FLUID MECHANICS
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SOME HISTORICAL NOTES

THE term hydrodynamics was introduced by **Daniel Bernoulli** (1700-1783) to comprise the two sciences of hydrostatics and hydraulics. He also discovered the famous theorem still known by his name.

d'Alembert (1717-1783) investigated resistance, discovered the paradox associated with his name, and introduced the principle of conservation of mass (equation of continuity) in a liquid.

Euler (1707 -1783) formed the equations of motion of a perfect fluid and developed the mathematical theory. This work was continued by Lagrange (1736-1813).

Navier (1785-1836) derived the equations of motion of a viscous fluid from certain hypothesis of molecular interaction. Stokes (1819-1903) also obtained the equations of motion of a viscous fluid. He may be regarded as having founded the modern theory of hydrodynamics.

Rankine (1820-1872) developed the theory of sources and sinks.
Helmholtz (1821-1894) introduced the term velocity potential, founded the theory of vortex motion, and discontinuous motion, making fundamental contributions to the subject.

Kirchhoff (1824-1887) and Rayleigh (1842-1919) continued the study of discontinuous motion and the resistance due to it.

Osborne Reynolds (1842-1912) studied the motion of viscous fluids, introduced the concepts of laminar and turbulent flow, and pointed out the abrupt transition from one to the other.

Joukowski (1847-1921) made outstanding contributions 'to aerofoil design and theory, and introduced the aerofoils known by his name.

Lanchester (1868-1945) made two fundamental contributions to the modern theory of flight; (i) the idea of circulation as the cause of lift,
(ii) the idea of tip vortices as the cause of induced drag.
He explained his theories to the Birmingham Natural History Society in 1894 but did not publish them till 1907 in his Aerodynamics.

CHAPTER-I

SOME SUBSIDIARY RESULTS

1.0 Learning Objectives: After reading this chapter, you should be able to learn "Some subsidiary results" which provides the brief resume of results on vectors, vector analysis, complex variables and boundary value problems etc. This makes the text self-contained with regard to the use of subsidiary results from various mathematical disciplines.

1.1 Brief Introduction of Vector analysis:

Since the use of vectors not only simplifies and condenses the exposition of fluid mechanics but also makes mathematical and physical concepts more tangible and easier to grasp, it is proposed to give the vectorial treatment of what follows further.

Throughout this manuscript, **bold face** type is used to denote vector quantities.

If **a, b** and **c** are any vector functions (of position), then with the vector notation

$$\boldsymbol{a} = \boldsymbol{i}a_1 + \boldsymbol{j}\,a_2 + \boldsymbol{k}\,a_3 \equiv (a_1, a_2, a_3,)\ ;\ \boldsymbol{b} \equiv (b_1, b_2, b_3), etc.$$

1.2 Rules of Vector Algebra*:*

$$\boldsymbol{a}\,.\,\boldsymbol{b} = \boldsymbol{b}\,.\,\boldsymbol{a} = ab\cos\theta = a_1b_1 + a_2\,b_2 + a_3b_3$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = a\,b\sin\theta\,\boldsymbol{n} = \sum i\,(a_2b_3 - a_3b_2)$$

$$\boldsymbol{a}\,.\,(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\,.\,\boldsymbol{b} + \boldsymbol{a}\,.\,\boldsymbol{c}\,;\ \boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{c}$$

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{a}) = \sum (a_1(b_2c_3 - b_3c_2)$$

$$\boldsymbol{a} \times (\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\,.\,\boldsymbol{c}\,) - \boldsymbol{c}(\boldsymbol{a}.\boldsymbol{b})$$

The **Vector operator ∇** (called *del)* is defined as:

$$\boldsymbol{\nabla} \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} = (\frac{\partial}{\partial x}\ ,\frac{\partial}{\partial y}\ ,\frac{\partial}{\partial z})$$

Then, if $\phi(x, y, z)$ and **a**(x,y,z) have continuous first partial derivatives in a region, we have the following definitions :

Important Results

(i) Let $\boldsymbol{q} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w, then$

$$|\mathbf{q}| = \sqrt{u^2 + v^2 + w^2} = q$$

D.C's are given b y l = cos α = $\frac{u}{|\boldsymbol{q}|}$, m = cos β = $\frac{v}{|\boldsymbol{q}|}, n = cos\,\gamma = \frac{w}{|\boldsymbol{q}|}$

where l, m, n, are components of a unit vector i.e. $l^2 + m^2 + n^2 = 1$

(ii) $\boldsymbol{a}.\boldsymbol{b} = ab\cos\theta\,,\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\, n$

(iii) $\nabla\phi = \boldsymbol{i}\frac{\partial\varphi}{\partial x} + \boldsymbol{j}\frac{\partial\varphi}{\partial y} + \boldsymbol{k}\frac{\partial\varphi}{\partial z}$, where ϕ is a scalar and

$\nabla \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$ is a vector (operator)

(iv) div $\mathbf{q} = \nabla.\mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \boldsymbol{q}$ = (u, v, w)

If $\nabla\cdot\boldsymbol{q} = 0, then \boldsymbol{q}$ is said to be solenoidal vector.

(v) $d\boldsymbol{r} = \boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz,$ $d\phi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$

and

$\nabla\phi = \boldsymbol{i}\frac{\partial\varphi}{\partial x} + \boldsymbol{j}\frac{\partial\varphi}{\partial y} + \boldsymbol{k}\frac{\partial\varphi}{\partial z},$

Therefore,

$d\phi = (\nabla\phi).\, d\boldsymbol{r}$

(vi) $\text{Curl}\boldsymbol{q} = \nabla\times\boldsymbol{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$

$= \boldsymbol{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

(vii) **(a)** Gradient of a scalar is a vector.

(b) Divergence of a scalar and curl of a scalar are meaningless.

(c) Divergence of a vector is a scalar and curl of a vector is a vector.

(viii) $\nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$

where ∇^2 is Laplacian operator.

(ix) Curl grad $\phi = 0$, div curl $\boldsymbol{q} = 0$

(x) Curl Curl $\boldsymbol{q} = graddiv\boldsymbol{q} - \nabla^2\boldsymbol{q}$

i.e. $\nabla^2\boldsymbol{q} = graddiv\boldsymbol{q} - curlcurl\boldsymbol{q}$

(xi) **Gauss's divergence theorem**

(a) $\boldsymbol{q}\cdot d\boldsymbol{S} = \int_V div\boldsymbol{q}dv$

(b) $\int_S \boldsymbol{n}\times\boldsymbol{q}dS = \int_V curl\,\boldsymbol{q}dv$

(xii) **Green's theorem**

(a) $\int_V \nabla\varphi\cdot\nabla\psi dV = \int_S \varphi\nabla\psi\cdot dS - \int_V \varphi\nabla^2\psi dV$

$= \int_S \psi\nabla\varphi\cdot dS - \int_V \psi\nabla^2\varphi\cdot dV$

(b) $\int_V (\varphi\nabla^2\psi - \psi\nabla^2\varphi)dV = \int_V \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\varphi n}\right)dS$

(xiii) **Stoke's theorem** $\int_C \boldsymbol{q}\cdot d\boldsymbol{r} = \int_S curl\,\boldsymbol{q}\cdot dS = \int_S curl\,\boldsymbol{q}\cdot\boldsymbol{n}dS$

(xiv) **Orthogonal curvilinear coordinates:**

Let there be three orthogonal families of surfaces

$$f_1(x, y, z) = \alpha,\ f_2(x, y, z) = \beta,\ f_3(x, y, z) = \gamma \quad (1)$$

where x, y, z are Cartesian co-ordinates of a point P(x, y, z) in space. The surfaces

$$\alpha = \text{constant},\ \beta\ \ \text{constant},\ \gamma = \text{constant} \quad (2)$$

form an orthogonal system in which every pair of surfaces is an orthogonal system. The values α, β, γ are called orthogonal curvilinear co-ordinates.
From three equations in (1), we can get

$$x = x(\alpha, \beta, \gamma),\ y = y(\alpha, \beta, \gamma),\ z = z(\alpha, \beta, \gamma)$$

The surfaces (2) are called co-ordinate surfaces.
Let **r** be the position vector of the point P(x, y, z)

i.e. $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = \boldsymbol{r}\,(\alpha, \beta, \gamma)$

A tangent vector to the α-curve (β = constant, γ = constant) at P is $\frac{\partial \boldsymbol{r}}{\partial \alpha}$. A unit tangent vector is

$$\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}/\partial \alpha}{|\partial \boldsymbol{r}/\partial \alpha|}$$

or $$\frac{\partial \boldsymbol{r}}{\partial \alpha} = h_1 \boldsymbol{e}_1$$

where $h_1 = \left|\frac{\partial \boldsymbol{r}}{\partial \alpha}\right| = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2}$

Similarly, $\boldsymbol{e}_2, \boldsymbol{e}_3$ are unit vectors along β-curve and γ-curve respectively such that

$$\frac{\partial \boldsymbol{r}}{\partial \beta} = h_2 \boldsymbol{e}_2, \frac{\partial \boldsymbol{r}}{\partial \gamma} = h_3 \boldsymbol{e}_3$$

Further, $$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \alpha} d\alpha + \frac{\partial \boldsymbol{r}}{\partial \beta} d\beta + \frac{\partial \boldsymbol{r}}{\partial \gamma} d\gamma$$

$$= h_1\, d\alpha\, \boldsymbol{e}_1 + h_2 d\beta \boldsymbol{e}_2 + h_3 d\gamma \boldsymbol{e}_3$$

Therefore,

$$(ds)^2 = d\boldsymbol{r}.d\boldsymbol{r} = h_1^2 d\alpha^2 + h_2^2 d\beta^2 + h_3^2 d\gamma^2$$

where $h_1\, d\alpha$, $h_2\, d\beta$, $h_3\, d\gamma$ are arc lengths along α, β and γ curves.
In orthogonal curvilinear co-ordinates, we have the following results.

(i) grad $\phi = \left(\frac{1}{h_1}\frac{\partial \varphi}{\partial \alpha}, \frac{1}{h_2}\frac{\partial \varphi}{\partial \beta}, \frac{1}{h_3}\frac{\partial \varphi}{\partial \gamma}\right)$

(ii) If $\boldsymbol{q} = (q_1, q_2, q_3), then$

$$\text{div}\ \boldsymbol{q} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 q_1) + \frac{\partial}{\partial \beta}(h_3 h_1 q_2) + \frac{\partial}{\partial \gamma}(h_1 h_2 q_3)\right]$$

(iii) If $Curl\ \boldsymbol{q} = \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ then

$$\xi_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial \beta}(h_3 q_3) - \frac{\partial}{\partial \gamma}(h_2 q_2)\right]$$

$$\xi_2 = \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial \gamma}(h_1 q_1) - \frac{\partial}{\partial \alpha}(h_3 q_3)\right]$$

$$\xi_3 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \alpha}(h_2 q_2) - \frac{\partial}{\partial \beta}(h_1 q_1)\right]$$

(iv) $$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}\left(\frac{h_2 h_3}{h_1}\frac{\partial \varphi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h_3 h_1}{h_2}\frac{\partial \varphi}{\partial \beta}\right) + \frac{\partial}{\partial \gamma}\left(\frac{h_1 h_2}{h_3}\frac{\partial \varphi}{\partial \gamma}\right)\right].$$

The Cartesian co-ordinate system (x, y, z) is the simplest of all orthogonal co-ordinate systems. In many problems involving vector field theory, it is convenient to work with other two most common orthogonal co-ordinates i.e. cylindrical polar co-ordinates and spherical polar co-ordinates denoted respectively by (r, θ, z) and (r, θ, ψ). For cylindrical co-ordinates, $h_1 = 1$, $h_2 = r$, $h_3 = 1$. For spherical co-ordinates, $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$.

1.3 A Note on Connectivity: A region of space is said to be connected if a path joining any two points of the same lies entirely in the given region.

When the two paths taken together form a reducible circuit, they are termed reconcilable. And when one circuit can be continuously varied so as to coincide with another circuit without leaving the region, the two circuits are called reconcilable.

A simply connected region(acyclic) region is one in which all paths connecting any two points within the region can be deformed into one another without passing outside the region. Obviously, in simply connected region, every circuit is reducible. i.e., it can be contracted to a point of the region without ever passing out of it.

Examples of simply connected region:

(i) The region between two concentric spheres.

(ii) Un-bounded space.

(iii) Region interior to sphere and region exterior to a sphere, etc.

Triply connected **Simply connected**

A region is said to be doubly connected if it can be made simply connected by the insertion on one barrier.

Examples of Doubly Connected region:

(i) Regions between two co-axial infinitely long cylinders.

(ii) Region exterior to an infinitely long cylinder.

(iii) Region interior to an anchor ring; region exterior to an anchor ring, etc.

In general, a region is said to be r-ply connected if it can be made simply connected by the insertion of (r-1) barriers.

The above definitions can also be expressed as under:

A domain is called simply connected, if the frontier thereof consists of a single continuum. Generally, a domain is called r-ply connected if the frontier of the same consists of r distinct continuum.

Note: Fortunately, the multiply-connected regions which occur in most hyderodynamical problems are of an extremely simple kind, and that it is not necessary to develop a formal topological theory (i.e. the study of figures which survive twisting and stretching: rubber sheet geometry).

1.4 Material or Total Derivative of a Functional Determinant (Jacobian): Let

$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial c} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} = \nabla x \times \nabla y \,.\nabla z$$

where the operator ∇ stands for $\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}\right)$.

We shall assume the validity of the operator

$$\frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) = \frac{\partial u}{\partial t}, etc.$$

Then the rule of differentiating products provides

$$\dot{J} = \nabla\dot{x} \times \nabla y \,.\nabla z + \nabla x \times \nabla\dot{y} \,.\nabla z + \nabla x \times \nabla y \,.\nabla\dot{z} \qquad \left(\dot{x} = \frac{dx}{dt}\right)$$

Now, $\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}$

Also $\frac{dJ}{dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \Rightarrow \frac{dJ}{dt} = J\, div\, \boldsymbol{q}$

1.5 Brief Resume of Complex Function Theory: A very powerful technique for dealing with two-dimensional problems in theoretical hydrodynamics is furnished by the properties of analytic functions (i.e. functions possessing derivatives for all values of z=x-iy in a region) of complex variable z. Thus, if f(z) is regular(analytic) in a domain D of the complex z-plane, and if we write

$$w = f(z) = \phi(x,y) + i\,\psi(x,y)$$

then it is shown in all texts on complex variable that if f(z) is to possess a unique derivative, then it is necessary as well as sufficient that

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y};\; \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad (1)$$

where it is supposed that these partial derivatives are continuous. These are called Cauchy-Riemann partial differential equations. The vector equivalent to (1) is

$$\boldsymbol{grad\,\phi} = (\boldsymbol{grad\,\psi}) \times \boldsymbol{k}$$

An alternative single equivalent expression to (1) is $\frac{\partial\phi}{\partial n} = \partial\psi/\partial s$ where n and s are perpendicular directions related to each other in the anti-clockwise sense.

If we eliminate ψ and ϕ in succession between equations (1), we get

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y} = -\frac{\partial^2\phi}{\partial y^2} \quad i.e. \nabla^2\phi = 0; \; likewise \, \nabla^2\psi = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. These two conjugate functions, $\phi \; and \; \psi$ are called velocity potential and stream function.

Since $\phi \; and \; \psi$ are harmonic functions (i.e. functions which satisfy the Laplace's equations $\nabla^2\phi = \nabla^2\psi = 0$) these will be the possible velocity potential and stream function, and provided the necessary boundary conditions for a problem are satisfied, these will yield a unique solution to the problem.

If w=f(z) provides the solution to a hydrodynamical problem, it is called the complex potential characterizing the given fluid flow.

It may be observed that equations(1) imply that the family of curves, $\phi(x, y) = constant$ and $\psi(x, y) = constant$ are orthogonal families.

We now include clear statements of pertinent definitions, principles, and theorems which are relevant to the study of Hydrodynamics.

Cauchy's Theorem: if f(z) is analytic within the region bounded by C (a simple closed curve) as well as on C, then

$$\int_C f(z)dz = \oint_C f(z)dz = 0$$

A simple consequence of this theorem is that $\int_{z_1}^{z_2} f(z)dz$ has a value independent of paths joining $z_1 \; and \; z_2$.

Cauchy's Integral Formulae: If f(z) is analytic within and on a simple closed positively oriented curve C, and a is any point interior to C, then

$$f(a) = \frac{1}{2\pi}\oint_C \frac{f(z)}{z-a}\,dz; \qquad f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where $f^n(a)$ is the nth derivative of f(z) at z=a.

Taylor's Series: Let f(z) be analytic inside and on a circle having its centre at z=a. Then for all points z in the circle

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

If a=0, there results Maclaurin series.

Singular Points: A singular point of a function f(z) is a value of z at which f(z) ceases to be analytic. If f(z) is analytic everywhere in some domain except at an interior point z=a, then z=a is called an isolated singular point.

Poles: If $f(z) = \frac{F(z)}{(z-a)^n}; F(a) \neq 0$,where F(z) is analytic everywhere in a region including z=a and if n is a positive integer, then f(z) has an isolated singularity at z=a. This isolated singularity is called a pole of order n. If n=1, the pole is called a simple pole; if n=2, it is called a double pole, and so on.

Laurent's Series: If f(z) is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles (positively oriented)C_1 *and* C_2 with centre a and respective radii $R_1 and\ R_2 (R_1 > R_2)$, then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\, dz; n = 0,1,2 \ldots.$$

$$a_{-n} = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{-n+1}} dz\, ; n = 1, 2, 3, \ldots.$$

The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the analytic part and the remainder $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ is called the principal part. If the principal part is zero, the Laurent series reduces to a Taylor's series.

Residues: If f(z) be single-valued and analytic inside and on a circle C except at the point z=a, chosen as the Centre of C, then Laurent series is given by

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

where

$$a_n = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z-a)^{n+1}}\, dz; n = 0, \pm 1, \pm 2 \ldots.$$

Clearly, $a_{-1} = \frac{1}{2\pi i} \int_C f(z) dz = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\}$
where n is order of the pole.
The coefficient a_{-1} is called the residue of f(z) at the pole z=a.
For simple poles(n=1) , $a_{-1} = Lim\ (z-a) f(z)\ \ as\ z \to a.$

Cauchy's Residue Theorem: If f(z) is analytic on the boundary C of a region R except a finite number of poles within R, then

$$\int_C f(z) dz = 2\pi i [\ sum of\ residues of\ f(z) at\ its\ poles]$$

Cauchy's theorem and Cauchy's integral formulae are special cases of this theorem.

1.6 Boundary Value Problems: Scientific problems are often formulated mathematically which led to partial differential equations and associated conditions called boundary conditions. Consequently, the existence and uniqueness of the problem is of fundamental importance, from a mathematical as well as physical point of view.

Two types of boundaries, (i) the open boundary (where the region of interest extends indefinitely in one or more directions, without any specifications of the solution in these directions.), and (ii) the closed boundary (where the region of interest is completely surrounded, with boundary conditions specified in all directions) are usually considered along with three types of boundary conditions.

(1) *Dirichlet's Conditions:* require the determination of the function satisfying Laplace equation in region R and taking prescribed values on the boundary C.

(2) Neumann's Conditions: require the determination of a function satisfying Laplace equations in R and taking prescribed values of normal derivative on the boundary C.

(3) *Cauchy's Conditions:* require the determination of the function satisfying Laplace equation in R and taking prescribed values of function as well as normal derivative on the boundary C.

Here R may be a simply-connected region bounded by a simple closed curve C, or R may be unbounded region $(y \geq 0)$.

The general partial differential equation of second order, viz.

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

is classified under three heads:

Hyperbolic if $S^2 - 4RT > 0,\ e.g\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}\ ;\ wave\ equation.$

Parabolic if $S^2 - 4RT = 0. e.g., \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}\ ;\ diffusion\ equation.$

Elliptic if $S^2 - 4RT < 0, e.g., \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0; Harmonic\ equation.$

1.7 Check Your Progress:

i) Prove

$div(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b}.curl\ \boldsymbol{a} - \boldsymbol{a}.curl\ \boldsymbol{b}$

ii) $\boldsymbol{if\ u = 1 + 2y - 3z, v = 4 - 2x + 5z, w = 6 + 3x - 5y}$**, q=(u,v,w).** Find curl **q.**

[Ans: $= -(10i + 6j + 4k)$]

iii) Show that the real and imaginary parts of the complex functions are harmonic functions.

1.8 Summary: In this chapter, we have revised the basic concepts and definitions of vectors, operations of dot and cross products of vectors, gradient, divergence, curl, complex variables, orthogonal curvilinear coordinates, Gauss divergence theorem, Green's theorem, Stokes Theorem, concepts of connectivity, Jacobian, Cauchy's theorem, Cauchy's Integral Formula, Taylor's series and other results and theorems on Complex function theory. The students also understood the different boundary value problems.

1.9 Keywords: Scalar, vector, gradient, divergence, curl, singular point, pole, parabolic, hyperbolic, elliptic.

1.10 Self-Assessment Test:

SA1: State Stokes's theorem.
SA2: Define Material and total derivative of a functional determinant.
SA3: State different boundary value problems.
SA4: State Cauchy-Riemann's equations in Cartesian coordinate system.
SA5: State Taylor's and Laurent's series and deduce the Taylor's series from the Laurent's series.

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CHAPTER-2

BASIC CONCEPTS

2.0 Learning Objectives:

When you finish your reading this chapter, you should be able to understand the fundamental ideas of fluid mechanics, identify the numerous fluid flow issues that can arise in real-world situations, able to differentiate between ideal fluids and real fluids and comprehend the fundamental characteristics of fluids.

2.1 Introduction:

The study of the motion and equilibrium of fluids is the focus of fluid mechanics, one of the oldest branches of physics and the basis for a great deal of other knowledge in the applied sciences and engineering. Nearly all engineering disciplines, as well as astrophysics, biology, biomedicine, meteorology, physical chemistry, plasma physics, and geophysics, are interested in the topic. The area of fluid mechanics has continuously expanded since the eighteenth century, when the study of hydraulics as a science was linked to the development of civil engineering and naval architecture. The advancement of mechanical, chemical, and aeronautical engineering over the past few decades, as well as recent space travel, have all contributed to the study of fluid mechanics becoming one of the most crucial foundational topics in engineering science.

The exotic regimes of hypervelocity flight and flow of electrically conducting fluids have been added to the fluid dynamics study frontier. Hypersonic flow and magneto fluid dynamics are two emerging areas of research as a result. To properly comprehend the underlying physical phenomena, it is now important to combine knowledge of fluid mechanics, electromagnetic theory, and thermodynamics.

The use of fluid mechanics principles in daily life is still possible despite the recent remarkable improvements in the field. The rules of fluid mechanics regulate fish motion and bird flight in the water and air, respectively. Baseball pitchers rely on the circulation principle to provide them access to a bewildering variety of pitches. The design of ships and aeroplanes used for air and ocean travel is based on the fluid mechanics theory. One day, fluid mechanics principles may even be able to describe destructive natural events like hurricanes and tornadoes. Since air and water make up a large portion of our environment, practically everything we do is somewhat related to the field of fluid mechanics.

In order to proceed in a logical manner with the discussion of fluid properties, it is necessary to differentiate between a solid and a fluid. Matter exists in three states: solid, liquid and gaseous. The latter two categories make up the fluid state. All matter, whether solid, liquid or gas is made up of small particles. These small particles are known as molecules. These molecules are in a state of movement. In solids, molecules are more closely packed together and do not move so vigorously. In liquids, molecules are packed closer with significant forces of attraction. A liquid, tends together in globules if taken in small quantities and forms a free surface in large volumes. In gases, molecules are relatively farther apart and have very weak forces of attraction. As the temperature increases, the difference in molecules becomes smaller until a liquid get transformed in a gas. Due to the difference in molecular spacing, the solids and the fluids behave differently when subjected to stresses.

2.1.1 Application Areas of Fluid Mechanics:

Since fluid mechanics is widely employed in daily life as well as in the design of contemporary engineering systems, from vacuum cleaners to supersonic aircraft, it is crucial to have a solid understanding of its fundamental concepts. For instance, fluid mechanics is important to the human body. Breathing apparatuses and dialysis systems are created utilizing fluid dynamics. The heart constantly pumps blood to all areas of the human body through the arteries and veins.

An ordinary house is, in some respects, an exhibition hall filled with applications of fluid mechanics. Fluid mechanics is largely used in the design of the water, natural gas, and sewage pipe systems for every home and every community. The fluid mechanics network's pipes and ducts operate similarly. The network of pipes and ducts used by heating and cooling systems is the same. A refrigerator consists of two heat exchangers where the refrigerant absorbs and rejects heat, tubes through which the refrigerant circulates, and a compressor that pressurizes the refrigerant. All of these components were designed with a significant contribution from fluid mechanics. Fluid mechanics is even used to operate common faucets.

Numerous fluid mechanics applications can be seen in cars as well. Fluid mechanics is used to analyze every part involved in moving gasoline from the fuel tank to the cylinders, including the fuel line, fuel pump, fuel injectors, and carburetors, as well as the mixing of fuel and air in the cylinders and the expulsion of combustion gases through the exhaust pipes.

The design of the hydraulic brakes, power steering, automatic gearbox, lubrication systems, cooling system for the engine block, which includes the radiator and water pump, as well as the tyres all make use of fluid mechanics. Recent model cars have a sleek, streamlined shape because to attempts to reduce drag through in-depth understanding of the flow across surfaces.

A wider range of applications include the design and analysis of aircraft, boats, submarines, rockets, jet engines, wind turbines, biomedical devices, cooling systems for electronic components, and transportation systems for moving water, crude oil, and natural gas. Fluid mechanics also plays a significant role in these fields. In order to ensure that the structures can sustain wind loads, it is also taken into account while designing buildings, bridges, and even billboards. The laws of fluid mechanics also regulate a number of other natural phenomena, including the rain cycle, weather patterns, the rise of ground water to the tops of trees, winds, ocean waves, and currents in vast bodies of water.

2.2 Fluid: The substance known as the fluid is described as an accumulation of molecules. It is a fluid that can flow since it is an isotropic substance (a fluid is considered to be isotropic with respect to a property if it remains the same in all directions; if it changes at a point, it is said to be anisotropic). The fluid's ongoing deformation under the influence of forces is exhibited in the fluid's tendency to flow. In other words, regardless of how little a shear stress may be, the fluid continuously deforms as it is applied. A fluid can be thought of as being made up of discrete particles, each larger than a molecule but infinitesimally small in comparison to the fluid's total volume.

We shall deal with the homogenous and macroscopic treatment of fluid. The fluid is regarded as a continuum which cannot support shear stress while at rest with regard to any coordinate system. It follows that any small volume element in the fluid is considered so large that it contains a very great
number of molecules. ***Fluid mechanics is a science which deals with the behaviour of fluids when subjected to a system of forces.***

2.2.1 Types of Fluid:

Ideal fluid or Perfect fluid (Frictionless, homogeneous and incompressible):
Liquids which are incompressible, i.e., their volume does not change when the pressure changes.
Hence, the ideal fluid is one which in incapable of sustaining any tangential stress or action in the form of a shear but the normal force or pressure acts between the adjoining layers of the fluid. The pressure at every point of an ideal fluid is equal in all direction, whether the fluid be at rest or in motion. The theory defines some concepts of the flow such as wave motion, the lift and the induced drag of an airfoil etc., but it fails to define the phenomena such as skin friction, drag of a body etc.

Real fluid or actual fluid (Viscous and compressible):
Fluids which are viscous and compressible i.e., when a fixed mass of fluid undergoes changes in volume, its density also changes. The ability for changes in volume of a mass of fluid is known as compressibility.
Hence, the real fluid is one in which both the tangential and normal forces exist.

Viscosity: It is also known as internal friction and is that characteristic of real fluid which is capable to offer resistance to shearing stress. The resistance is , comparatively, small (not negligible) for fluids such as water and gases but it is quite large for other fluids such as oil, glycerine, paints varnish, coal-tar etc.

2.3 Fluid Properties: A continuous fluid can have some properties that are independent of its motion. These characteristics are referred to as the fluid's fundamental properties.
Properties are considered to be either *intensive* or *extensive*. **Intensive properties** are those that are independent of the mass of the system, such as temperature, pressure, and density. **Extensive properties** are those whose values depend on the size—or extent—of the system. Total mass, total volume *V*, and total momentum are some examples of extensive properties.
We'll discuss about a few of a fluid's characteristics.

