we observe that

$$\int \varphi = \int a_1 \chi_{E_i} + \int a_2 \chi_{E_2} + \cdots + \int a_n \chi_{E_n}$$

$$= a_1 \int \chi_{E_i} + a_2 \int \chi_{E_2} + \cdots + a_n \int \chi_{E_n}$$

$$= a_1 mE_1 + a_2 mE_2 + \cdots + \cdots + a_n mE_n$$

$$= \sum_{k=1}^{N} a_i \, mE_i$$

Hence for any representation of φ, we have

$$\int \ \varphi = \sum_{k=1}^{N} a_i \, mE_i$$

Let f be a bounded real- valued function and E a measurable set of finite measure. By analogy with the Riemann integral we consider for simple functions φ and ψ the numbers

$$inf_{\psi \geq f} \int_E \ \psi$$

and

$$sup_{\varphi \leq f} \int_E \ \varphi,$$

and **ask when these two numbers are equal.**

The answer is given by the following proposition :

Theorem 2.2.5. Let f be defined and bounded on a measurable set E with mE finite. In order that

$$inf_{\psi \geq f} \int_E \ \psi\,(x)dx = \ sup_{\psi \leq f} \int_E \ \varphi(x)dx$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f be measurable.

Proof. Let f be bounded by M and suppose that f is measurable. Then the sets

$$E_k = \{x \mid \frac{KM}{n} \geq f(x) > \frac{(K-1)M}{n}\}, -n \leq K \leq n,$$

are measurable, disjoint and have union E. Thus

$$\sum_{k=-n} m\, E_k = mE$$

The simple function defined by

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} K\, \chi_{E_k}(x)$$

and

$$\varphi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (K-1)\, \chi_{E_k}(x)$$

Satisfy

$$\varphi_n(x) \leq f(x) \leq \psi_n(x)$$

Thus

$$inf\ \int_E\ \psi\,(x)dx \leq \int_E\ \psi_n\,(x)dx = \frac{M}{n} \sum_{k=-n}^{n} k\, mE_k$$

and

$$sup\ \int_E\ \varphi\,(x)dx \geq \int_E\ \varphi_n\,(x)dx = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, mE_k\ ,$$

whence

$$0 \leq inf \int_E\ \psi\,(x)dx - \ sup_E \int \varphi(x)\, dx \leq \frac{M}{n} \sum_{k=-n}^{n} m\, E_k = \frac{M}{n} mE.$$

Since n is arbitrary we have

$$inf \int_E\ \psi\,(x)dx - \ sup\ \int_E\ \varphi(x)\, dx = 0$$

and the condition is sufficient.

Suppose now that

$$\inf \int_E\ \psi\,(x)dx = \ \sup\ \int_E\ \varphi(x)dx.$$

Then given n there are simple functions φ_n and ψ_n such that

$$\varphi_n(x) \leq f(x) \leq \psi_n(x)$$

and

$$\int \psi_n(x)dx - \int \phi_n(x)dx < \frac{1}{n} \qquad \text{(i)}$$

Then the functions

$$\psi^* = \ inf\ \psi_n$$

and
$$\varphi * = \ sup\ \varphi_n$$

are measurable and

$$\varphi^*(x) \leq f(x) \leq \psi^*(x)\ .$$

Now the set

$$\Delta = \{x|\ \phi^*\{x) < \psi^*\{x)\}$$

is the union of the sets

$$\Delta_V\{x|\ \ \varphi^*(x) < \psi^*(x) - \frac{1}{V}\}.$$

But each Δ_V is contained in the set $\{x|\varphi_n(x) < \psi_n(x) - \frac{1}{V}\}$, and this latter set by (i) has measure less than $\frac{V}{n}$. Since n is arbitrary, $m\Delta_V = 0$ and so $= 0$. Thus $\varphi *= \psi^*$ except on a set of measure zero, and $\varphi *= f$ except on a set of measure zero. Thus f is measurable and the condition is also necessary.

Definition 2.2.6. If f is a bounded measurable function defined on a measurable set E with mE finite, we define the Lebesgue integral of f over E by

$$\int_E\ f\ (x)dx = \ inf\ \int_E\ \psi\ (x)dx$$

for all simple functions $\psi \geq f$.

By the previous theorem, this may also be defined as

$$\int_E\ f\ (x)dx = \ sup\ \int_E\ \varphi\ (x)dx$$

for all simple functions $\varphi \leq f$.

We sometimes write the integral as $\int_E\ f$. If $\mathrm{E} = [a, b]$ we write $\int_a^b f$ instead of $\int_{[a,b]} f$

Definition and existence of the Lebesgue integral for bounded functions.

Definition 2.2.7. Let F be a bounded function on E and let E_k be a subset of E. Then we define $M[f, E_k]$and $m[f, E_k]$ as

$$M[f; E_k] = l.u.b\ _{x \in E_k} f(x)$$

$$m[f, E_k] = g.l.b_{x \in E_k} f(x)$$

Definition 2.2.8. By a measurable partition of E we mean a finite collection $P = \{E_1, E_2, \dots, E_n\}$ of measurable subsets of E such that

$$\bigcup_{k=1}^{n} E_k = E$$

and such that $m(E_j \cap E_k) = 0\ \ (j, k = 1, \dots, n\ , j \neq k)$. The sets $E_1,\ E_2, \dots, E_n$ are called the **components of P.**

If P and Q are measurable partitions, then Q is called a refinement of P if every component of Q is wholly contained in some component of P.

Thus a measurable partition P is a finite collection of subsets whose union is all of E and whose intersections with one another have measure zero.

Definition 2.2.9. Let f be a bounded function on E and let $P = \{E_1, \dots, E_n\}$ be any measurable partition E. We define the upper sum $U[f, P]$ as

$$U[f; P] = \sum_{k=1}^{n} M\ [f; E_k].\, mE_k$$

Similarly, we define the lower sum $L[f; P]$ as

$$L[f; P] = \sum_{k=1}^{n} m\ [f; E_k].\, mE_k$$

As in the case of Riemann integral, we can see that every upper sum for f is greater than or equal to every lower sum for f. We then define the Lebesgue upper and lower integrals of a bounded function f on E by

$$inf\ U[f; P] \text{ and } sup\ L[f; P]$$

respectively taken over all measurable position of E. We denote them respectively by

$$\int_E^{-} f \quad and \quad \int_{\bar{E}} f$$

Definition 2.2.10. We say that a bounded function f on E is Lebesgue integrable on E if

$$\int_E^{-} f \quad = \int_{\bar{E}} f$$

Also we know that if ψ is a simple function, then

$$\int_E \psi = \sum_{k=1}^{n} a_k \, mE_k$$

Keeping this in mind, we see that

$$\int_E^{-} f \quad = inf \int_E^{-} \psi(x) \, dx$$

for all simple functions $\psi(x) \geq f(x)$. Similarly

$$\int_{\bar{E}} f = sup \int_{\bar{E}} \varphi(x) dx$$

for all simple functions $\varphi(x) \leq f(x)$.

Now we use the theorem:

"Let f be defined and bounded on a measurable set E with mE finite. In order that

$$inf_{\ f \leq \Psi} \int_E \psi\, (x) dx = sup_{\, f \geq \varphi} \int_E \varphi(x) dx$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f is measurable." And our definition of **Lebesgue integration** takes the form :

"If f is a bounded measurable function defined on a measurable set E with mE finite, we define the (Lebesgue) integral of f over E by

$$\int_E f\,(x) dx = inf \int_E \psi\,(x) dx$$

for all simple functions $\psi \geq f$."

The following theorem shows that the Lebesgue integral is in fact a generalization of the Riemann integral.

Theorem 2.2.11. Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx$$

Proof. Since f is a bounded function defined on $[a, b]$ and is Riemann integrable, therefore,

$$R\int_a^{\bar{b}} f(x)dx = inf_{\varphi \geq f} \int_a^b \varphi(x)dx$$

And

$$R\int_{\underline{a}}^b f(x)dx = sup_{\psi \leq f} \int_a^b \psi(x)dx$$

for all step functions φ and ψ and then

$$R\int_a^{\bar{b}} f(x)dx = R\int_{\bar{a}}^b f(x)dx$$

$$\Rightarrow inf_{\varphi \geq f} \int_a^b \varphi(x)dx = sup_{\psi \leq f} \int_a^b \psi(x)dx \qquad \text{(i)}$$

Since every step function is a simple function, we have

$$R\int_a^{\bar{b}} f(x)dx = sup_{\psi \leq f} \int_a^b \psi(x)dx \leq inf_{\varphi \geq f} \int_a^b \varphi(x)dx \leq R\int_{\bar{a}}^b f(x)dx$$

Then (i) implies that

$$sup_{\psi \leq f} \int_a^b \psi(x)dx = inf_{\varphi \geq f} \int_a^b \varphi(x)dx$$

and this implies that f is measurable also.

Comparison of Lebesgue and Riemann integration

(1) The most obvious difference is that in Lebesgue's definition we divide up the interval into subsets while in the case of Riemann we divide it into subintervals.

(2) In both Riemann's and Lebesgue's definitions we have upper and lower sums which tend to limits. In the Riemann case the two integrals are not

necessarily the same and the function is integrable only if they are the same. In the Lebesgue case the two integrals are necessarily the same, their equality being consequence of the assumption that the function is measurable.

(3) Lebesgues's definition is more general than Riemann. We know that if function is the R- integrable then it is Lebesgue integrable also, but the converse need not be true. For example the characteristic function of the set of irrational points have Lebesgue integral but is not R- integrable.

Let χ be the characteristic function of the irrational numbers in [0,1]. Let E_1 be the set of irrational numbers in [0,1], and let E_2 be the set of rational numbers in [0,1]. Then $P = [E_1, E_2]$ is a measurable partition of [0,1]. Moreover, χ is identically 1 on E_1 and χ is identically 0 on E_2.

Hence

$$M[\chi, E_1] = m[\chi, E_1] = 1,$$

While

$$M[\chi, E_2] = m[\chi, E_2] = 0.$$

Hence

$$U[\chi, P] = 1. mE_1 + 0. mE_2 = 1.$$

Similarly

$$L[\chi, P] = 1.\ mE_1 + 0. ME_2 = 1.$$

Therefore,

$$U\ [\chi\ , P] = L[\chi, P].$$

Therefore, it is Lebesgue integrable.

For Riemann integration

$$M[\chi, J] = 1, \quad m[\chi, J] = 0$$

for any interval $J \subset [0,1]$

$$U[\chi, J] = 1, \quad L[\chi, J] = 0 \ .$$

The function is not Riemann- integrable.

Theorem 2.2.12. If f and g are bounded measurable functions defined on a set E of finite measure, then

(i) $\int_E \ a f = a \int_E \ f$

(ii) $\int_E \ (f + g) = \int_E \ f + \int_E \ g$

(iii) If $f \leq g$ a.e., then

$$\int_E \ f \leq \int_E \ g$$

(iv) If $f = g$ a.e., then

$$\int_E \ f = \int_E \ g$$

(v) If $A \leq f(x) \leq B$, then

$$AmE \leq \int_E \ f \leq BmE.$$

(vi) If A and B are disjoint measurable sets of finite measure, then

$$\int_{A \cup B} f = \int_A \ g + \int_B \ f$$

Proof. We know that if ψ is a simple function then so is a . Hence

$$\int_E af = \inf_{\psi \geq f} \int_E a\psi = a \inf_{\psi \geq f} \int \psi = a \int_E f$$

which proves (i).

To prove (ii)

let ε denote any positive real number. There are simple functions $\phi \leq f$, $\psi \geq f$, $\xi \leq g$ and $\eta \geq g$ satisfying

$$\int_E \phi(x)dx > \int_E f - \in, \qquad \int_E \psi(x)dx < \int_E f + \in,$$

$$\int_E \xi(x)dx > \int_E g - \in, \qquad \int_E \eta(x)dx < \int_E g + \in,$$

Since $\phi + \xi \leq f + g \leq \psi + \eta$, we have

$$\int_E (f+g) \geq \int_E \phi + \xi = \int_E \phi + \int_E \xi > \int_E f + \int g - 2 \in$$

$$\int_E (f+g) \leq \int_E (\psi + \eta) = \int_E \psi + \int_E \eta < \int_E f + \int_E g + 2 \in$$

Since these hold for every $\in > 0$, we have

$$\int_E (f+g) = \int_E f + \int_E g$$

To prove (iii) it suffices to establish

$$\int_E (g - f) \geq 0$$

For every simple function $\psi \geq g - f$, we have $\psi \geq 0$ almost everywhere in

E. This means that

$$\int_E \psi \geq 0$$

Hence we obtain

$$\int_E (g-f) = \inf_{\psi \geq (g-f)} \int_E \psi(x)dx \geq 0 \qquad (1)$$

which establishes (iii).

Similarly we can show that

$$\int_E (g-f) = \sup_{\psi \leq (g-f)} \int_E \psi(x)dx \leq 0 \qquad (2)$$

Therefore, from (1) and (2) the result (iv) follows.

To prove (v) we are given that

$$A \leq f(x) \leq B$$

Applying (iv) we get

$$\int_E f(x)dx \leq \int_E B\,dx = B\int_E d\,x = BmE$$

That is,

$$\int_E f \leq BmE$$

Similarly we can prove that $\int_E f \geq AmE$.

Now we prove (vi).

We know that

$$x_{A\cup B} = x_A + x_B$$

Therefore,

$$\int_{A\cup B} f = \int_{A\cup B} \chi_{A\cup B} f = \int_{A\cup B} f(\chi_A + \chi_B)$$

$$= \int_{A\cup B} f_{\chi_A} + \int_{A\cup B} f_{\chi_B}$$

$$= \int_A f + \int_B f$$

which proves the theorem.