(a) Density: The density ρ represents a quantitative expression of the idea of mass. It is defined as the mass of the fluid contained within a unit volume. Consider δm be the mass of the fluid in a small volume δv surrounding that point, then, mathematically the density at a point is defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v},$$

In physical sense $\delta v \to \epsilon^3$ in which ϵ is large in comparison with the mean distance between molecules. In other words, ϵ^3 is the infinitesimal volume over which the substance can be taken as continuum. In fact, the limit $\delta v \to 0$ implies that after a certain stage the continuum hypothesis will breakdown and so the limit does not exist and the ratio will starts fluctuating rapidly. The density is an index of the inertial characteristics.

The density of the fluid depends on the space coordinates and the temperature i.e., $\rho = f(x, y, z)$.The density of water at $4^0 C, is\ \ 1g/cm^3$ or $1000kg/m^3$.

For gases, the density is a function of pressure and temperature. Under ideal conditions, the equation of state for an ideal gas $p = \rho\ RT$ provides a solution for density.

(b) **Specific weight:** The specific weight γ of a fluid is defined as the weight per unit volume. Thus $\gamma = \rho g$.

(c) Specific Volume: The specific volume of a fluid is defined as the volume per unit mass and is clearly the reciprocal of density.

(d) **Pressure:** When a fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel. Such a force per unit area is known as pressure.

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

where δA is an elementary area around P and δF is the normal force due to fluid on δA.

We know that the pressure at every point of an ideal fluid is equal in all directions whether the fluid be at rest or in motion. It does not depend on the orientation of the plane. If it varies with the orientation (as in real fluids in motion) then the average of all such values at that point is taken. It follows that an element δA of a very small area, free to rotate about its centre will have a force of constant magnitude acting on either side of it.

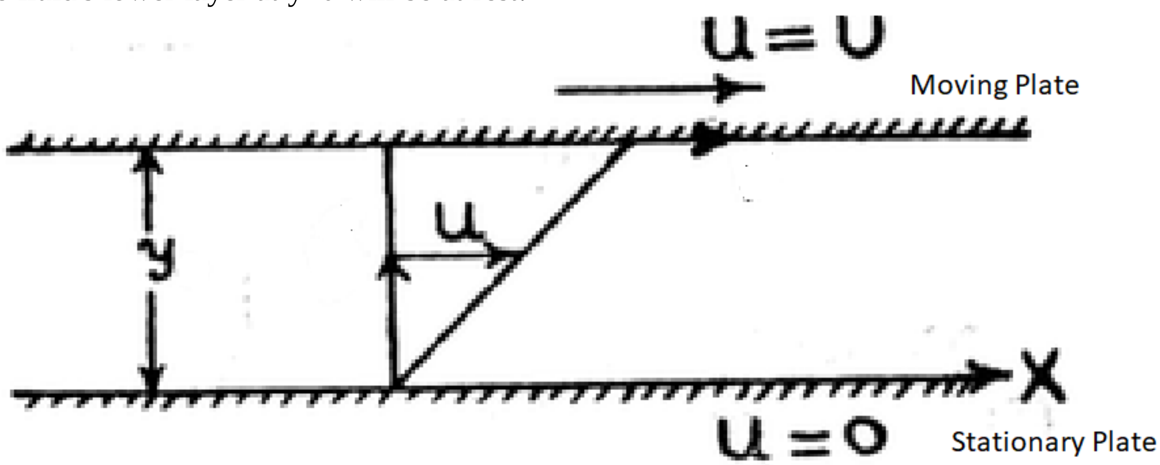
(e) **Specific Gravity:** The specific gravity S of a substance is the ratio of its specific weight of a fluid to the specific weight of an equal volume of water at standard conditions ($4^0 C\ or\ 68^0 F$).

(f) Viscosity (Internal Friction): Compared to syrup and heavy oil, water and air flow much more easily. This shows that the fluid has a characteristic that regulates the rate of flow. This characteristic of a fluid is called viscosity. As a result, a fluid's viscosity refers to its ability to resist changing its shape in some way. Viscosity is a quality that all existent fluids have to varied degrees.

Each component of the fluid is subjected to stress from the surrounding components of the fluid. There are two components to the stress at each area of the element's surface: pressure and shear stress, which are known as normal and tangential to the surface, respectively. Shear stresses only happen in moving fluids, but pressure is applied to both moving and stationary fluids. It is this characteristic that allows fluids to be separated from solids. Viscosity refers to the quality that causes shear stresses. Viscosity arises when there is a relative motion between different fluid layers. It is possessed by all real fluids. Its magnitude is expressed by a coefficient which relates the size of the shear stress at a point in a fluid to the rate of shear strain which causes it.

Consider a fluid that is initially at rest between two parallel plates that are spaced apart by a little distance (h) along the y-axis and extended indefinitely in other directions in order to better understand the nature of viscosity. Consider a situation where the lower plate is kept at rest and the top plate is moving with a velocity U in the x direction. Due to viscosity, the fluid will move as well.

The fluid exhibits a linear velocity profile between the plates (provided no pressure gradient exists along the plate in the direction of motion). The fact that there is no relative velocity between a fluid and a solid surface for any fluid is a fact supported by experimental observations. As a result, the fluid's upper layer at y=h will be moving with the plate moving at a velocity U, while the fluid's lower layer at y=0 will be at rest.



If we consider a small element of the fluid , the shear stress τ on the top(which is numerically the same as the bottom in this case) is given by

$$\tau = \mu \frac{du}{dy} \qquad (1)$$

where μ is a constant of proportionality which is called the coefficient of viscosity or the coefficient of dynamic viscosity. (1) is known as Newton's law of viscosity and the fluids obeying this is called Newtonian fluids. The viscosity of a liquid decreases rapidly with increasing temperature whereas the viscosity of gas increases with temperature. The viscosity of fluids also depends on pressure, but this dependence is usually of little importance compared to the temperature variation in problems of fluid dynamics.

(i) If τ=0 then $\mu = 0$ the equation (1) represents an ideal fluid or perfect fluid.

(ii) if $\frac{du}{dy} = 0 \; then \; \mu = \infty$ the equation (1) represents the elastic bodies.

(iii) A fluid for which the constant of proportionality does not change with the rate of deformation is said to be Newtonian and is represented by a straight line.

(iv) If the viscosity varies with the rate of deformation, then it represents Non-Newtonian fluids. Non-Newtonian fluids are those in which there is no shear stress and there exists a non-linear relation between $\tau \; and \frac{du}{dy}$.The main classes of non-Newtonian fluids are Binghan plastics, Pseudoplastic and Dilatants.

Viscosity of a fluid is practically independent of pressure and depends upon the temperature only.

(g) Temperature: Suppose two bodies of different heat content are brought into contact while isolated from all other bodies. Then some thermal energy will move from one body into the other body. The body from where the thermal energy moves is said to be at a higher temperature while the body into which the energy flows is said to be at a lower temperature. When two bodies are in thermal equilibrium then they are said to have a common property, known as temperature T.

(h) Thermal conductivity: The well-known Fourier's heat conduction law states that the conductive heat flow per unit area (or heat flux) q_n is proportional to the temperature decrease per unit distance in a direction normal to the area through which the heat is flowing. Thus

$$q_n \propto -\frac{\partial T}{\partial n} \qquad so\ that\ \ q_n = -k\frac{\partial T}{\partial n}$$

where k is said to be the thermal conductivity.

(g) Specific heat: The specific heat C of a fluid is defined as the amount of heat required to raise the temperature of a unit mass of the fluid by one degree. Thus $C = \frac{\partial Q}{\partial T}$ $, where\ \delta Q$ is the amount of heat added to raise the temperature by δT. The value of the specific heat depends on two well-known processes- the constant volume process and the constant pressure process. The specific heats of the above processes are denoted and defined as

Specific heat at constant volume $= C_v = \left(\frac{\partial Q}{\partial T}\right)_v$

Specific heat at constant pressure$= C_p = \left(\frac{\partial Q}{\partial T}\right)_p$

Rate of these two specific heats is denoted by $\gamma. Thus\ \ \gamma = \frac{C_p}{C_v}$.

(h) **Surface Tension:** If a small capillary tube is inserted into a beaker containing mercury then the surface of the mercury in the capillary tube is convex and its level is lower than the outside level of the mercury. On the other hand, if water is considered instead of mercury than the surface of water in the capillary is concave and its level is higher than the outside level. This phenomenon depends upon the nature of two immiscible fluids and the temperature.

It is typical to assume that the condition present at a liquid's free surface or at the boundary between two immiscible fluids (which do not mix) possesses simply the equilibrium property of a uniform surface tension. The boundary may separate two media of the same phase but with distinct constitutions, or it may separate two media of the same phase, whether they are solid, liquid, or gaseous. The stress between two adjacent segments of the free surface, measured at per unit length of the common boundary line, relies only on the properties of the two fluids and the temperature. The free surface acts as though it were in a condition of uniform tension. This property of the surface which exert a tension is called the surface tension and is denoted by σ_s.

$$\sigma_s = Stretching\ force\ /unit\ length$$

(i) **Vapour Pressure:** All liquids have a tendency to evaporate when exposed to the atmosphere. The rate at which the evaporation occurs is dependent on the molecular activity of the liquid, which is a function of temperature and the condition of the atmosphere adjoining the liquid. Consider a closed bottle partly filled with a liquid and maintained at a constant temperature. The number of vapour molecules in the air above the liquid increases when the liquid evaporates, simultaneously a small number of vapour molecules re-enter the liquid. Thus, the concentration of vapour molecules above the liquid surface increases, with the passage of time, to such an extent that the rate at which molecules enter the liquid is equal to the rate at which molecules leave the liquid. Hence the air above the liquid surface is saturated with vapour molecules. The pressure on the liquid surface exerted by the vapour molecules is called Vapour Pressure. The vapour pressure is dependent on temperature. The phenomenon of boiling a liquid is closely related to the vapour pressure. When the pressure above a liquid equals the vapour pressure of the liquid, boiling occurs.

At certain locations throughout the system during the flow of liquids, it is feasible that extremely low pressures will be created. The pressures may be less than or equal to the vapour pressures in such situations. A liquid enters an unstable condition when its pressure is decreased to a level that is just a little below the saturated vapour pressure at the liquid's temperature. At this point, the liquid usually starts to create vapour pockets all over it. Cavitation is the term for such pockets' appearance. For example, when the water is heated slowly, bubbles formed near the bottom are known cavities formed in the water. Vortices in rivers are called cavities.

(j) Bulk Modulus of Elasticity and Compressibility: Several fluids, when subjecting to increasing or decreasing pressure, undergo a change in the density. At a constant temperature, an increase or decrease in the relative density is proportional to the increase or decrease in the pressure.

$$dp \propto \frac{d\rho}{\rho} \quad or\ dp = k\frac{d\rho}{\rho}$$

Where the constant of proportionality k is called the bulk modulus of elasticity. In other words, the bulk modulus of elasticity is defined as the ratio of the net increase of pressure on an element of fluid to the unit strain produced by the pressure change. The inverse of the bulk modulus of elasticity is called the compressibility of the fluid. The compressibility of a fluid is the ratio of the relative change of the volume to the change in applied pressure.

2.4 Types of Flows:

(i) Laminar and Turbulent flows: A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles don not intersect, is said to be laminar. On the other hand, a flow, in which each fluid particle does not trace a definite curve and the curves traced out by fluid particles intersect, is said to be turbulent.

(ii) Steady and Unsteady flows: A flow in which properties and conditions (P) associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with the time, is said to be steady. Mathematically, we may write $\frac{\partial P}{\partial t}$=0. Here P may be velocity, density, pressure, temperature etc. On the other hand, a flow, in which properties and conditions associated with the motion of the fluid depend on the time so that the flow pattern varies with time, is said to unsteady.

(iii) Uniform and Non-Uniform Flows: A flow in which the fluid particles possess equal velocities at each section of the channel or pipe is called uniform. On the other hand, a flow, in which the fluid particles possess different velocities at such section of the channel or pipe is called non-uniform. These terms are usually considered in connection with flow in channels.

(iv) Rotational and Irrotational Flows: A flow, in which the fluid particles go on rotating about their own axis, while flowing is said to be rotational and a flow in which the fluid particles do not rotate about their own axes, while flowing is said to be irrotational.

(v) Barotropic Flow: A flow is said to be barotropic when the pressure is function of density.

Example: A plate at a distance of 0.2 cm from the fixed plate moves at 2m/sec. and requires a force of 40 dynes/cm² to maintain this speed. Determine the coefficient of viscosity of the fluid between the plates.

Solution: The velocity gradient becomes

$$\frac{du}{dy} = 2 \times \frac{100}{0.2} = 10^3, \;\; F = 40\; dynes/cm^2$$

$$\mu = \frac{F}{du/dy} = \frac{40}{10^3} = 4 \times 10^{-2}\, poise.$$

Example: A plate weighing 150N measures $\mathbf{80 \times 80}$ cm. It slides down an inclined plane over an oil film of 1.2 mm thick. For an inclination of $\boldsymbol{\frac{\pi}{6}}$ and a velocity of 20cm/sec., calculate the velocity of the fluid.

Solution: Shear stress $\tau = \frac{Force}{area} = \left(\frac{150 \sin\frac{\pi}{6}}{0.80 \times 0.80}\right) = 117.19\; N/m^2$

Rate of deformation $\frac{du}{dy} = \frac{20}{0.12} = 175\; rad/sec$

From Newton's law of viscosity, we have

$\tau = \mu \frac{du}{dy} \Rightarrow \mu = \frac{\tau}{\frac{du}{dy}} = \frac{117.19}{175} = 0.67\; N - s/m^2.$

2.5 Check Your Progress

i) Determine the coefficient of viscosity μ of a fluid, the rate at which the fluid is moving out of a circular pipe of radius a and length *l* is measured when the pressures on the two sides of the pipe is p_1 and p_2.

[Ans: 0.02 poise]

ii) Determine the coefficient of viscosity of a fluid, the fluid is made to rotate between two long co-axial cylinders of radius $r_2\; and\; r_2 (r_2 > r_1)$.If the inner cylinder rotates with angular velocity ω while the outer is at rest, then the torque T on a unit length of each cylinder is $T = 4\pi\mu\omega r_1^2 r_2^2 / (r_2^2 - r_1^2)$, where radii of the cylinders are 3 cm and 3.5 cm, the inner cylinder rotates at a speed of 120 rpm. And the torque is $5.35 \times 10^2 dynes - cm$.

[Ans:0.60 poise]

iii) . A liquid compressed in a cylinder has a volume of 0.4 cc. at $6.8 \times 10^7 dynes/cm^2$ and a volume of 0.396 cc. at $1.36 \times 10^8 dynes/cm^2$.What is its bulk modulus?

[Ans: $6.8 \times 10^9\; dynes/cm^2$]

iv) . Find the shape of the surface of a fluid under a gravitational field and bounded on one side by a vertical plane wall.

v) . Bulk modulus of water is $2.2 \times 10^{10} dynes/cm^2$. Find the change in the volume when 100 cc of water is subjected to an increase of pressure by $7.7 \times 10^6\; dynes/cm^2$.

[Ans: 0.035cc]

2.6 Summary: We covered the fundamental characteristics that relate to the analysis of fluid flow in this chapter. We defined density and specific gravity and discussed about intensive and extensive properties. The properties of vapour pressure and its different manifestations, the specific temperatures of ideal gases and incompressible solids, and the coefficient of compressibility are then discussed. We also touched about the characteristic of viscosity, which dominates most elements of fluid flow and surface tension. The various important forms of flows are also covered in this chapter.

2.7 Keywords: extensive, intensive, ideal fluid, real fluid, density, Bulk modulus, temperature, thermal conductivity, viscosity, vapor pressure, steady flows, unsteady flows, laminar flows.

2.8 Self-Assessment Test

SA 1. What is hydrodynamics?

SA2. Define viscosity of the fluid.

SA3. Discuss the areas of fluid mechanics?

SA4. Define (i) Ideal fluid and real fluid

(ii) Specific heat

(iii) Bulk Modulus and compressibility of the fluid.

(iv) Thermal conductivity.

(v) Density, specific weight and specific volume.

SA5. Differentiate between Newtonian and non-Newtonian fluids.

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CHAPTER -3

KINEMATICS OF FLUIDS

3.0 Learning Objectives: After reading this chapter, the students should be able to understand the flow lines, role of the material derivative in transforming between Lagrangian and Eulerian descriptions, velocity potential functions, translation, deformation and rotation of fluid elements, distinguish between rotational and irrotational regions of flow based on the flow of vorticity property. and applications of Reynold's transport theorem.

3. 1 General Consideration of Fluid:

Fluid Dynamics

Fluid dynamics or hydrodynamics is the science treating the study of fluids in motion. By the term fluid, we mean a substance that flows i.e. which is not a solid. Fluids may be divided into two categories

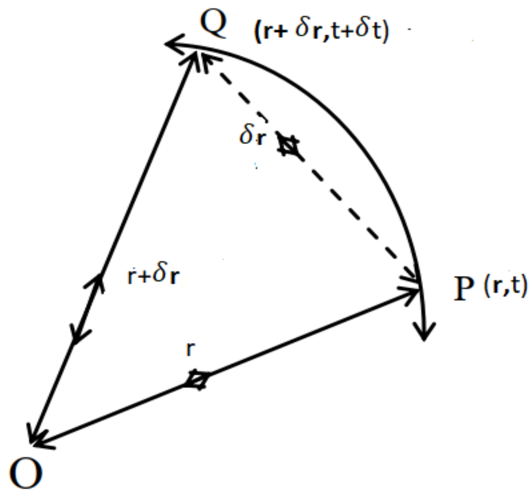
(i) liquids which are incompressible i.e. their volumes do not change when the pressure changes

(ii) gases which are compressible i.e. they undergo change in volume whenever the pressure changes. The term hydrodynamics is often applied to the science of moving incompressible fluids. However, there are no sharp distinctions between the three states of matter i.e. solid, liquid and gases. The term hydrodynamics is often applied to the science of moving incompressible fluids.

In **microscopic** view of fluids, matter is assumed to be composed of molecules which are in random relative motion under the action of intermolecular forces. In solids, spacing of the molecules is small, spacing persists even under strong molecular forces. In liquids, the spacing between molecules is greater even under weaker molecular forces and in gases, the gaps are even larger.

If we imagine that our microscope, with which we have observed the molecular structure of matter, has a variable focal length, we could change our observation of matter from the fine detailed microscopic viewpoint to a longer-range **macroscopic** viewpoint in which we would not see the gaps between the molecules and the matter would appear to be continuously distributed. We shall take this macroscopic view of fluids in which physical quantities associated with the fluids within a given volume V are assumed to be distributed continuously and, within a sufficiently small volume δV, uniformly. This observation is known as **Continuum hypothesis**. It implies that at each point of a fluid, we can prescribe a unique velocity, a unique pressure, a unique density etc. Moreover, for a continuous or ideal fluid we can define a **fluid particle** as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

3.2 Velocity of Fluid at a Point: Suppose that at time t, a fluid particle is at the point P having position vector $\boldsymbol{r}(\boldsymbol{i.e.}\, \boldsymbol{OP} = \boldsymbol{r})$



and at time t + δt the same particle has reached at point Q having position vector $\boldsymbol{r} + \delta\boldsymbol{r}$ The particle velocity **q** at point P is

$$\boldsymbol{q} = \underset{\delta t \to 0}{Lt} \frac{(\boldsymbol{r} + \delta\boldsymbol{r}) - \boldsymbol{r}}{\delta t} = \underset{\delta t \to 0}{Lt} \frac{\delta\boldsymbol{r}}{\delta t} = \frac{d\boldsymbol{r}}{dt}$$

where the limit is assumed to exist uniquely. Clearly **q** is in general dependent on both $\bar{r}$ and t, so we may write

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r}, t) = \boldsymbol{q}(x, y, z, t),$$

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ (P has co-ordinates (x, y, z))

Suppose,

$$\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$$

and since

$$\boldsymbol{q} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k},$$

therefore

$$u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}.$$

Remarks. (i) A point where $\boldsymbol{q} = \bar{0}$, is called a stagnation point.

(ii) When the flow is such that the velocity at each point is independent of time i.e. the flow pattern is same at each instant, then the motion is termed as steady motion, otherwise it is unsteady.

Flux across any surface: The flux i.e. the rate of flow across any surface S is defined by the integral

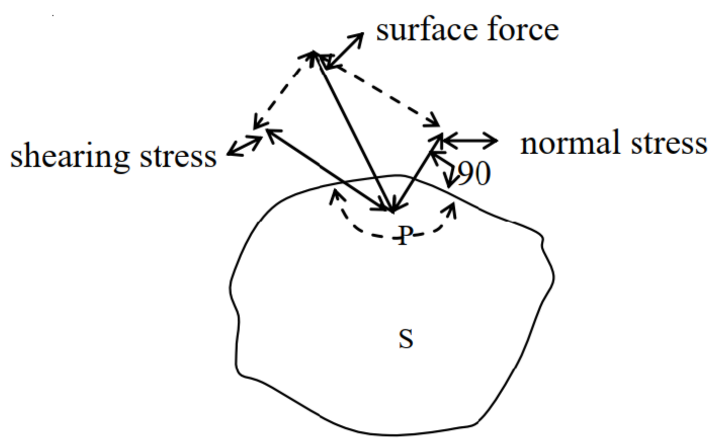
$$\int_S \rho(\boldsymbol{q} \cdot \boldsymbol{n})dS$$

where ρ is the density, $\bar{q}$is the velocity of the fluid and $\hat{n}$is the outward unit normal at any point of S.
Also, we define

Flux = density × normal velocity × area of the surface.

3.3 Stress and Coefficient of Viscosity:

(a) Stresses: Two types of forces act on a fluid element. One of them is **body force** and other is **surface force.** The body force is proportional to the mass of the body on which it acts while the surface force is proportional to the surface area and acts on the boundary of the body.
Suppose **F** is the surface force acting on an elementary surface area (dS) at a point P of the surface S.



Let F_1 and F_2 be resolved parts of **F** in the directions of tangent and normal at P. The normal force per unit area is called the **normal stress** and is also called **pressure**. The tangential force per unit area is called the **shearing stress**.
i.e.,

If a force $\delta \boldsymbol{F}$ acts on a disc area $\delta S\, \hat{n}$, then the stress vector is defined by

$$\mathbf{T} = \lim(\delta \boldsymbol{F}/\delta S) = d\boldsymbol{F}/dS$$

Obviously, the vector **T** depends on $\hat{n}$. The component of stress in the direction $\hat{n}$ is called Normal stress and component of **T** perpendicular to $\hat{n}$, i.e., in the plane of area δS is called shear stress.
Normal stress tends to pull the disc away from the surface while shear stress tends to shear the disc off the surface while sliding on it tangentially.
Fluids for which shear stress is negligible are called inviscid, ideal, or perfect while those for which shear stress is dominant are called viscous or real fluids.
(b) Viscosity: It is the internal friction between the particles of the fluid which offers resistance to the deformation of the fluid. The friction is in the form of tangential and shearing forces (stresses).

Fluids with such property are called **viscous** or **real** fluids i.e., in viscous fluid both the tangential and normal forces exist and those not having this property are called **inviscid or ideal or perfect fluids i.e.,** An inviscid fluid is a continuous fluid substance which cannot exert any shearing stress however small. A perfect fluid is also known as frictionless or non-viscous.

Actually, all fluids are real, but in many cases, when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored.

From the definition of body force and shearing stress, it is clear that body force per unit area at every point of surface of an ideal fluid act along the normal to the surface at that point. Thus ideal fluid does not exert any shearing stress.

Thus, we conclude that viscosity of a fluid is that property by virtue of which it is able to offer resistance to shearing stress. It is a kind of molecular frictional resistance.

Coefficient of Viscosity: Consider a fluid element OACB, sheared in one plane by a single shear stress T. The shear strain angle $\delta\theta$, which results under T, will continuously grow with time as long as stress T is maintained, the upper surface moving at relative speed δu, larger than the lower surface. Commonly occurring fluids, such as water, air, oil, etc. show a linear relation between applied shear and the resulting strain rate, i.e.

$$\mathrm{T} \propto \left(\frac{\delta\theta}{\delta t}\right) \;\; i.e., \mathrm{T} = \mu \left(\frac{d\theta}{dt}\right) \tag{1}$$

$$\text{Now } \delta\theta = \tan\delta\theta = \frac{\delta u \delta t}{\delta y} \quad implies \frac{\delta\theta}{\delta t} = \frac{\delta u}{\delta y} \tag{2}$$

From (1) and (2), using limits we get $\mathrm{T} \propto (\frac{du}{dy})$ which we write

$$\mathrm{T} = \mu\frac{d\theta}{dt} = \mu\frac{du}{dy} \qquad [Newton\ law\ of\ viscosity]$$

The constant of proportionality, written μ, is called the *Coefficient of Viscosity.* Eq.(3) is the relation between shear strain rate $\frac{d\theta}{dt}$ and velocity gradient $\frac{du}{dy}$ and the applied stress $\mathrm{T}(=\mathrm{T}_{yx})$.

3.4 Flow Lines: Stream Lines, Path Lines and Streak Lines:

Stream Lines: A stream line (often written $\Psi - line$) is a curve drawn in the fluid such that, at any time, the direction of the tangent at any point of the curve coincides with the direction of the velocity of the fluid particle at that point. Thus if u, v, w be the components of the velocity of the fluid particle at $\mathrm{P}(x, y, z)$, the direction ratios of the tangent being $d\boldsymbol{r} = (dx, dy, dz)$ at that point, the differential equations of stream lines are

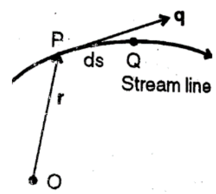
$$\boldsymbol{q} \times d\boldsymbol{r} = \boldsymbol{0}$$

Or

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1}$$

where

$$\boldsymbol{q} = ui + vj + wk$$



Stream lines form a doubly infinite set at any time t. They are generally not material curve : a stream line through r_0 at time t_0 does not, in general, consist of the same particles as the stream line which goes through r_0 at any other time t. The aggregate of all stream lines is called the stream-pattern.

The appearance and form of the stream pattern is altered completely if a uniform velocity is superimposed on the fluid as a whole, e.g., the solutions of

$$\frac{dx}{u+u_0} = \frac{dy}{v} = \frac{dz}{w} \qquad (2)$$

Differ markedly from those of (1). Thus, the stream lines due to a fixed sphere in an infinite uniform stream are very different from those occasioned by the motion of a sphere in a still stream; although the two systems are dynamically equivalent.

The point where **q**=0 is such that the stream lines are not well-defined there at due to various singularities occur there. Such a point is known as critical point or stagnation point.

In terms of components, the differential equations of stream lines are:

In Cartesian coordinates (x, y, z)

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad dr = (dx, dy, dz)$$

For cylindrical coordinates $(r, \theta, z), \boldsymbol{q} = (q_r, q_\theta, q_z)$

$$\frac{dr}{q_r} = r\frac{d\theta}{q_\theta} = \frac{dz}{q_z} \qquad d\boldsymbol{r} = (dr, rd\theta\,, dz)$$

For spherical polar $(r, \theta, \phi), \boldsymbol{q} = (q_r, q_\theta, q_\phi)$

$$\frac{dr}{q_r} = r\frac{d\theta}{q_\theta} = r\sin\theta\ \frac{d\phi}{q_\phi} \qquad d\boldsymbol{r} = (dr, r\ d\theta, rsin\theta\ d\phi)$$

Note: Stream lines are curves whose tangents are everywhere parallel to the velocity vector **q**. In unsteady flow, **q(r,**t) at point **r,** will change both its magnitude and its direction with time, so it is meaningful to consider only the instantaneous stream lines when the flow is unsteady.

The projection of Ψ-line in the planes $z = 0, x = 0, y = 0$ are, as per its definition

$$\frac{dy}{dx} = \frac{v}{u}, \frac{dz}{dy} = \frac{w}{v}, \frac{dx}{dz} = \frac{u}{w}$$

$$yielding \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

The easiest way to solve these equations is to use parametrization, say $x = x(s), y = y(s), z = z(s)$, where parameter s=0 at some reference point and whose value increases along the $\Psi - line$. We can then express above as

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds \ or \ \frac{dx_i}{ds} = q_i(x_i, t)$$ [s is not necessarily arc-length]

Solution of these equations, when Ψ −line goes through $\boldsymbol{r} = \boldsymbol{r_0}$ is

$$x_i = x_i(\boldsymbol{r_0}, t, s) \quad i = 1,2,3.$$

As s varies, this set traces out the $\Psi - line$ through (x_0, y_0, z_0).

Path lines: A path line or trajectory is a curve which a fluid particle describes during its motion i.e., a path line is a line traced by a particle in the fluid.
The path line shows the direction of the velocity of the fluid particle at any instant of time. Such a line is obtained by using the position of an element as a function of time. The path line are given by

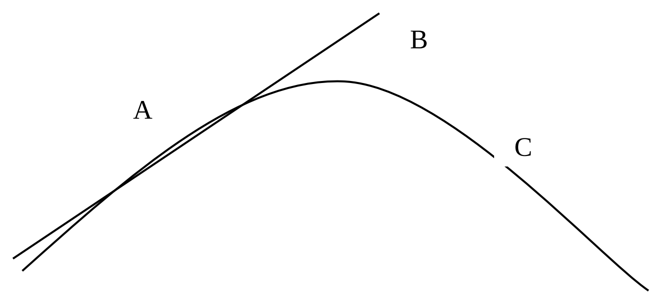
$$\boldsymbol{q} = \frac{d\boldsymbol{r}}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = u(x, y, z, t), \frac{dy}{dt} = v(x, y, z, t), \frac{dz}{dt} = w(x, y, z, t)$$

Path lines form a triply infinite set.
In general, the path lines vary with each fluid particle. It represents the direction of velocity of a single particle of fluid at various time. For steady motion, the stream lines coincide with the paths of the fluid particle, this is not so for unsteady motion.

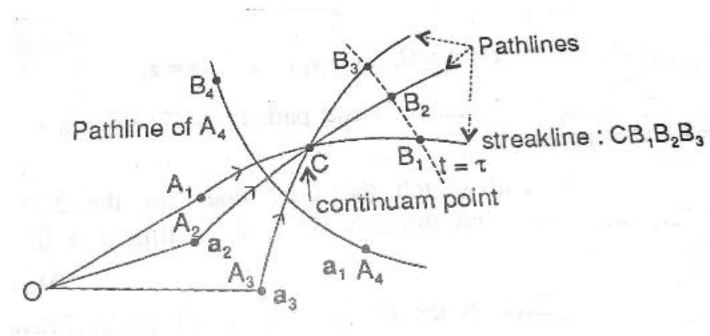
Difference Between the Stream Lines and Path Lines:

Consider a particular stream line and take any three consecutive points A, B and C on it. Since the velocity **q** is a function of **r** and t, any particle through A at time t will move along AB, but when



it reaches B in time δt, BC shall no longer be the direction of velocity at B . Consequently, the particle will not move in the direction of the new velocity at B. However, in the case of steady motion, the stream lines remain unchanged as the time passes, and so these are the same as the actual paths of the fluid particle. In passing we may note that steam lines reveal how each fluid particle is moving at a given instant, whereas the path line show how a given particle is moving at each instant.
Stream surface: A stream surface is a surface made by the steam lines passing through an arbitrary line in the fluid region at any instant of time.
Stream tube: The stream lines drawn through each point of a closed curve enclose a tubular surface in the fluid, called a stream tube or tube of flow. A stream tube of infinitesimal cross-section is called a stream filament.

Streak lines: A streak line is a line on which lie all those fluid elements that at some earlier instant passed through a particular point in space. It is a line making the position of a set of fluid particles that had passed through a fixed point in the flow field. A streak line is defined as the locus of different particles passing through fixed point. A streak line connects the locations of the particles at one instant moving with the fluid which passed through a particular point.



Let C be any point of the continuum. This point is traversed by an infinite number of particles, each with its own path line. Consider three fluid particles A_1, A_2, A_3 labelled by their position vectors $\boldsymbol{a}_!, \boldsymbol{a_2}, \boldsymbol{a_3}$ respectively at time t=0. As these particles describe their separate path lines, these fluid particles will arrive at C at different times and continue to move to occupy the points $B_!, B_2, B_3$ respectively, at some latter time t. These points, together with continuum point C, line on a curve which is called streak line associated with the point C. If a dye is injected at C, a thin strand of colour will appear along this streak line $CB_1B_2B_3$ at time t. Obviously, the streak line $CB_1B_2B_3$ emanating from C will alter its shape with time. A fourth fluid blob A_4, which at time t=0 lies on the path line A_2C will in general, have a different path line A_4B_4 which may never pass through the point C.

Equation of Streak line:

Consider a fluid particle (x_0, y_0, z_0) passes a fixed point $\boldsymbol{r_1}(x_1, y_1, z_1)$ in the course of time. By Lagrangian description of fluid flow, we have

$$x_1 = f_1(x_0, y_0, z_0, t); y_1 = f_2(x_0, y_0, z_0, t); z_1 = f_3(x_0, y_0, z_0, t)$$

Solving the equations for x_0, y_0, z_0 we have

$$x_0 = F_1(x_1, y_1, z_1, t); y_0 = F_2(x_1, y_1, z_1, t); z_0 = F_3(x_1, y_1, z_1, t).$$

Since a streak line is the locus of the positions (x, y, z) of the particles which have passed through the fixed point $(x_1, y_1. z_1)$, therefore, the equation of the streak line at an instant of time t is given by

$$x = G_1(x_0, y_0, z_0, t); y = G_2(x_0, y_0, z_0, t); z = G_3(x_0, y_0, z_0, t).$$

Hence the streak line passing through the fixed point (x_1, y_1, z_1) at time t is given by

$$x = G_1(F_1, F_2, F_3, t); y = G_2(F_1, F_2, F_3, t); z = G_3(F_1, F_2, F_3, t).$$

Example: Find the stream lines and the paths of the particles for the two-dimensional velocity field

$$\boldsymbol{u} = \frac{\boldsymbol{x}}{\boldsymbol{1+t}}, \boldsymbol{v} = \boldsymbol{y}, \boldsymbol{w} = \boldsymbol{0}$$

Solution: The stream lines at time t are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds$$

Or

$$\frac{dx}{ds} = \frac{x}{1+t}, \frac{dy}{ds} = y, \frac{dz}{ds} = 0$$

Thus keeping t constant,(i.e., at particular instant), the stream line through $r_0(a,b,c)$ *is*

$$\log x = \frac{1}{1+t}s + \log a \Rightarrow x = a\, e^{\frac{s}{1+t}}$$

$$\log y = s + \log b \Rightarrow y = be^s$$

$$z = c$$

This is a curve (in the plane z=c); $\frac{y}{b} = \left(\frac{x}{a}\right)^{1+t}$

The particle paths are solutions of

$$\frac{dx}{dt} = \frac{x}{1+t}, \frac{dy}{dt} = y, \frac{dz}{dt} = 0$$

These are $x(a(1+t), y = be^t, z = c$

or the curves in the plane z=c given by $y = b\, e^{\frac{x-a}{a}}$

Example: Find the path lines and streak lines for the velocity field

$$\boldsymbol{q} = (x/t, \text{y}, 0)$$

Solution: (i) For a fluid particle that was initially at $\boldsymbol{r_0} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ and now is at **r**=(x,y,z), the path lines $\frac{dr}{dt} = \boldsymbol{q}$ are

$$\frac{dx}{dt} = \frac{x}{t}, \frac{dy}{dt} = y, \frac{dz}{dt} = 0$$

Therefore, $\int_a^x \frac{dx}{x} = \int_{t_0}^t \frac{dt}{t} \Rightarrow \log\left(\frac{x}{a}\right) = \log\left(\frac{t}{t_0}\right) \Rightarrow x = \frac{at}{t_0}$

$$\int_b^y \frac{dy}{y} = \int_{t_0}^t dt \Rightarrow \log\left(\frac{y}{b}\right) = (t - t_0) \Rightarrow y = be^{t-t_0}$$

$$\frac{dz}{dt} = 0 \Rightarrow z = const. = c$$

Thus the path lines are $x = \frac{at}{t_0},\ y = b\, e^{t-t_0}, z = c$ (1)

(ii) Streak line is the curve traced out by the fluid particles which were initially at $\boldsymbol{r_0} = (a,b,c)$ and now pass through the fixed point $\boldsymbol{r_1} = (x_1, y_1, z_1)$ at time T.

If the fluid particle $\boldsymbol{r_0}$ passes through $\boldsymbol{r_1}$ at time T, then equation (1) yield

$$x_1 = \frac{aT}{t_0}, y_1 = b\, e^{T-t_0}, z_1 = c$$

Thus, $$a = \frac{x_1 t_0}{T}, b = \frac{y_1 e^{t_0}}{e^T}, c = z_1 \qquad (2)$$

Eliminating a, b, c between equations (1) and (2) yields the equation of streak lines through $\boldsymbol{r_1}$ at time t

$$x = \frac{x_1 t}{T}, y = \frac{y_1 e^t}{e^T}, z = z_1 \qquad (3)$$

For the steady flow, streak lines (3) and path lines (2) coincide with the stream lines $x^t = ky, z = constant.$

Example: The velocity vector q is given by $\boldsymbol{q} = \boldsymbol{i}x - \boldsymbol{j}y$. Determine the equation of the stream lines.

Solution: From the definition of the stream line $\boldsymbol{q} \times d\boldsymbol{r} = \boldsymbol{0},$ we have

$$(ix - jy) \times (idx + jdy) = 0$$

or $$(xdy + ydx)\boldsymbol{k} = 0$$

or $$\frac{dx}{x} = -\frac{dy}{y}$$

By integrating, we obtain $$log x + \log y = \log c$$

or $$xy = c$$

which represents the rectangular hyperbolas where c is arbitrary constant.

Example: The velocity q in a three-dimensional flow field for an incompressible fluid is given by

$$\boldsymbol{q = 2xi - yj - zk}$$

Determine the equations of the streamlines passing through the point $(\mathbf{1}, \mathbf{1}, \mathbf{1})$.

Solution: The equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$\Rightarrow$ $$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z}$$

From first two factors, we have

$$\frac{dx}{2x} = \frac{dy}{-y} \Rightarrow \frac{dx}{x} + \frac{2dy}{y} = 0$$

By integrating, $$\log x + 2\log y = \log A$$

or $xy^2 = A,$ where A is an integration constant.

From first and third factors, $$\frac{dx}{2x} = \frac{dz}{-z} \Rightarrow \frac{dx}{x} + 2\frac{dz}{z} = 0$$

By integrating, we have $xz^2 = B,$ where B is an integration constant.

At the point (1,1,1), A=1=B

Hence the required streamlines are

$$xy^2 = 1 \; and \; xz^2 = 1$$

Example: The velocity components in a two-dimensional flow field for an incompressible fluid are given by

$$\boldsymbol{u = e^x cosh\, y \ \ and\ v = -e^x sinh\, y}$$

Determine the equation of the streamlines for this flow.

Solution: The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{e^x \cosh y} = \frac{dy}{-e^x \sinh y}$$

or $\qquad dx + \coth y\, dy = 0$

By integrating $\qquad x + \log \sinh y = \log c \ \Rightarrow \sinh y = ce^{-x}$

where log c is constant of integration.

3.5 The Eulerian and Lagrangian Methods:

Now we describe two methods by which the general problem of Hydro-dynamics can be dealt with. These are Eulerian (Flux) and Lagrangian Methods and refer to 'Local time-rate' of change and 'individual time-rate' of change.

(1) **Euler's method:** In this method we select any point fixed in space occupied by the fluid and observe the changes which take place in velocity, density and pressure as the fluid passes through this point. Obviously, the point being fixed *x, y, z and t* are independent variables and so $\dot{x}, \ddot{x},\ etc.$ are meaningless in this method.

Let us consider any scalar point function

$$\phi(x, y, z, t) = \phi(\boldsymbol{r}, t)$$

associated with a fluid in motion. Then keeping the point *P (x, y,z)*, as fixed, the change is

$$\phi(\boldsymbol{r}, t + \delta t) - \phi(\boldsymbol{r}, t)$$

Whence the local time-rate of change, $\partial\phi/\partial t$ is

$$\frac{\partial\phi}{\partial t} = \lim_{\delta t \to 0} \frac{\phi(\boldsymbol{r}, t + \delta t) - \phi(\boldsymbol{r}, t)}{\delta t}$$

A similar expression can be established for a vector point function, i.e.,

$$\frac{\partial \boldsymbol{f}}{\partial t} = \lim_{\delta t \to 0} \frac{\boldsymbol{f}(\boldsymbol{r}, t + \delta t) - \boldsymbol{f}(\boldsymbol{r}, t)}{\delta t}$$

(2) **Lagrangian Methods:** In this method we seek to determine the history of every fluid particle, i.e. we select any particle of the fluid and purse it on its onward course making observations of changes in velocity, density and pressure at each instant and at each point.

Thus the expressions $\dot{x}, \ddot{x},\ etc.$ have definite significance, and to specify a particular fluid-particle we need its initial position coordinates, say *(a,b,c)* or (r_o) so that there are altogether four independent variables (a, b, c, t) in Cartesian treatment and $(\boldsymbol{r_0}, t)$ in vector treatment.

Let us now consider any scalar point function $\phi(x, y, z, t), i.e., \phi(\boldsymbol{r}, t)$ associated with a fluid in motion. Then keeping the particle fixed, the change is

$$\phi(\boldsymbol{r} + \delta r, t + \delta t) - \phi(r, t)$$

The change δr in the position of the particle during the time δt depends upon **q,** the velocity of the particle at time t. Thus

$$\delta \boldsymbol{r} = \boldsymbol{q}\delta t$$

Then $$\lim_{\delta t \to 0} \frac{\phi(\boldsymbol{r}+\boldsymbol{q}\delta t, t+\delta t)-\phi(\boldsymbol{r},t)}{\delta t} = \frac{d\phi}{dt}$$

is the individual time-rate of change.

3.5.1 Relationship Between the Lagrangian and Eulerian Method:

To relate these two methods, we establish a relation between points in space and the particle parameters.

I. Lagrangian to Eulerian: Let Q be some quantity defined in terms of Lagrangian description

$$Q = Q(a, b, c, t) \tag{1}$$

We shall express a, b, c in terms of the coordinates x,y,z of a point in space. In Lagrangian method, it is defined as

$$x = f_1(a, b, c, t), y = f_2(a, b, c, t), z = f_3(a, b, c, t) \tag{2}$$

By solving these relations to obtain a, b, c in terms of Eulerian variables x, y, z and t, we have

$$a = g_1(x, y, z, t), b = g_2(x, y, z, t), c = g_3(x, y, z, t) \tag{3}$$

From (1) and (3) , we get

$$Q = Q[g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t)] \tag{4}$$

which represents the Eulerian description.

II. Eulerian to Lagrangian: Let Q be some quantity defined in terms of Eulerian description, we then have

$$Q = Q(x, y, z, t) \tag{5}$$

We shall express x,y,z in terms of the particle parameter a, b, c. Let u,v,w are the velocity components at the point (x,y,z) at any instant t, which is defined as

$$u = F_1(x, y, z, t), v = F_2(x, y, z, t), w = F_3(x, y, z, t) \tag{6}$$

Again from the Lagrangian description, we have

$$u = \frac{\partial x}{\partial t},\ v = \frac{\partial y}{\partial t},\quad w = \frac{\partial z}{\partial t} \tag{7}$$

$where\ x, y, z$ are functions of the variables a, b, c and t.

From (6) and (7), the velocity components of a fluid element is given by

$$\frac{\partial x}{\partial t} = F_1(x, y, z, t), \frac{\partial y}{\partial t} = F_2(x, y, z, t), \frac{\partial z}{\partial t} = F_3(x, y, z, t) \tag{8}$$

Which represents the first order linear differential equation. By integrating, we have

$$x = f_1(x_0, y_0, z_0, t), y = f_2(x_0, y_0, z_0, t), z = f_3(x_0, y_0, z_0, t) \tag{9}$$

Where x_0, y_0 , z_0 are the initial values of *x,y,z* at an initial instant $t = t_0$ assumed to be constants of integration. Choosing the particle parameters *a,b,c* equal to x_0, y_0, z_0 respectively. Thus, we have

$$x = f_1(a, b, c, t), y = f_2(a, b, c, t), z = f_3(a, b, c, t) \tag{10}$$

From the relations (5) and (10), we have

$$Q = Q[f_1(a, b, c, t), f_2(a, b, c, t), f_3(a, b, c, t)]$$

Which represents the Lagrangian description.

3.6 Relation between the local and individual time-rates:

Let (u, v, w) be the components of velocity **q** along the coordinate axes, so that

$$\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}; \text{ where } \frac{dx}{dt} = u, etc.$$

Now $$\phi = \phi(x, y, z, t)$$

Therefore,
$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} + \frac{\partial\phi}{\partial t}$$
$$= u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial t}$$
$$= (u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}).\left(\boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z}\right) + \frac{\partial\phi}{\partial t}$$
$$= \boldsymbol{q}.(\nabla\,\phi) + \frac{\partial\phi}{\partial t}$$

Thus,
$$\frac{d\phi}{dt} = \boldsymbol{q}.(\nabla\,\phi) + \frac{\partial\phi}{\partial t} = (\boldsymbol{q}.\nabla)\phi + \frac{\partial\phi}{\partial t}$$

A similar expression for a vector point function **f** can be established in the form

$$\frac{d\boldsymbol{f}}{dt} = (\boldsymbol{q}.\nabla)\boldsymbol{f} + \frac{\partial\boldsymbol{f}}{\partial t}$$

3.7 Substantial Derivative or Material Derivative following the Fluid Motion:

Consider some property of the fluid (e.g., temperature, density, fluid boundary, fluid velocity) typified by some function $G(x, y, z, t,)$; is a scalar (or vector) point function.

Then

$$G = G(x, y, z, t) = G(\boldsymbol{r}, t)$$

The position vector **r** may depend upon time 't' and hence we may calculate $\frac{dG}{dt}$.

Now $$G + \delta G = G(r + \delta r\,, t + \delta t)$$

Therefore,
$$\delta G = G(\boldsymbol{r} + \delta\boldsymbol{r}, t + \delta t) - G(\boldsymbol{r}, t)$$
$$= [\,G(\boldsymbol{r} + \delta\boldsymbol{r}, t + \delta t) - G(\boldsymbol{r}, t + \delta t)] + [G(\boldsymbol{r}, t + \delta t) - G(\boldsymbol{r}, t)]$$

i.e. $$\delta G = \delta\boldsymbol{r}.\nabla G(\boldsymbol{r}, t + \delta t) + \delta t\,\partial G(\boldsymbol{r}, t)/\partial t \quad \text{[to first order]}$$

Dividing both sides by δt and proceeding to limits, we obtain

$$\frac{dG}{dt} = \boldsymbol{q}.\nabla G + \partial G/\partial t \qquad \left[\text{As } \frac{d\boldsymbol{r}}{dt} = \boldsymbol{q}\right] \qquad (1)$$

This equation indicates the time rate of change of the quantity G as a fluid particle moves about but is written in terms of quantity observed at a point.

The **operator** $\frac{d}{dt} \equiv (\boldsymbol{q}.\boldsymbol{\nabla}) + \partial/\partial t$ is known as Substantial derivative or material derivative or differentiation following the motion of the fluid. Often d/dt is denoted by D/Dt . $\frac{\partial}{\partial t}$ =Local derivative , $q.\nabla$ = convective derivative and G is associated with change of physical quantity due to motion of fluid particle.

Notes:

1) The term $(\boldsymbol{q}.\nabla)G$ represents the rate of change of G at a fixed time t due to the change of position from one point to the other and the term $\frac{\partial G}{\partial t}$ gives the rate of change of G at fixed point.
2) $\frac{d\rho}{dt} = 0$ implies incompressible fluid but not steady flow, but $\frac{\partial \rho}{\partial t} = 0$ implies ρ is independent of t at a fixed point.
Similarly, the fluid boundary $f(\boldsymbol{r}, t) = 0$ always consists of the same fluid particles, we must have $\frac{df}{dt} = 0$.
3) If G is replaced by the velocity vector **q** , we obtain particle acceleration, viz.

$$\boldsymbol{a} = \frac{d\boldsymbol{q}}{dt} = (\boldsymbol{q}.\nabla)\boldsymbol{q} + \frac{\partial \boldsymbol{q}}{\partial t}$$

Since $\boldsymbol{q}.\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ $[as\ \boldsymbol{q} = (u, v, w)]$, the acceleration components (a_x, a_y, a_z) are given by $\boldsymbol{a_x} = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ with two more expressions for $\frac{dv}{dt}, \frac{dw}{dt}$ $etc.$

Remarks:

(i) The Eulerian method is sometimes also called the flux method.

(ii) Both Lagrangian and Eulerian methods were used by Euler for studying fluid dynamics.

(iii) Lagrangian method resembles very much with the dynamics of a particle

(iv) The two methods are essentially equivalent, but depending upon the problem, one has to judge whether Lagrangian method is more useful or the Eulerian.

Example: ***The velocity components for a two-dimensional flow system can be given in the Eulerian system by***

$$\boldsymbol{u = 2x + 2y + 3t; v = x + y + t/2}$$

Find the displacement of a fluid particle in the Lagrangian system.

Solution: The velocities may be expressed in terms of the displacements as

$$u = \frac{dx}{dt} = 2x + 2y + 3t; v = \frac{dy}{dt} = x + y + t/2$$
$$\frac{dx}{dt} - 2x - 2y = 3t; \frac{dy}{dt} - x - y = t/2$$

The solutions of the simultaneous differential equations can be determined by operator method as follows:

$(D - 2)x - 2y = 3t$ (1)

$-x + (D - 1)y = \frac{1}{2}t$ (2)

Eliminating x from (1) and (2), we have

$$D(D - 3)y = 2t + 1/2$$

whose solution is given by

$y = a + be^{3t} - \left(\frac{7}{18}\right)t - \left(\frac{1}{3}\right)t^2$ (3)

Substituting the value of y in the equation (2), we have

$x = -a + 2be^{3t} + \left(\frac{1}{3}\right)t^2 - \left(\frac{7}{9}\right)t - \frac{7}{18}$ (4)

The arbitrary constants a and b are determined by using the initial conditions

: $x = x_0, y = y_0$ at $t = t_0 = 0$ in (3) and (4), we have

$$y_0 = a + b;\ x_0 = -a + 2b - 7/18$$

Thus $a = -\frac{1}{3}\left(x_0 - 2y_0 + \frac{7}{18}\right), b = \frac{1}{3}\left(x_0 + y_0 + \frac{7}{18}\right)$

Substituting the values of a and b in (3) and (4), we get the required solution.

Example: For a two-dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u = x + y + 2t; v = 2y + t$. Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.

Solution: Proceed as in example above, we find

$x = Ae^t + Be^{2t} - \frac{1}{4}(6t + 5)$ (1)

$y = Be^{2t} - \frac{1}{4}\ (2t + 1)$ (2)

where A and B are arbitrary constants.

Initially $x = x_0, y = y_0$ at $t = t_0 = 0$; Then $A = x_0 - y_0 + 1; B = y_0 + \frac{1}{4}$

Hence the solution (1) and (2) can be written in the form

$$x = F_1(x_0, y_0, t), y = F_2(x_0, y_0, t)$$

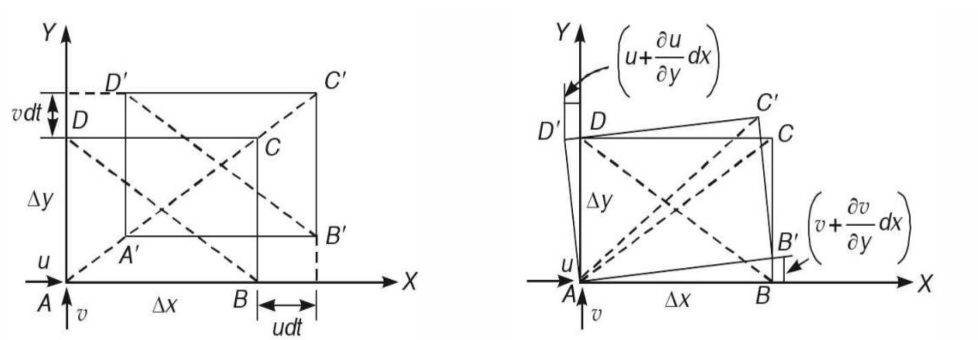
where $F_1 = (x_0 - y_0 + 1)e^t + (y_0 + \frac{1}{4})e^{2t} - \frac{1}{4}(6t - 1)$; $F_2 = (y_0 + \frac{1}{4})e^{2t} - \frac{1}{4}\ (2t + 1)$

This determines the Lagrange coordinates as a function of the initial positions x_0, y_0 and the time t.

3.8 Translation, Deformation and Rotation of Fluid Element:

(i) **Translation motion:** When the fluid particle moves without changing its shape, then that fluid element is said to under goes translation motion.

Let ABCD be the position of fluid element in the rest position and after sometime it take the position $A'B'C'D'$ due to translation motion without changing the shape.



(i) Translation (ii) Rotation

(ii) **Rotation Motion:** When the fluid element rotate about one point without changing the shape, then fluid element is said to be under goes rotation. The angular velocity of rotation is given by

$$w = -\frac{1}{2}\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

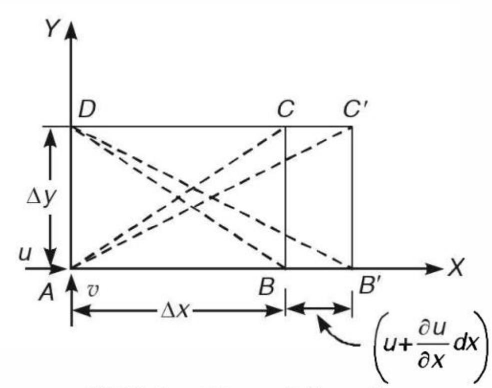
$$= \frac{1}{2} curl\, \boldsymbol{q} = \frac{1}{2}\, \nabla \times \boldsymbol{q}$$

$$= \frac{\mathbf{1}}{\mathbf{2}}\left[\, \boldsymbol{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \boldsymbol{j}\, \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right. \quad (1)$$

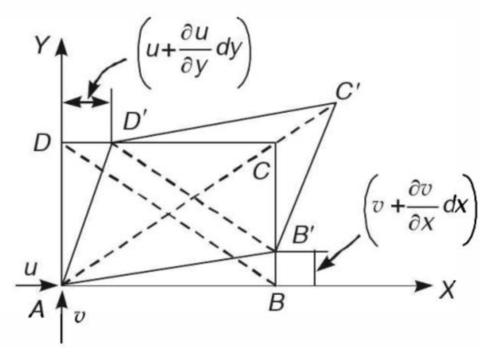
(iii) **Rate of Deformation:** A fluid element is said to be undergoes deformation if the distance between two nearest fluid element changes. There are two types of rates of deformation.

(a) Linear deformation: In this case distance between two particles change in straight line due to motion.

(b) Angular deformation: In this case the distance between two particles changes, when particle rotate about one angular fixed point.