Lebesgue Bounded Convergence Theorem

Theorem 2.2.13. Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $\langle f_n \rangle$ is uniformly bounded, that is, there exists a real number M such that $|f_n(x)| \leq M$ for all $n \in N$ and all $x \in E$. If

$$lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \text{ in } E,$$

then

$$\int_E f = lim_{n\to\infty} \int_E f_n.$$

Proof. We shall apply Egoroff's theorem to prove this theorem. Accordingly for a given $\in > 0$, there is an N and a measurable set $E_0 \subset E$ such that $mE_0^c < \frac{\in}{4M}$ and for $n \geq N$ and $x \in E_0$ we have

$$|f_n(x) - f(x)| < \frac{\in}{2m(E)}$$

Then we have

$$|\int_E f_n - \int_E f| = |\int_E (f_n - f)| \leq \int_E |f_n - f|$$

$$= \int_{E_0} |f_n - f| + \int_{E_0{}^c} |f_n - f|$$

$$< \frac{\in}{2m(E)} . m(E_0) + \frac{\in}{4M} . 2M$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

Hence

$$\int_E f_n \to \int_E f.$$

<u>The integral of a non- negative function</u>

Definition 2.2.14. If f is a non- negative measurable function defined on a measurable set E, we define

$$\int_E f = sup_{h\le f} \int_E h$$

where h is a bounded measurable function such that $m\{x \mid h(x) \neq 0\}$ is finite.

Theorem 2.2.15. If f and g are non- negative measurable functions, then

(i) $\int_E cf = c\int_E f, c > 0$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If $f \le g$ a.e., then and $\int_E f \le \int_E g$.

Proof. The proof of (i) and (iii) follow directly from the theorem concerning properties of the integrals of bdd functions.

We prove (ii) in detail.

If

$$h(x) \le f(x) \text{ and } k(x) \le g(x),$$

we have

$$h(x) + k(x) \le f(x) + g(x),$$

and so

$$\int_E (h+k) \le \int_E (f+g)$$

$$i.e. \int_E h + \int_E k \le \int_E (f+g)$$

Taking suprema, we have

$(iv)\ \int_E f + \int_E g \le \int_E (f+g)$

On the other hand, let **l be a bounded measurable function** which vanishes outside a set of finite measure and which is not greater than $(\mathrm{f}+\mathrm{g})$. Then we define the functions h and k by setting

$$h(x) = min\left(f(x), l(x)\right)$$

and

$$k(x) = l(x) - h(x)$$

We have

$$h(x) \le f(x),$$

$$k(x) \le g(x),$$

while h and k are bounded by the bound l and vanish where l vanishes. Hence

$$\int_E l = \int_E h + \int_E k \le \int_E f + \int_E g$$

and so taking supremum, we have

$$sup \le \int_E f + \int_E g$$

that is,

(v) $\int_E f + \int_E g \ge \int_E (f+g)$

From (iv) and (v), we have

$$\int_E (f+g) = \int_E f + \int_E g.$$

Fatou's Lemma 2.2.16. If $\langle f_n \rangle$ is a sequence of non- negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on aset E, then

$$\int_E f \le \underline{lim} \int_E f_n$$

Proof. Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E′| of finite measure. Define a function h_n by setting

$$h_n(x) = \min\{h(x), f_n(x)\}$$

Then h_n is bounded by the bounds for h and vanishes outside E′. Now $h_n(x) \to h(x)$ for each x in E'.

Therefore by "**Bounded Convergence Theorem**" we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \underline{\lim} \int_E f_n$$

Taking the supremum over h, we get

$$\int_E f \leq \underline{\lim} \int_E f_n.$$

Lebesgue Monotone Convergence Theorem

Theorem 2.2.16.. Let $\langle f_n \rangle$ be an increasing sequence of non- negative measurable functions and let $f = \lim \quad f_n$. Then

$$\int f = \underline{\lim} \int f_n$$

Proof. By Fatou's Lemma we have

$$f \leq \underline{\lim} \int f_n$$

But for each n we have $f_n \leq f$, and so $\int f_n \leq \underline{\lim} \quad f_n$. But this implies

$$\overline{\lim}\, f_n \leq \int f$$

Hence

$$\int f = \lim \int f_n$$

Definition 2.2.17. A non- negative measurable function f is called integrable over the measurable set E if

$$\int_E f < \infty$$

Theorem 2.2.18. Let f and g be two non- negative measurable functions. If f is integrable over E and $g(x) < f(x)$ on E, then g is also integrable on E, and

$$\int_E (f-g) = \int_E f - \int_E g$$

Proof. Since

$$\int_E f = \int_E (f-g) + \int_E g$$

and the left hand side is finite, the term on the right must also be finite and so g is integrable.

Theorem 2.2.19. Let f be a non- negative function which is integrable over a set E. Then given $\in > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$$\int_A f < \in.$$

Proof. If $|f| \leq K$, then

$$\int_A f \leq \int_A K = KmA$$

Set $\delta < \frac{\in}{K}$. Then

$$\int_A f < K . \frac{\in}{K} = \in.$$

Set $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ otherwise. Then each f_n is bounded and f_n converges to f at each point. By the monotone convergence theorem there is an N such that

$$\int_E f_N > \int_E f - \frac{\in}{2}, \text{ and } \int_E (f - f_N) < \frac{\in}{2}.$$

Choose $\delta < \frac{\in}{2N}$. If $mA < \delta$, we have

$$\int_A f = \int_A (f - f_N) + \int_A f_N$$

$$< \int (f - f_N) + NmA \qquad \text{(since } \int_A f_N \leq$$

$\int_A N = NmA$) E

$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

The General Lebesgue Integral

We have already defined the positive part f^+ and negative part f of a function as

$$f^+ = max\,(f, 0)$$

$$\overline{f} = max\,(-\overline{f}, 0)$$

Also it was shown that

$$f = f^+ - \overline{f}$$

$$|f| = f^+ + \overline{f}$$

With these notions in mind, we make the following definition.

Definition 2.2.20. A measurable function f is said to be integrable over E if f^+ and $\bar{f}$ are both integrable over E. In this case we define

$$\int_E f = \int_E f^+ - \int_E \overline{f}$$

Theorem 2.2.21. Let f and g be integrable over E. Then

(i) The function $f + g$ is integrable over E and

$$\int_E (f + g) = \int_E f + \int_E g$$

(ii) If $f \leq g$ a.e., then

$$\int_E f \leq \int_E g$$

(iii) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof. By definition, the functions $f^+, \overline{f}, \ \ g^+, \overline{g}$ are all integrable. If

$$h = f + g,$$

then

$$h = (f^+ - \overline{f)} + (g^+ - \overline{g})$$

and hence

$$h = (f^+ + g^+) - (\overline{f} + \overline{g}) \ .$$

Since $f^+ + g^+$ and $\overline{f} + \overline{g}$ are integrable therefore their difference is also integrable. Thus h is integrable.

We then have

$$\int_E h = \int_E ([f^+ + g^+) - (\overline{f} + \overline{g})]$$

$$= \int_E (f^+ + g^+) - \int_E (\overline{f} + \overline{g})$$

$$= \int_E f^+ + \int_E g^+ - \int_E \overline{f} - \int_E \overline{g}$$

$$= \left(\int_E f^+ - \int_E \overline{f}\right) + \left(\int_E g^+ - \int_E \overline{g}\right)$$

That is,

$$\int_E (f + g) = \int_E f + \int_E g$$

Proof of (ii) follows from part (i) and the fact that the integral of a non- negative integrable function is non- negative.

For (iii) we have

$$\int_{A\cup B} f = \int f\,\chi_{A\cup B}$$

$$= \int f\,\chi_A + \int f\,\chi_B$$

$$= \int_A f + \int_B f$$

It should be noted that $f + g$ is not defined at points where $f = \infty$ and $g = -\infty$ and where $f = -\infty$ and $g = \infty$. However, the set of such points must have measure zero, since f and g are integrable. Hence the integrability and the value of $\int(f + g)$ is independent of the choice of values in these ambiguous cases.

Theorem 2.2.22. Let f be a measurable function over E. Then f in integrable over E iff $|f|$ is integrable over E. Moreover, if f is integrable, then

$$|\int_E f| \leq \int_E |f|$$

Proof. If f is integrable then both f^+ and f^- are integrable. But $|f| = f^+ + f^-$. Hence integrability of f^+ and f^- implies the integrability of $|f|$.

Moreover, if f is integrable, then since

$$f(x) \leq |f(x)| = |f|(x),$$

the property which states that if $f \leq g$ a.e. , then $\int f \leq \int g$ implies that

$$\int f \leq \int |f| \qquad \text{(i)}$$

On ther other hand since $-f(x) \leq |f(x)|$, we have

$$-\int f \leq \int |f| \qquad \text{(ii)}$$

From (i) and (ii) we have

$$|\int f| \le \int |f| .$$

Conversely, suppose f is measurable and suppose $|f|$ is integrable. Since

$$0 \le f^+(x) \le |f(x)|$$

It follows that f^+ is integrable. Similarly f^- is also integrable and hence f is integrable.

Lemma 2.2.23. Let f be integrable. Then given $\in > 0$ there exists $\delta > 0$ such that

$$|\int_A f| < \in$$

whenever A is a measurable subset of E with mA $< \delta$.

Proof. When f is non- negative, the lemma has been proved already. Now for arbitrary measurable function f we have $= f^+ - f^-$. So by that we have proved already, given $\in > 0$, there exists $\delta_1 > 0$ such that

$$\int_A f^+ < \frac{\in}{2},$$

when $mA < \delta_1$. Similarly there exists $\delta_2 > 0$ such that

$$\int_A f^- < \frac{\in}{2},$$

when $< \delta_2$. Thus if $< \delta = min\,(\delta_1, \delta_2)$, we have

$$|\int_A f \le \int_A |f| = \int_A f^+ + \int_A \overline{f} < \frac{\in}{2} + \frac{\in}{2} = \in$$

This completes the proof.

Lebesgue Dominated Convergence Theorem

Theorem 2.2.24. Let a sequence $\langle f_n \rangle$, n ϵ N of measurable functions be dominated by an integrable function g, that is,

$$|f_n(x)| \le g(x)$$

holds for every $n \epsilon N$ and every $x \epsilon E$ and let $\langle f_n \rangle$ converges pointwise to a function f, that is, $f(x) = lim_{n\to\infty} f_n(x)$ for almost all x in E. Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof. Since $|f_n| \le g$ for every $n \epsilon N$ and $f(x) = lim \quad f_n(x)$, we have $|f| \le g$. Hence f_n and f are integrable. The function $g - f_n$ is non- negative, therefore by Fatou's Lemma we have

$$\int_E g - \int_E f = \int_E (g - f) \le \underline{lim} \int_E (g - f_n)$$

$$= \int_E g - \overline{lim} \int_E f_n$$

whence

$$\int_E f \ge \overline{lim} \int_E f_n$$

Similarly considering $g + f_n$ we get

$$\int_E f \le \overline{lim} \int_E f_n$$

Consequently, we have

$$\int_E f = lim \quad \int_E f_n.$$

2.3 <u>Check Your Progress</u>

Q.1. If $f \epsilon L[a, b]$ and if $f(x) \ge 0$ a.e. in $[a, b]$ then $\int_a^b f \ge 0$.

Q.2. If f is a bounded function in $L[a, b]$ then $|f| \epsilon L[a, b]$ and

$|\int_a^b f| \le \int_a^b |f|$.

Q.3. Show that the function defined on E=[0,∞) is follows:

$f(x) = \frac{\sin x}{x}$, for $x \ne 0$ and $f(0) = 0$, is not lebesgue integrable in E.

Q.4. Give an example to show that the integral of a no where zero function can be zero.

Q.5. If E is a measurable subset of $[a, b]$ and f is a bounded measurable function of $f \epsilon L[a, b]$ such that $f(x) \geq 0$ a.e. on E then $\int_E f \geq 0$.

2.4 Summary

In this section, we will give a brief overview of measure theory, which leads to a general notion of an integral called the Lebesgue integral. Integrals, as we saw before, are important in probability theory since the notion of expectation or average value is an integral.

2.5 Keywords

Lower Lebesgue Sum, Upper Legesgue Sum, Upper Lebegue Integral, Lower Lebegue Integral Sum, Lebesgue Integral, L-Integral of Non-Negative Function.

2.6 Self-Assessment Test

1. If $f(x)=\begin{cases} 0; & if\ 0 \leq x \leq 1 \\ 1; & if\ [1 \leq x < 2] \cup [3 \leq x < 4]. \\ 2; & if\ [2 \leq x < 3] \cup [4 \leq x < 5] \end{cases}$

   Show that $\int_0^5 f(x)dx = 6$.

2. Let $f : R \rightarrow R$ be a function defined by

$$f(x)=\begin{cases} 0, & x \notin [0,1] \\ 1, & x \in [0,1]\ and\ rational \\ -1, & x \in [0,1]\ and\ irrational \end{cases}.$$

   Show that f is L-integrable.

3. Show that the function f defined on interval [a,b] by

$$f(x) = \begin{cases} 0; & if\ \ x\ is\ irrational \\ 1; & if\ \ x\ is\ rational \end{cases}$$

is L – integrable but not R- integrable.

4. If f is L- integral, show that

$$\int f dx = -\int f(-x) dx.$$

5. Show that f : [0,1] → R is R- integrable if and only if the discontinuities of f form a set of lebesgue measure zero.

2.7 Answers to check your progress

A.1. Let f be a L-integrable function on $[a, b]$.

We know that if $f(x) \geq 0$ for all $x\epsilon[a, b]$ then $U[f, P] \geq 0$ for every partition P. Therefore,

L$\int_a^{\bar{b}} f = U[f; P] \geq 0$

Now, since $f\epsilon L[a, b]$(given) then

$$\int_a^b f = \text{L}\int_a^{\bar{b}} f \geq 0$$

Hence, $\int_a^b f \geq 0$.

A.2. Let f is a bounded function in L[a,b]

$f\epsilon L[a, b] \Rightarrow f$ is measurable

$\Rightarrow |f|$ is also measurable

$\Rightarrow |f|\epsilon L[a, b]$.

Now, it remains to prove that $f(x) \leq g(x)$ a.e. on $E \Rightarrow \int_E f \leq \int_E g$.

We know that

$$f(x) \leq |f(x)| \text{ for all } x\epsilon[a, b]$$

$$\Rightarrow \int_a^b f \leq \int_a^b |f| \qquad (1)$$

Further, we know that

$$-f(x) \leq |f(x)| = |f|(x)$$

$$\Rightarrow \int_a^b f \leq \int_a^b |f|$$

(2)

From (1) and (2) we conclude that

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

A.3. Consider $\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{n\pi} \frac{|\sin x|}{x} dx$

$$= \sum_{r=1}^{n} \int_0^{n\pi} \frac{|\sin\{t+(r-1)\pi\}|}{\{t+(r-1)\pi\}} dt$$

$$\ge \sum_{r=1}^{n} \int_0^{n\pi} \frac{|\sin\{t+(r-1)\pi\}|}{\{r\pi\}} dt$$

$$= \frac{1}{\pi} \sum_{r=1}^{n} \frac{1}{r} \int_0^{\pi} |\sin t\,| dt$$

$$= \frac{1}{\pi} \sum_{r=1}^{n} \frac{1}{r} \int_0^{\pi} \sin t\, dt$$

$$= \frac{2}{\pi} \sum_{r=1}^{n} \frac{1}{r}.$$

Which implies that $\lim_{n\to\infty} \int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx \ge \frac{2}{\pi}\sum_{r=1}^{n}\frac{1}{r}$.

But we know that $\sum_{r=1}^{n}\frac{1}{r}$ is a divergent series, therefore $\sum_{r=1}^{n}\frac{1}{r} = \infty$

So, $\int_0^{\infty} |f(x)| dx \ge \infty \Rightarrow \int_0^{\infty} |f(x)| dx = \infty$

$\Rightarrow |f|$ is not L-integrable.

Hence, $f(x)$ is not L-integrable.

A.4. Let us define a function $f: \mathbf{Q} \to \mathbf{R}$ such that $f(x) = 1$ for all $x \in \mathbf{Q}$.

Clearly, the function f defined above is no where zero.

But we know that $m(\mathbf{Q}) = 0$. (because **Q** is a countable set and measure of a countable set is zero)

Then, by first mean value theorem

$$1.m(\mathbf{Q}) \le \int_{\mathbf{Q}} f \le 1.m(\mathbf{Q})$$

$$\Rightarrow 0 \le \int_{\mathbf{Q}} f \le 0$$

$$\Rightarrow \int_{\mathbf{Q}} f = 0.$$

A.5. We know that $\int_E f = \int_a^b f\chi_E$, χ_E is the characterstic function of E.

Is is given that $f \ge 0$ a.e. on E.

Also by definition of characterstic function $\chi_E \geq 0$ on [a,b].

$\Rightarrow f\chi_E \geq 0$ a.e. on [a,b].

$\Rightarrow \int_a^b f\chi_E \geq 0$

$\Rightarrow \int_E \; f \geq 0$

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Lesson: Function of Bounded Variation

Structure:

3.5 Keywords

3.6 Self-Assessment Test

3.7 Answers to check your progress

3.8 References/ Suggested Readings

3.1 Introduction

The "fundamental theorem of the integral calculus" is that differentiation and integration are inverse processes. This general principle may be interpreted in two different ways.

If $f(x)$ is integrable, the function

$$F(x) = \int_a^x f(t)dt \qquad \text{(i)}$$

is called the indefinite integral of f(x) ; and the principle asserts that

$$F'(x) = f(x) \qquad \text{(ii)}$$

On the other hand, if F(x) is a given function, and f(x) is defined by (ii), the principle asserts that

$$\int_a^x f\,(t)dt = F(x) - F(a) \qquad \text{(iii)}$$

The main object of this chapter is to consider in what sense these theorems are true.

From the theory of Riemann integration (ii) follows from (i) if x is a point of continuity of f. For we can choose h_0 so small that $|f(t) - f(x)| < \in$ for $|t - x| \leq h_0$; and then

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h}\{f(t) - f(x)\}\,dt\right| \leq \in \ (|h| < h_0),$$

by the mean- value theorem. This proves (ii).

We shall show that more generally this relation holds almost everywhere. Thus differentiation is the inverse of Lebesgue integration.

The problem of deducing (iii) from (ii) is more difficult and even using Lebesgue integral it is true only for a certain class of functions. We require in the first place that $F'(x)$ should exist at any rate almost everywhere and as we shall see this is not necessarily so. Secondly, if $F'(x)$ exists we require that it should be integrable.

Differentiation of Monotone Functions

Definition 3.2.1. Let C be a collection of intervals. Then we say that C covers a set E in the sense of Vitali, if for each $\in > 0$ and x in E there is an interval I ϵ C such that x ϵ I and $l(I) < \in$.

Now we prove the following lemma which will be utilized in proving a result concerning the differentiation of monotone functions.

Vitali Lemma

Lemma 3.2.2. Let E be a set of finite outer measure and C a collection of intervals which cover E in the sense of Vitali. Then given $\in > 0$ there is a finite disjoint collection $\{I_1, \dots, I_n\}$ of intervals in C such that

$$m^*\left[E - \bigcup_{n=1}^{N} I_n\right] < \in.$$

Proof. It suffices to prove the lemma in the case that each interval in C is closed, for otherwise we replace each interval by its closure and observe that the set of endpoints of $I_1, I_2, \dots, I_N$ has measure zero.

Let O be an open set of finite measure containing E. Since C is a Vitali covering of E, we may suppose without loss of generality that each I of C is contained in O. We choose a sequence $\langle I_n \rangle$ of disjoint intervals of C by induction as follows :

Let I_1 be any interval in C and suppose $I_1, \dots, I_n$ have already been chosen. Let k_n be the supremum of the lengths of the intervals of C which do not meet any of the intervals $I_1, \dots, I_n$. Since each I is contained in O, we have $k_n \le mO < \infty$. Unless

$$E \subset \cup_{i=1}^{n} I_i,$$

we can find I_{n+1} in C with $l(I_{n+1}) > \frac{1}{2} k_n$ and I_{n+1} disjoint from $I_1, I_2, \dots, I_n$.

Thus we have a sequence $\langle I_n \rangle$ of disjoint intervals of C, and since $\cup I_n \subset O$, we have $l(I_n) \leq mO < \infty$. Hence we can find an integer N such that

$$\sum_{N+1}^{\infty} l\,(I_n) < \frac{\in}{5}$$

Let

$$R = E - \bigcup_{i=1}^{N} I_n$$

It remains to prove that $m^*R < \in$.

Let x be an arbitrary point of R. Since $\bigcup_{i=1}^{N} I_n$ is a closed set not containing x, we can find an interval I in C which contains x and whose length is so small that I does not meet any of the intervals $I_1, I_2, \ldots, I_N$. If now $I \cap I_i = \phi$ for $\leq N$, we must have $l(I) \leq k_N < 2l(I_{N+1})$. Since $lim \quad l\,(I_n) = 0$, the interval I must meet at least one of the intervals I_n. Let n be the smallest integer such that I meets I_n. We have $n > N$, and $l(I) \leq k_{n-1} \leq 2l(I_n)$. Since x is in I, and I has a point in common with I_n, it follows that the distance from x to the midpoint of I_n is at most

$$l(I) + \frac{1}{2}l(I_n) \leq \frac{5}{2}l(I_n).$$

Let J_m denote the interval which has the same midpoint as I_m and five times the length of I_m. Then we have $x \in J_m$. This proves

$$R \subset \bigcup_{N+1}^{\infty} J_n$$

Hence

$$m^*R \leq \sum_{N+1}^{\infty} l\,(J_n) = 5m \sum_{N+1}^{\infty} l\,(J_n) < m \in.$$

The Four Derivatives of a Function

Whether the differential coefficients

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists or not, the four expressions

$$D^{+}f(x) = \overline{\lim_{h\to 0+}}\frac{f(x+h)-f(x)}{h}$$

$$D^{-}f(x) = \overline{\lim_{h\to 0+}}\frac{f(x)-f(x-h)}{h}$$

$$D_{+}f(x) = \underline{\lim_{h\to 0+}}\frac{f(x+h)-f(x)}{h}$$

$$D_{-}f(x) = \underline{\lim_{h\to 0+}}\frac{f(x)-f(x-h)}{h}$$

always exist. These derivatives are known as Dini Derivatives of the function f. $D^{+}f(x)$ and $D_{+}f(x)$ are called upper and lower derivatives on the right and $D^{-}f(x)$ and $D_{-}f(x)$ are called upper and lower derivatives on the left.

Clearly we have $D^{+}f(x) \geq D_{+}f(x)$ and $D^{-}f(x) \geq D_{-}f(x)$.

If

$$D^{+}f(x) = D_{+}f(x),$$

the function f is said to have a right hand derivative.

If

$$D^{-}f(x) = D_{-}f(x),$$

the function is said to have a left hand derivative.

If

$$D^{+}f(x) = D_{+}f(x) = D^{-}f(x) = D_{-}f(x) \neq \pm\infty,$$

we say that f is differentiable at x and define $f'(x)$ to be the common value of the derivatives at x.

Theorem 3.2.3. Every non- decreasing function f defined on the interval $[a, b]$ is differentiable almost everywhere in $[a, b]$. The derivative $f'1$ is measurable and

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

Proof. We shall show first that the points x of the open interval (a, b) at which not all of the four Dini- derivatives of f are equal form a subset of measure zero. It suffices to show that the following four subsets of (a, b) are of measure zero:

$$A = \{x \epsilon (a, b) | D_- + f(x) < D^\wedge + f(x)\},$$

$$B = \{x \epsilon (a, b) \big| D_- + f(x) < D^\wedge - f(x)\},$$

$$C = \{x \epsilon (a, b) | D_- + f(x) < D^\wedge - f(x)\},$$

$$D = \{x \epsilon (a, b) | D_- + f(x) < D^\wedge + f(x)\}.$$

To prove $m^* A = 0$, consider the subsets

$$A_{u,v} = \{x \epsilon (a, b) | D\_f(x) < u < v < D^+ f(x)\}$$

of A for all rational numbers u and v satisfying $u < v$. Since A is the union of this countable family $\{A_{u,v}\}$, it is sufficient to prove $m^*(A_{u,v}) = 0$ for all pairs u, v with $u < v$.

For this purpose, denote $\alpha = m^*(A_{u,v})$ and let $\in$ be any positive real number. Choose an open set $U \supset A_{u,v}$ with $m^* U < m + \in$. Set x be any point of $A_{u,v}$. Since $D\_ f(x) < u$, there are arbitrary small closed intervals of the form $[x - h, x]$ contained in U such that

$$f(x) - f(x - h) < uh.$$

Do this for all $x \epsilon A_{u,v}$ and obtain a Vitali cover C of $A_{u,v}$. Then by Vitali covering theorem there is a finite sub collection $\{J_1, J_2, \ldots, J_n\}$ of disjoint intervals in C such that

$$m^* \left(A_{u,v} - \bigcup_{i=1}^{n} J_i \right) < \in$$

Summing over these n intervals, we obtain

$$\sum_{i=1}^{n}[f(x_i)-f(x_i-h_i)] < u\sum_{i=1}^{n} h_i$$

$$< um^*U$$

$$< u(\alpha+\in).$$

Suppose that the interiors of the intervals $J_1, J_2,\ldots,J_n$ cover a subset F of $A_{u,v}$. Now since $D^+f(y) > v$, there are arbitrarily small closed intervals of the form $[y, y+k]$ contained in some of the intervals J_i $(i = 1, 2,\ldots,n)$ such that

$$f(y+k)-f(y) > vk$$

Do this for all $y \in F$ and obtain a Vitali cover D of F. Then again by Vitali covering lemma we can select a finite subcollection $[K_1, K_2, \ldots, K_m]$ of disjoint intervals in D such that

$$m^*\left[F-\bigcup_{i=1}^{m} K_i\right] < \in$$

Since $m * F > \alpha - \in$, it follows that the measure of the subset H of F which is covered by the intervals is greater than $\alpha - 2 \in$. Summing over these intervals and keeping in mind that each K_i is contained in a J_n, we have

$$\sum_{i=1}^{n}\{f(x_i)-f(x_i-h_i)\} \geq \sum_{i=1}^{m}[f(y_i+k_i)-f(y_i)]$$

$$> v\sum_{i=1}^{m} k_i$$

$$> v(\alpha - 2 \in)$$

so that

$$v(\alpha - 2 \in) < u(\alpha+\in)$$

Since this is true for every $\in > 0$, we must havev $\alpha \leq u\alpha$. Since $u < v$, this implies that $\alpha = 0$. Hence $m^*A = 0$. Similarly, we can prove that $m^*B = 0$,

$m^*C = 0$ and $m^*D = 0$.

This shows that

$$g(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is defined almost everywhere and that f is differentiable whenever g is finite. If we put

$$g_n(x) = n\left[f\left(x+\tfrac{1}{n}\right) - f(x)\right] \textit{ for } x \in [a,b],$$

where we re- define $f(x) = f(b)$ for $x \geq b$. Then $g_n(x) \to g(x)$ for almost all x and so g is measurable since every g_n is measurable. Since f is non- decreasing, we have $g_n \geq 0$. Hence, by Fatou's lemma

$$\int_a^b g \leq \underline{lim}\int_a^b g_n = \underline{lim}\, n\int_a^b [f\left(x+\frac{1}{n}\right) - f(x)]dx$$

$$= \underline{lim}\, n\, [\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx\,]$$

$$= \underline{lim}\, n\, [\int_a^b f(x)dx + \int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx]$$

$$= \underline{lim}\, n\, [\int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx]$$

$$\leq f(b) - f(a)$$

(Use of $f(x) = f(b)$ for $x \geq b$ for first interval and f non- decreasing in the 2^{nd} integral).

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable almost everywhere and $g(x) = f'(x)$ almost everywhere. This proves the theorem.

<u>Functions of Bounded Variation</u>

Let f be a real- valued function defined on the interval [a,b] and let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be any partition of *[a,b]*.