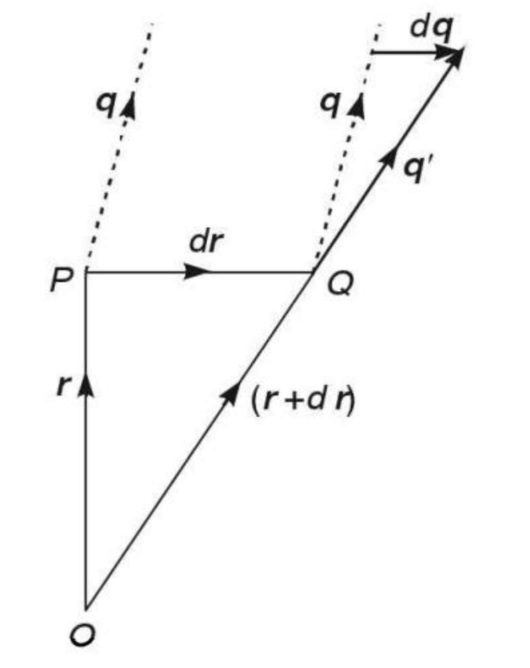


(*iii*) Rate of linear deformation



(*iv*) Rate of angular deformation

Expression for Translation, Rotation and Rate of Deformation:



Let a fluid particle $P(x.y.z)$ whose position vector **r** with respect to origin O. $Q(x+\delta x, y+\delta y, z+\delta z)$ be the position of that fluid particle at any time t whose position vector is $\boldsymbol{r}+\boldsymbol{\delta r}$, **q** is the velocity at P, $\boldsymbol{q}+\boldsymbol{dq}$ be the velocity at Q. Then

$$\boldsymbol{q}' = \boldsymbol{q}+\boldsymbol{dq}$$

$$= \boldsymbol{q}+\frac{\partial \boldsymbol{q}}{\partial x}dx+\frac{\partial \boldsymbol{q}}{\partial y}dy+\frac{\partial \boldsymbol{q}}{\partial z}dz$$

$$= \boldsymbol{q}+\boldsymbol{i}\left[\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz\right]+\boldsymbol{j}\left[\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz\right]+\boldsymbol{k}[\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz]$$

$$[As\ \boldsymbol{q}=\boldsymbol{i}u+\boldsymbol{j}v+\boldsymbol{k}w]$$

$$= \boldsymbol{q}+\boldsymbol{i}\left[\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dz-\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dy\right\}+\left\{\frac{\partial u}{\partial x}dx+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dy+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dx\right\}\right]$$

$$+\boldsymbol{j}\left[\left\{\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx-\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dz\right\}\right.$$

$$\left.+\left\{\frac{\partial v}{\partial y}dy+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dx+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right\}\right]$$

$$+\boldsymbol{k}[\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial u}{\partial z}\right)dy-\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dx\right\}$$

$$+\left\{\frac{\partial w}{\partial z}dz+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dx+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dy\right\}]$$

Thus $\boldsymbol{q}' = \boldsymbol{q}+\boldsymbol{w}\times\boldsymbol{dr}+\boldsymbol{D}$ **(2)**

where

$$\boldsymbol{w}\times\boldsymbol{dr}=\frac{1}{2}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) & \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) & \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\\ dx & dy & dz\end{vmatrix}$$

$$= \boldsymbol{i}\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dz-\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dy\right\} + \mathrm{j}\left\{\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx-\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dz\right\}$$
$$+\boldsymbol{k}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial u}{\partial z}\right)dy-\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dx\right\}$$
$$d\boldsymbol{r} = idx + jdy + kdz$$

$$D = \boldsymbol{i}\left\{\frac{\partial u}{\partial x}dx+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dy+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dx\right\}$$
$$+\boldsymbol{j}\left\{\frac{\partial v}{\partial y}dy+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dx+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right\}$$
$$+\boldsymbol{k}\left\{\frac{\partial w}{\partial z}dz+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)dx+\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)dy\right\}$$
$$= \boldsymbol{i}\left(\in_{xx}dx+\in_{xy}dy+\in_{xz}dz\right)+\boldsymbol{j}\left(\in_{yy}dy+\in_{yx}dx+\in_{yz}dz\right)+\boldsymbol{k}\left(\in_{zz}dz+\in_{zx}dx+\in_{zy}dy\right)$$
$$= \boldsymbol{i}(\in_x.d\boldsymbol{r})+\boldsymbol{j}(\in_y.d\boldsymbol{r})+\boldsymbol{k}(\in_z.d\boldsymbol{r})$$

Where $\in_x = \boldsymbol{i}\in_{xx}+\boldsymbol{j}\in_{xy}+\boldsymbol{k}\in_{xz}, \in_y = \boldsymbol{i}\in_{yx}+\boldsymbol{j}\in_{yy}+\boldsymbol{k}\in_{yz}, \in_z = \boldsymbol{i}\in_{zx}+\boldsymbol{j}\in_{zy}+\boldsymbol{k}\in_{zz}$ are strain rate tractions of the fluid elements in the x, y, z direction.

Equation (2) represents the most general mode of motion of a fluid element. The first term **q** represents the linear motion of all parts of the fluid element without changing the shape of the element. Hence the first term represents the pure translatory part of the motion. The second term $w \times dr$ represents the pure rotation of the fluid element. The third term D represents the rate of deformation (rate of strain term) and so the third term D gives the deformation of the fluid element.

If D=0 then it represents the rigid body. Hence, we can say that the most general motion of a fluid element can be expressed as the combination of translation, rotation and deformation of the fluid element.

Example: ***Velocity field at point is given by*** $\boldsymbol{u = 1 + 2y - 3z, v = 4 - 2x + 5z, w = 6 + 3x - 5y}$***.Show that it represents a rigid body motion.***

Solution: The general motion of fluid element is given by

$$\boldsymbol{q' = q + w \times dr + D}$$

Where $\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j} + z\boldsymbol{k}$ = **translation** velocity

$$= (1 + 2y - 3z)\boldsymbol{i} + (4 - 2x + 5z)\boldsymbol{j} + (6 + 3x - 5y)\boldsymbol{k}$$

$\boldsymbol{w \times dr}$ = **rotation velocity**

Now $\boldsymbol{w} = \frac{1}{2}\, curl\, \boldsymbol{q} = \frac{1}{2}\, \nabla \times \boldsymbol{q} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} =$

$$\frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+2y-3z & 4-2x+5z & 6+3x-5y \end{vmatrix}$$

$$= \frac{1}{2}[i(-5-5) + j(-3-3) + k(-2-2)] = -(5i+3j+2k)$$

$$w \times dr = \begin{vmatrix} i & j & k \\ -5 & -3 & -2 \\ dx & dy & dz \end{vmatrix} = i(2dy-3dz) + j(5dz-2dx) + k(3dx-5dy)$$

D=rate of deformation

$= \boldsymbol{i}\big(\in_{xx}\, dx + \in_{xy}\, dy + \in_{xz}\, dz\big) + \boldsymbol{j}\big(\in_{yy}\, dy + \in_{yx}\, dx + \in_{yz}\, dz\big) + \boldsymbol{k}(\in_{zz}\, dz + \in_{zx}\, dx + \in_{zy}\, dy)$

Here

$$\in_{xx} = \frac{\partial u}{\partial x} = 0 = \in_{yy} = \in_{zz}$$

$$\in_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\in_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\in_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0$$

Put these values in D, we get D=0

This D is called rate of deformation. As D=0 then it represents the rigid body. Hence we can say that the most general motion as a fluid element can be expressed as the combination of translation, rotation and not deformation of the fluid element.

Example: ***What type of the motion do the following velocity components constitute?***

$$u = a + by - cz; v = d - bx + ez\, ; w = f + cx - ey$$

where a, b,c,d,e,f are arbitrary constants.

Solution: We know that general motion of fluid element is given by

$$\boldsymbol{q}' = \boldsymbol{q} + w \times dr + D$$

Where **q**(translation velocity)$=u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} = (a+by-cz)\boldsymbol{i} + (d-bx+ez)\boldsymbol{j} + (f+cx-ey)\boldsymbol{k}$

$$w \times dr = rotation\ velocity$$

$$w = \frac{1}{2}\, curl\, \boldsymbol{q} = \frac{1}{2}\nabla \times \boldsymbol{q}$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a+by-cz & d-bx+ez & f+cx-ey \end{vmatrix}$$

$$= \frac{1}{2}[i(-2e)+j(-2c)+k(-2b)] = -(ei+cj+bk)$$

Therefore, $w \times dr = \begin{vmatrix} i & j & k \\ -e & -c & -b \\ dx & dy & dz \end{vmatrix} = i(bdy-cdz)+j(edz-bdx)+k(dx-edy)$

D=rate of deformation

$$= i\left(\epsilon_{xx}\ dx + \epsilon_{yy}\ dy + \epsilon_{zz}\ dz\right) + j\left(\epsilon_{yx}\ dx + \epsilon_{yy}\ dy + \epsilon_{yz}\ dz\right) + k\left(\epsilon_{zx}\ dx + \epsilon_{zy}\ dy + \epsilon_{zz}\ dz\right)$$

Here $\epsilon_{xx} = \frac{\partial u}{\partial x} = 0, \epsilon_{yy} = \frac{\partial v}{\partial y} = 0, \epsilon_{zz} = \frac{\partial w}{\partial z} = 0$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) = \frac{1}{2}(b-b) = 0$$

$$\epsilon_{yz} = 0; \epsilon_{zx} = 0$$

Therefore, D=0

Hence the motion of fluid element consistent translation and rotation not deformation. As D=0 so it represents the rigid body motion.

Example: ***Give a velocity field with components*** $u = cx + 2\,w_0 y + u_0; v = cy + v_0; w = -2cz + w_0$ *where* c, u_0, v_0 *and* w_0 ***are constants with the above velocity components at a point p(x,y,z), determine the velocity components at neighbouring point*** $\boldsymbol{Q(x+dx, y+dy, z+dz)}$ ***and determine the different types of motion which are involved.***

Solution: The general motion of fluid element is given by

$$\boldsymbol{q}' = \boldsymbol{q} + w \times dr + D$$

Where **q**(translation velocity)=$u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} = (cx+2w_0 y+u_0)i + (cy+v_0)j + (-2cz + w_0)k$

$$w \times dr = rotation\ velocity$$

$$w = \frac{1}{2}\ curl\ \boldsymbol{q} = \frac{1}{2}\nabla \times \boldsymbol{q}$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ cx+2w_0 y+u_0 & cy+v_0 & -2cz+w_0 \end{vmatrix} = -w_0 k$$

$$w \times dr = \begin{vmatrix} i & j & k \\ 0 & 0 & -w_0 \\ dx & dy & dz \end{vmatrix} = i(w_0 dy) + j(-w_0 dx) = (w_0 dy)i - (w_0 dx)j$$

D=rate of deformation

$$= i\left(\epsilon_{xx}\ dx + \epsilon_{yy}\ dy + \epsilon_{zz}\ dz\right) + j\left(\epsilon_{yx}\ dx + \epsilon_{yy}\ dy + \epsilon_{yz}\ dz\right) + k\left(\epsilon_{zx}\ dx + \epsilon_{zy}\ dy + \epsilon_{zz}\ dz\right)$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = c; \epsilon_{yy} = \frac{\partial v}{\partial y} = c; \epsilon_{zz} = \frac{\partial w}{\partial z} - 2c$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = w_0$$

$$\epsilon_{yz} = 0 \,;\, \epsilon_{zx} = 0$$

Therefore, $D = (cdx + w_0 dy)i + (w_0 dx + cdy)j - (2cdz)k$

Hence point Q has the three types of motion (i) translation velocity (ii) rotational velocity (iii) rate of strain velocity.

3.9 Particle Acceleration: Since the velocity field vector **q** is a function of both position and time, i.e., of four independent variables, we may write it as, say

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r}, t) \qquad (1)$$

Suppose that the value of the velocity at time $t + \delta t$ when the particle has moved to a neighbouring position is $\boldsymbol{q} + \boldsymbol{\delta q}$. Then

$$\delta \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r} + \delta r, t + \delta t) - \boldsymbol{q}(\boldsymbol{r}, t)$$
$$= [\boldsymbol{q}(\boldsymbol{r} + \boldsymbol{\delta r}, t + \delta t\,) - q(\boldsymbol{r}, t + \delta t\,)] - [\boldsymbol{q}(\boldsymbol{r}, t + \delta t) - \boldsymbol{q}(\boldsymbol{r}, t)] \qquad (2)$$

Now, to first order of approximations

$$\boldsymbol{q}(\boldsymbol{r} + \boldsymbol{\delta r}, t + \delta t\,) - q(\boldsymbol{r}, t + \delta t\,) = (\delta r.\nabla)\,\boldsymbol{q}(\boldsymbol{r}, t + \delta t) \qquad (3)$$
$$\boldsymbol{q}(\boldsymbol{r}, t + \delta t) - \boldsymbol{q}(\boldsymbol{r}, t) = \delta t\ \partial \boldsymbol{q}(r, t)\ /\partial t \qquad (4)$$

The acceleration **a** of the fluid particle at a point being $Lim\ (\delta \boldsymbol{q}\ /\delta t\ )\ as\ \delta t \to 0$;we divide (2) by δt, use (3) and (4) and proceed to the limits. These yields

$$\boldsymbol{a} = \frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}.\nabla)\boldsymbol{q} \qquad \textbf{(5)}$$

NOTES:

(1) The expression (5) is in reality the Lagragian acceleration. In the Eulerian concepts, it is composed of two factors: one a temporal acceleration ($\partial \boldsymbol{q}/\partial t$) at the point, and the other convective acceleration, $(\boldsymbol{q}.\nabla)\boldsymbol{q}$, resulting from flow entering the fluid element from regions having different velocities.

(2) **Lagrange's acceleration relation:** Since

$$(\boldsymbol{q}.\nabla)\boldsymbol{q} = \nabla\left(\frac{1}{2}\boldsymbol{q}^2\right) + \omega \times \boldsymbol{q}\,, \qquad (\omega = curl\ q)$$

$\therefore$ $$\boldsymbol{a} = \frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \omega \times \boldsymbol{q}$$

(6)

The acceleration vector given by (6) is *Lagrange's acceleration relation* and its chief merit is that whereas the form (5) is not invariant under a change of coordinate system, the form (6) is invariant under change of coordinate system.

The vector $\boldsymbol{q} \times \omega$ is called *Lamb vector*.

(3) **Particle acceleration in curvilinear coordinates.** With velocity components (q_1, q_2, q_3) in the (α, β, γ) −directions and using the vector definitions

$$\nabla = \left(\frac{1}{h_1}\frac{\partial}{\partial \alpha}, \frac{1}{h_2}\frac{\partial}{\partial \beta}, \frac{1}{h_3}\frac{\partial}{\partial \gamma}\right), \qquad curl\ \boldsymbol{q} = \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$$

where $\omega_1 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \beta}(h_3 q_3) - \frac{\partial}{\partial \gamma}(h_2 q_2)\right], etc.$

we get the acceleration components $(a_1, a_2, a_3)$from Lagrange's acceleration relation

$$\boldsymbol{a} = \left(\frac{d\boldsymbol{q}}{dt}\right) = \frac{\partial \boldsymbol{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \omega \times \boldsymbol{q}$$

$$a_1 = \frac{\partial q_1}{\partial t} + \frac{1}{h_1}\frac{\partial}{\partial \alpha}(q_1^2 + q_2^2 + q_3^2) + (\omega_2 q_3 - \omega_3 q_2) \qquad (7)$$

with similar expressions for $a_2\ and\ a_3$.

(4) **Particle acceleration in cylindrical coordinates.** With velocity components (u, v, w) in the (r, θ, z) − directions and using the vector definitions

$$\boldsymbol{q}(u, v, w); q^2 = u^2 + v^2 + w^2; \nabla = \left(\frac{\partial}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial z}\right)$$

$$curl\ \boldsymbol{q} = \left[\frac{1}{r}\frac{\partial \omega}{\partial \theta} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \theta}\right]$$

In the Lagrange acceleration relation, we get

$$\boldsymbol{a} = \frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)(u^2 + v^2 + w^2) + (\omega_1, \omega_2, \omega_3) \times (u, v, w)$$

(i)

Putting $\frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z} = \frac{d}{dt}$

and splitting the three components in (i), we get

$$\boldsymbol{a} = \left(\frac{du}{dt} - \frac{v^2}{r}, \frac{dv}{dt} + \frac{uv}{r}, \frac{dw}{dt}\right) \qquad (8)$$

(5) **Particle acceleration in space polar coordinates.** With velocity components (u, v, w) in (r, θ, ϕ)-directions and using the vector definitions

$$\boldsymbol{q} = (u, v, w), q^2 = u^2 + v^2 + w^2; \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{rsin\theta}\frac{\partial}{\partial \phi}\right)$$

$curl\ \boldsymbol{q} = \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, where

$$\omega_1 = \frac{1}{rsin\theta}\left[\frac{\partial}{\partial\theta}(wsin\theta) - \frac{\partial v}{\partial\phi}\right], \qquad \omega_2 = \frac{1}{r}\left[\frac{1}{sin\theta} r\frac{\partial u}{\partial\phi} - \frac{\partial}{\partial r}(r\,w)\right],$$

$$\omega_3 = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial\theta}\right];$$

in the Lagrange acceleration relation, we get

$$\boldsymbol{a} = \frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\left(\frac{\partial}{\partial r}\ ,\frac{1}{r}\frac{\partial}{\partial\theta}\ ,\frac{1}{rsin\theta}\ \frac{\partial}{\partial\phi}\right)(u^2 + v^2 + w^2) + (\omega_1, \omega_2, \omega_3) \times (u, v, w)$$

(i)

Putting $\frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + \frac{w}{rsin\theta}\frac{\partial}{\partial\phi} = \frac{d}{dt}$

and splitting the three components in (i), we get

$$\boldsymbol{a} = \left(\frac{du}{dt} - \frac{v^2+w^2}{r}\ ,\frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r}\ ,\frac{dw}{dt} + \frac{vw\cot\theta}{r} + \frac{uw}{r}\right) \qquad (9)$$

Example: ***Determine the acceleration of a fluid particle from the following flow field***

$$\boldsymbol{q} = i(Axy^2t) + j(Bx^2yt) + k(Cxyz)$$

Solution: We have acceleration

$$\boldsymbol{a} = \frac{d\boldsymbol{q}}{dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\boldsymbol{q} = a_x i + a_y j + a_z k;\ \boldsymbol{q} = ui + vj + wk$$

Comparing with given equation

$$u = Axy^2t, v = Bx^2yt,\ w = Cxyz \qquad (1)$$

Then components of acceleration along x, y, z axes

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Using (1), we have the components of acceleration in x,y,z axis:

$$a_x = Axy^2t \times\ (Ay^2t) + Bx^2yt \times (2Axyt) + Cxyz \times (0) + Axy^2$$
$$= A^2xy^4t + 2ABx^3y^2t^2 + Axy^2 = Axy^2(Ay^2t^2 + 2Bx^2t^2 + 1)$$

$$a_y = Axy^2t \times (2Bxyt) + Bx^2yt \times (Bx^2t) + Cxyz \times (0) + Bx^2y$$
$$= 2ABx^2y^3t^2 + B^2x^4yt^2 + Bx^2y$$
$$a_z = Axy^2t \times (Cyz) + Bx^2yt \times (Cxz) + Cxyz \times (Cxy) + 0$$
$$= ACxy^3zt + BCx^3yzt + C^2x^2y^2z$$

Example: ***The velocity components in spherical polar coordinates $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$ of a flow are***

$$\boldsymbol{q_r = \left(\frac{r^2}{t^2}\right) sin\phi, q_\theta = \frac{r}{t} cot\,\theta\, cosec\,\phi, \quad q_\phi = \frac{r}{t} sin\,\theta\, cos\,\phi} \qquad \textbf{(1)}$$

Determine the components of acceleration of a fluid particle.

Solution: $\boldsymbol{a} == \left(\frac{dq_r}{dt} - \frac{q_\theta^2 + q_\phi^2}{r}, \frac{dq_\theta}{dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{dq_\phi}{dt} + \frac{q_\theta q_\phi \cot\theta}{r} + \frac{q_r q_\phi}{r}\right)$;

$$\boldsymbol{\frac{d}{dt}} = \left(\frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r sin\theta}\frac{\partial}{\partial \phi}\right)$$

Let $a_r,\ a_\theta, a_\phi$ be the components of acceleration, then

$$a_r = \frac{\partial q_r}{\partial t} + q_r \frac{\partial q_r}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_r}{\partial \theta} + \frac{q_\phi}{r sin\,\theta}\frac{\partial q_r}{\partial \phi} - \frac{q_\theta^2 + q_\phi^2}{r}$$

(2)

$$a_\theta = \frac{\partial q_\theta}{\partial t} + q_r \frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_\phi}{r sin\,\theta}\frac{\partial q_\theta}{\partial \phi} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}$$

(3)

$$a_\phi = \frac{\partial q_\phi}{\partial t} + q_r \frac{\partial q_\phi}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\phi}{\partial \theta} + \frac{q_\phi}{r sin\,\theta}\frac{\partial q_\phi}{\partial \phi} + \frac{q_\theta q_\phi \cot\theta}{r} + \frac{q_r q_\phi}{r} \qquad (4)$$

Using (1) in (2), (3) and (4) we obtain

$$a_r = -\frac{2r^2}{t^3}\sin\phi + \left(\frac{r^2 \sin\phi}{t^2}\right)\left(\frac{2r}{t^2}\right)\sin\phi + \frac{1}{t}(\cos\phi)\left(\frac{r^2}{t^2}\cos\phi\right) - \frac{r}{t^2}(\cot^2\theta\, cosec^2\phi + \sin^2\theta\cos^2\phi)$$

$$a_\theta = -\frac{r}{t^2}\cot\theta\, cosec\phi$$
$$+ \left(\frac{r^2}{t^2}\right)\sin\phi\frac{1}{t}(cot\theta\ cosec\ \phi) + \frac{1}{t}(\cot\theta\ cosec\phi)\frac{1}{t}(-r\ cosec^2\theta\ cosec\ \phi)$$
$$+ \frac{1}{t}(\cos\phi)\frac{1}{t}(r\cot\theta\ cosec\phi\cot\phi) + \frac{r^2}{t^3}\cot\theta - \frac{r}{t^2}\sin^2\theta\cos^2\phi\cot\theta$$

$$a_\phi = -\frac{r}{t^2}\sin\theta\cos\phi + \left(\frac{r^2}{t^2}\sin\phi\right)\left(\frac{1}{t}\sin\theta\cos\phi\right) + \left(\frac{1}{t}\cot\theta\ cosec\ \phi\right)\left(\frac{r}{t}\cos\theta\cos\phi\right)$$
$$+ \frac{1}{t}\cos\phi\left(-\frac{r}{t}\sin\theta\sin\phi\right) + \frac{r^2}{t^3}\sin\theta\sin\phi\cos\phi + \frac{r}{t^2}\sin\theta\cot^2\theta cot\phi$$

Example: ***The velocity component of a flow in cylindrical polar coordinates are*** $(\boldsymbol{r^2 z\cos\theta}$, $\boldsymbol{rz\sin\theta},\ \boldsymbol{z^2 t})$. ***Determine the components of the acceleration of a fluid particle.***

Solution: Let q_r, q_θ, q_z be the components of velocity in cylindrical polar coordinates (r, θ, z). Then we have

$$q_r = r^2 z\cos\theta, q_\theta = rz\sin\theta, q_z = z^2 t$$

Let $a_r, a_\theta\ and\ a_z$ be the components of acceleration. Then

$$\boldsymbol{a} = \left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}, \frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}, \frac{dq_z}{dt}\right); \frac{d}{dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial\theta} + q_z\frac{\partial}{\partial z}$$

$$a_r = \frac{\partial q_r}{\partial t} + q_r\frac{\partial q_r}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_r}{\partial\theta} + q_z\frac{\partial q_r}{\partial z} - \frac{q_\theta^2}{r}$$
$$= rz^2(2r^2\cos^2\theta - 3\sin^2\theta + rt\cos\theta)$$

$$a_\theta = \frac{\partial q_\theta}{\partial t} + q_r \frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\theta}{\partial \theta} + q_z \frac{\partial q_\theta}{\partial z} + \frac{q_r q_\theta}{r}$$
$$= z^2 r\, sin\theta(3\, r\cos\theta + t)$$

$$a_z = \frac{\partial q_z}{\partial t} + q_r \frac{\partial q_z}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_z}{\partial \theta} + q_z \frac{\partial q_z}{\partial z}$$
$$= z^2(1 + 2t^2 z)$$

3.10 The Velocity Potential or Velocity Function-Potential Flow:

Suppose that the fluid velocity at time t is $\boldsymbol{q} = (u, v, w)$ and further suppose that the expression $u\,dx + v\,dy + w\,dz$ is an exact differential, say $(-d\Phi)$, where $\Phi(\mathrm{x,y,z,t})$ is a scalar function.

Then $$-d\Phi = udx + v\,dy + w\,dz \tag{1}$$

But $$\Phi = \Phi(x, y, z, t) \tag{2}$$

So $$d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz + \frac{\partial \Phi}{\partial t}dt \tag{3}$$

From (1) and (3), we get

$$udx + v\,dy + w\,dz = -(\frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz + \frac{\partial \Phi}{\partial t}dt)$$

Comparing up terms, we have

$$u = -\frac{\partial \Phi}{\partial x}, \qquad v = -\frac{\partial \Phi}{\partial y}, \qquad w = -\frac{\partial \Phi}{\partial z}$$
$$\therefore \boldsymbol{q} = -\nabla\,\Phi \tag{4}$$

and $\frac{\partial \Phi}{\partial t} = 0 \quad \Rightarrow \Phi = \Phi(x, y, z)$ $i.e.$ Φ is a function of x,y,z.

This Φ is called velocity *potential.* The negative sign in the equation (4) is a convention. It ensures that the flow takes from the higher potential to lower potential.

3.11 Vorticity: If **q** be the velocity vector of a fluid particle, then the vector quantity

$$\Omega = \nabla \times \boldsymbol{q} = curl\,\boldsymbol{q}$$

Is called the *vorticity vector* or simply the *vorticity* and is a measure of the angular velocity of an infinitesimal element. The components of spin are given by (ξ, η, ζ), where

$$\Omega = \xi i + \eta j + \zeta k = \nabla \times \boldsymbol{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$$

Thus, we have

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Note:

1. In two dimensional Cartesian coordinates, the vorticity is given by
$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$
2. In two dimensional polar coordinates, the vorticity is given by
$$\Omega = \frac{\mathrm{q_\theta}}{r} + \frac{\partial q_\theta}{\partial r} - \frac{1}{r}\frac{\partial q_r}{\partial q_\theta}$$

3. The vorticity components in cylindrical polar coordinates (r, θ, z) are given by

$$\Omega_r = \frac{1}{r}\frac{\partial q_z}{\partial \theta} - \frac{\partial q_\theta}{\partial z}, \quad \Omega_\theta = \frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}, \quad \Omega_z = \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r} - \frac{1}{r}\frac{\partial q_r}{\partial \theta}$$

4. The vorticity components in spherical polar coordinates (r, θ, ϕ) are given by

$$\Omega_r = \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} - \frac{1}{rsin\theta}\frac{\partial q_\theta}{\partial \phi} + \frac{q_\phi}{r}\cot\theta \;, \Omega_\theta = \frac{1}{rsin\theta}\frac{\partial q_r}{\partial \phi} - \frac{\partial q_\phi}{\partial r} - \frac{q_\phi}{r}, \Omega_\phi = \frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r} - \frac{1}{r}\frac{\partial q_r}{\partial \theta}$$

Vortex Line: A vortex line is a curve drawn in the fluid such that the tangent to it at each point is in the direction of the vorticity vector Ω at that point. The vortex line is often abbreviated into $\Omega - line$.

The definition of the vortex line implies that its analytical expression is given by $dr \times \Omega = 0$, or its equivalent in Cartesian form by the differential equations

$$dr \times \Omega = (idx + jdy + kdz) \times (i\,\xi + j\,\eta + k\,\zeta) = 0$$

$$(\eta\, dz - \zeta\, dy)i + (\zeta\, dx - \xi\, dz)j + (\xi\, dy - \eta\, dx)k = 0$$

$$\eta\, dz - \zeta\, dy = 0, \qquad \zeta dx - \xi\, dz = 0, \xi\, dy - \eta\, dx = 0$$

$$\therefore\ \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

Vortex tube and Vortex filament: The vortex lines drawn through each point of a closed curve enclose a tubular space in the fluid called a vortex tube. A vortex tube of infinitesimal cross-section is called a vortex filament or simply a vortex.

3.12 Rotational and Irrotational Motion:

The motion of a fluid is said to be irrotational when the vorticity vector Ω of every fluid particle is zero so that $\xi = 0, \eta = 0, \zeta = 0$. When the vorticity vector is different from zero, the motion is said to be rotational.

Rotational motion is also called vortex motion. The definition implies that in an irrotational motion of the fluid, there are no vortex lines.

Since $\Omega = curl\, \boldsymbol{q}$ and $\Omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$

We conclude that the motion is irrotational if $curl\, \boldsymbol{q} = 0$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}\,, \ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}\,, \ \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

when the motion is irrotational i.e. when curl **q**=0, then **q** must be of the form $(-grad\phi)$ for some scalar point function $\phi(say)$ because $curl\ grad\ \Phi = 0$. Thus velocity potential exists whenever the fluid motion is irrotational. Again, notice that when velocity potential exists, the motion is irrotational because $\boldsymbol{q = -grad\,\phi \ \Rightarrow -curl\ grad\ \Phi = 0.}$

Thus, the fluid motion is irrotational if and only if the velocity potential exists.