By the variation of f over the partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b], we mean the real number

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

and then

$$V_a^b(f) = \text{ sup } \{V(f,P)\} \text{ for all possible partitions P of } [\text{a, b}]\}$$
$$= sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is called the total variation of f over the interval [a,b]. If $V_a^b(f) < \infty$, then we say that f isafunction of bounded variation and we write f ϵ BV.

Lemma 3.2.4. Every non- decreasing function f defined on the interval [a,b] is of bounded variation with total variation

$$V_a^b(f) = f(b) - f(a).$$

Proof. For every partition $P = \{x_1, \ldots, x_n\}$ of [a,b] we have

$$V(f,P) = \sum_{i=1}^{n} | f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})$$
$$= f(b) - f(a)$$

This implies the lemma.

Jordan Decomposition Theorem

Theorem 3.2.5. A function $f: [a,b] \rightarrow R$ is of bounded variation if and only if it is the difference of two non- decreasing functions.

Proof. Let $f = g - h$ on [a,b] with g and h increasing. Then for any, subdivision we have

$$\sum_{i=1}^{n} | f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})] + \sum_{i=1}^{n} [h(x_i) - h(x_{i-1})]$$
$$= g(b) - g(a) + h(b) - h(a)$$

Hence

$$V_a^b(f) \leq g(b) + h(b) - g(a) - h(a),$$

which proves that f is of bounded variations.

On the other hand, let f be of bounded variation. Define two functions $g, h : [a, b] \to R$ by taking

$$g(x) = V_a^x(f), \qquad h(x) = V_a^x(f) - f(x)$$

for every $x \epsilon [a, b]$. Then $f(x) = g(x) - h(x)$.

The function g is clearly non- decreasing. On the other hand, for any two real numbers x and y in $[a, b]$ with $x \leq y$, we have

$$h(y) - h(x) = \left[V_a^y(f) - f(y)\right] - [V_a^x(f) - f(x)]$$

$$= V_x^y(f) - [f(y) - f(x)]$$

$$\geq V_x^y(f) - V_x^y(f) = 0$$

Hence h is also non- decreasing. This completes the proof of the theorem.

Examples 3.2.6. (1) If f is monotonic on [a,b], then f is of bounded variation on $[a, b]$ and

$$V(f) = |f(b) - f(a)|,$$

where $V(f)$is the total variation.

(2) If f' exists and is bounded on $[\mathrm{a}, \mathrm{b}]$, then f is of bounded variation.

For if

$$|f'(x)| \leq M$$

we have

$$\sum_{i=1}^{n} |\, f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} M\,(x_i - x_{i-1}) = M(b - a)$$

no matter which partition we choose.

(3) f may be continuous without being of bounded variation. Consider

$$f(x) = \begin{cases} x\sin\dfrac{\pi}{x} & (0 < x \leq 2) \\ 0 & (x = 0) \end{cases}$$

Let us choose the partition which consists of the points $0, \frac{2}{2^{n-1}}, \frac{2}{2^{n-3}}, \dots, \frac{2}{5}, \frac{2}{3}, 2$ Then the sum in the total variation is

$$[2+\frac{2}{3}]+\left[\frac{2}{3}+\frac{2}{5}\right]+\ +\left[\frac{2}{2^{n-3}}+\frac{2}{2^{n-1}}\right]+\frac{2}{2^{n-1}}$$
$$>\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},$$

and this can be made arbitrarily large by taking n large enough, since $\Sigma\frac{1}{n}$ diverges. Since

$$|f(x)-f(a)|\leq V(f)$$

for every x on [*a*,*b*] it is clear that every function of bounded variation is bounded.

The Differentiation of an Integral

Let f be integrable over [*a*,*b*] and let

$$F(x)=\int_a^x f(t)dt$$

If f is positive, $h>0$, then we see that

$$F(x+h)-F(x)=\int_x^{x+h} f\,(t)dt\geq 0$$

Hence, integral of a positive function is non-decreasing.

We shall show first that F is a function of bounded variation. Then, being function of bounded variation, it will have a finite differential coefficient F' almost everywhere. Our object is to prove that $F'(x)=f(x)$ almost everywhere in [*a*,*b*]. We prove the following lemma:

Lemma 3.2.7. If f is integrable on [*a*,*b*], then the function F defined by $F(x)=\int_a^x f(t)dt$ is a continuous function of bounded variation on [a,b].

Proof. We first prove continuity of F. Let x_0 be an arbitrary point of [*a*,*b*]. Then

$$|F(x) - F(x_0)| = |\int_{x_0}^{x} f\,(t)dt|$$

$$\leq \int_{x_0}^{x} |f(t)|dt$$

Now the integrability of f implies integrability of $|f|$ over [a,b]. Therefore, given $\in > 0$ there is a $\delta > 0$ such that for every measurable set $A \subset [a,b]$ with measure less than δ, we have

$$\int_A |f| < \in.$$

Hence

$$|F(x) - F(x_0)| < \in \text{ whenever } |x - x_0| < \delta_1$$

and so f is continuous.

To show that F is of bounded variation, let $a = x_0 < x_1 < \cdots < x_n = b$ be any partition of [a,b]. Then

$$\sum_{i=1}^{n} |\, F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} |\int_a^{x_i} f\,(t)dt - \int_a^{x_{i-1}} f\,(t)dt\, |$$

$$= \sum_{i=1}^{n} |\int_{x_{i-1}}^{x_i} f\,(t)dt\, |$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|dt$$

$$= \int_a^b |f(t)|dt$$

Thus

$$V_a^b F \leq \int_a^b |f(t)|dt < \infty$$

Hence F is of bounded variation.

Lemma 3.2.8. If f is integrable on $[\mathrm{a}, \mathrm{b}]$ and

$$\int_a^x f\,(t)dt = 0$$

for all $x \in$ [a,b], then $f = 0$ almost everywhere in [a,b].

Proof. Suppose $f > 0$ on a set E of positive measure. Then there is a closed set $F \subset E$ with $m\,F > 0$. Let O be the open set such that

$$O = (a, b) - F$$

Then either $\int_a^b f \neq 0$ orelse

$$0 = \int_a^b f = \int_F f + \int_O f$$

$$= \int_F f + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f\,(t)dt \qquad (1)$$

Because O is the union of a countable collection $\{(a_n, b_n)\}$ of open intervals. But, for each n,

$$\int_{a_n}^{b_n} f\,(t)dt = \int_a^{b_n} f\,(t)dt - \int_a^{a_n} f\,(t)dt$$

$$= F(b_n) - F(a_n) = 0 \qquad \text{(by hypothesis)}$$

Therfore, from (1), we have

$$\int_F f = 0$$

But since $f > 0$ on F and > 0 , we have $\int_F f > 0$.

We thus arrive at a contradiction. Hence $f = 0$ almost everywhere.

Lemma 3.2.9. If f is bounded and measurable on $[a, b]$ and

$$F(x) = \int_F^x f(t)dt + F(a),$$

then $F'(x) = f(x)$ for almost all x in [a,b].

Proof. We know that an integral is of bounded variation over [a,b] and so $F'(x)$ exists for almost all x in [a,b]. Let $|f| \leq K$. We set

$$f_n(x) = \frac{F(x+h) - F(x)}{h}$$

With $h = \frac{1}{n}$. Then we have

$$f_n(x) = \frac{1}{h}[\int_a^{x+h} f\,(t)dt - \int_a^{x} f(t)dt]$$

$$= \frac{1}{h}\int_x^{x+h} f\,(t)dt$$

$$\Rightarrow |f_n(x)| = |\frac{1}{h}\int_x^{x+h} f\,(t)dt|$$

$$\leq \frac{1}{h}\int_x^{x+h} |\,f(t)|dt \leq \frac{1}{h}\int_x^{x+h} K\,dt$$

$$= \frac{K}{h}.h = K$$

Moreover,

$$f_n(x) \to F'(x) \text{ a.e.}$$

Hence by the theorem of bounded convergence, we have

$$\int_a^c F'(x)dx\ = lim \quad \int_a^c f_n(x)dx\ = lim_{h\to 0}\frac{1}{h}\int_a^c[F(x+h) - F(x)]\,dx$$

$$= lim_{h\to 0}[\frac{1}{h}\int_{a+c}^{c+h} F(x)dx\ \text{-}\ \frac{1}{h}\int_a^c F(x)dx]$$

$$= lim_{h\to 0}[\frac{1}{h}\int_c^{c+h} F(x)dx\ \text{-}\ \frac{1}{h}\int_a^{a+h} F(x)dx]$$

$$= F(c) - F(a)$$ (since F is continuous)

$$= \int_a^c f(x)dx$$

Hence

$$\int_a^c[F'(x)\text{-}\,f(x)]dx\ = 0$$

for all c ϵ [a,b], and so

$$F'(x) = f(x) \ a.e.$$

by using the previous lemma.

Now we extend the above lemma to unbounded functions.

Theorem 3.2.10. Let f be an integrable function on [a,b] and suppose that

$$F(x) = F(a) + \int_a^x f(x)dt$$

Then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Proof. Without loss of generality we may assume that $f \geq 0$ (or we may write "From the definition of integral it is sufficient to prove the theorem when $f \geq 0$).

Let f_n be defined by $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ if $f(x) > n$. Then $f - f_n \geq 0$ and so

$$G_n(x) = \int_a^x (f - f_n)$$

is an increasing function of x, which must have a derivative almost everywhere and this derivative will be non- negative. Also by the above lemma, since f_n is bounded (by n), we have

$$\frac{d}{dx}\left(\int_a^x f_n\right) = f_n(x) \ a.e. \qquad \text{(i)}$$

Therefore,

$$F'(x) = \frac{d}{dx}\left(\int_a^x f\right) = \frac{d}{dx}\left(G_n + \int_a^x f_n\right)$$

$$= \frac{d}{dx} G_n + \frac{d}{dx}\int_a^x f_n$$

$$\geq f_n(x) \ a.e. \qquad \text{(using (i))}$$

Since n is arbitrary, making $n \to m$ we see that $F'(x) \geq f(x)$ a.e.

Consequently,

$$\int_a^b F'(x)dx \ \geq \int_a^b f(x)dx$$

$$= F(b) - F(a) \text{ (using the hypothesis of the theorem)}$$

Also since $F(x)$ is an increasing real valued function on the interval [a,b], we have

$$\int_a^b F'(x)dx \le F(b) - F(a) = \int_a^b f(x)dx$$

Hence

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

$$\Rightarrow \int_a^b [F'(x) - f(x)dx = 0$$

Since $F'(x) - f(x) \ge 0$, this implies that $F'(x) - f(x) = 0$ a.e. and so $F'(x) = f(x)$ a.e.

Definition 3.2.11 A real- valued function f defined on [a,b] is said to be absolutely continuous on [a,b] if, given $\varepsilon > 0$ there is a $\delta > 0$ such that

Absolute Continuity

$$\sum_{i=1}^{n} | f(x_i') - f(x_i)| < \in$$

for every finite collection $\{(x_i, x_i')\}$ of non- overlapping intervals with

$$\sum_{i=1}^{n} | x_i' - x_i| < \delta$$

An absolutely continuous function is continuous, since we can take the above sum to consist of one term only. Moreover, if

$$F(x) = \int_a^x f(t)dt,$$

then

$$\sum_{i=1}^{n} | F(x_i{}') - F(x_i)| = \sum_{i=1}^{n} | \int_a^{x_i{}'} f\,(t)dt - \int_a^{x_i} f\,(t)dt|$$

$$= \sum_{i=1}^{n} | \int_{x_i}^{x_i{}'} f\,(t)dt\, |$$

$$\leq \sum_{i=1}^{n} \int_{x_i}^{x_i{}'} | f(t)x\, |dt = \int_E \ |f(t)|dt,$$

where E is the set of intervals $(x, x_i{}')$

$$\to 0 \ as \sum_{i=1}^{n} | x_i{}' - x_i| \to 0.$$

The last step being the consequence of the result.

"Let $\in > 0$. Then there is a $\delta > 0$ such that for every measurable set $E \subset [a, b]$ with $mE < \delta$, we have

$$\int_A \ | f \,| < \in".$$

Hence every indefinite integral is absolutely continuous.

Lemma 3.2.11. If f is absolutely continuous on [a,b], then it is of bounded variation on [a,b].

Proof. Let δ be a positive real number which satisfies the condition in the definition for $\in = 1$. Select a natural number

$$n > \frac{b-a}{\delta}$$

Consider the partition $I = \{x_0, x_1, \dots, x_n\}$ of [a,b] defined by

$$x_i = x_0 + \frac{i(b-a)}{n}$$

for every $i = 0,1\, ,\dots, n$. Since $|x_i - x_{i-1}| < \delta$, it follows that

$$V_{x_{i-1}}^{x_i}(f) < 1$$

This implies

$$V_a^b(f) = \sum_{i=1}^{n} V_{x_{i-1}}^{x_i}(f) < n$$

Hence f is of bounded variation.

Corollary 3.2.12. If f is absolutely continuous, then f has a derivative almost everywhere.

Lemma 3.2.13. If f is absolutely continuous on [a,b] and $f(x) = 0$ a.e., then f is constant.

Proof. We wish to show that $f(a) = f(c)$ for any c ϵ [a,b]. Let $E \subset (a,c)$ be the set of measure $c - a$ in which $f'(x) = 0$, and let ∈ and η be arbitrary positive numbers. To each x in E there is an arbitrarily small interval $[x, x + h]$ contained in [a,c] such that

$$|f(x + h) - f(x)| < \eta h$$

By Vitali Lemma we can find a finite collection $\{[x_k, y_k]\}$ of non- overlapping intervals of this sort which cover all of E except for a set of measure less than δ, where δ is the positive number corresponding to ∈ in the definition of the absolute continuity of f . If we label the x_k so that $x_k \le x_{k+1}$, we have (or if we order these intervals so that)

$$a = y_0 \le x_1 < y_1 \le x_2 < \cdots . < y_n \le x_{n+1} = c$$

and

$$\sum_{k=0}^{n} | x_{k+1} - y_k| < \delta$$

Now

$$\sum_{k=0}^{n} | f(y_k) - f(x_k)| \le \eta \sum_{k=1}^{n} (y_k - x_k)$$

$$< \eta(c - a)$$

by the way to intervals $\{[x_k, y_k]\}$ were constructed, and

$$\sum_{k=0}^{n} | f(x_{k+1}) - f(y_k)| < \in$$

by the absolute continuity of f . Thus

$$|f(c) - f(a)| = | \sum_{k=0}^{n} [f(x_{k+1}) - f(y_k)] + \sum_{k=1}^{n} [f(y_k) - f(x_k)] |$$
$$\leq \in + \eta(c - a)$$

Since $\in$ and η are arbitrary positive numbers, $f(c) - f(a) = 0$ and so $(c) = f(a)$. Hence f is constant.

Theorem 3.2.14. A function F is an indefinite integral if and only if it is absolutely continuous.

Proof. We know that if F is an indefinite integral then F is absolutely continuous. Suppose on the other hand that F is absolutely continuous on [a,b]. Then F is of bounded variation and we may write

$$F(x) = F_1(x) - F_2(x),$$

where the functions F_i are monotone increasing. Hence $F'(x)$ exists almost everywhere and

$$|F'(x)| \leq F_1'(x) + F_2'(x)$$

Thus

$$\int |F'(x)| dx \leq F_1(b) + F_2(b) - F_1(a) - F_2(a)$$

and $F'(x)$ is integrable. Let

$$G(x) = \int_a^x F'(t) dt$$

Then G is absolutely continuous and so is the function $f = F - G$. But by the above lemma since

$$f'(x) = F'(x) - G'(x) = 0 \ a.e.,$$

we have f to be a constant function. That is,

$$F(x) - G(x) = A \text{ (constant)}$$

or

$$F(x) = \int_a^x F'(t)dt = A$$

or

$$F(x) = \int_a^x F'(t)dt + A$$

Taking $x = a$, we have $A = F(a)$ and so

$$F(x) = \int_a^x F'(t)\, dt + F(a)$$

Thus $F(x)$ is indefinite integra1 of $F'(x)$.

Corollary 3.2.15. Every absolutely continuous function is the indefinite integral of its derivative.

3.3 Check Your Progress

Q.1. Let f be a function defined by $f(x) = \begin{cases} x \sin\frac{1}{x}; & for\ x \neq 0 \\ 0; & x = 0 \end{cases}$. Find

$D^+ f(0), D_+ f(0), D^- f(0)$ *and* $D_- f(0)$

Q.2. If the function $f(x)$ assumes its maximum at c, show that

$D^+ f(c) \leq 0, D_- f(0) \leq 0.$

Fill in the blanks in following question.

Q.3. Let $f: [a, b] \to R$ be a fuction which satisfies Lipschitz condition then show

that it is absolutely continuous.

Sol. Let $f: [a, b] \to R$ be the given function which satisfies Lipschitz

Condition, i.e., for any constant M.

$$| f(x) - f(y) | \leq M | x - y |, \forall\, x, y \in [a, b] \qquad \ldots(1)$$

Now for given $\varepsilon > 0$, take $\delta = \dfrac{\varepsilon}{M}$

Where, $(]a_i, b_i[) : i \in N$ is a finite non-overlapping collection of pairwise disjoint intervals. Then from (1)

$$\Rightarrow \sum_{r=1}^{n} | f(b_r) - f(a_r) | \le M \sum | b_r - a_r | < M \frac{\varepsilon}{M} = \varepsilon$$

$$\Rightarrow \sum_{r=1}^{n} | f(b_r) - f(a_r) | < \varepsilon$$

Q.4. Show that if $f'(x)$ exists and is ------------------ on $[a,b]$ then f is of bounded variation on $[a,b]$.

Sol. It is given that $f'(x)$ exists and bounded so that there exist $m > 0$ such that $|f(x)| \le m$ on $[a,b]$.

$$\Rightarrow \left| \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right| \le m$$

$$\Rightarrow \left| f(x_i) - f(x_{i-1}) \right| \le \ldots\ldots\ldots\ldots\ldots\ldots$$

$\Rightarrow \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \le m \sum \left(x_i - x_{i-1} \right) = m(b-a)$, for any patition P of $[a,b]$.

Hence, $f \in BV[a,b]$.

Q.5. Show that the function $f(x) = \begin{cases} x^p \sin \frac{1}{x}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$, $p \ge 2$ is of bounded variation on [0,1].

3.4 <u>Summary</u>

In this chapter we discuss functions of bounded variation and three related topics. Begin by defining the variation of a function and what it means for a function to be of bounded variation, then develop some properties of functions of bounded variation. Consider algebraic properties as well as more abstract properties such as realizing that every function of bounded variation can be written as the difference of two increasing functions. Examine the definition of the Riemann-Stieltjes integral and see when functions of bounded variation are Riemann-Stieltjes integrable.

3.5 Keywords

Absolute Continuous Function, Function of Bounded Variation, Lebesgue Point, Vittali's Cover of a Set, Indefinite Integral, Convex Function.

3.6 Self-Assessment Test

1. Show that the function f defined on [0,1] by

$$f(x) = \begin{cases} x\cos\dfrac{\pi x}{2}; & for\ 0 < x \le 1 \\ 0; & x = 0 \end{cases}$$

is continuous but not of bounded variation on [0,1].

2. Let f be a function of bounded variation, then show that *f(x)* exists a.e.

3. Show that if $F(x) = F(a) + \int_a^x f(t)dt$, $then\ F'(x) = f(x)$ a.e.

4. If f is integrable on [a,b] and $\int_a^x f(t)dt = 0$ for all $x \in$ [a,b]. Show that $f(t)$= 0 a.e. in [a,b].

5. Show that every increasing function on [a,b] is of bounded variation and every function of bounded variation on [a,b] is almost everywhere differentiable on [a,b]

3.7 Answers to check your progress

A.1. By the definition of Dini's derivatives, we have

$$D^{+} f(0) = \overline{\lim_{h\to 0^{+}}} \frac{f(0+h)-f(0)}{h} = \overline{\lim_{h\to 0^{+}}} \left\{ \frac{h\sin\frac{1}{h} - 0}{h} \right\} = \overline{\lim_{h\to 0}} \sin\frac{1}{h} = 1$$

$$D_{+} f(0) = \underline{\lim_{h\to 0^{+}}} \frac{f(0+h)-f(0)}{h} = \underline{\lim_{h\to 0}} \sin\frac{1}{h} = -1$$

$$D^{-} f(0) = \overline{\lim_{h\to 0^{-}}} \frac{f(0-h)-f(0)}{-h} = \overline{\lim_{h\to 0^{-}}} \left\{ \frac{-h\sin(-\frac{1}{h}) - 0}{-h} \right\} = \overline{\lim_{h\to 0}} \sin(-\frac{1}{h}) = 1$$

$$D_{-} f(0) = \underline{\lim}_{h \to 0^{-}} \frac{f(0-h) - f(0)}{-h} = \underline{\lim}_{h \to 0} (-\sin \frac{1}{h}) = -1.$$

A.2. It is given that the function $f(x)$ assumes its maximum value at $x = c$. Therefore,

$$f(c+h) \le f(c) \Rightarrow f(c+h) - f(c) \le 0$$

And

$$f(c-h) \le f(c) \Rightarrow f(c-h) - f(c) \le 0$$

which implies that

$$\frac{f(c+h) - f(c)}{h} \le 0 \quad and \quad \frac{f(c-h) - f(c)}{h} \ge 0$$

Hence

$$D^{+} f(c) = \overline{\lim}_{h \to 0^{+}} \frac{f(c+h) - f(c)}{h} \le 0$$

Similarly $D_{-} f(c) \ge 0$

A.3. (i) $\sum_{r=1}^{n} (b_r - a_r) < \delta = \dfrac{\varepsilon}{M}$

(ii) $| f(b_r) - f(a_r) | \le M(b_r - a_r), \forall r$

A.4. (i) Bounded

(ii) $m(x_i - x_{i-1})$

(iii) $V_{ab}(f)$

A.5. We know that the finction $f'(x)$ exists and bounded then f is of bounded variation.

Now, $Rf'(0) = \lim_{n \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{n \to 0} \dfrac{(0+h)^p \sin \frac{1}{h} - 0}{h}$

$$= \lim_{n \to 0} h^{p-1} \sin \frac{1}{h} = 0$$

And $Lf'(0)=\lim\limits_{n\to 0}\frac{f(0-h)-f(0)}{-h}=\lim\limits_{n\to 0}\frac{(-h)^p\sin(-\frac{1}{h})-0}{-h}=0$.

So $Rf'(0)=Lf'(0)=0.$

$\Rightarrow f'(0)=0\Rightarrow f'(x)$ exists.

And $f'(x)=x^p\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)+px^{p-1}\sin\frac{1}{x}$

$$=x^{p-2}\left[px\sin\frac{1}{x}-\cos\frac{1}{x}\right],\ 0<x\le 1$$

$\Rightarrow f'(x)$ is bounded for $0\le x\le 1$.

Here $f(x)$ is of bounded variation on $[0,1]$.

3.8 References/ Suggested Readings

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MAL-512: M. Sc. Mathematics (Measure and Integration Theory)

Lesson No. 4 **Author: Dr. Vizender Singh**

Lesson: Lebesgue L^p-space

Structure:

4.1 Introduction

Many of the classical spaces in analysis consist of measurable functions and most of the important norms on such spaces have been defined by integrals. The Lebesgue L^p-spaces arc among such important classes. A complete understanding of these spaces needs a thorough understanding of the Lebesgue theory of measure and integration which we have developed in the preceding chapters. We are now fully prepared to introduce the L^p spaces. These spaces have remarkable properties and are of enormous importance in analysis as well as its applications.

***L^p*- space**

If f is a measurable function on E, then If L^p is so for each $p,\ 0 < p < \infty$. Designate by L^p(E), the class of all p- integrable functions over E, i.e.,

$$L^p(\mathrm{E})=\{\ f: \int_E\ |f|^p < \infty\ \}$$

Examples 1. Let $E = [0,16]$ and $f: E \to$ R be a function defined by $f(x) = (x)^{-1/4}$. Then $f \in L^1(E)$ but $f \notin L^4$(E).

2. Let $E = \left[0, \frac{1}{2}\right]$ and the function $f: E \to$ R be defined by

$$f(x) = [x \log 2\ (\frac{1}{x})]^{-1}$$

Then $f \in L^1(E)$.

3. Let E =(0, ∞) and the function $f: E \to$ R be defined by

$$f(x) = (1 + x)^{-1/2}.$$

Then L^p(E), for each p, $2 < p < \infty$.

It is easy to verify that $L^p(E)$ is a linear space over R. Indeed, we observe that:

1. $g \in L^p(E) \Rightarrow f + g \in L^p$(E), since

$$|f + g|^p \le 2^p \max\{|f|^p, |g|^p\}$$

$$\leq 2^p \{|f|^p + |g|^p\}.$$

2. $f \in L^p(E)$ and $\in$ R $\Rightarrow \alpha f \in L^p(E)$.

Furthermore, if$f \in L^p(E)$, then the inequalities

$$\begin{cases} 0 \leq f^+ \leq |f| \\ 0 \leq f^- \leq |f| \end{cases}$$

imply that f^+, f^- and $|f|$ also in $L^p(E)$.

In order to define $L^\infty(E)$, let f be a real- valued and measurable function on a set E with $m(E) > 0$. A real number M is said to be an essential bound for the function if

$$|f(x)| \leq M \text{ a. e. on } E.$$

A function is said to be essentially bounded if it has an essential bound. In other words, a function defined on E is essentially bounded if it is bounded except possibly on a set of measure zero. The essential supremum off on E is defined by

$$\text{ess sup } |f(x)| = \inf \{\text{M} : |f(x)| \leq M \text{ a.e. on } E\},$$

or equivalently

$$\text{ess sup } |\text{f(x)}| = \inf \{\text{M} : \text{m}(\{x \in \text{E} : |\text{f(x)}| > M \}) = 0\}.$$

If f does not have any essential bound, then its essential supremum is defined to be ∞.

Let us designate by $L^\infty(E)$ the class of all those measurable functions defined on E which are essentially bounded on E, i.e.

$$L^\infty(E) = \{f \text{: ess sup } |f| < \infty\}.$$

It is not difficult again to verify that $L^\infty(E)$ is a linear space over R.

Examples 1. Every bounded function on E in $L^\infty(E)$.

2. The function : $[a, b] \rightarrow$ R given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ \infty & \text{if } x \text{ is rational} \end{cases}$$

is in $L^{\infty}[a, \mathrm{b}]$.

Let us now define a function $||\,.\,||_p : L^p(E) \to$ R, $0 < p \leq \infty$, as follows:

$$||\,f||_p = \left(\int_E \ |f|^p\right)^{1/p}, 0 < p < \infty$$

$$||f||_\infty = \text{ess sup } |f|.$$

Lemma. 4.2.1. Let $\in L^{\infty}(E)$. Then:

(a) $|f(x)| \leq ||f||_\infty a.e.$ on E.

(b) $||f||_\infty = \sup\{M : m(\{x \in E : |f(x)| \geq M\}) \neq 0\}$.

Proof. (a) Let $\eta = \|f\|_\infty$. Note that

$$\{x \in E : |f(x)| \geq \eta\} = \bigcup_{n=1}^{\infty}\{x \in E : |f(x)| > \eta + \frac{1}{n}\}.$$

Since the union of a countable collection of sets of measure zero has measure zero, we have

$$m(\{x \in E : |f(x)| \geq \|f\|_\infty\}) = 0$$

$$\Rightarrow |f(x)| \leq \|f\|_\infty \text{ a.e. on } E.$$

(b) It is obvious from the definition of $\|f\|_\infty$ on E.