Note 1: We may observe that whenever velocity potential exists, the system of surfaces given by the differential equation

$$\boldsymbol{q}.\boldsymbol{dr} = \boldsymbol{0} \ \ or\ \ udx + vdy + wdz = 0$$

(1)

possess solution, $\Phi(r) = constant,$ for

$$0 = \boldsymbol{q}.\boldsymbol{dr} = -\nabla\,\Phi.\,dr = -d\Phi \;\Rightarrow \Phi(r) = constant.$$

The surfaces $\Phi(r) = constant$ are called the equipotential. Further, these surfaces cut the stream lines $\boldsymbol{q} \times \boldsymbol{dr} = \boldsymbol{0}$ orthogonally since the velocity vector which is parallel to dr for the stream lines, is perpendicular to dr in (1).

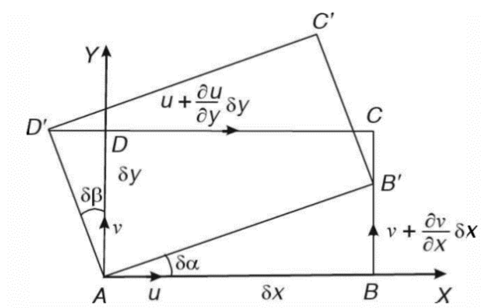
Note 2: Vortex is flow in circles about a central point. It is termed free when motion is such that the tangential velocity **q** is inversely proportional to the radius, i.e., $\boldsymbol{q} \propto \frac{\mathbf{1}}{\boldsymbol{r}}$ implies $\mathbf{q}\boldsymbol{r} = constant.$ Motion is irrotational and vorticity is zero; stream lines are circles and circulation is constant.

A vortex is termed forces when the motion is the result of some external force, and the motion is such that $\boldsymbol{q} \propto \boldsymbol{r}$ **.** Hence vorticity is constant.

3.13 The angular velocity vector: Rotational Flow:

Consider a rectangular element in two-dimensional flow such that $AB = \delta x$ and $AD = \delta y$ as shown in figure. Upon rotating about A during a small interval δt, let the element assume the shape indicated by $A'B'C'D'$ in figure, $B'$and D' approximately lying on BC and CD produced.

Let $u\ and\ v$ be the components of velocity at A. Then the components of velocity along BC and DC are respectively $v(x + \delta x, y) = v + \frac{\partial v}{\partial x}\delta x$ and $u(x, y + \delta y\,) = u + \frac{\partial u}{\partial y}\delta y.$



Therefore, velocity of B relative to A along BC $= \frac{\partial v}{\partial x}\delta x$

and velocity of D relative to A along DC$= \frac{\partial u}{\partial y}\delta y$

$$\therefore BB' = \frac{\partial v}{\partial x}\delta x\ \delta t \text{ and } DD' = -\frac{\partial u}{\partial y}\delta y\ \delta t$$

Hence, the angular velocity of AB about z-axis i.e., perpendicular to the plane through A

$$= \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{tan\delta\alpha}{\delta t} \qquad \because \delta\alpha\ is\ small, \delta\alpha = \tan\delta\alpha$$

$$= \lim_{\delta t \to 0} \frac{BB'/\delta x}{\delta t} = \lim_{\delta t \to 0} \frac{\frac{\partial v}{\partial x}\delta x\ \delta t}{\delta x\ \delta t} = \frac{\partial v}{\partial x}$$

Again, the angular velocity of AD about z-axis

$$= \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{tan\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{DD'/\delta y}{\delta t} = -\lim_{\delta t \to 0} \frac{\frac{\partial u}{\partial y}\delta y\ \delta t}{\delta y\ \delta t} = -\frac{\partial u}{\partial y}$$

Let ω_z denote the average of the angular velocities of AB and AD. Then, we have

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \qquad (1)$$

The average angular velocity components ω_x , ω_y *and* ω_z, in the case of three-dimensional flows may be obtained in a similar manner as follows:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),\quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (2)$$

Hence the angular velocity vector $\boldsymbol{\omega}$ of a fluid element is given by

$$\boldsymbol{\omega} = \boldsymbol{i}\,\omega_x + \boldsymbol{j}\omega_y + \boldsymbol{k}\,\omega_z$$

$$= \frac{1}{2}[\boldsymbol{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)]$$

Thus, $\boldsymbol{\omega} = \frac{1}{2}\, curl\, \boldsymbol{q}$ *or* $2\,\boldsymbol{\omega} = curl\, \boldsymbol{q}$ (3)

But the vorticity vector Ω is given by $\Omega = curl\ \boldsymbol{q}$ (4)

From (3) and (4), we have $\boldsymbol{\Omega = 2\,\omega}$

Thus, the curl of the velocity of any particle of a rigid body equal twice the angular velocity.

Note: 1. $\boldsymbol{\omega}$ is also called the rotation. The condition for the two-dimensional flow to be irrotational is that the rotation w_z is everywhere zero i.e., $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$.

Again, the condition for irrotational in three-dimensional flow is that

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z};\ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x};\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Note 2. A flow, in which the fluid particle also rotates (i.e. possess some angular velocity) about their own axes, while flowing, is said to be a rotational flow. Again, a flow, in which the fluid particles do not rotate about their own axes, and retain their original orientations, is said to be an irrotational flow.

3.14 Reynold's Transport Theorem:

Let V denote the Lagrangian region which moves with the fluid and **r** be any general point of V. Since V consists of the same fluid particles, **r=r(t)** yields the position vector of a typical fluid particle of the region V. Let F(**r**, t) denote some scalar field (e.g. temperature or density) associated with the fluid. Reynold's theorem states that

$$\frac{d}{dt}\int_V F(\boldsymbol{r},t)dV = \int_V \{\frac{\partial F}{\partial t} + \nabla.(\,F\boldsymbol{q})\}dV = \int_V \left(\frac{DF}{Dt} + F\ div\ \boldsymbol{q}\right)dV$$

(1)

Proof: Let $G(t) = \int_V F(\boldsymbol{r},t)dV$ (2)

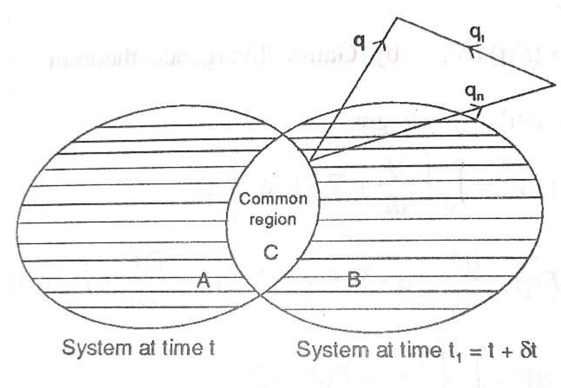


Figure shows the region at time t and also at time $t_1 = t + \delta t. Since\ \delta t$ is vanishingly small, there is always a region C (say) common to both locations. Thus at time t, system occupies volume $V = A \cup C$, while at time t_1, it extends over $B \cup C$. The common part C is the same at times t and t_1. However, the specific property F in it may be different since F depends on time. Thus the rate of change of G becomes

$$\frac{dG}{dt} = \lim_{\delta t \to 0} \frac{G(t_1) - G(t)}{\delta t} = \lim_{\delta t \to 0} \frac{[G_C(t_1) + G_B(t_1)] - [G_A(t) + G_C(t)]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{G_C(t_1) - G_C(t)}{\delta t} + \lim_{\delta t \to 0} \frac{G_B(t_1) - G_A(t)}{\delta t} = \frac{\partial G}{\partial t} + H \qquad (3)$$

$$where \frac{\partial G}{\partial t} = \frac{\partial}{\partial t} \int_V F\, dV = \int_V \frac{\partial F}{\partial t}\, dV \qquad \text{(constant location)}$$

(4)

The term H expresses contribution due to flow of fluid, it needs elaboration. The volumetric flow rate of fluid, dQ passing through a differential area dS is given by

$$dQ = \boldsymbol{q} \,.\, \hat{\boldsymbol{n}}\, \mathrm{dS} \qquad [dV = (\boldsymbol{q}.\hat{\boldsymbol{n}})\, dt\, dS, dQ = \frac{dV}{dt}$$

Where $\boldsymbol{q} \,.\, \hat{\boldsymbol{n}} > \boldsymbol{0} \Rightarrow$ outflow from C to B; $\boldsymbol{q} \,.\, \hat{\boldsymbol{n}} < \boldsymbol{0} \Rightarrow$ inflow from A to C.

The flux of fluid characteristic F through dS is

$$dH = F\, dQ = F(\boldsymbol{q} \,.\, \hat{\boldsymbol{n}})\, dS$$

The total contribution through closed surface S, by summing, is

$$H = \lim_{\delta t \to 0} \frac{G_B(t_1) - G_A(t)}{\delta t} = \oint_S F(\boldsymbol{q} \,.\, \hat{\boldsymbol{n}})\, dS$$

$$= \int_V \nabla.(\mathrm{F}\mathbf{q})\mathrm{dV}, \qquad \text{[by Gauss Divergence Theorem]}$$

(5)

From (1), (3),(4) and (5) we get

$$\frac{d}{dt}\int_V F(\boldsymbol{r}, t) dV = \int_V \left\{\frac{\partial F}{\partial t} + \nabla.(F\boldsymbol{q})\right\} dV$$

(6)

Since $\frac{\partial F}{\partial t} + \nabla\,.\,(F.q) = \frac{\partial F}{\partial t} + \boldsymbol{q}.\nabla F + F(\nabla.\boldsymbol{q}) = \frac{DF}{Dt} + F(div\,\boldsymbol{q})$

$$\therefore \frac{d}{dt}\int_V F(\boldsymbol{r}.t) dV = \int_V \left\{\frac{DF}{Dt} + F\,(div\,\boldsymbol{q})\right\} dV \qquad (7)$$

Equations (6) and (7) are the combined statement (2) of Reynold’s transport theorem.

3.15 Check Your Progress:

i) Determine the streamlines and the path of the particles

$$u = \frac{x}{1+t}, \qquad v = \frac{y}{1+t}, w = \frac{z}{1+t}$$

[Ans: $x = Ay, y = bz;\ x = a(1+t), y = b(1+t), z = c(1+t)$.

ii) Find the equation of the stream lines for the flow $\boldsymbol{q} = -\boldsymbol{i}(3y^2) - \boldsymbol{j}(6x)$ at the point (1,1).

[Ans: $3x^2 = y^3 + 2$

iii) Determine the rates of strain and explain the nature of rates of strain for the following velocity components:

a) $u = cx, v = 0, w = 0$ b) $u = u(x, y), v = v(x, y), w = 0$ c) $u = 2cy, v = 0, w = 0$

d) $u = c, v = 0, w = o$

[Hint: Find $\in_{xx}, \in_{yy}, \in_{zz}, \in_{xy}, \in_{yz}, \in_{zx}$. Again, find the kind of motion i.e., translational, rotational, or rate of deformation]

iv) Given the velocity field **q**=$\boldsymbol{i}Ax^2yt + \boldsymbol{j}By^2zt + \boldsymbol{k}Czt^2$, determine the acceleration of a fluid particle of fixed identity.

[Ans: $A(2Ax^3y^2 + Bx^2y^2zt), B(y^2z + 2By^3z^2Cy^2zt^3), C * 2zt + Czt^4)$]

v) Determine the acceleration at the point (2,1,3) at t=0.5 sec, $u = yz + t, v = xz - t\ and\ w = xy$.

[Ans: $19.5 m/sec^2$,$13.5 m/sec^2$,$6.5 m/sec^2$]

vi) Determine the vorticity components when velocity distribution is given by

$$\mathbf{q} = \boldsymbol{i}Ax^2yt + \boldsymbol{j}By^2zt + \boldsymbol{k}Czt^2$$

where A,B and C are constants.

[Ans: $-By^2t, 0, -Ax^2t$]

vii) The velocity in the flow fluid is given by $\boldsymbol{q} = \boldsymbol{i}(Az - By) + \boldsymbol{j}(Bx - Cz) + \boldsymbol{k}(Cy - Ax)$ where A,B,C are non-zero constant. Determine the equation of the vortex line.

viii) Show that the velocity potential $\phi = \left(\frac{a}{2}\right) \times (x^2 + y^2 - 2z^2)$ satisfies the Laplace equation. Also determine the stream lines.

3.16 Summary: Fluid kinematics deals with describing the motion of fluids without necessarily considering the forces and moments that cause the motion. In this chapter, several kinematic concepts related to flowing fluids are introduced. We discuss the material derivative and its role in transforming the conservation equation from the Lagrangian description of fluid flow to the Eulerian description. After that the various ways to visualize flow fields-stream lines, streak lines, path lines are discussed. The fundamental kinematic properties of fluid motion and deformation-rate of translation, rate of rotation have been explained. Finally, we discussed the Reynold's Transportation theorem.

3.17 Keywords: Velocity, Acceleration, Lagrangian method, Eulerian method, Velocity potential, vorticity, vortex line, stream lines, streak lines, path lines, Rotational flow.

3.18 Self-Assessment Test:

SA1: Obtain equation of motion in terms of stress components of a fluid which flowing with velocity (u,v,w).

SA2:Show that general motion of fluid element is made up of three parts namely, pure translation, rotation and rate of deformation.

SA3: Show that the velocity field $q_r = 0, q_\theta = Ar + \frac{B}{r}, q_z = 0$ satisfy the equation of motion $\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = 0, where\ A\ and\ B$ are arbitrary constants.

SA4: Give examples of irrotational and rotational flows.

SA5: Show that $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two-dimensional fluid. Show that the streamlines at time 't' are the curves $(x-t)^2 - (y-t)^2 = constant$, and the paths of the fluid particles have the equations.

SA6: Show that the following velocity field is a possible case of irrotational flow of an incompressible flow $u = yzt, v = zxt, w = xyt$.

SA7: Prove that acceleration of the fluid element of fixed identity can be represented by the material derivative of the velocity vector.

SA8: Differentiate between the Lagrangian approach and Eulerian approach of the fluid motion.

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CHAPTER -4

EQUATION OF CONTINUITY

4.0 Learning Objectives: After reading this chapter, the students should be able to apply the conservation of mass equation to balance the incoming and outgoing flow rates in a flow system, to express the equation of continuity in Lagrangian and Eulerian method and their equivalency

4.1 Equation of Continuity or Conservation of mass:

Physical quantities are said to be *conserved* when they do not change with regard to time during a process. The law of conservation of mass state as 'mass is neither created nor destroyed.' The mathematical expression of the law of conservation of mass is known as the *equation of continuity*.

By continuity we mean physical quantity. The fluid always remains a continuum i.e., as a continuously distributed matter. When a region of fluid contains neither sources nor sinks i.e., there is no creation or annihilation of the fluid then the amount of fluid within the region is conserved in accordance with the principle of conservation of matter. The general conservation principle is defined as follows:

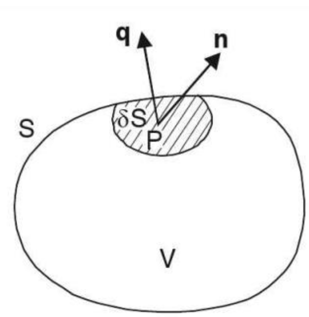
$$In - Out + Source - Sink = Accumulation,$$

where each term represents a rate for a differential element of volume.

4.1.1 Equation of Continuity (Vector form) by Euler's Method:

Let ρ denotes the density of the fluid at a point P($\mathbf{r}$) of the mass of the fluid contained in any closed surface S fixed in space and containing a volume element V. The continuity equation is based upon the following maxim.

The rate at which the mass of fluid inside any volume is increasing is equal to the source rate of mass within the volume minus the rate at which mass flows out through the surface of the volume.



Now, if mass of the fluid within this surface is m then

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int_V \rho \, dv = \int_V \frac{\partial \rho}{\partial t} \, dv \tag{1}$$

Since volume does not vary with time. Further, if R is the source rate of mass per unit volume, then the fluid mass generated is

$$\int_V R dv \tag{2}$$

Let δS be small surface element, **n** is the unit normal to the surface, **q** is the velocity of fluid at P(r) then component of velocity along the normal $= \boldsymbol{n}.\boldsymbol{q}$

The rate of mass across $\delta S = density \times velocity \times area = \rho\,(\boldsymbol{n}.\boldsymbol{q})\delta S$

The total rate at which mass flow out $= \int_S \rho\,(\boldsymbol{n}.\boldsymbol{q}) dS \; = \int_V div\,(\rho \boldsymbol{q}) dv$

{By divergence theorem}

The above maxim now provides the mathematical formulation

$$\int_V \frac{\partial \rho}{\partial t} \, dv = \int_V R dv - \int_V div\,(\rho \boldsymbol{q}) dv \tag{3}$$

or
$$\int_V \left[\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{q}) - R\right] dv = 0 \tag{4}$$

The result (4) will hold for any arbitrarily chosen volume V. Hence the integrand itself must vanish and the continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{q}) = R \tag{5}$$

For the very special but important case, when R=0, the source-free equation of continuity is

$$\frac{\partial \rho}{\partial t} + div\,(\rho\, \boldsymbol{q}) = 0 \tag{6}$$

This is the equation of continuity by Euler's method.

Note: 1. In the absence of sources within the surface, i.e., when R=0, the continuity maxim reads thus:

The increase in the mass of the fluid within the fixed surface during the time δt must be equal to the excess of the mass that flows in over the mass that flows out in the same interval δt.

Note:2. The forms (3) and (5) or (6) are known as the integral and differential forms of the equation of continuity.

Cor.1. Since $\nabla.(\rho\, \boldsymbol{q}) = \rho\, \nabla.\boldsymbol{q} + (\boldsymbol{q}.\nabla)\rho$ the equation of continuity may be written as

$$\frac{\partial \rho}{\partial t} + \rho\, \nabla.\boldsymbol{q} + (\boldsymbol{q}.\nabla)\rho \; = 0 \quad or \quad \frac{d\rho}{dt} + \rho\,(\nabla.\boldsymbol{q}) = 0 \tag{7}$$

$$\left[\text{since } \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{q}.\nabla)\rho\right]$$

Since $\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$; $\therefore \;\; \nabla.(\rho \boldsymbol{q}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$

the equation of continuity (7) can be put in the Cartesian form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Or
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Further if equation (7) is divided by ρ, we have $\frac{1}{\rho}\frac{d\rho}{dt} + (\nabla.\boldsymbol{q}) = 0 \Rightarrow \frac{d}{dt}(\log \rho) + (\nabla.\boldsymbol{q}) = 0.$

Cor.2. If the fluid be incompressible and motion is steady then density is constant i.e., $\frac{\partial\rho}{\partial t}=0$, so the equation of continuity reduces to

$$\nabla.\boldsymbol{q}=0 \quad or \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{8}$$

Thus, the velocity **q** is solenoidal. Obviously $(\nabla.\boldsymbol{q})$ gives the rate of volume expansion of a fluid element. For this reason, it may be called dilatation or expansion.

Cor.3. If the fluid is incompressible and motion is irrotational then there exists velocity potential Φ, i.e. $\boldsymbol{q}=-\boldsymbol{grad}\,\Phi$, and hence

$$div\,\boldsymbol{q}=\boldsymbol{0} \;\Rightarrow \boldsymbol{\nabla^2}\Phi=0 \qquad \text{by (8)} \qquad \text{[Laplace equation]}$$

From equation (5), it becomes

$$\nabla^2\Phi=R \qquad \text{[Poisson's equation]}$$

Note: Since $div\,\boldsymbol{q}=-(d\rho/dt)/\rho$, we can interpret $div\,\boldsymbol{q}$ as the relative rate at which the density is decreasing. Thus, $div\,\boldsymbol{q}>0 \Rightarrow \frac{d\rho}{dt}<0$ and consequently an attenuation of the fluid at the point considered: hence the term divergence.

Example: ***A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is $\boldsymbol{u_0}$ to become $\boldsymbol{\rho_0\Phi(vt-x)}$. Prove that the velocity u (at time and distance x from the origin) is given by***

$$\boldsymbol{v+\frac{(u_0-v)\Phi(vt)}{\Phi(vt-x)}}$$

Solution: The equation of continuity in the present case is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0 \;\; or \;\; \frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{1}$$

Since $\quad \rho=\rho_0\Phi(vt-x)=\rho_0\Phi(z) \quad (say)$

$\therefore \quad \frac{\partial\rho}{\partial x}=\rho_0\frac{d\Phi}{dz}\frac{\partial z}{\partial x}=-\rho_0\frac{d\Phi}{dz}=-\rho_0\Phi'(z)$

$$\frac{\partial\rho}{\partial t}=\rho_0\frac{d\Phi}{dz}\frac{\partial z}{\partial t}=\rho_0\,v\,\Phi'(z)$$

With these (1) reduces to

$$v\,\Phi'(z)-u\,\Phi'(z)-\Phi(z)\frac{\partial u}{\partial z}=0 \qquad [\partial x=-\partial z]$$

or $\quad \frac{du}{v-u}-\frac{d\Phi}{\Phi}=0$

Integrating this equation, we get

$$\log(v-u)+\log\Phi=\log A$$

Or $\quad (v-u)\Phi=A$

At any time, t, when $x=0, u=u_0, so\ that\ \Phi(vt-x)=\Phi(vt)$

Hence $\quad A=(v-u_0)\Phi(vt)$

$\therefore \quad (v-u)\Phi=(v-u_0)\Phi(vt)$

$$u=v+\frac{(u_0-v)\Phi(vt)}{\Phi(vt-x)}$$

Example: ***If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that***

$$\left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{x}}\right)^{2}+\left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{y}}\right)^{2}=\boldsymbol{q}\ \nabla^{2}\boldsymbol{q}$$

Solution: Since the motion is irrotational and density is constant, we have

$$q^2 = \Phi_x^2 + \Phi_y^2;\ \ \Phi_{xx} + \Phi_{yy} = \nabla^2\Phi = 0 \tag{1}$$

To get the form of $q\nabla^2 q$, we have to appeal to Laplacian expansion

$$\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2(\nabla a).(\nabla b)$$

Where we put $a = b = q$ to obtain

$$\nabla^2 q^2 = 2[q\nabla^2 q + (\nabla q)^2] \tag{2}$$

$$\nabla^2 q^2 = \nabla^2\left(\Phi_x^2 + \Phi_y^2\right) = \nabla^2\Phi_x^2 + \nabla^2\Phi_y^2$$

$$= 2[\Phi_x\nabla^2\Phi_x + (\nabla\Phi_x)^2] + 2[\Phi_y\nabla^2\Phi_y + \left(\nabla\Phi_y\right)^2] \qquad \text{by (2)}$$

$$= 2[(\nabla\Phi_x)^2 + \left(\nabla\Phi_y\right)^2] \qquad \text{since } \nabla^2\Phi_x = \frac{\partial}{\partial x}\nabla^2\Phi = 0\ \ etc.$$

$$= 2[\left(i\,\Phi_{xx} + j\Phi_{xy}\right)^2 + \left(i\Phi_{xy} + j\Phi_{yy}\right)^2]$$

$$= 4[\Phi_{xx}^2 + \Phi_{xy}^2] \qquad \text{(since by (1) } \Phi_{xx} = -\Phi_{yy}) \tag{3}$$

Further, taking the gradient of the first of (1), we get

$$q\nabla q = \Phi_x\nabla\Phi_\mathrm{x} + \Phi_y\nabla\Phi_y = \Phi_x\left(i\Phi_{xx} + j\Phi_{xy}\right) + \Phi_y(i\Phi_{yx} + j\Phi_{yy})$$

$$= i\left(\Phi_x\Phi_{xx} + \Phi_y\Phi_{xy}\right) + j(\Phi_x\Phi_{xy} + \Phi_y\Phi_{yy})$$

$$(q\nabla q)^2 = (\Phi_x^2 + \Phi_y^2)(\Phi_{xx}^2 + \Phi_{xy}^2)$$

Or

$$(\nabla q)^2 = \Phi_{xx}^2 + \Phi_{xy}^2 \tag{4}$$

From (3) and (4), we get

$$\nabla^2 q^2 = 4\,(\nabla q)^2$$

Therefore, from (2) $\quad 4\,(\nabla q)^2 = 2[q\nabla^2 q + (\nabla q)^2]$

i.e., $\quad (\nabla q)^2 = q\nabla^2 q$

which is the required result.

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4.2. Equation of Continuity in Cartesian Co-ordinates: - Let (x, y, z) be the rectangular Cartesian co-ordinates.

Let $\quad \boldsymbol{q} = u\boldsymbol{i} + vj + w\boldsymbol{k}$ (1)

and $\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\ \boldsymbol{k}$ (2)

Then, the equation of continuity $\frac{\partial\rho}{\partial t} + div(\rho\boldsymbol{q}) = 0$ can be written as

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{3}$$

i.e., $\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$ (4)

which is the required equation of continuity in Cartesian co-ordinates.

Corollary (1). If the fluid motion is steady, then $\frac{\partial\rho}{\partial t} = 0$ and the equation (3) becomes

$$\frac{\partial}{\partial x}(eu) + \frac{\partial}{\partial y}(ev) + \frac{\partial}{\partial z}(ew) = 0 \qquad (5)$$

Corollary (2). If the fluid is incompressible, then ρ = constant and the equation of continuity is $\nabla\cdot\mathbf{q} = 0$

i.e. $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (6)$$

Corollary (3). If the fluid is incompressible and of potential kind, then equation of continuity is

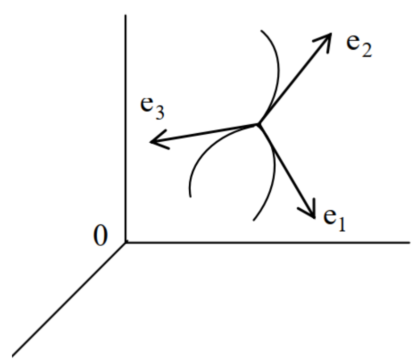
$$\nabla^2\phi = 0$$

i.e. $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$, where $\bar{q} = -\nabla\varphi$.

4.3. Equation of Continuity in Orthogonal Curvilinear Co-Ordinates: Let (u_1, u_2, u_3) be the orthogonal curvilinear co-ordinates and $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$be the unit vectors tangent to the co-ordinate curves.

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be the unit vectors tangent to the co-ordinate curves.

Let $\boldsymbol{q} = q_1e_1 + q_2e_2 + q_3e_3$ (1)



The general equation of continuity is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{q}) = 0 \qquad (2)$$

We know from vector calculus that for any vector point function **f** = (f_1, f_2, f_3),

$$\nabla\cdot\boldsymbol{f} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3f_1) + \frac{\partial}{\partial u_2}(h_3h_1f_2) + \frac{\partial}{\partial u_3}(h_1h_2f_3)\right] \qquad (3)$$

where h_1, h_2, h_3 are scalars.

Using (3), the equation of continuity (2) becomes

$$\frac{\partial\rho}{\partial t} + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3\rho q_1) + \frac{\partial}{\partial u_2}(h_3h_1\rho q_2) + \frac{\partial}{\partial u_3}(h_1h_2\rho q_3)\right] \qquad (4)$$

Corollary (1). When motion of fluid is steady, then equation (4) becomes

$$\frac{\partial}{\partial u_1}(h_2h_3\rho q_1)+\frac{\partial}{\partial u_2}(h_3h_1\rho q_2)+\frac{\partial}{\partial u_3}(h_1h_2\rho q_3)=0 \tag{5}$$

Corollary (2). When the fluid is incompressible, the equation of continuity is (ρ = const)

$$\frac{\partial}{\partial u_1}(h_2h_3q_1)+\frac{\partial}{\partial u_2}(h_3h_1q_2)+\frac{\partial}{\partial u_3}(h_1h_2q_3)=0 \tag{6}$$

Corollary (3). When fluid is incompressible and irrotational then ρ = const $\bar{q}=-\nabla\varphi=-\left(\frac{1}{h_1}\frac{\partial}{\partial u_1},\frac{1}{h_2}\frac{\partial}{\partial u_2},\frac{1}{h_3}\frac{\partial}{\partial u_3}\right)\phi$ and the equation of continuity becomes

$$\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\varphi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)=0 \tag{7}$$

Now, we shall write equation (4) in cylindrical & spherical polar co-ordinates.

4.4. Equation Of Continuity In Cylindrical Co-Ordinates (r, θ, z) . Here,

$$u_1\equiv r, u_2\equiv\theta, u_3\equiv z \ and \ h_1=1, h_2=r, h_3=1$$

The equation of continuity becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(r\rho q_3)\right]=0$$

i.e.