Theorem 4.2.2 Let E be a measurable set with m(E) < ∞. Then $L^{\infty}(E) \subset L^p(E)$. For each p with $1 \leq p < \infty$ Furthermore, if $f \in L^{\infty}(E)$

$$\|f\|_\infty = \lim_{n\to\infty} \|f\|_p.$$

Proof. Let $f \in L^{\infty}(E)$ and $\eta = \|f\|_\infty$. Then

$$|f(x)|^p \leq \eta^p \text{ a.e. on } E$$

$$\Rightarrow \int_E \ |f(x)|^p \leq \eta^p . m(E) .$$

Therefore, $f \in L^p(E)$ and consequently $L^\infty(E) \subset L^p(E)$. Furthermore, we note that

$$\|f\|_p \leq \eta[m(E)]^{1/p}.$$

Since $[m(E)]^{1/p} \to 1$ as p $\to \infty$, we obtain

$$\lim_{p\to\infty} \sup \|f\|_p \leq \eta$$

On the other hand, let $|f(x)| \geq \alpha$ on a set F with $m(F) > 0$. Then

$$\|f\|_p \geq \eta[m(F)]^{1/p}$$

$$\Rightarrow \lim_{p\to\infty} \inf \|f\|_p \geq \alpha.$$

This verifies that (cf. Lemma 2.3)

$$\sup\{\alpha : m(\{x \in E : |f(x)| \geq \alpha\}) \neq 0\} \leq \lim_{p\to\infty} \inf \|f\|_p.$$

Henoe. the result fonows from

$$\eta \leq \lim_{p\to\infty} \inf \|f\|_p \leq \lim_{p\to\infty} \sup \|f\|_p$$

Corollary 4.2.3. $L^\infty(E) \subset \bigcap_{p\geq 1} L^p(E)$, and the norm on $L^\infty(E)$ is equal to the limit of $\|f\|_p$ as $p \to \infty$ provided $m(E) < \infty$.

The Holder and Minkowski Inequalities

Lemma 4.2.4. Let $0 < \lambda < 1$. Then

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$$

holds good for every pair of nonnegative real numbers α and β with equality only if $\alpha = \beta$.

Proof. The inequality is trivial if either $\alpha = 0$ or $\beta = 0$. Hence, assume that $\alpha > 0$ and $\beta > 0$. Consider the function φ defined for a nonnegative real number t by

$$\varphi(t) = (1-\lambda) + \lambda t - t^{-\lambda}.$$

Then,

$$\varphi'(t) = \lambda\left(1 - t^{\lambda-1}\right).$$

and so $t = 1$ is the only possible point for the extrema of φ. It is verified that φ attains its maximum at $t = 1$. Thus

$$\varphi(t) \geq \varphi(1) = 0$$
$$\Rightarrow 1 - \lambda + \lambda t \geq t^{\lambda}.$$

Setting $t = \alpha/\beta$, the inequality follows. The equality holds good only for $t = 1$ which is obtained only if $\alpha = \beta$.

Riesz-Hölder Inequality

Theorem 4.2.5 Let p and q be nonnegative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq ||f||_p ||g||_q.$$

Equality holds iff, for some non zero constant A and B, we have A$|f|^p$= B$|g|^q$ a.e.

Proof. When $p = 1$, then $q = \infty$, and the inequality is available trivially in this case. Indeed, if $||g||_\infty = M$, then $|g| \leq M$ a.e., and so

$$|fg| \leq M|f| \text{ a.e.}$$

Thus $fg \in L^1$, and by integrating, we get

$$\int |fg| \leq M \int |f| \leq ||f||_1 ||g||_\infty.$$

Now, assume that $1 < p < \infty$ and consequently $1 < q < \infty$. The inequality is trivial if either $f = 0$ a.e. or $g = 0$ a.e. So assume $f \neq 0$ a.e. and $g \neq 0.$ a.e. so assume that $f \neq 0$ and $g \neq 0$ This gives that $||f||_p > 0$ and $||g||_q > 0$. Now, applying lemma 3.1 with

$$\begin{cases} \lambda = \dfrac{1}{p} \\ \alpha = (\dfrac{|f(t)|}{||f||_p})^p \\ \beta = (\dfrac{|g(t)|}{||g||_q})^q \end{cases}$$

we obtain

$$\frac{|f(t)}{||f||_p}\frac{g(t)|}{||g||_q} \leq \frac{\mathrm{I}}{p}\frac{|f(t)|^p}{||f||_p} + \frac{1}{q}\frac{|g(t)|^q}{||g||_q} \tag{2}$$

This gives that $fg \in L^1$, and by integrating, we find that

$$\frac{\int \ |fg|}{||f||_p||g||_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\int \ |fg| \leq ||f||_p||g||_q. \tag{3}$$

Equality in (2) would occur if $\alpha = \beta$, and, consequently, in (3) if $\alpha = \beta$ holds a.e. In other words, if

$$||g||_q{}^q|f(t)|^p = ||f||_p{}^p|g(t)|^q$$

Note. The inequality (3) is homogeneous, i.e., it holds for *af* and *bg* with a, $b \in$ R whenever it is so for f and g.

Riesz-Minkowski Inequality

Theorem 4.2.6 Let $1 \leq p \leq \infty$. Then for every pair *f*, $g \in L^p$, the following inequality holds:

$$||f + g||_p \leq ||f||_p + ||g||_p$$

Proof. The case for p = 1 is straight forward. If $p = \infty$, we note that

$$\begin{cases} |f| = ||f||_\infty \\ |g| = ||g||_\infty \end{cases}$$

$$\Rightarrow |f + g| \leq ||f||_\infty + ||g||_\infty$$

and hence the result follows in this case also. Thus, we now assume that

$1 < p < \infty$.

Since L^p is a linear space, $f + g \in L^p$. Also, we have

$$\int |f+g|^p \leq \int |f+g|^{p-1}|f| + \int |f+g|^{p-1}|g|\,.$$

Let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, since $(p-1)q = p$, observe that

$$\int (|f+g|^{p-1})^q = \int |f+g|^p,$$

and therefore $|f+g|^{p-1} \in L^p$. As such, by Theorem 3.2, both

$$|f+g|^{p-1}|f| \text{ and } |f+g|^{p-1}|g|$$

belong to L^1, and the Riesz- Holder Inequality leads to

$$\int |f+g|^{p-1}|f| \leq ||f||_p||\,(|f+g|^{p-1})||_q$$

and

$$\int |f+g|^{p-1}|g| \leq ||g||_p||\,(|f+g|^{p-1})||_q$$

But

$$||\,(|f+g|^{p-1})||_q = \left(\int |f+g|^{(p-1)q}\right)^{1/q} = (||f+g||_p)^{p/q}$$

Since $(p-1)q = p$. Hence

$$\int |f+g|^p \leq (||f||_p + ||g||_p)(||f+g||_p)^{p/q}$$

If $||f+g||_p$ is nonzero finite, the result follows by dividing both sides by $||f+g||_p^{p/q}$. In case $||f+g||_p = 0$, there is nothing to prove while in case $||f+g||_p = \infty$, we either have $||f||_p = \infty$ or $||g||_p = \infty$ in view of the relation $|f+g| \leq |f| + |g|$, and the result is obviously true again.

Remark. Equality holds in Theorem 3.3 if and only if one of the functions f and g is a multiple of the other.

Note. For the special case $p = q = 2$, the inequality in Theorem 3.2 is known as the Cauchy Schwarz Inequality. Cauchy (1821) first proved the inequality (Cauchy's inequality) for square summable sequences. Indeed, if $\{a_n\}$ and

$\{b_n\}$ are sequences of numbers (real or complex) such that $\sum_{n=1}^{\infty}|a_n|^2 < \infty$ and $\sum_{n=1}^{\infty}|b_n|^2 < \infty$, then

$$\sum_{n=1}^{\infty}|a_n b_n| \leq \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2}$$

This inequality was generalized to integrals by A. Schwarz (1885). However, the same generalization had already been obtained by a Russian mathematician Victor Bunyakovsky (1859) which remained unnoticed by Western mathematicians. O. Hölder (1989) extended Cauchy's inequality for the general values of$p$ and q by establishing, for sequences $\{a_n\}$ and $\{b_n)$ with $\sum_{n=1}^{\infty}|a_n|^{\mathrm{p}} < \infty$ and $\sum_{n=1}^{\infty}|b_n|^{\mathrm{p}} < \infty$, that

$$\sum_{n=1}^{\infty}|a_n b_n| \leq \left(\sum_{n=1}^{\infty}|a_n|^{\mathrm{p}}\right)^{1/p}\left(\sum_{n=1}^{\infty}|b_n|^{\mathrm{q}}\right)^{1/q}$$

where $(1/\mathrm{p}) + (1/\mathrm{q}) = 1$. The latter inequality is then generalized to the case of integrals by F. Riesz (1910).

Completeness of L^p-spaces

We are now prepared to show that $\|\cdot\|_p$ defines a norm on L^p. In fact, for $1 \leq p \leq \infty$, the function $\|\cdot\|_p : L^p \to \mathrm{R}$ satisfies the following conditions:

1. $||f||_p \geq 0$
2. $||f||_p = 0$ if and only if $f = 0$ a.e.
3. $||\alpha f\|_p = |\alpha|\|f\|_p$, α a real number
4. $||f + g||_p \leq ||f\|_p + \|g\|_p$.

Conditions (1) and·(3) are immediate from the definition of $||.\|_p$; condition while condition (4) is available from the Riesz-Minkowski Inequality.

Unfortunately the definition of $||\cdot||$ on L^p fails to satisfy the norm requirement that $\|f\|_p = 0 \Rightarrow f = 0$. As such, $\|\cdot\|_p$ is not a norm on L^p. However, to avoid this difficulty, we do not distinguish between equivalent functions, i.e. the functions that are equal almost everywhere. In that situation, we regard the space L^p consisting of equivalence classes of functions; for example, 0 will represent the class of functions each of which is equivalent to zero. Thus $\|\cdot\|_p$ now defines a norm on, regarded as the space of equivalence classes, and therefore, L^p becomes a normed space. If one refers to L^p as a normed space, it is in reality the space of equivalence classes $\tilde{f}$ of functions. But this should not pose any problem since, for any $\tilde{f} \in L^p$, the norm is given by

$$||\tilde{f}|| = \|g\|_p,$$

where g is any function in the equivalence class $\tilde{f}$, and this definition does not depend on the choice of the function g in the class $\tilde{f}$. In actual practice, the equivalence classes are relegated to the background, and the elements of L^p are thought of as functions, where any two functions are regarded identical if they are equivalent.

The norm $\|\cdot\|_p$ on L^p induces in a natural way a metric don it given by

$$d(f, g) = \|f - g\|_p.$$

Riesz-Fischer Theorem

Theorem 4.2.7. The normed spaces $L^p, 1 \leq p \leq \infty$, are complete.

Proof. To prove the result for the case $p = \infty$, let $\{f_n\}$ be a Cauchy sequence in L^∞. Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

except on a set $A_{n,m} \subset [a, \mathrm{b}]$ with $m(A_{n,m}) = 0$. If $A = \bigcup_{n,m} A_{n,m}$, then $m(A) = 0$ and

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

,for all n and m, and for all $\mathrm{x} \in [a, \mathrm{b}] - A$. Therefore, it follows that $\{f_n\}$ converges uniformly to a bounded limit f outside A and the result is proved by observing the fact that the convergence in L^∞ is equivalent to uniform convergence outside a set of measure zero. Now, assume $1 \leq p < \infty$. It is enough to show that each absolutely summable sequence in L^p is summable in

L^p to some element in L^p.

Let $\{f_n\}$ be a sequence in L^p with

$$\sum_{n=1}^{\infty} ||f_n||_p = M < \infty.$$

Define a sequence $\{g_n\}$ of functions, where

$$g_n(x) = \sum_{k=1}^{n} |\, f_k(x)|.$$

Observe, for each x that $\{g_n(x)\}$ is an increasing sequence of (extended) real numbers and as such must converge to an extended real number $g(x)$ (say), i.e., $g_n(x) \to g(x)$, for each $x \in [a, \mathrm{b}]$. Since the functions g_n are measurable, the function g is so. Also, in view of Riesz- Minkowski Inequality, we note that

$$||g_n||_p \leq || \textstyle\sum_{k=1}^{\infty} |f_k(x)| \,||_p$$

$$\leq \sum_{k=1}^{\infty} ||f_k||_p < M$$

$$\Rightarrow \int \quad |g_n|^p \leq M^p(b-a).$$

Therefore, since $g_n \geq 0$, by Fatou' s Lemma, we have

$$\int \quad g^p \leq Mp(b-a).$$

This verifies that g^p is integrable, and hence $g(x)$ is finite a.e. on $[a, b]$ Thus, we find that, for each x for which $g(x)$ is finite, the sequence therefore must be summable to a real number (say) $s(x)$. Let us set $\{f_n(x)\}$ is an absolutely summable sequence of real numbers,and

$$s(x) = 0,$$

for those x where $g(x) = \infty$ a.e. of the partial sums. Then the function s so defined is the limit

$$s_n(x) = \sum_{k=1}^{n} f_k\,(x).$$

i.e., $s_n(x) \to s(x)$ a.e. Hence$s$ is a measurable function. Further

$$|s_n(x)| \leq \sum_{k=1}^{n} |\,f_k(x)|$$

$$= g_n(x)$$

$$\leq g(x),$$

which implies $|s(x)| \leq g(x)$. Therefore, $s \in L^p$ since $g \in L^p$, and

$$|s_n(x) - s(x)|^p \leq 2^p\big(g(x)\big)^p.$$

But $2^p g^p$ is an integrable function and $|s_n(x) - s(x)|^p \to 0$ a.e. So, by the Lebesgue Dominated Convergence Theorem, we have

$$\int\ |s_n - s|^p \to 0$$

$$\Rightarrow ||s_n - s||_p \to 0$$

Hence the sequence $\{f_n\}$ is summable in L^p and has the sum s in L^p. This proves the theorem completely.

Remark 4.2.8. It is worthwhile to point out that the space $C[a, b]$ is a normed space under the norm $||.||_p$but not a Banach space. However, it can be noted that the completion of $C[a, b]$ under $||.||_p$ is $L^p[a, b]$, for each p with 1$< p <$ ∞.

Theorem **4.2.7(A**) If $0 < p < l$, then L^p is a complete metric space with metric ρ defined by

$$\rho(f, g) = ||f - g||_p^p,$$

$\forall\, f, g \in L^p$.