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho q_1)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3)=0 \tag{8}$$

Corollary (1). When the fluid motion is steady, then equation (8) becomes

$$\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+r\frac{\partial}{\partial z}(\rho q_3)=0 \tag{9}$$

Corollary (2). For incompressible fluid, equation of continuity is

$$\frac{\partial}{\partial r}(rq_1)+\frac{\partial}{\partial\theta}(q_2)+r\frac{\partial q_3}{\partial z}=0 \tag{10}$$

Corollary (3). When the fluid is incompressible and is of potential kind, then equation (8) takes the form

$$\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\varphi}{\partial z}\right)=0 \tag{11}$$

where $\boldsymbol{q}=-\nabla\phi$; ∇ is expressed in cylindrical co-ordinates.

4.5 Equation Of Continuity in Spherical Co-Ordinates (r, θ, ψ). Here,

$$(u_1,u_2,u_3)\equiv(r,\theta,\psi) \quad and \quad h_1=1, h2=r, h3=r\sin\theta$$

The equation of continuity becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, eq_1)+\frac{\partial}{\partial\theta}(r\sin\theta\, eq_2)+\frac{\partial}{\partial\psi}(r\rho q_3)\right]=0$$

$$\Rightarrow \quad \frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial\theta}(\sin\theta\,\rho q_2)+r\frac{\partial}{\partial\psi}(\rho q_3)\right]=0 \tag{12}$$

Corollary (1). For steady case, equation (12) becomes

$$\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial\theta}(\sin\theta\,.\,\rho q_2)+r\frac{\partial}{\partial\psi}(\rho q_3)=0 \tag{13}$$

Corollary (2). For incompressible fluid, we have

$$\sin\theta\frac{\partial}{\partial r}(r^2q_1)+r\frac{\partial}{\partial\theta}(\sin\theta\,.\,q_2)+r\frac{\partial q_3}{\partial\psi}=0 \tag{14}$$

Corollary (3). When fluid is incompressible and of potential kind, then equation of continuity is

$$\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \varphi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{\sin\theta}.\frac{\partial \varphi}{\partial \psi}\right) = 0 \qquad (15)$$

where $\boldsymbol{q} = -\nabla\phi$; ∇ is expressed in spherical co-ordinates.

4.6 Symmetrical Forms of Motion and Equation Of Continuity for them. We have the following three types of symmetry which are special cases of cylindrical and spherical polar co-ordinates.

(i) Cylindrical Symmetry: - In this type of symmetry, with suitable choice of cylindrical polar co-ordinates (r, θ, z), every physical quantity is independent of both θ and z so that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0 \quad and \quad \boldsymbol{q} = \boldsymbol{q}(r,t)|Not\ \boldsymbol{r}$$

For this case, the equation of continuity in cylindrical co-ordinates, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r) = 0 \qquad (1)$$

If the flow is steady, then equation (1) becomes

$$\frac{\partial}{\partial r}(\rho q_1 r) = 0 \qquad \Rightarrow \rho q_1\, r = \text{constant} = F(t), \text{(say)}.$$

Further, if the fluid is incompressible then $q_1\, r$ = constant = G(t), (say).

(ii) Spherical Symmetry: - In this case, the motion of fluid is symmetrical about the centre and thus with the choice of spherical polar co-ordinates (r, θ, ψ), every physical quantity is independent of both θ & ψ. so that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \psi} = 0 \quad and\ \boldsymbol{q} = \boldsymbol{q}(r,t)$$

The equation of continuity, for such symmetry, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}.\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \qquad (2)$$

For steady motion, it becomes

$$\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \qquad \Rightarrow \rho q_1\, r^2 = \text{const} = F(t), \text{(say)}$$

and for incompressible fluid, it has the form $q_1\, r^2$ = constant = G(t), (say).

(iii) Axial Symmetry: - (a) In cylindrical co-ordinates (r, θ, z), axial symmetry means that every physical quantity is independent of θ , i.e.,$\frac{\partial}{\partial \theta} = 0$ and thus the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r) + \frac{\partial}{\partial z}(\rho q_3 r)\right] = 0$$

(b) In spherical co-ordinates (r, θ, ψ), axial symmetry means that every physical quantity is independent of ψ i.e. $\frac{\partial}{\partial \psi} = 0$ and the equations of continuity, for this case, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho q_2 \sin\theta) = 0.$$

4.7 Equation of Continuity in the Lagrangian Method: Let us consider a fluid particle of infinitesimal volume dv and density ρ at time t. Since the mass of the fluid-particle is invariant as it moves about, we must have

$$\frac{d}{dt}(\rho dv) = 0$$

Hence
$$\rho dv = constant = \rho_0 dv_0 \; (say) \qquad (1)$$

Where $\rho_0 dv_0$ refers to the mass of the particle in its initial position at $t = t_0$.
In Cartesian rectangular coordinates, let

$$dv = dx\, dy\, dz; \;\; dv_0 = da\, db\, dc\,;$$

Then since $x, y, z\; are\; functions\; of\; a, b, c$

$$dx\, dy\, dz = \frac{\partial(x\,, y, z)}{\partial(a, b, c)}\, da\, db\, dc$$

Hence the principle of continuity gives

$$\rho\, \frac{\partial(x\,, y, z)}{\partial(a, b, c)}\, da\, db\, dc = \rho_0\, da\, db\, dc$$

Or
$$\rho\, \frac{\partial(x\,, y, z)}{\partial(a, b, c)} = \rho_0$$

Or
$$\rho J = \rho_0 \quad \text{where } J = \frac{\partial(x\,, y, z)}{\partial(a, b, c)}$$

Note: It is not necessary that $r_0 = ai + bj + ck$ should be the initial position vector. Any variable vector which can serve to identify a particle and which changes continuously from one particle to another will serve the purpose.

4.8 Equivalence of the Eulerian and the Lagrangian forms of the Equation of Continuity:

Equation of the continuity in Lagrangian form is

$$\rho J = \rho_0$$

Where
$$J = \frac{\partial(x, y, z)}{\partial(a, b, c)}$$

The components of velocity in the two systems are connected by $u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}$

Now since

$$J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$\frac{dJ}{dt} = \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \end{vmatrix}$$

$$\frac{dJ}{dt} = \frac{\partial(u,y,z)}{\partial(a,b,c)} + \frac{\partial(x,v,z)}{\partial(a,b,c)} + \frac{\partial(x,y,w)}{\partial(a,b,c)} \qquad (1)$$

Now,

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial b}$$

$$\frac{\partial u}{\partial c} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c}$$

Now eliminating $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ from the above three equations provide

$$\begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} - \frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} - \frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} = 0$$

Splitting this determinant into two, we get

$$\frac{\partial u}{\partial x}\,\frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial(u,y,z)}{\partial(a,b,c)}$$

Or $\qquad \frac{\partial u}{\partial x}\ J = \frac{\partial(u,y,z)}{\partial(a,b,c)}$

Similarly, $\qquad \frac{\partial v}{\partial y}\ J = \frac{\partial(x,v,z)}{\partial(a,b,c)}; \qquad \frac{\partial w}{\partial z}\ J = \frac{\partial(x,y,w)}{\partial(a,b,c)}$

Adding these three equations, and using (1) we obtain

$$\frac{dJ}{dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Or
$$\frac{dJ}{dt} = J\ div\ \boldsymbol{q} \qquad (2)$$

Step-1 Lagrangian equation of continuity

$$\rho J = \rho_0 \ \Rightarrow \frac{d}{dt}(\rho J) = 0 \ \Rightarrow \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$$

$$\Rightarrow J\frac{d\rho}{dt} + \rho\, J\ div\ \boldsymbol{q} = 0 \qquad \text{by (2)}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho\ div\ \boldsymbol{q} = \boldsymbol{0} \qquad \text{[Eulerian equation of continuity]}$$

Step-2 Eulerian equation continuity

$\Rightarrow \qquad \frac{d\rho}{dt} + \rho \, div \, \boldsymbol{q} = \boldsymbol{0}$

$\Rightarrow \qquad \frac{d\rho}{dt} + \rho \frac{1}{J}\frac{dJ}{dt} = 0 \qquad$ by (2)

$\Rightarrow \qquad J\frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0 \Rightarrow \frac{d}{dt}\,(\rho J) = 0$

Integrating, we get $\quad \rho \, J = \rho_0 \quad$ (say)

$\Rightarrow$ Lagrangian equation of continuity.

4.9 Kinematically Possible Incompressible Fluid Motion:

If the velocity vector $\boldsymbol{q} = (u, v, w)$ be kinematically possible for an incompressible fluid motion, then the equation of continuity must be satisfied. If in addition the motion is irroatational, then $curl\, \boldsymbol{q} = 0$, or in Cartesian form

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0;\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0\ ;\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (1)$$

Evidently, in such a case the velocity potential ϕ necessarily exists and is given by $\boldsymbol{q} = -\boldsymbol{grad}\, \phi$,

$$or \qquad u = \frac{\partial \phi}{\partial x}\ , v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}$$

In case the equations (1) are not satisfied, i.e., $curl\, \boldsymbol{q} \neq \boldsymbol{0}$, then the motion is vertical (rotational) and velocity potential cannot exist.

The stream lines, if needed, are easily obtained by solving the differential equations

$$\boldsymbol{q} \times d\boldsymbol{r} = \boldsymbol{0} \ \ \boldsymbol{or}\ \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Example: ***Show that*** $\boldsymbol{u = -\frac{2xyz}{(x^2+y^2)^2}}$; $\boldsymbol{v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}}$; $\boldsymbol{w = \frac{y}{x^2+y^2}}$ ***are the velocity components of possible liquid motion. Is this motion irrotational?***

Solution: $\quad \frac{\partial u}{\partial x} = 2yz\,\frac{3x^2-y^2}{(x^2+y^2)^3};\frac{\partial v}{\partial y} = 2yz\frac{y^2-3x^2}{(x^2+y^2)^3};\frac{\partial w}{\partial z} = 0$

The equation of continuity for the incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substitution led to

$$\frac{2yz(3x^2 - y^2)}{(x^2 + y^2)^3} + 2yz\frac{y^2 - 3x^2}{(x^2 + y^2)^3} + 0 = 0$$

Which is satisfied.

For the motion to be possible it is evidently necessarily that the equation of continuity should be satisfied.

For irrotational motion

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0;\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0;\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

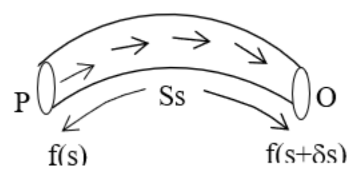
Here $\frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2}$; $\frac{\partial w}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}, etc.$

Thus, all the three equations referred to above are satisfied. Hence the motion is irrotational.

Example: If σ(s) is the cross-sectional area of a stream filament, prove that the equation of continuity is

$\boldsymbol{\frac{\partial}{\partial t}(\rho\sigma) + \frac{\partial}{\partial s}(\rho\sigma q) = 0}$**, where δs is an element of arc of the filament and q is the fluid speed.**

Solution. Let P and Q be the points on the end sections of the stream filament.



The rate of flow of fluid out of volume of filament is

$$(\rho q\sigma)_Q - (\rho q\sigma)_P = \frac{\partial}{\partial s}(\rho q\sigma)_P Ss$$

where we have retained the terms up to first order only, since δs is infinitesimally small

Now, the fluid speed is along the normal to the cross-section. At time t, the mass within the segment of filament is ρσδs and its rate of increase is

$$\frac{\partial}{\partial t}(\rho\sigma\delta s) = \frac{\partial}{\partial t}(\rho\sigma)\delta s \tag{2}$$

Using law of conservation of mass, we have from (1) & (2)

$$\frac{\partial}{\partial t}(\rho\sigma)\delta s + \frac{\partial}{\partial s}(\rho q\sigma)\delta s = 0 \qquad | \text{ Total rate} = 0$$

i.e.

$$\frac{\partial}{\partial t}(\rho\sigma) + \frac{\partial}{\partial s}(\rho\sigma q) = 0 \tag{3}$$

which is the required equation at any point P of the filament.

Deduction: - For steady incompressible flow, $\frac{\partial}{\partial t}(\rho\sigma) = 0$ and equation (3) reduces to

$$\frac{\partial}{\partial s}(\rho\sigma q) = 0 \Rightarrow \frac{\partial}{\partial s}(\sigma q) = 0 \Rightarrow \sigma\, q = \text{constant}$$

which shows that for steady incompressible flow product of velocity and cross-section of stream filament is constant. This result means that the volume of fluid a crossing every section per unit time is constant

$$\left(\sigma q = c \Rightarrow \sigma\frac{dis\tan ce}{t} = c \Rightarrow \frac{volume}{t} = c\right)$$

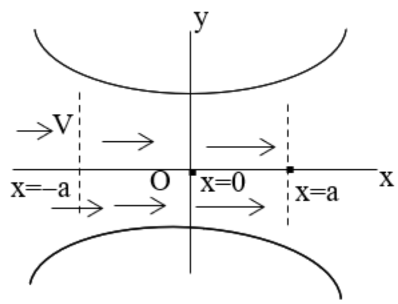
Example: ***Liquid flows through a pipe whose surface is the surface of revolution of the curve y = a + k x^2/a about the x-axis (–a ≤ x ≤ a). If the liquid enters at the end x = –a of the pipe with***

velocity V, show that the time taken by a liquid particle to transverse the entire length of the pipe from x = –a to x = a is

$$\frac{2a}{V(1+k)^2}\left(1+\frac{2}{3}k+\frac{1}{5}k^2\right)$$

Assume that k is so small that the flow remains appreciably one-dimensional throughout.

Solution. Here, $y = a + k\frac{x^2}{a}$



Therefore, area of the section distant x from 0 is = $\pi\, y^2 = \pi\left(a+\frac{kx^2}{a}\right)^2$

Area at x = –a is $\pi\,(a + ka)^2 = \pi\, a^2\,(1+k)^2$.

Applying the equation of continuity at the two sections (i.e. expressing equal rates of volumetric flow across the two sections i.e. equating flux), we get

$$\pi a^2(1+k)^2\, V = \pi\left(a+\frac{kx^2}{a}\right)^2 \dot{x},$$

where $\dot{x}$ (= dx/dt) is the velocity at the section distant x from 0.

Therefore, $dt = \frac{1}{V(1+k)^2}\left(1+\frac{kx^2}{a^2}\right)^2 dx$

Thus, the required time is

$$T = 2\int_0^a \frac{1}{V(1+k)^2}\left(1+\frac{kx^2}{a^2}\right)^2 dx$$

$$= \frac{2}{V(1+k)^2}\int_0^a \left(1+\frac{k^2x^4}{a^4}+\frac{2kx^2}{a^2}\right) dx$$

$$= \frac{2a}{V(1+k)^2}\left(1+\frac{2}{3}k+\frac{1}{5}k^2\right)$$

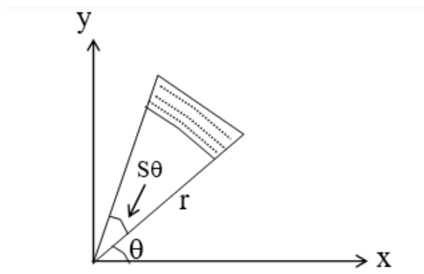
Hence the result

Example: ***A mass of a fluid moves in such a way that each particle describes a circle in one plane about a fixed axis, show that the equation of continuity is***

$$\boldsymbol{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho \omega) = 0,}$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

Solution. Here, the motion is in a plane i.e., we have a two-dimensional case and the particle describe a circle



Therefore, z = constant, r = constant

$$\Rightarrow \qquad \frac{\partial}{\partial z} = 0, \frac{\partial}{\partial r} = 0 \tag{1}$$

i.e. there is only rotation.

We know that the equation of continuity in cylindrical co-ordinates (r, θ, z) is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho q_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3) = 0 \tag{2}$$

Using (1), we get

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2) = 0$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho r\omega) = 0, \text{ where } q = q_2 = r\omega.$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho \omega) = 0$$

Hence the result

Example: ***A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders, show that the equation of continuity is***

$$\boldsymbol{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0}$$

where v_θ, v_z are the velocities perpendicular and parallel to z.

Solution. We know that the equation of continuity in cylindrical co-ordinates (r, θ, z) is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0, \text{ where } \bar{q} = (v_r, v_\theta, v_z)$$

Since the lines of motion (path lines) lie on the surface of cylinder, therefore the component of velocity in the direction of dr is zero i.e., $v_r = 0$

Thus, the equation of continuity in the present case reduces to

$$\frac{\partial e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial v}(e v_\theta) + \frac{\partial}{\partial z}(e v_z) = 0$$

Hence the result

Example: ***The particles of a fluid move symmetrically in space with regard to a fixed Centre, prove that the equation of continuity is***

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}.\frac{\partial}{\partial r}(r^2 u) = 0.$$

where u is the velocity at a distance r

Solution. First, derive the equation of continuity in spherical co-ordinates. Now, the present case is the case of spherical symmetry, since the motion is symmetrical w.r.t. a fixed centre.
Therefore, the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}.\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \Big| \because \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \psi} = 0$$

$\Rightarrow$ $\frac{\partial \rho}{\partial t} + \frac{1}{r^2}.\frac{\partial}{\partial r}(\rho q_1 r^2) = 0$, where $q_1 \equiv u$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{1}{r^2}.\frac{\partial \rho}{\partial r}ur^2 + \frac{1}{r^2}.\rho.\frac{\partial}{\partial r}(ur^2) = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + u.\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0$$

Hence the result

Example: ***If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is***

$$\frac{\partial \partial}{\partial t} + \frac{\partial}{\partial r}(\rho q_r) + \frac{2\rho}{r}q_r + \frac{\cos ec\theta}{r}\frac{\partial}{\partial \psi}(\rho q_\psi) = 0$$

Solution. First derive the equation of continuity in spherical co-ordinates (r, θ, ψ) as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2 \sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(\rho q_1 r^2) + r\frac{\partial}{\partial \theta}(\rho q_2 \sin\theta) + r\frac{\partial}{\partial \psi}(\rho q_3)\right] = 0$$

In the present case, it is given that lines of motion lie on the surfaces of cones, therefore velocity perpendicular to the surface is zero i.e. $q_2 = 0$
Therefore, the equation of continuity becomes.

$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_r r^2) + \frac{1}{r\sin\theta}.\frac{\partial}{\partial \psi}(\rho q_\psi) = 0$ where $(q_1, q_2, q_3) \equiv (q_r, q_\theta, q_\psi)$

$$\Rightarrow \quad \frac{\partial \rho}{\partial r} + \frac{1}{r^2}\left[r^2\frac{\partial}{\partial r}(\rho q_r) + \rho q_r(2r)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial \psi}(\rho q_\psi) = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho q_r)\frac{2\rho}{r}q_r + \frac{\cos ec\theta}{r}\frac{\partial}{\partial \psi}(\rho q_\psi) = 0$$

Hence the result

Example: Show that polar form of equation of continuity for a two-dimensional incompressible fluid is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta} = 0$$

If u = $\frac{-\mu\cos\theta}{r^2}$***, thenfind v and the magnitude of the velocity q, where q= (u, v)***

Solution. First derive the equation of continuity in polar co-ordinates (r, θ) in two dimensions as

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_1) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2) = 0|z = 0$$

In the present case ρ = constant

Therefore, the equation of continuity reduces to

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru) + \frac{\rho}{r}\frac{\partial}{\partial\theta}(v) = 0, where\ \boldsymbol{q} = (q_1, q_2, q_3) \equiv (\text{u, v, w})$$

i.e. $$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial\theta} = 0$$

Hence the result.

Now u = $\frac{-\mu\cos\theta}{r^2} \Rightarrow \frac{\partial}{\partial r}\left(\frac{-\mu\cos\theta}{r^2}r\right) + \frac{\partial v}{\partial\theta} = 0$

$\Rightarrow \frac{\mu\cos\theta}{r^2} + \frac{\partial v}{\partial\theta} = 0 \Rightarrow \frac{\partial v}{\partial\theta} = \frac{-\mu\cos\theta}{r^2}$

Integrating w.r.t θ, we get

$$\text{v} = \frac{-\mu\sin\theta}{r^2}$$

and thus $|\bar{q}| = q = \sqrt{u^2 + v^2} = \frac{\mu}{r^2}$

4.10 Check Your Progress:

i) If every particle moves on the surface of a sphere, prove that the equation of continuity is $\frac{\partial\rho}{\partial t}\ cos\theta + \frac{\partial}{\partial\theta}(\rho\ \omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)$=0, ρ being the density, θ , ϕ the latitude and longitude of any element and $\omega\ and\ \omega'$ the angular velocities of the element in latitude and longitude respectively.

ii) Show that in a two-dimensional incompressible steady flow field the equation of continuity is satisfied with the velocity components in rectangular coordinates given by $u(x,y) = \frac{k(x^2-y^2)}{(x^2+y^2)^2}, v(x,y) = \frac{2kxy}{(x^2+y^2)^2}$, where k is arbitrary constant.

iii) Consider a two-dimensional incompressible steady flow field with velocity components in spherical coordinates (r,θ,ϕ) given by

$$v_r = c_1\left(1 - \frac{3}{2}\frac{r_0}{r} + \frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta, v_\phi = 0, v_\theta = -c_1\left(1 - \frac{3}{4}\frac{r_0}{r} - \frac{1}{4}\frac{r_0^3}{r^3}\right)sin\theta, r \geq r_0 > 0\,.$$

iv) Determine the constants $l, m\ and\ n$ in order that velocity

$$\boldsymbol{q} = \frac{\{x + lr\}\boldsymbol{i} + (y + mr\}\boldsymbol{j} + (z + nr)\boldsymbol{k}}{r(x + r)}$$

where $r = \sqrt{(x^2 + y^2 + z^2)}$ may satisfying the equation of continuity for a liquid.

v) Each particle of a mass of liquid moves in a plane through the axis of z; find the equation of continuity.

4.11 Summary: This chapter deals with the law of conservation of mass or equation of continuity which is commonly used in the Fluid Mechanics. The expressions of equation of continuity in vector form, in Cartesian forms have been obtained. The equation of continuity in orthogonal curvilinear coordinate system is obtained and further particularly in Cartesian, spherical and cylindrical coordinate system has been obtained. Also the equations of continuity in Lagrangian and Eulerian form are obtained and also discussed their equivalency.

4.12 Keywords: incompressible fluid, equation of continuity, axial symmetry, spherical symmetry, cylindrical symmetry.

4.13 Self-Assessment Test:

SA1: Derive the equation of continuity in Euler form from Lagrangian form and Lagrangian form from Eulerian form.

SA2: What is meant by incompressible fluid? Derive the equation of continuity in Cartesian coordinates.

SA3: Does the three-dimensional incompressible flow given by

$$u(x,y,z) = \frac{kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \qquad v(x,y,z) = \frac{ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}, w(x,y,z) = \frac{kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

satisfy the equation of continuity? k is an arbitrary constant. Thus show that the above motion is kinematically possible for incompressible fluid.

SA4: Dows the two-dimensional incompressible flow given by

$$v_r = \frac{c_1}{r^2} + c_2 \cos\theta, v_\theta = -c_2 \sin\theta, \qquad v_\phi = 0$$

where $c_! \; and \; c_2$ are arbitrary constants and r>0, satisfy the equation of continuity.

SA5: Does the velocity distribution $\boldsymbol{q} = (5x, 5y, -10z)$ satisfy the law of conservation of mass for incompressible flow?

SA6: Does the one-dimensional incompressible flow given by $u(y) = ay^2 + by + c, v = w = 0 \;\; and \; a, b, c$ are constants, satisfy the equation of continuity?

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CHAPTER-5

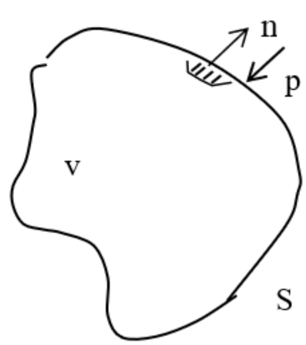
EQUATIONS OF MOTION

5.0 Learning Objectives: After reading this chapter, the students should be able to understand the flux method using Euler's dynamical equation, understand the use and limitations of Bernoulli's theorem and equations and apply it to solve a variety of flow problems and should be able to study the impulsive motion.

5.1 Euler's Equation of Motion for a Perfect Fluid:(Equation of Conservation of Momentum)

To obtain Euler's dynamical equation, we shall make use of Newton's second law of motion.

Consider a region v of fluid bounded by a closed surface S which consists of the same fluid particles at all times. Let **q** be the velocity and ρ be the density of the fluid.
Then ρ dv is an element of mass within S and it remains constant.



The linear momentum of volume v is

$$\boldsymbol{M} = \int_v \boldsymbol{q} \ \ \rho dv \qquad\qquad | \text{ mass} \times \text{velocity} = \text{momentum.}$$

Rate of change of momentum is

$$\frac{d\boldsymbol{M}}{dt} = \frac{d}{dt}\int_v \boldsymbol{q} \ \ \rho dv = \int_v \frac{d\boldsymbol{q}}{dt} \rho dv + \int_v \boldsymbol{q} \ \frac{d}{dt}(\rho dv)$$

$$\frac{d\boldsymbol{M}}{dt} = \int_v \frac{d\boldsymbol{q}}{dt} \rho dv \qquad (1)$$

Since the mass $(\rho\, dv)$ remains constant.
The fluid within v is acted upon by two types of forces
The first type of forces are the surface forces which are due to the fluid exterior to v.
Since the fluid is ideal, the surface force is simply the pressure p at a point of the surface element dS directed along the inward normal at all point of S, then The total surface force on S is

$$\int_S \boldsymbol{p}(-\boldsymbol{n})\boldsymbol{dS} = -\int_S \boldsymbol{pndS} = \int_v \nabla p dv \quad \text{(By Gauss div. Theorem)} \tag{2}$$

The second type of forces are the body forces which are due to some external agent. Let F be the body force per unit mass acting on the fluid. Then $\boldsymbol{F\,\rho dv}$ is the body force on the element of mass $\boldsymbol{\rho dv}$ and the total body force on the mass within v is

$$\int_v \boldsymbol{F\,\rho dv} \tag{3}$$

By Newton's second law of motion, we have

Rate of change of momentum = total force

$$\Rightarrow \qquad \int_v \frac{d\boldsymbol{q}}{dt}\,\boldsymbol{\rho\,dv} = \int_v \boldsymbol{F\,\rho dv} - \int_v \nabla \boldsymbol{p}\,\boldsymbol{dv}$$

$$\Rightarrow \qquad \int_v \left(\frac{d\boldsymbol{q}}{dt}\,\boldsymbol{\rho} - \boldsymbol{F\,\rho} + \nabla \boldsymbol{p}\right)\boldsymbol{dv} = \mathbf{0}$$

Since dv is arbitrary, we get

$$\frac{d\boldsymbol{q}}{dt}\,\boldsymbol{\rho} - \boldsymbol{F\,\rho} + \nabla \boldsymbol{p} = \mathbf{0}$$

$$\text{or} \qquad \frac{d\boldsymbol{q}}{dt} = \boldsymbol{F} - \frac{1}{\rho}\,\nabla \boldsymbol{p} \tag{4}$$

which holds at every point of the fluid and is known as ***Euler's dynamical equation*** for an ideal fluid.

Remark. The above method for obtaining the Euler's equation of motion, is also known as **flux method**.

5.1.1 Other Forms of Euler's Equation of Motion. (i) We know that

$$\frac{d}{dt} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{q}\cdot\boldsymbol{\nabla},$$

therefore, equation (4) becomes.

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}\cdot\boldsymbol{\nabla})\boldsymbol{q} = \boldsymbol{F} - \frac{1}{\rho}\boldsymbol{\nabla}\boldsymbol{p} \tag{5}$$

But $\qquad (\boldsymbol{q}.\nabla)\boldsymbol{q} = \nabla\left(\frac{1}{2}\,\boldsymbol{q}^2\right) + \boldsymbol{\Omega}\times\boldsymbol{q}, \quad \boldsymbol{\Omega} = \boldsymbol{curl\,q}$

Therefore, Euler's equation becomes

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla\left(\frac{1}{2}\,\boldsymbol{q}^2\right) + \boldsymbol{\Omega}\times\boldsymbol{q} = \boldsymbol{F} - \frac{1}{\rho}\,\nabla \boldsymbol{p} \tag{6}$$

Equation (6) is called Lamb's hydrodynamical equation. The chief advantage of (6) is that it is invariant under a change of coordinate system.

(ii) Cartesian Form. Let $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, $\boldsymbol{F}$ = (X,Y,Z) and ∇p = $\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$,

then equation (5) gives

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=X-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}&=Y-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}&=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \tag{7}$$

Equation (7) are the required equations in Cartesian form.