<u>Convergence of L^p-spaces</u>

Definition 4.2.9. A sequence $\{f_n\}$ of functions in L^p, $1 \leq p \leq \infty$, is said to converge to $f \in L^p$ in the norm of L^p iff for each $\varepsilon > 0$, there exists a positive integer N such that $||f_n - f||_p \to 0$. This type of convergence is usually referred to as convergence in the mean of order p when $1 \leq p <\infty$ and nearly uniform convergence when $p = \infty$.

As usual, one may define a p- mean Cauchy sequences of functions in L^p.

Theorem 4.2.10. Let $\{f_n\}$ be a sequence in L^p which converges in the mean of order p to f in L^p. Then:

a. If the sequence $\{f_n\}$ converges In the mean oforder p to g, then

$$f = g \text{ a.e.}$$

b. The sequence $\{f_n\}$ is p- mean Cauchy sequence.

c. $\lim_{p\to\infty} ||f_n||_p = ||f||_p$, in particular the sequence $\{f_n\}$ is bounded with respect to the norm $||f||_p$.

Remark 4.2.11. The converse of (b) is true, since L^p is a complete normed space, while that of (c) need not be true.

In general, convergence in the mean implies nor is implied by the pointwise convergence or convergence almost everywhere

Examples

1. For each $n \in$ N, consider the function f_n: (0, 1) → R given by

$$f_n = \begin{cases} n & \text{if} \quad 0 < x \leq 1/\text{n} \\ 0 & \text{if} \quad 1/\text{n} < x < 1. \end{cases}$$

It can easily be ver fied that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in (0,1)$ while $||f_n||_p \to \infty$ as $n \to \infty$ for $p > 1$.

2. For each $n \in \mathbf{N}$, consider the function $f_n : \mathbf{R} \to \mathbf{R}$ given by

$$f_n = x_{[n,n+1]}.$$

Note that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in$ R. On the other hand

$$f_n \in L^p,\ 1 \leq p \leq \infty$$

$$\Rightarrow ||f_n||_p = 1, \forall n \in \mathbf{N}$$

$$\Rightarrow\ ||f_n||_p \nrightarrow 0 \text{ as } n \to \infty \text{ for any } p,\ 1 \leq p \leq \infty.$$

Towards the relationship between pointwise convergence and convergence in the mean of order p, we prove that the following results

Theorem 4.2.12. Let $\{f_n\}$ be a sequence in L^p, $1 \le p < \infty$, such that $f_n \to f$ a.e. and that $f \in L^p$. If $\lim_{n\to\infty} ||f_n||_p = ||f||_p$ then $lim_{n\to\infty} ||f_n - f||_p = 0$.

Proof. We assume, without any loss of generality, that each $f_n \ge 0$ a.e. so that f is also so since the result in the general case follows by considering

$$f = f^+ - f^-.$$

For any pair of nonnegative real numbers a and b, we have

$$|a-b|^p \le 2^p \ \ |a|^p + |b|^p\ , \ 1 \le p < \infty.$$

Taking $a = f_n$ and $b = f$, we get

$$2^p\ (|f_n|^p + |f|^p - \left|f_{n.} - f\right|^{\mathrm{p}} \ge 0 \text{ a.e.}$$

Thus, using Fatou's Lemma and tho hypothesis, we get

$$2^{\mathrm{p}} \int |f|^{\mathrm{p}} = \int \lim_{\mathrm{n}\to\infty} [2^{\mathrm{p}-1}(|\mathrm{f_n}|^{\mathrm{p}} + |f|^{\mathrm{p}}) - \left|f_{n.} - f\right|^{\mathrm{p}}]$$

$$\le \lim_{\mathrm{n}\to\infty} \inf \int [2^{\mathrm{p}-1}(|f_n|^p + |f|^{\mathrm{p}}) - \left|f_{n.} - f\right|^{\mathrm{p}}]$$

$$= 2^{\mathrm{p}-1} \lim_{\mathrm{n}\to\infty} \int |f_n|^p + 2^{\mathrm{p}-1} \int |f|^{\mathrm{p}} + \lim_{\mathrm{n}\to\infty} \inf(-\int \left|f_{n.} - f\right|^{\mathrm{p}})$$

$$= 2^{\mathrm{p}} \int |f|^{\mathrm{p}} - \lim_{\mathrm{n}\to\infty} \sup \int \left|f_{n.} - f\right|^{\mathrm{p}}.$$

Since $\int |f|^{\mathrm{p}} < \infty$, it follows that

$$\lim_{\mathrm{n}\to\infty} \sup \int \left|f_{n.} - f\right|^{\mathrm{p}} \le 0.$$

Therefore

$$\lim_{\mathrm{n}\to\infty} \sup \int \left|f_{n.} - f\right|^{\mathrm{p}} = \lim_{\mathrm{n}\to\infty} \inf \int \left|f_{n.} - f\right|^{\mathrm{p}} = 0,$$

so that

$$\lim_{\mathrm{n}\to\infty} \int \left|f_{n.} - f\right|^{\mathrm{p}} = 0$$

Hence

$$\lim_{\mathrm{n}\to\infty} ||f_{n.} - f||_p = 0$$

Bounded Linear Functional on L^p Spaces

Let p and q be two conjugate exponents. If $\in L^q$, $1 \leq q \leq \infty$, it follows from the Riesz- Hölder Inequality that $f \cdot g \in L^1$ for each $f \in L^p$. As such, for a fixed $g \in L^q$, one can define a function $F_g: L^p \to$ R by

$$F_g(f) = \int \ f\, g.$$

Clearly, F_g is a linear functional on the Banach space L^p. In fact, we now prove that it is bounded also.

Theorem 4.2.13. Let p and q $(1 \leq p, q \leq \infty)$ be two conjugate exponents and $g \in L^q$. Then the linear functional defined by

$$F_g(f) = \int \ f\, g.$$

is a bounded linear functional on L^p such that $||F_g|| = ||g||_q$.

Proof. First consider the case when $p = \infty$ and $q = 1$. Observe, by the Riesz- Hölder Inequality that

$$|F_g(f)| = ||g||_1 \ ||f||_\infty, \forall f \in L^p.$$

Thus, it follows that F_g is a bounded linear functional on L^p and that

$$||F_g|| \leq ||g||_1.$$

To prove the reverse inequality, let

$$f = \text{sgn } g.$$

Clearly $f \in L^\infty$ and satisfies $\|f||_\infty = 1$. Therefore

$$F_g(f) = \int \ f\, g = \int \ |g| = ||g||_1$$

$$\Rightarrow ||F_g|| = ||g||_1.$$

Let us now consider the case when $\text{J} < p < \infty$. Again, by the Riesz- Hölder Inequality,

$$\left|F_g(f)\right| \leq \left||g|\right|_q \left||f|\right|_p, \qquad \forall f \in L^p.$$

Therefore, F_g is a linear functional on L^p and satisfies $||F_g|| \leq ||g||_{\mathrm{q}}$. Further, to obtain the reverse inequality, let

$$f = |g|^{\mathrm{q}-1} \operatorname{sgn} g.$$

Clearly, f is a measurable function and $|f| = |g|^{\mathrm{p}(\mathrm{q}-1)} = |g|^{\mathrm{q}}$. This verifies that $f \in L^p$. Also, since

$$f - g = (\ |g|^{\mathrm{q}-1} \operatorname{sgn} g)g = |g|^{\mathrm{q}},$$

$$sgn\ g(x) = \begin{cases} 1 & if\ g(x) \geq 0 \\ -1 & if\ g(x) < 0 \end{cases}$$

we note that

$$F_g(f) = \int \ fg = \int \ |g|^{\mathrm{q}}$$

$$= \ (\textstyle\int \ |g|^{\mathrm{q}})^{1/p}\ (\textstyle\int \ |g|^{\mathrm{q}})^{1/q}$$

$$= \ (\textstyle\int \ |f|^{\mathrm{q}})^{1/p}\ (\textstyle\int \ |g|^{\mathrm{q}})^{1/q}$$

$$\Rightarrow \left||f|\right|_p \left||g|\right|_q$$

$$\Rightarrow ||F_g|| \geq ||g||_q$$

Hence the proof is complete

<u>Riesz Representation Theorem</u>

Theorem 4.2.14 Let F be a bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function g in L^q such that

$$F \ = \int \ f\,g,$$

and that $||F|| = ||g||_q$.

The proof of Theorem 7.2 needs the following lemma.

Lemma 4.2.15. Let g be an integrable function on a, b and K be a constant such that

$$\left|\int \quad f\, g\right| \leq K\; ||f||_p,$$

for all bounded measurable functions f. Then $g \in L^q$ and $||g||_q \leq K$.

Proof. First we consider the case when $p = 1$ and $q = \infty$. Let $\varepsilon > 0$ be given, and let

$$E = \{x \in [a, b] : |g\,(x)| \geq K{+}\varepsilon\}$$

Set f - (sgn g)χ_E. Then is a bounded measurable function such that

$$||f||_1 = m(\mathrm{E}).$$

Therefore

$$\mathrm{Km(E)} = K||f||_1 \geq |\int \quad fg\,|$$

$$= |\int \quad g(\mathrm{sgn}\; g)\chi_E$$

$$= \int_B \;|g| \geq K{+}\varepsilon \;\mathrm{m(E)}$$

$\Rightarrow m$(E) $= 0$, since $\varepsilon > 0$ is arbitrary.

Hence $||g||_\infty \leq K.$

Let us now assume that $1 < p < \infty$. Define a sequence $\{g_n\}$ of bounded measurable functions, where

$$g_n(x) = \begin{cases} g(x) & \text{if} \quad |g(x)| \leq n \\ 0 & \text{if} \quad |g(x)| > n \end{cases}$$

If we get $f_n = |g_n|^{q/p}\mathrm{sgn}\; g_n$, then each f_n is a bounded measurable function such that

$$||f_n||_p = \left(||g_n||_\mathrm{q}\right)^{q/p} \text{ and } |g_n|^q = f_n \cdot g_n \Rightarrow f_n \cdot g.$$

Therefore

$$\left(||g_n||_\mathrm{q}\right)^q = \int f_n g \leq K||f_n||_p = K\left(||g_n||_\mathrm{q}\right)^{q/p}$$

$$\Rightarrow \left(||g_n||_{\mathrm{q}}\right)^{q-q/p} \leq K$$

$$\Rightarrow ||g_n||_{\mathrm{q}} \leq K$$

$$\Rightarrow \int |g_n|^q \leq K^{\mathrm{q}}$$

But $|g_n|^q \to |g|^q$a. e. Thus by Fatou's lemma

$$\int \quad |g|^q \leq \lim_{n\to\infty} \inf \int \quad |g_n| \leq K^{\mathrm{q}}.'$$

we have Hence, $g \in L^{\mathrm{q}}$ and $||g||_{\mathrm{q}} \leq K$.

Proof of theorem $\mathbf{4.2.14}$. We shall obtain the proof of this theorem in four stages

Stage 1. Suppose $f = \chi_t$, t ϵ [a,b] where X_t denotes the characterstic function of the interval $[a$, t].

Set φ(t) $= \chi_t$ Clearly, φ defines a real- valued function on $[a, b]$. We first show that φ is an absolutely continuous fimction. Let $\{(x_i,x_i')\}$ be any finite collection of non-overlapping subintervals of [a,b] such that $\sum_i \ |x_i' - x_i|$<δ.

$$f = \sum_i \ (\chi_{x_i'} - \chi x_i)\,\mathrm{sgn}\{\varphi\left(x_i'\right) - \varphi(x_i)\},$$

Then

$$||f||_p \ < \delta,$$

and so

$$\sum_i |\,\varphi(x_i') - \varphi(x_i)| = F(f)$$

$$\leq \left||F|\right|.\,||f||_p$$

$$< ||F|| \cdot \delta^{1/p}.$$

Thus $\sum_i \ |\varphi(x_i') - \varphi(x_i)| < \varepsilon$, for any finite collection of intervals of total length less than δ ($= \frac{\epsilon^p}{||F||^p}$) and as such p is absolutely continuous on [a, b].

Then by Theorem 4.7.2, there is an integrable function g on [a, b] such that

$$\varphi(\mathrm{t}) = \int_0^t g, \forall\, t \in [a, \mathrm{b}]$$

Therefore

$$\mathrm{F}(\chi_t) = \int g\chi_t.$$

Stage 2. Suppose f is a step fuction. Since every step function on [a, b] can be expressed as a linear combination of the form $\sum \ c_l\, \chi_{ti}$ with the exception of a finite number of points and F is a linear functional, we have

$$F(f) = \int \ g\, f.$$

Stage 3. Suppose is a bounded measurable function on [a, b] then there is a sequence $\{\psi_n\}$ of step functions such that $\psi_n \to f$ a.e. Since the sequence $\{|f - \psi_n|^p\}$ is uniformly bounded and $f - \psi_n \to 0$ a.e., the Bounded Convergence Theorem gives $||f - \psi_n||_p \to 0$ as $n \to \infty$, and therefore

$$|F(f) - F(\psi_n)| = |F(f - \psi_n)|$$
$$\leq \big||F|\big|\, ||f - \psi_n||_p$$
$$\Rightarrow F(f) = \lim_{n\to\infty} F\,(\psi_n)$$
$$= \lim_{n\to\infty} \int \ g\, \psi_n.$$

But, since $|g\psi_{\mathrm{n}}| \leq \eta|g|$, where η is the un form bound of $\{\psi_n\}$ by Lebesgue Dominated Convergence Theorem, we have

$$\int \ f\, g = \lim_{n\to\infty} \int \ g\, \psi_n.$$

Hence $\int \ f\, g = F(f)$, for each bounded measurable function f. Furthermore, since

$$|F(f)| \leq ||F|| \; ||f||_p,$$

we have $g \in L^q$ and $||g||_q \leq ||\mathrm{F}||$,

in view of Lemma.