(iii) Equations of Motion in Cylindrical Co-ordinates. (r, θ, z).

Here,

$$q = (u, v, w),\ dr = (\mathrm{dr},\ \mathrm{r d\theta},\ \mathrm{dz})$$

$$\nabla \mathrm{p} = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Let $\quad F = (F_r\ , F_\theta\ , F_z)$

Also, the acceleration components in cylindrical co-ordinates are

$$\frac{dq}{dt} = \left(\frac{du}{dt} - \frac{v^2}{r}, \frac{dv}{dt} + \frac{uv}{r}\frac{dw}{dt}\right)$$

Thus, the equation of motion

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p. \text{ becomes}$$

$$\left.\begin{aligned}\frac{du}{dt}-\frac{v^2}{r}&=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\ \frac{dv}{dt}+\frac{vu}{r}&=F_\theta-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}\\ \frac{dw}{dt}&=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \tag{8}$$

(iv) Equations of Motion in Spherical co-ordinates (r, θ, ϕ).

Here,

$$q = (u\ v\ w), \qquad dr = (dr, r\ d\theta, r\sin\theta\ d\phi)$$

$$\nabla \mathrm{p} = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}\right)$$

Let $\quad F = (F_r,\ F_\theta, F_\phi)$

The components of acceleration in spherical co-ordinates are

$$\frac{dq}{dt} = \left(\frac{du}{dt} - \frac{v^2+w^2}{r}, \frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r}, \frac{dw}{dt} + \frac{vw\cot\theta}{r}\right)$$

Thus, the equation of motion take the form

$$\frac{du}{dt} - \frac{v^2+w^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{dw}{dt} + \frac{vw\cot\theta}{r} = F_\phi - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} \tag{9}$$

Remark: - The two equations, the equation of continuity and the Euler's equation of motion, comprise the equations of motion of an ideal fluid. Thus the equations

$$\frac{\partial \rho}{\partial t} + div(\rho q) = 0\ ; \quad \frac{\partial q}{\partial t} + (q.\nabla)q = F - \frac{1}{\rho}\nabla p$$

are fundamental to any theoretical study of ideal fluid flow. These equations are solved subject to the appropriate boundary and initial conditions dictated by the physical characteristics of the flow.

(v) Equation of motion referred to rotating axes: If the axes rotate with uniform angular velocity $\omega = (\omega_!, \omega_2, \omega_3)$, then the expression for acceleration is

$$\frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}'.\nabla)\boldsymbol{q} = \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{q} + (\boldsymbol{q}'.\nabla)\boldsymbol{q}$$

Hence the Euler's equation of motion then is

$$\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{q} + (\boldsymbol{q}'.\nabla)\boldsymbol{q} = \boldsymbol{F} - \frac{\mathbf{1}}{\rho}\, \boldsymbol{\nabla p}$$

Here **q** and **q'** are the absolute and relative velocities.

Definition: The velocity vector **q** is called Beltrani vector if **q** is parallel to Ω=curl **q** i.e., if $\boldsymbol{q} \times \Omega = 0$.

Definition: A fluid is said to be barotropic if $p = f(\rho)$.

Def: Conservative Field of Force: In a conservative field of force, the work done by a force in taking a unit mass from a point A to a point B is independent of the path, i.e.,

$$\int_{ACB} F.dr = \int_{ADB} F.dr = -\chi$$

Hence χ is a scalar function and known as force potential function. It can be proved that $F = -\nabla\chi$.

Cor. Acceleration potential: When the body forces are conservative so that $F = -\nabla\, \boldsymbol{\chi}$ and the fluid is barotropic, i.e., density is a function of pressure, so that $\rho^{-1}\,\nabla p = \nabla \int \frac{dp}{\rho}$, then Euler's equation of motion may be expressed as

$$\frac{d\boldsymbol{q}}{dt} = -\nabla\chi - \nabla \int \frac{dp}{\rho}$$

Or $$\boldsymbol{a} = -\nabla\left(\chi - \int \frac{dp}{\rho}\right) = -grad\ \Phi \qquad \text{(say)}$$

This result shows that the acceleration vector **a** possesses acceleration potential $\Phi = \chi - \int \frac{dp}{\rho}$

5.2 Cauchy Pressure Equation: Integrals of the equation of motion:

To obtain the solution of Euler's equation of motion, which is non-linear, we will have to entertain simplifying assumptions. Firstly we assume that the external forces form a conservative system so that $F = -\nabla\Omega$. Secondly we assume that the fluid is barotropic ($p = f(\rho)$) so that

$$\frac{1}{\rho}\nabla p = \nabla \int \frac{dp}{\rho} = \nabla P \qquad (say)$$

Since $$\frac{d\boldsymbol{q}}{d\boldsymbol{t}} = \frac{\partial \boldsymbol{q}}{\partial t} + \omega \times \boldsymbol{q} + \nabla\left(\frac{1}{2}\,\boldsymbol{q}^2\right) \qquad \text{[Langrage acceleration relation]}$$

The equation of motion can be set as

$$\frac{\partial \boldsymbol{q}}{\partial t} + \omega \times \boldsymbol{q} + \nabla\left(\frac{1}{2}\,\boldsymbol{q}^2\right) = -\nabla\Omega - \nabla P$$

or $$\frac{\partial \boldsymbol{q}}{\partial t} = \boldsymbol{q} \times \boldsymbol{w} - \nabla(\Omega + P + \tfrac{1}{2}q^2) \qquad (1)$$

This last expression is the final result. Since this result, as it stands, is not very useful, several special cases will be considered.

Special cases:

(i) When the motion is irrotational , $\omega = curl\ \boldsymbol{q} = 0, and\ \boldsymbol{q} = -\nabla\,\phi$ and (1) reduces to

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) = \nabla H \qquad [H = \Omega + P + \tfrac{1}{2}q^2]$$

(2)

Since the operators ∇ and $\frac{\partial}{\partial t}$ are interchangeable. The solution of (2) , viz, **grad** $(H - \frac{\partial\phi}{\partial t}) = 0$ is

$$\Omega + \int\frac{dp}{\rho} + \frac{1}{2}\,q^2 - \frac{\partial\phi}{\partial t} = C(t) = constant \qquad (3)$$

The constant C(t) shall be a function time t only.

Equation (3) is known as pressure equation.

(ii) **Bernoulli's Equation:** When the motion is steady as well as irrotational, $\frac{\partial \boldsymbol{q}}{\partial t} = \ 0\ \ and\ \omega = 0$;

(1) reduces to

$$\nabla\left(\chi + P + \frac{1}{2}q^2\right) = 0$$

The solution of this equation is

$$\chi + P + \tfrac{1}{2}q^2 = C \qquad (4)$$

If the fluid is incompressible and homogeneous, $\rho = constant,$ then $P = \int\frac{dp}{\rho} = p/\rho,$ therefore, (4) becomes

$$\chi + \frac{p}{\rho} + \frac{1}{2}q^2 = C \qquad (5)$$

Here C is an absolute constant i.e., is independent of time also. Equation (5) is the Bernoulli's equation for unsteady, irrotational of an incompressible fluid. It may be remarked that Bernoulli's theorem is still true even if the velocity potential ϕ does not exist.

5.3 Lagrange's Equation of Motion. *To obtain Lagrange's equation of motion.*

Let initially a fluid element be at (a, b, c) at time t = t_0 when its volume is dV_0 and density is ρ_0. After a lapse of time t, let the same fluid element be at (x, y, z) when its volume is dV and density is ρ . The equation of continuity is

$$\rho J = \rho_0 \qquad (1)$$

where $J = \frac{\partial(x,y,z)}{\partial(a,b,c)}$

The components of acceleration are

$$\ddot{x} = \frac{\partial^2 x}{\partial t^2}, \ddot{y} = \frac{\partial^2 y}{\partial t^2}, \ddot{z} = \frac{\partial^2 z}{\partial t^2}$$

Let the body force $\boldsymbol{F}$be conservative so that we can express it in terms of a body force potential function $\boldsymbol{\chi}$ as

$$\boldsymbol{F} = -\nabla\boldsymbol{\chi} \tag{2}$$

By Euler's equation of motion,

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{F} - \frac{1}{\rho}\nabla p = -\nabla\chi - \frac{1}{\rho}\nabla p \tag{3}$$

Its Cartesian equivalent is

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= -\frac{\partial\chi}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial^2 y}{\partial t^2} &= -\frac{\partial\chi}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial^2 z}{\partial t^2} &= -\frac{\partial\chi}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \tag{4}$$

We note that a, b, c, t are the independent variables and our object is to determine x, y, z in terms of a, b, c, t and so investigate completely the motion.

To deduce equations containing only differentiations w.r.t. the independent variables a, b, c, t we multiply the equations in (4) by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$ and add to get

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial\chi}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \tag{5}$$

Similarly, we get

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial\chi}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \tag{6}$$

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial\chi}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \tag{7}$$

These equations (5), (6), (7) together with equation (1) represent Lagarange's Hydrodynamical Equations.

5.4 Bernoulli's Theorem: ***Statement:*** *For the steady motion of an inviscid barotropic fluid under conservative body forces, the pressure at a point is given by*

$$\int\frac{dp}{\rho} + \frac{1}{2}q^2 + \chi = \mathrm{C}, \mathrm{C}\text{ being absolute constant.}$$

Proof: Let AB be streamline drawn in the fluid of density ρ and let δs be an element of it. When δs as axis, draw a small cylinder of cross-section k. Consider steady motion of the fluid within this cylinder. Also, let **q** be the velocity and let the component of the body force in the direction of motion be F.

Forces are conservative implies $F = -\nabla\chi$.

Motion is steady implies $\frac{\partial q}{\partial t} = 0$, density is a function of pressure only implies that there exists a realtion of the type

$$P = \int_c^p \frac{dp}{\rho} \text{ so that } \nabla P = \frac{1}{\rho}\nabla p.$$

By Euler's equation, $\frac{d\boldsymbol{q}}{dt} = -\nabla\chi - \nabla P$

Or $\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}.\nabla)\boldsymbol{q} = -\nabla(\chi + P)$

$\Rightarrow$ $\nabla(\chi + P) + (\boldsymbol{q}.\nabla)\boldsymbol{q} = \boldsymbol{0}$

But $\nabla(\boldsymbol{q}.\boldsymbol{q}) = \boldsymbol{2}[\boldsymbol{q} \times curl\, \boldsymbol{q}) + (\boldsymbol{q}.\nabla)\boldsymbol{q}]$

$\nabla(\chi + P) + \frac{1}{2}\nabla q^2 - \boldsymbol{q} \times curl\, \boldsymbol{q} = \boldsymbol{0}$

Or $\nabla(\chi + P + \frac{1}{2}q^2) = \boldsymbol{q} \times curl\, \boldsymbol{q}$ **(1)**

Multiplying (1) scalarly by **q** and noting that

$$\boldsymbol{q}.(\boldsymbol{q} \times curl\, \boldsymbol{q}) = (\boldsymbol{q} \times \boldsymbol{q}).curl\, \boldsymbol{q} = \boldsymbol{0}. For\ \boldsymbol{q} \times \boldsymbol{q} = \boldsymbol{0}$$

We obtain $\mathbf{q}.\nabla\left(\chi + P + \frac{1}{2}q^2\right) = 0$

The solution of this is $\chi + P + \frac{1}{2}q^2 = constant = C$

Or $\chi + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = C$

Where C is a constant for the particular stream line (or vortex line) chosen, but varies from one stream line to the other.

NOTES: (1) If the motion is irrotational, velocity potential exists. In this particular case, C is an absolute constant.

(2) If ρ is constant, there result the simplest case:

$$\frac{1}{2}q^2 + \frac{p}{\rho} + \chi = C$$

(3) If the body force is due to gravity, $\chi = gh$ where h is the position (height) above some fixed horizontal datum plane. The result may then be written as

$$\frac{1}{2}q^2 + \frac{p}{\rho} + gh = constant$$

Or in the language of hydraulics

$$velocity\ head + pressure\ head + position\ head = total\ head$$

Where the total head is constant along any stream line.

5.5 Equation of impulsive motion: To find the relation between impulsive pressure and change of velocity.

Let $\bar{\omega}$ denote the impulsive pressure and I the extraneous impulse per unit mass of fluid. Let $\boldsymbol{q_1}$ ***and*** $\boldsymbol{q_2}$ be the velocities just before and just after the impulsive action.

Newton's second law for impulsive motion applied to the fluid within closed surface S states:

$$Extraneous\ Impulse = Change\ of\ Momentum$$

Then, if **n** is inward unit normal we must have

$$\int_S \bar{\omega}\, \boldsymbol{n}\, ds + \int_V I\, \rho dv = \int_V \rho(\boldsymbol{q_2} - \boldsymbol{q_1}) dv$$

But $\int_S \overline{\omega}\, \boldsymbol{n}\, ds = -\int_V grad\, \overline{\omega}\ dv$ [by Gauss Theorem]

$$\therefore \quad \int_V [\, I\, \rho - \nabla\, \overline{\omega} - \rho(\boldsymbol{q_2} - \boldsymbol{q_1})]dv = 0$$

Since the surface is arbitrary, we must have

$$I = \left(\frac{1}{\rho}\right) \nabla\, \overline{\omega} = (\boldsymbol{q_2} - \boldsymbol{q_1}). \qquad (1)$$

Cor.1. *Interpretation of potential as impulsive pressure.* Let us suppose that ϕ is the velocity potential of a motion generated from rest by impulsive pressure $\overline{\omega}$ and that external impulses are non-operative, then

$$I = 0; q_1 = 0,; q_2 = -\nabla\phi;$$

With these values, the above equation (1) reduces to

$$\frac{1}{\rho} \nabla\, \overline{\omega} = \nabla\phi.$$

If ρ be constant, integration provides the result

$$\overline{\omega} = \rho\phi + constant \qquad (2)$$

The constant may be omitted, as an extra pressure, constant throughout the fluid, produces no effect on the motion.

Cor.2. In case of a liquid, ρ is constant and if the external impulses are superficial to the liquid, I=0. Then, taking the divergence of both members of (1) and using $div\, \boldsymbol{q_1} = 0\ and\ div \boldsymbol{q}_2 = 0$, we get

$$\nabla^2\, \overline{w} = 0 \qquad \text{[Laplace equation] (3)}$$

Cor.3. For the liquid motion started from rest by impulsive pressure alone we obtain from (1) $\boldsymbol{q} = -\boldsymbol{grad}\, \left(\frac{\overline{\omega}}{\rho}\right)$; hence the motion is necessarily irrotational and velocity potential ϕ exists and is given by $\phi = \frac{\overline{w}}{\rho}$.

Cor.4. In the absence of external impulses, (1) produces

$$\boldsymbol{q_2} - \boldsymbol{q_1} = -\boldsymbol{\nabla}\, (\frac{\overline{w}}{\rho}) \qquad (4)$$

Now, if the fluid motion before the action of the instantaneous forces is irrotational, i.e., $\boldsymbol{q_1} = -\nabla\phi_1$, then obviously, $q_2 = -\nabla\, [\frac{\overline{\omega}}{\rho} + \phi_1]$ so that the fluid motion remains irrotational after these forces have ceased to operate. Setting $\boldsymbol{q_2} = -\boldsymbol{\nabla}\phi_2$ we immediately obtain

$$\phi_2 = \phi_1 + \frac{\overline{\omega}}{\rho} + C \qquad (5)$$

If $\boldsymbol{q_2}(= \boldsymbol{\nabla}\phi_2)$ is constant, (4) provides

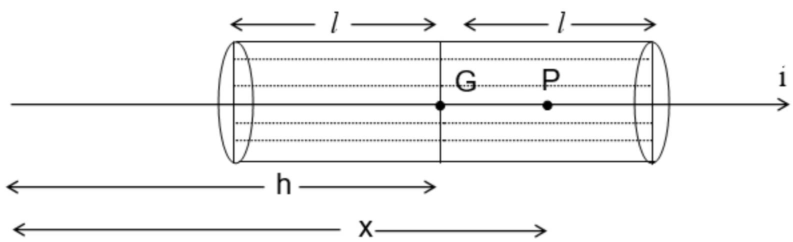
$$\boldsymbol{q_1} = \nabla\left(\frac{\overline{\omega}}{\rho}\right) \Rightarrow curl\, \boldsymbol{q} = 0 \qquad (6)$$

Thus, the given irrotational motion can be established completely throughout the fluid after the action of impulsive pressure $\overline{\omega} = (\rho\, \phi_1 + C)$ and that it is impossible to create or destroy by rotational motion any combination of instantaneous pressure forces.

Example: ***A homogeneous incompressible liquid occupies a length 2l of a straight tube of uniform small bore and is acted upon by a body force which is such that the fluid is attracted to a fixed point of the tube, with a force varying as the distance from the point. Discuss the motion and determine the velocity and pressure within the liquid.***

Solution. We note that the small bore of the tube permits us to ignore any variation of velocity across any cross-section of the tube and to suppose that the flow is unidirectional.

We u be the velocity along the tube and p be the pressure at a general point P at distance x from the centre of force O. Also, let h be the distance of the centre of mass G of the fluid, as shown in the figure.



Equations of motion of the fluid are:

(i) Equation of Continuity: Here, $\boldsymbol{q} = (\boldsymbol{u}, \mathbf{0}, \mathbf{0})$

Therefore, equation of continuity becomes

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(t) \qquad \textbf{(1)}$$

(ii) Euler's Equation: In this case, it becomes

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + X - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad |\text{using (1)} \qquad (2)$$

where $-\mu x\hat{\imath}$is the body force per unit mass, μ being a positive constant.

We observe that equation (2) can be written as

$$\frac{du}{dt} = -\mu x - \frac{1}{\rho}\frac{dp}{dx} \qquad (3)$$

Integrating w.r.t. x, we get

$$x\frac{du}{dt} = -\mu\frac{x^2}{2} - \frac{p}{\rho} + C \qquad (4)$$

where C is a constant and at most can be a function of t only. w.r.t. (x, y, z)

Let Π be the pressure at the free surfaces x = h–*l* and x = h + *l* of the liquid

Then using these boundary conditions, equation (4) becomes

$$(h-l)\frac{du}{dt} = -\frac{1}{2}\mu(h-l)^2 - \frac{\Pi}{\rho} + C$$

$$(h+l)\frac{du}{dt} = -\frac{1}{2}\mu(h-l)^2 - \frac{\Pi}{\rho} + C$$

which on subtraction give

$$\frac{du}{dt} = -\mu h \tag{5}$$

But in the fluid motion all fluid particles move with the same velocity u and u = $\frac{dh}{dt}$

∴ Equation (5) becomes

$$\frac{d^2h}{dt^2} = -\mu h \tag{6}$$

Now, we solve the different equation (6), which can be written as

$$(D^2 + \mu)\, h = 0$$

Here auxiliary equation is

$$D^2 + \mu = 0 \;\Rightarrow D = \pm\sqrt{\mu}\, i$$

Therefore, the solution of (6) is

$$h = A\cos(\sqrt{\mu}\, t + \epsilon)$$

where A and $\in$ are constants which can be determined from initial conditions.

To Calculate Pressure:– We have from (3) & (5)

$$-\mu x - \frac{1}{\rho}\frac{dp}{dx} = -\mu h$$

$$\Rightarrow \frac{1}{\rho}\frac{dp}{dx} = \mu(h - x)$$

Integrating w.r.t. x , we get

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{2(-1)} + D \tag{7}$$

The boundary condition x = h – l, p = Π gives

$$\frac{\Pi}{\rho} = \mu.\frac{l^2}{-2} + D$$

i.e. $\qquad D = \Pi/\rho + \frac{\mu l^2}{2}$

Therefore, equation (7) becomes

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{-2} + \Pi/\rho + \frac{\mu l^2}{2}$$

$$= \frac{\Pi}{\rho} - \frac{\mu}{2}[(h - x)^2 - l^2]$$

$$= \frac{\Pi}{\rho} - \frac{\mu}{2}[(h - x + l)(h - x - l)]$$

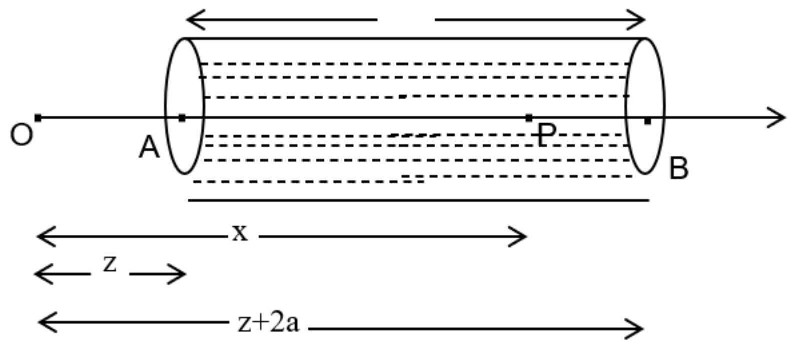
Example: ***A quantity of liquid of density ρ occupies a length 2a of a long straight tube of inform small cross-section and is under the action of a force Kx per unit mass towards a fixed-point O. Show that when the nearer free surface is at a distance z from O, the pressure at a distance x exceeds the atmospheric pressure by***

$$k\rho(x-z)\left(a - \frac{x}{2} + \frac{z}{2}\right)$$

Solution. The equation of continuity for incompressible fluids ($div\, q = 0$) in the present case of one-dimensional flow is

$$\frac{\partial}{\partial x}(\rho u) = 0 \quad i.e. \frac{\partial u}{\partial x} = 0 \tag{1}$$

Thus, u is a function of time t only.



Euler's equation of motion becomes

$$\frac{\partial u}{\partial t} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

i.e. $$\frac{du}{dt} = -kx - \frac{1}{\rho}\frac{dp}{dx} \qquad (2)$$

where $\boldsymbol{F} = X\boldsymbol{i} = -kx\boldsymbol{i}, \nabla p = \frac{\partial p}{\partial x}\boldsymbol{i} = \frac{dp}{dx}\boldsymbol{i}.$

Integrating (2) w.r.t. x , we get

$$x\frac{du}{dt} = -\frac{kx^2}{2} - \frac{p}{\rho} + A \qquad (3)$$

Since p = Π at x = z , x = z + 2a; therefore, equation (3) becomes

$$z\frac{du}{dt} = \frac{-kz^2}{2} - \frac{\Pi}{\rho} + A$$

and

$$(z+2a)\frac{du}{dt} = \frac{-k(z+2a)^2}{2} - \Pi/\rho + A.$$

Subtracting these, we get

$$2a\frac{du}{dt} = -\frac{k}{2}(4a^2 + 4az) = -2ak\,(z + a)$$

$$\Rightarrow \frac{du}{dt} = -k\,(z + a) \qquad (4)$$

From (2) & (4), we get

$$kx + \frac{1}{\rho}\frac{dp}{dx} = k\,(z + a)$$

i.e. $$\frac{dp}{dx} = \rho k\,(z + a - x)$$

Integrating w.r.t. x, we get

$$p = \frac{-k\rho}{2}(z + a - x)^2 + B \qquad (5)$$

The boundary condition p = Π at x = z gives

$$B = \Pi + \frac{k\rho a^2}{2}$$

Therefore, the pressure p is

$$p = \Pi + \frac{k\rho a^2}{2} - \frac{k\rho}{2}(z + a - x)^2$$

$= \Pi + \frac{k\rho}{2}\,[a^2 - (z + a - x)^2]$

$= \Pi + \frac{k\rho}{2}\,[(2a + z - x)\,(x - z)]$

$\Rightarrow\; p - \Pi = k\rho\,(a + z/2 - x/2)\,(x - z)$

Hence the result

Example: ***Air obeying Boyle's law is in motion in a uniform tube of small cross section. Prove that if ρ be the density and V be the velocity at a distance x from a fixed point at a time t, then***

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}[\rho(V^2 + k)].\ \textit{where k is a constant}$$

Solution. At time t, let p be the pressure and V be the velocity at a distance x from the end of the tube. Then, by Boyle's law

$$p = k\rho \tag{1}$$

Equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0 \qquad |\mathbf{q} = (V, 0, 0) \tag{2}$$

and the equation of motion is

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad |\mathbf{F} = 0 \text{ at free surface.} \tag{3}$$

Differentiating (2) w.r.t. t, we get

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial t}V + \rho\frac{\partial V}{\partial t}\right) = 0 \tag{4}$$

From equations (1) and (3), we get

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \tag{5}$$

Using equations (2) and (5) in (4), we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left[-V\frac{\partial}{\partial x}(\rho V) - \rho V\frac{\partial V}{\partial x} - k\frac{\partial \rho}{\partial x}\right] = 0$$

i.e. $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\rho V^2) + k\frac{\partial \rho}{\partial x}\right] = 0$$

i.e. $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(V^2 + k^2)\rho\right] = 0$$

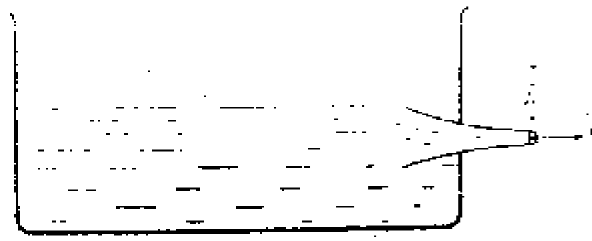
i.e. $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}[V^2 + k)\rho]$$

Hence the result.

Example: ***Prove Bernoulli's theorem, that in a steady motion, $\int \frac{dp}{\rho} + \frac{1}{2}q^2 + \chi$; is constant along a stream line. Deduce the theorem of Torricelli.***

Solution: Consider the efflux of liquid from a small orifice in one of the walls of a vessel kept filled to a constant level (giving steady motion). Let h be the depth of the vena contracta (the contraction), q is the speed of efflux there at, and **Π** the atmospheric pressure. Then, by Bernoulli's theorem

$$\frac{\Pi}{\rho} + gh = \frac{\Pi}{\rho} + \frac{1}{2}q^2 \qquad (1)$$



Because velocity is practically zero at the free surface of the water in the vessel, and the pressure is **Π**, both there and on the walls of the escaping jet. Hence (1) yields

$$q^2 = 2gh \qquad \text{(Torricelli's theorem)}$$

Example: ***A long straight pipe of length L has a slowly tapering circular cross section. It is inclined so that its axis makes and angle α to the horizontal with its smaller cross-section downwards. The radius of the pipe at its upper end is twice that of at its lower end and water is pumped at a steady rate through the pipe to emerge at atmospheric pressure. It the pumping pressure is twice the atmospheric pressure, show that the fluid leaves the pipe with a speed U given by***

$$U^2 = \frac{32}{15}\left[gL\sin\alpha + \frac{\Pi}{\rho}\right],$$

where Π is atmospheric pressure

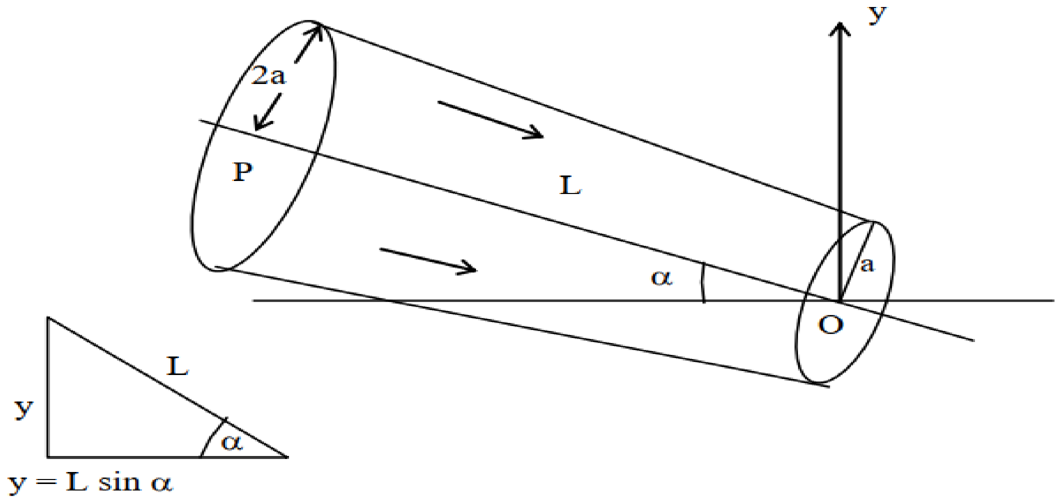
Solution. The assumption that the pipe is slowly tapering means that any variation in the velocity over any cross-section can be ignored. Let the velocity at the wider and of the pipe be V and the emerging velocity be U (velocity at the lower end). The only body force is that of gravity, so $\bar{F} = -g\hat{j}$ and consequently χ = g y

$$\left|\begin{array}{l} \because \bar{F} = -\nabla\chi \Rightarrow -qj = -\nabla\chi = -\frac{\partial\chi}{\partial x}i - \frac{\partial\chi}{\partial y}j - \frac{\partial\chi}{\partial z}k \\ \Rightarrow -g = -\frac{\partial\chi}{\partial y} \Rightarrow \chi = gy \end{array}\right.$$

Bernoulli's equation, $\frac{p}{\rho} + \frac{1}{2}q^2 + \chi = C$ | ∵ For water ρ is const.

becomes $\frac{p}{\rho} + \frac{1}{2}q^2 + gy = C$ (1)

Applying this equation of the two ends of the pipe, we get



$$\frac{2\Pi}{\rho} + \frac{1}{2}.V^2 + gL\sin\alpha = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad (2) \qquad |\text{for lower end } y = 0$$

Let a and 2a be the radii of the lower and upper ends respectively, then by the principle of conservation of mass

$$\pi(2a)^2V = \pi\, a^2\, U$$

$$\Rightarrow\ V = \frac{U}{4} \qquad (3)$$

From (2) and (3), we obtain

$$\Pi + \frac{1}{2}\rho\left(\frac{U^2}{16}\right) + g\rho L\sin\alpha = \frac{1}{2}\rho U^2$$

$$\Rightarrow \quad \frac{1}{2}\rho\left(U^2 - \frac{U^2}{16}\right) = \Pi + g\rho L\sin\alpha.$$

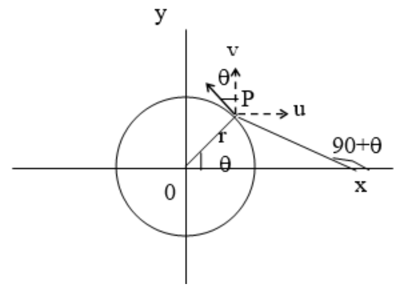
$$\Rightarrow \frac{15}{32}\rho U^2 = \Pi + g\rho L\sin\alpha.$$

$$\Rightarrow \quad U^2 = \frac{32}{15}\left[gL\sin\alpha + \frac{\Pi}{\rho}\right]$$

Hence the result.