Stage 4. Finally, suppose $f \in L^p$ is any arbitrary function. Let $\varepsilon > 0$ be given. Then, there is a step function ψ such that $||f - \psi||_p < \varepsilon$. Since ψ is bounded, we have

$$F(\psi) = \int \psi\, g.$$

Therefore

$$|F(f) - \int f\, g| = |F(\mathrm{f}) - \mathrm{F}(\psi) + \int \psi g - \int f\, g|$$

$$\leq |F(f - \psi)| + |\int (\psi - f)\, g|$$

$$\leq ||\mathrm{F}||.\, ||\mathrm{f} - \psi||_p + ||g||_q .\, ||\mathrm{f} - \psi||_p$$

$$< (||F|| + ||g||_q)^{\varepsilon}$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$,

$$F(f) = \int f\, g.$$

The equality $||F|| = ||g||_q$ follows from Theorem 4.1.13

<u>Convex Functions</u>

Definition 4.2.16. A function ϕ defined an open interval (a, b) is said to be convex if for each $x, y \in (a, b)$ and λ, μ such that λ, $\mu \geq 0$ and $\lambda + \mu = 1$, we have

$$\phi(\lambda x + \mu y) \leq \lambda\phi(x) + \mu\phi(y)$$

The end points a, b can take the values $(-\infty,\infty)$ respectively. Ifwe take $\mu = 1 - \lambda, \lambda \geq 0$, then $\lambda + \mu = 1$ and so ϕ will be convex if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1 - \lambda\phi(y)) \qquad \text{(i)}$$

If we take a $< s < t < u < b$ and

$$\lambda = \frac{t-s}{u-s}, \qquad \mu = \frac{u-t}{u-s}, \qquad u = x, \qquad s = y,$$

then

$$\lambda + \mu = \frac{t-s+u-t}{u-s} = \frac{u-s}{u-s} = 1$$

and so (i) reduces to

$$\phi\left[\frac{t-s}{u-s}u + \frac{u-t}{u-s}s\right] \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(s)$$

Or

$$\phi \leq \frac{t-s}{u-s}\phi(u) + \frac{u-t}{u-s}\phi(u) \qquad \text{(ii)}$$

Thus the segment joining $(s, \phi(s))$ and $(u, \phi(n))$ is never below the graph of ϕ. A function ϕ is sometimes said to be convex on (a,b) it for all x, y ϵ (a, b),

$$f\left[\frac{x+y}{2}\right] \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

(Clearly this definition is consequence of major definition taking $\lambda = \mu = \frac{1}{2}$.)

If for all positive numbers λ, μ satisfying $\lambda + \mu = 1$, we have

$$\phi(\lambda x + \mu y) < \lambda\phi(x) + \mu\phi(y),$$

then ϕ is said to be Strictly Convex.

Theorem 4.2.17. A differentiable function ϕ is convex on (a,b) if and only if ϕ' is a monotonically increasing function. If ϕ'' exists on (a,b), then lβ is convex if and only if $\phi'' \geq 0$ on (a, b) and strictly convex if $\psi'' > 0$ on (a,b).

Proof. Suppose first that ϕ is differentiable and convex and let $a < s < t < u < v < b$. Then applying Theorem 5 to $a < s < t < u$, we get

$$\frac{\phi(t) - \phi(s)}{t-s} \leq \frac{\phi(u) - \phi(s)}{u-s} \leq \frac{\phi(u) - \phi(t)}{u-t}$$

and a $< t < u < v$, we get

$$\frac{\phi(u)-\phi(t)}{u-t} \le \frac{\phi(v)-\phi(t)}{v-t} \le \frac{\phi(v)-\phi(u)}{v-u}$$

Hence

$$\frac{\phi(t)-\phi(s)}{t-s} \le \frac{\phi(v)-\phi(u)}{v-u}$$

If $t \to s$, $\frac{\phi(t)-\phi(s)}{t-s}$ decreases to $\phi'(\mathrm{s})$ and if $u \to v$, $\frac{\phi(v)-\phi(u)}{v-u}$ increases to $\phi'(\mathrm{v})$. Hence $\phi'(\mathrm{v}) \ge \phi'(\mathrm{s})$ for all $\mathrm{s} <$ v and so ϕ' is monotonically increasing function. Further, if ϕ'' exists, it can never be negative due to monotonicity of ϕ'.

Conversely, let $\psi'' \ge 0$. Our aim is to show that ψ is convex. Suppose, on the contrary, that ϕ is not convex on (a, b) . Therefore, there are points a $< s < t < u < b$ such that

$$\frac{\phi(\mathrm{t})-\phi(\mathrm{s})}{\mathrm{t}-\mathrm{s}} > \frac{\phi(\mathrm{u})-\phi(\mathrm{t})}{\mathrm{u}-\mathrm{s}}$$

that is, slope of chord over (s,t) is larger than the slope of the chord over (t,u). But slope of the chord over (s,t) is equal to $\phi'(\alpha)$, for some $\alpha \in (s, \mathrm{t})$ and slope of the chord over (t,u) is $\phi'(\beta)$, $\beta \in$ (t,u). But $\phi'(\alpha) > \phi'(\beta)$ implies ϕ' is not monotone increasing and so ψ'' cannot be greater than zero. We thus arrive at a contradiction. Hence ϕ is convex.

If $'' > 0$, then ϕ is strictly convex, for otherwise there would exist collinear points of the graph of ϕ and we would have $\phi'(\alpha) = \phi'(\beta)$ for appropriate α and β with $\alpha < \beta$. But then $\phi'' = 0$ at some point between α and β which is a contradiction to $'' > 0$. This completes the proof.

Theorem 4.2.18. If ϕ is convex on (a,b), then ϕ is absolutely continuous on each closed subinterval of (a,b).

Proof. Let $[c, d] \subset (a, b)$. If x, y $\in$ [c, d], then we have $a < c \le x \le y \le d < b$ and we have

$$\frac{\phi(c)-\phi(a)}{c-a} \le \frac{\phi(y)-\phi(x)}{y-x} \le \frac{\phi(b)-\phi(d)}{b-d}$$

Thus

$$|\phi(y)-\phi(x)| \le M|x-y|, x, y \in [c, d]$$

and so ϕ is absolutely continuous there.

Definition 4.2.19. Let ϕ be a convex function on (a,b) and $x_0 \in (a,b)$. The line

$$y = m(x - x_0) + \phi(x_0) \qquad \text{(i)}$$

through $(x_0, \phi(x_0))$ is called a **Supporting Line** at x_0 if it always lie below the graph of ϕ, that is, if

$$\phi(x) \geq m(x - x_0) + \phi(x_0) \qquad \text{(ii)}$$

The line (i) is a supporting line if and only if its slope m lies between the left and right hand derivatives at x_0. Thus, in particular, there is at least one supporting line at each point.

Theorem 4.2.20. (Jensen Inequality). Let ϕ be a convex function on $(-\infty, \infty)$ and let f be an integrable function on [0,1]. The

$$\int \phi(f(t))dt \geq \phi \int (f(t))dt$$

Proof. Put

$$\alpha = \int_0^1 f(t)dt$$

Let $y = m(x - \alpha) + \phi(\alpha)$ be the equation of supporting line at α. Then

$$\phi(f(t)) \geq m(f(t) - \alpha) + \phi(\alpha)$$

Integrating both sides with respect to t over $[0,1]$, we have

$$\int_0^1 \phi(f(t))dt \geq m\left[\int \ f(t)dt - \int \ f(t)dt\right] + \int_0^1 \phi(x)dt$$

$$= 0 + \phi(\alpha)\int_0^1 d\,t$$

$$= \phi(\alpha) = \phi[\textstyle\int_0^1 f(t)dt\ ].$$

4.3 <u>Check Your Progress</u>

Q.1 Prove that $\| f + g \|_1 \leq \| f \|_1 + \| g \|_1$.

Q.2. If $f \in L^2[0,1]$, show that $|\int_0^1 f(x)dx| \leq \left[\int_0^1 |f(x)|^2\, dx\right]^{\frac{1}{2}}$.

Q.3. Let f(x) be a real valued function defined on [a,b] such that

$$f(x)=\begin{cases}1, & x\in Q\\ \infty, & x\notin Q\end{cases}\ \forall x\in[a,b].$$

Show that $f\in L^{\infty}[a,b]$.

Fill in the blanks

Q.4 Every convex function on an open interval is continuous.

Sol. If $a<x_1<x<x_2<b$, the convexity of a function ϕ implies

$$\phi(x)\le\frac{x_2-x}{x_2-x_1}\phi(x_1)+\phi(x_2)\frac{x-x_1}{x_2-x_1} \qquad \text{(i)}$$

If we make $x\to x_1$ in (i), we obtain-----------------------------;and if we take $x_2\to x$ we obtain $\phi(x)\le\phi(x+0)$. Hence $\phi(x)=\phi(x+0)$ for all values of x in (a,b). Similarly $\phi(x-0)=\phi(x)$ for all values of x. Hence

-------------------------------------, and so ϕ is continuous.

Q.5. Let ϕ be convex on (a,b) and $a<s<t<u<b$, then

$$\frac{\phi(t)-\phi(s)}{t-s}\le\frac{\phi(u)-\phi(s)}{u-s}\le\frac{\phi(u)-\phi(t)}{u-t}$$

If ϕ is strictly convex, equality will not occur.

Solution. Let $a<s<t<u<b$ and suppose ϕ is convex on (a,b). Since

----

therefore, convexity of ϕ yields

$$\phi\left[\frac{t-s}{u-s}u+\frac{u-t}{u-s}s\right]\le\frac{t-s}{u-s}\phi(u)+\frac{u-t}{u-s}\phi(s)$$

or

$$\phi(t)\le\frac{t-s}{u-s}\phi(u)+\frac{u-t}{u-s}\phi(s) \qquad \text{(ii)}$$

or

----

or

$$(u-s)\left(\phi(t)-\phi(s)\right) \leq (t-s)\phi(u)+u\phi(s)-t\phi(s)-u\phi(s)+s\phi(s)$$

or

----

or

$$\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(s)}{u-s} \qquad \text{(iii)}$$

This proves the first inequality. The second inequality can be proved similarly. If ϕ is strictly converse, equality shall not be there in (ii) and so it cannot be in (iii). This completes the proof of the theorem.

4.4 Summary

This chapter presents the study of L^p-spaces for $1 \leq p \leq \infty$. These spaces, known as "Lebesgue spaces," often occur in several branches of mathematical physics. A characterization of measurable transformations inducing composition operators is provided in the chapter, along with the characterization of the operators on L^p that are composition operators; several examples are also presented to illustrate the theory.

4.5 Keywords

L^P- Space, Norm, Banach Space, Complete Space, Distance Function, Conjugate Number, Cauchy Sequence.

4.6 Self-Assessment Test

1. Let f_n: $\boldsymbol{R}\to\boldsymbol{R}$ be a function such that

$$f_n = \begin{cases} 2^n; & x \in [2^n, 2^{n-1}] \\ 0\,; & otherwise \end{cases} \quad \text{for all } n$$

If $f(x)$= $\lim f_n(x)$, show that $\int f \neq lim \int f_k$.

2. If $f,g \in L^p$, then show that $f\,g \in L^{p/2}$.

3. Illustrate that L^1 consists precisely of the lebesgue integrable function on [0,1].

4. Prove the inequality $\left| \int_0^{\pi} x^{\frac{-1}{4}} sinx\, dx \right| \leq \pi^{3/4}$.

5. Let $f,g \in L(-\infty,\infty)$. define

$$h(x)=\int_{-\infty}^{\infty} f(x-y)g(y)dy,\ -\infty<x<\infty,$$

Prove that $h \in L^1(-\infty,\infty)$.

4.7 Answers to Check Your Progress

A.1 We know that $|f+g| \leq |f|+|g|$.

On integrating both the sides we get

$$\int |f+g|\,dx \leq \int |f|\,dx + \int |g|\,dx$$

Which shows that

$$\| f+g \|_1 \leq \| f \|_1 + \| g \|_1$$

A.2. Using Schwarz's inequality for $f,g \in L^2[0,1]$, we can write

$$\| fg \| \leq \| f \|_2 + \| g \|_2$$

$$\Rightarrow \int_0^1 |fg|\,dx \leq \left[\int_0^1 |f|^2\,dx \right]^{\frac{1}{2}} \left[\int_0^1 |g|^2\,dx \right]^{\frac{1}{2}}$$

If we take $g(x)=1, \forall x$, then we get

$$\int_0^1 |f|\,dx \leq \left[\int_0^1 |f|^2\,dx \right]^{\frac{1}{2}}$$

$$\Rightarrow \left|\int_0^1 f dx\right| \le \int_0^1 |f| dx \le \left[\int_0^1 |f|^2 dx\right]^{\frac{1}{2}}$$

Hence, $$\left|\int_0^1 f dx\right| \le \left[\int_0^1 |f|^2 dx\right]^{\frac{1}{2}}.$$

A.3 Since, we know that $m\{x \in [a,b] : |f(x) > M, M \ge 1|\} = m\{$ Set of all rational in [a,b]}=0

Therefore, for all $M \ge 1, |f(x)| \le M$ a.e. on [a,b]

$$\Rightarrow ess.\sup|f(x)| = \inf\{M : |f(x)| \le M \ \ a.e. \, on \, E\}$$

$$= \inf\{M, M \ge 1\} = 1$$

$$\Rightarrow ess.\sup|f(x)| < \infty.$$

Hence $f \in L^\infty[a,b]$.

A.4 (i) $\phi(\mathrm{x}_1 + 0) \le \phi(\mathrm{x}_1)$

(ii) $\phi(x - 0) = \phi(x + 0) = \phi(x)$

A.5 (i) $\frac{t-s}{u-s} + \frac{u-t}{u-s} = \frac{t-s+u-t}{u-s} = \frac{u-s}{u-s} = 1$

(ii) $(u\text{-}s)\ \phi(t) \le (t-s)\phi(u) + (u-t)\phi(s)$

(iii) $(u-s)\big(\phi(t) - \phi(s)\big) \le (t-s)\ \big(\phi(u) - \phi(s)\big)$.

4.8 <u>References/ Suggested Reading</u>

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