Example: ***A perfect incompressible fluid is moving steadily around the outside of a fixed cylinder of radius a and vertical axis oz. The fluid particles are transversing horizontal circles with centre on oz, the speed at distance r from oz being $\frac{a}{r}$. Show that the motion is irrotational. If the surface of the fluid is open to the atmosphere, and the origin o is chosen so that on the free surface, z = 0 when r = a, prove by means of Bernoulli's equation or otherwise, that the equation of the free surface is***

$$2gz = a - \frac{a^2}{r^2}$$



Solution. Let $\mathbf{q} = (u, v)$

$\therefore$ u $= \frac{a}{r}\cos(90+\theta) = -\frac{a}{r}\sin\theta = -\frac{ay}{r^2}$, $r^2 = x^2 + y^2$ i.e. $x = r\cos\theta$, $y = r\sin\theta$

v $= \frac{a}{r}\cos\theta = \frac{ax}{r^2}$

$$\text{curl } \boldsymbol{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-ay}{r^2} & \frac{ax}{r^2} & 0 \end{vmatrix} = \boldsymbol{ka}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)\right]$$

$$= \boldsymbol{ka}\left[\frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4}\right] = 0$$

So, the motion is irrotational.

Now Bernoulli's pressure equation is

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = C$$

$$\left| \begin{array}{l} \boldsymbol{F} = -\nabla\chi \\ \Rightarrow -g\boldsymbol{k} = -\frac{\partial\chi}{\partial z}\boldsymbol{k} \\ \Rightarrow \chi = gz \end{array} \right.$$

or $$\frac{p}{\rho} + \frac{1}{2}\frac{a^2}{r^2} + gz = C$$

when r = a, z = 0, p = 0 so that C $= \frac{1}{2}$, therefore,

$$\frac{p}{\rho} + \frac{1}{2}\frac{a^2}{r^2} + gz = \frac{1}{2}$$

The equation of the free surface (p = 0) is

$$2gz = 1 - \frac{a^2}{r^2}$$

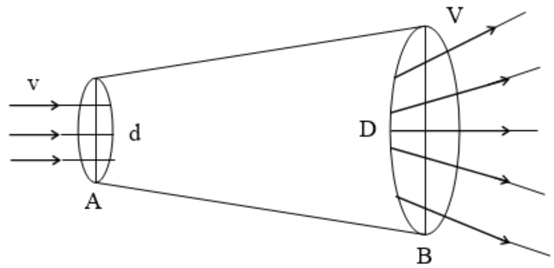
Hence the result.

Example: ***Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. If V and v be the corresponding velocities of the stream and if the motion be supposed to be that of divergence from the vertex of the cone, prove that***

$$\frac{v}{V} = \frac{D^2}{d^2}e^{(v^2-V^2)/2k},$$

where k is the pressure divided by density and supposed constant

Solution.



Since the stream steadies down soon after and the external forces like gravity are neglected; therefore, by Bernoulli's equation, we have

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 = \text{constant} \qquad | \text{ i.e., } \rho \text{ is not const.} \qquad (1)$$

and by Bayle's law, we have

$$p = k\rho \qquad (2)$$

Therefore, (1) becomes

$$k\int \frac{d\rho}{\rho} + \frac{1}{2}q^2 = C$$

$$\Rightarrow k \log \rho + \frac{1}{2}q^2 = C$$

If at the sections A and B, $\rho = \rho_1$, $\rho = \rho_2$ respectively and q = v at A, q = V at B, then we have

$$k \log \rho_1 + \frac{1}{2}v^2 = C,$$

$$k \log \rho_2 + \frac{1}{2}V^2 = C$$

Subtracting, we get

$$2k \log (\rho_2/\rho_1) = v^2 - V^2.$$

$$\Rightarrow \rho_2/\rho_1 = e^{(v^2 - V^2)/2k} \qquad (3)$$

Now, by equation of continuity, we have

Flux across A = Flux across B.

$$\Rightarrow \pi\left(\frac{1}{2}d\right)^2 \rho_1 v = \pi\left(\frac{1}{2}D\right)^2 \rho_2 V$$

$$\Rightarrow \qquad \rho_2/\rho_1 = \frac{d^2}{D^2}\frac{v}{V} \qquad (4)$$

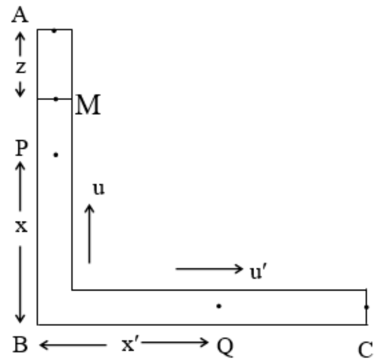
From (3) & (4), we get

$$\frac{v}{V} = \frac{D^2}{d^2}e^{(v^2 - V^2)/2k}$$

Hence the result.

Example: ***A straight tube ABC, of small bore, is bent so as to make the angle ABC a right angle and AB equal to BC. The end C is closed and the tube is placed with end A upwards and AB vertical, and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half. Also find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmospheric being neglected.***

Solution. Let AB = BC = a



When the liquid in AB has fallen through a distance z at time t, then let P be any point in the vertical column such that

AM = z, BP = x, BM = a–z

If u and p be the velocity and pressure at P, then equation of motion is

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} + \boldsymbol{u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = -\boldsymbol{g} - \frac{\boldsymbol{1}}{\boldsymbol{\rho}}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} \qquad (1) \qquad |u \equiv u(x, t)$$

and equation of continuity is

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \boldsymbol{0} \text{ i.e., } u = u(t)$$

Therefore, equation (1) becomes

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = -\boldsymbol{g} - \frac{\boldsymbol{1}}{\boldsymbol{\rho}}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}}$$

Integrating. w.r.t. x, we get

$$x\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = -\boldsymbol{g}\boldsymbol{x}. - \frac{\boldsymbol{1}}{\boldsymbol{\rho}}\boldsymbol{p} + \boldsymbol{C} \qquad (2)$$

Using the boundary condition p = 0 at x = a–z, we get

$$C = (\boldsymbol{a} - \boldsymbol{z})\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} + \boldsymbol{g}(\boldsymbol{a} - \boldsymbol{z})$$

Therefore, equation (2) becomes

$$x\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = -\boldsymbol{g}\boldsymbol{x} - \frac{\boldsymbol{p}}{\boldsymbol{\rho}} + (\boldsymbol{a} - \boldsymbol{z})\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} + \boldsymbol{g}(\boldsymbol{a} - \boldsymbol{z})$$

i.e. $$\frac{\boldsymbol{p}}{\boldsymbol{\rho}} = -(\boldsymbol{x} - \boldsymbol{a} + \boldsymbol{z})\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} + \boldsymbol{g}\right) \qquad (3)$$

Now, we take a point Q in BC, where BQ = x′ and let u′, p′ be the velocity and pressure at Q, then

$$\frac{\boldsymbol{p'}}{\boldsymbol{\rho}} = -(\boldsymbol{x'} - \boldsymbol{a})\frac{\partial \boldsymbol{u'}}{\partial \boldsymbol{t}} \qquad (4) \qquad | \text{ z = 0 and g is not affecting}$$

Equating the pressure at B, when x = 0, x′ = 0, we get

$$(a-z)\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} + \boldsymbol{g}\right) = \boldsymbol{a}\frac{\partial \boldsymbol{u'}}{\partial \boldsymbol{t}} \qquad | \text{ From (3) \& (4)}$$

$$= -a\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} \qquad | \because u' = -u$$

Initially, when C is just opened, then z = 0, t = 0 and we have

$$a\left[\left(\frac{\partial u}{\partial t}\right)_{t=0} + g\right] = -a\left(\frac{\partial u}{\partial t}\right)_{t=0}$$

$$\Rightarrow \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{-g}{2}\, i.e. \left(\frac{\partial u}{\partial t}\right)_0 = -g/2 \tag{5}$$

Therefore, from equation (3), initially, the pressure at P is given by

$$\frac{p_0}{\rho} = -(x-a)\left[\left(\frac{\partial u}{\partial t}\right)_0 + g\right] \,|\, p_0 \equiv (p)_{t=0}$$

$$= \frac{-g}{2}(x-a)$$

$$\Rightarrow \qquad p_0 = \frac{1}{2}\rho g(a-x) \tag{6}$$

But when the end C is closed, the liquid is at rest and the hydrostatic pressure at P is

$$p_1 = \rho g\, h = \rho g\,(a-x) \qquad [h = AP = a-x] \tag{7}$$

From (6) and (7), we get

$$p_0 = \frac{1}{2}p_1$$

Thus, the pressure is diminished to one-half.

Now, from (4), initial pressure at Q is given by

$$\frac{p'_0}{\rho} = -(x'-a)\left(\frac{\partial u'}{\partial t}\right)_{t=0} = (x'-a)\left(\frac{\partial u}{\partial t}\right)_{t=0} = (a-x')\frac{g}{2}$$

$$\Rightarrow \quad p_0' = \frac{1}{2}\rho g(a-x')$$

When the end C is closed, the initial pressure (hydrostatic) p_2 at Q (or B or C) is $\rho g a$.

Therefore, instantaneous change in pressure

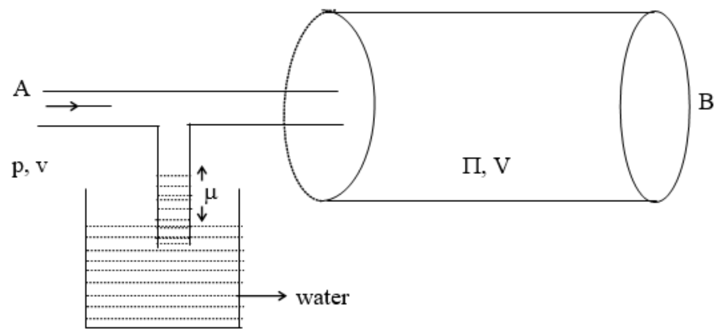
$$= p_2 - p_0' = \rho g a - \frac{1}{2}\rho g(a-x') = \frac{1}{2}\rho g(a+x')$$

Example: ***A stream in a horizontal pipe after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth*** $\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ ***below the pipe, s being the delivery per second.***

Solution. If v be the velocity in the tube of sectional area A and p be the pressure there, while V and Π being the corresponding quantities at the sectional area B, then Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant, gives} \qquad \because \rho \text{ is constant for water}$$

$$\frac{p}{\rho} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} + \frac{1}{2}V^2 \tag{1}$$



By the equation of continuity, flux across the sections A and B are equal.

i.e. Av = BV = s, where s is delivery per second

$$\Rightarrow v = \frac{s}{A}, V = \frac{s}{B}$$

Therefore, from equation (1), we get

$$\frac{p}{\rho} + \frac{1}{2}\frac{s^2}{A^2} = \frac{\Pi}{\rho} + \frac{1}{2}\frac{s^2}{B^2}$$

$$\Rightarrow \frac{1}{2}s^2\left(\frac{1}{A^2} - \frac{1}{B^2}\right) = \frac{1}{\rho}(\Pi - p) \qquad (2)$$

Now, if h be the height through which water is sucked up, then

ρg h = difference of pressures = $\Pi - p$

Therefore, from (2), we get

$$\frac{1}{2}s^2\left(\frac{1}{A^2} - \frac{1}{B^2}\right) = \frac{1}{\rho}\rho gh = gh.$$

$$\Rightarrow h = \frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$$

Hence the result.

Example: ***A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π. Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is***

$$\Pi + \frac{\rho}{2}\left[\frac{d^2(R^2)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Solution. In the incompressible liquid, outside the sphere, the fluid velocity $\bar{q}$ will be radial and thus will be a function of r, the radial distance from the centre of the sphere (the origin), and time t only.

The equation of continuity in spherical polar co-ordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2u) = 0 \qquad (1)$$

$$\left| \begin{array}{l} \because q = (u, 0, 0), u = u(r, t), \nabla \equiv \left(\frac{\partial}{\partial r}, 0, 0\right) \\ \nabla \cdot q = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u). \\ i.e.\ spherical\ symmetry. \end{array} \right.$$

$\Rightarrow r^2u$ = constant = f(t)

On the surface of the sphere,

$$r = R,\ u = \dot{\boldsymbol{R}}$$

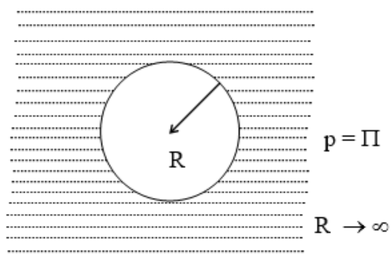
Therefore,

$$f(t) = R^2\,\dot{\boldsymbol{R}}$$

and thus

$$r^2u = R^2\dot{\boldsymbol{R}} \qquad (2)$$

Therefore,



We observe that $u \to 0$ as $n \to \infty$, as required.

From (1), it is clear that curl $\mathbf{q} = \overline{0}$

$\Rightarrow$ the motion is irrotational and $\bar{\boldsymbol{q}} = -\nabla\phi$

$$\Rightarrow\ u = -\frac{\partial\varphi}{\partial r} \quad \Rightarrow \quad -\frac{\partial\varphi}{\partial r} = \frac{f}{r^2} \qquad \text{From (2)}$$

$$\Rightarrow\ \phi = f/r \qquad (3)$$

The pressure equation for irrotational non-steady fluid motion in the absence of body forces is

$$\frac{p}{\rho} + \frac{1}{2}\boldsymbol{q}^2 - \frac{\partial\varphi}{\partial t} = \boldsymbol{C}(\boldsymbol{t})$$

$$\text{i.e.}\quad \frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 - \frac{\partial\varphi}{\partial t} = \boldsymbol{C}(\boldsymbol{t}). \qquad (4)$$

where C(t) is a function of time t.

As $r \to \infty$, $p \to \Pi$, $u = f/r^2 \to 0$, $\phi \to 0$

so that $C(t) = \Pi/\rho$ for all t (5)

Therefore, from (2), (3), (4) and (5), we get

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\partial}{\partial t}(f/r) - \frac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2 \qquad (6)$$

But $$\frac{\partial f}{\partial t} = \frac{d}{dt}(R^2\dot{R}) = \ddot{R}R^2 + 2R \rightleftarrows \dot{R}^2$$

At the surface of the sphere, we have r = R and equation (6) gives

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{R}\left(2R\dot{R}^2 + \ddot{R}R^2\right) - \frac{1}{2}\dot{R}^2$$

$$\Rightarrow \quad \frac{p}{\rho} = \frac{\Pi}{\rho} + 2\dot{R}^2 + R\ddot{R} - \frac{1}{2}\dot{R}^2$$

$$= \frac{\Pi}{\rho} + \frac{1}{2}(3\dot{R}^2 + 2R\ddot{R}) \qquad (7)$$

Now,

$$\frac{d^2(R^2)}{dt^2} + \left(\dot{R}\right)^2 = \frac{d}{dt}(2R\dot{R}) + (\dot{R})^2$$
$$= (2R\ddot{R} + 2\dot{R}^2) + \dot{R}^2$$
$$= 2R\,\ddot{R} + 3\dot{R}^2$$

Therefore, from (7), we obtain

$$p = \Pi + \frac{1}{2}\rho\left[\frac{d^2(R^2)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Hence the result.

Example: ***An infinite mass of homogeneous fluid is at rest subject to a uniform pressure Π, and contains a spherical cavity of radius a, filled with gas at a pressure mΠ. Prove that if the inertia of the gas be neglected, and Boyle's law supposed to hold throughout the ensuring motion, the radius of the sphere will oscillate between the values a and n a, where n is determined by the equation.***

$$1 + 3\,m \log n - n^3 = 0$$

If m be nearly equal to one, the time of oscillation will be $2\pi\sqrt{\frac{a^2\rho}{2\Pi}}$, ρ being the density of the fluid.

Solution. In the incompressible fluid outside the spherical cavity, the fluid velocity $\bar{q}$ will be radial and will be a function of r, the radial distance from the centre of cavity (the origin), and time t only. The continuity equation. div $\bar{q}$ = 0 in spherical polar co-ordinates, becomes (spherical symmetry)

$$\frac{1}{r^2}\frac{d}{dr}(r^2u) = 0 \Rightarrow r^2u = cons\,tan\,t = f(t) \tag{1}$$

where u = u(r, t) $\quad | \ \bar{q} = (u, 0, 0),\ \nabla \equiv \left(\frac{\partial}{\partial r}, 0, 0\right)$

We observe that $u \to 0$ as $r \to \infty$, as required.

Clearly curl $\bar{q} = \bar{0}$ so that the motion is irrotational and $\bar{q} = -\nabla\varphi$, where ϕ is velocity potential.

$\Rightarrow \quad u = -\frac{\partial\varphi}{\partial r} \Rightarrow \frac{f(t)}{r^2} = \frac{-\partial\varphi}{\partial r}$

$\Rightarrow \quad \phi = -f/r \qquad (2)$

The pressure equation for irrotational non-steady fluid motion, in the absence of body forces, is

$$p/\rho + \frac{1}{2}u^2 - \frac{\partial\varphi}{\partial t} = C(t) \tag{3}$$

where C(t) is a function of time t.

As $r\to\infty$, $p = \Pi$, $u = f/r^2 \to 0$, $\phi\to 0$,

so that $C(t) = \Pi/\rho$ for all t.

Putting the values of ϕ and C(t) in (3), we get

$$\frac{p}{\rho} + \frac{1}{2}u^2 - \frac{1}{r}\frac{\partial f}{\partial t} = \frac{\Pi}{\rho} \tag{4}$$

Now, when the cavity expands to radius r, Boyle's law provides pV = constant (V = volume) so that

$$\frac{4}{3}\pi a^3 m\Pi = \frac{4}{3}\pi r^3 p$$

$$\Rightarrow p = m\,\Pi\frac{a^3}{r^3} \qquad (5)$$

Now, $\frac{\partial f}{\partial t} = \frac{d}{dt}(r^2 u) = 2ru^2 + r^2\frac{du}{dt}$

$$= 2ru^2 + r^2 u.\frac{du}{dr} \qquad \left|\frac{du}{dt} = \frac{du}{dr}\frac{ds}{dt} = u\frac{du}{dr}\right. \qquad (6)$$

From (4), (5) and (6), we get

$$m\,\Pi\,\frac{a^3}{\rho r^3} + \frac{1}{2}u^2 - \left(2u^2 + ru\frac{du}{dr}\right) = \Pi/\rho$$

$\Rightarrow \quad 3u^2 r^2 + 2r^3 u\frac{du}{dr} = \frac{2\Pi}{\rho}.\left(\frac{ma^3}{r} - r^2\right)$ | Multiplying both sides by r^2 to make it exact

$$\Rightarrow \quad \frac{d}{dr}(r^3u^2) = \frac{2\Pi}{\rho}\left(\frac{ma^3}{r} - r^2\right)$$

Integrating w.r.t. r, we get

$$r^3 u^2 = \frac{2\Pi}{\rho}\left(ma^3\log r - \frac{r^3}{3}\right) + A \qquad (7)$$

where A is constant of integration, determined by the fact that at r = a, u = 0 (Initial conditions) and hence

$$A = \frac{-2\Pi}{\rho}\left(ma^3\log a - \frac{1}{3}a^3\right)$$

Thus, from (7), we get

$$r^3 u^2 = \frac{2\Pi a^3}{3\rho}\left[3m\log\left(\frac{r}{a}\right) - \left(\frac{r}{a}\right)^3 + 1\right] \qquad (8)$$

Now, u shall be zero again where r = na, i.e.$\frac{r}{a} = n$, provided. n is given by the equation

$$1 + 3m\log n - n^3 = 0$$

Hence proved the result.

Now, we consider the special case, when m = 1. Let r = a +x, where x is small.

Then $\dot{x} = u = \dot{r}$ and thus from (8), we get

$$(\dot{x})^2(a+x)^3 = \frac{2\Pi a^3}{3\rho}\left[3\log\left(1+\frac{x}{a}\right) - \left(1+\frac{x}{a}\right)^3 + 1\right] \qquad \left|\frac{r}{a} = 1 + \frac{x}{a}\right.$$

or $$(\dot{x})^2[1 + 3y + 3y^2 + \ldots] = \lambda\left[3\left(y - \frac{y^2}{2} + \ldots\right) - (1 + 3y + 3y^2 + \ldots) + 1\right]$$

where $\lambda = \frac{2\Pi}{3\rho}, y = \frac{x}{a}$ and y^3 is neglected.

Thus $$(\dot{x})^2(1 + 3y + 3y^2) = \lambda\left(-9\frac{y^2}{2}\right)$$

$\Rightarrow (\dot{x})^2 = \lambda\left(-9\frac{y^2}{2}\right)(1 + 3y + 3y^2)^{-1} = -9\lambda\frac{y^2}{2}$ | Neglecting y^3

$$\Rightarrow (\dot{x})^2 = -9\frac{2\Pi}{3\rho}\frac{1}{2}\left(\frac{x}{a}\right)^2 = -3\frac{\Pi}{\rho}\left(\frac{x^2}{a^2}\right)$$

Differentiating w.r.t. t, we get

$$\ddot{x} = -\left(\frac{3\Pi}{\rho a^2}\right).x$$

i.e. $$\ddot{x} = -\mu x, \mu = \frac{3\Pi}{\rho a^2}$$

This represents a S.H.M. of periodic time $2\pi\sqrt{\frac{a^2\rho}{2\Pi}}$ $\Big| \because \boldsymbol{T = \frac{2\pi}{\sqrt{m}}}$

Hence proved the result.

Example: ***Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius 'a' whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid, distance r from the centre, is***

$$\Pi(\boldsymbol{\log r - \log b})/(\boldsymbol{\log a - \log b})$$

Solution. In the incompressible liquid outside the cylinder |z| = b, the fluid velocity **q** will be radial and will be a function of r, the radial distance from the centre of the cylinder (the origin) and time t only, where r < a.

The equation of continuity div **q**= 0 in cylindrical co-ordinates. (Cylindrical symmetry), becomes

$$\boldsymbol{\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0} \Rightarrow ru = \text{const.} = f(t) = R\dot{\boldsymbol{R}} \text{ (say)} \tag{1}$$

$$\Big| \boldsymbol{q = (u, 0, 0), \nabla = \left(\frac{\partial}{\partial r}, 0, 0\right)}$$

We note that $\mathbf{q}\to 0$ as $r\to\infty$, as required.

Clearly curl **q** = 0 $\Rightarrow$ $\mathbf{q} = -\nabla\phi$

i.e. $-\boldsymbol{\frac{\partial\varphi}{\partial r} = f/r = u}$

$$\Rightarrow \qquad \phi = -f \log r \tag{2}$$

The pressure equation for irrotational non-steady fluid motion, in the absence of body forces, is

$$\boldsymbol{\frac{p}{\rho} + \frac{u^2}{2} - \frac{\partial\varphi}{\partial t} = C(t)} \qquad \boldsymbol{|q = (u, 0, 0)} \tag{3}$$

Initially, t = 0, u = 0 so that equation (3), on using (2), yields

$$\boldsymbol{\frac{p}{\rho} + \dot{f}(0) \log r = C(0)} \tag{4}$$

Now, p = Π when r = a, p = 0 when r = b (since the cylinder is annihilated $\Rightarrow$ no pressure), so that

$$\boldsymbol{\frac{\Pi}{\rho} + \dot{f}(0) \log a = C(0)} \tag{5}$$

and 0 + $\dot{\boldsymbol{f}}(\mathbf{0})$ log (b) = C(0) $\Rightarrow$ C(0) = $\dot{\boldsymbol{f}}(\mathbf{0})$ log b

$\therefore$ From (4) & (5), we get

$$\dot{\boldsymbol{f}}(\mathbf{0}) (\log r - \log b) = -\boldsymbol{\frac{p}{\rho}}$$

and $\dot{\boldsymbol{f}}(\mathbf{0})$ (log a – log b) = $\boldsymbol{\frac{-\Pi}{\rho}}$

Dividing these two we get

$$p = \Pi\,\boldsymbol{\frac{\log r - \log b}{\log a - \log b}}$$

Hence the result.

Example: ***An infinite mass of ideal incompressible fluid is subjected to a force $\mu r^{-7/3}$ per unit mass directed towards the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere r = a in it, show that the cavity will be completely filled after an interval of time $\pi a^{5/3}(10\mu)^{-1/2}$.***

Solution. The motion is entirely radial and consequently irrotational and the present case is the case of spherical symmetry. The equation of continuity is

$$\frac{1}{r^2}\frac{d}{dr}(r^2 u) = 0 \Rightarrow r^2 u = \text{constant} = f(t) \qquad (1)$$

On the surface of the sphere, r = R, $\dot{R}$= v (say)

Therefore,

$$r^2\,\dot{r} = f(t) = R^2\dot{R}$$

$$\Rightarrow \dot{f}(t) = R^2\ddot{R} + \dot{R}2R\dot{R} = R^2\frac{dv}{dt} + 2Rv^2$$

$$\Rightarrow \frac{\dot{f}(t)}{R} = 2v^2 + R\frac{dv}{dt} = 2v^2 + R\frac{dv}{dR}\frac{dR}{dt}$$

$$= 2v^2 + Rv\frac{dv}{dR} \qquad (2)$$

The Euler's equation of motion, in radial direction, using $\dot{r} = u$, is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

But $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t}\left(\dfrac{f(t)}{r^2}\right) = \dfrac{\dot{f}(t)}{r^2}, F_r = -\mu r^{-7/3}$

So, we need to integrate the Euler's equation

$$\frac{\dot{f}(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}u^2\right) = \frac{-\mu}{r^{7/3}} - \frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) \qquad (3)$$

Let us assume that the cavity has radius R at time t and its velocity then is $\dot{R} = v$. Integrating (3) over the whole liquid (r = R to r = ∞) at time t, we obtain

$$\left[\frac{-\dot{f}(t)}{r}\right]_R^\infty + \left[\frac{1}{2}u^2\right]_v^0 = \frac{3\mu}{4}\left[\frac{1}{r^{4/3}}\right]_R^\infty - \left[\frac{p}{\rho}\right]_R^\infty$$

Since the fluid is at rest at infinity, $u_\infty = 0$. Also $p_\infty = 0$, $p_R = 0$ (cavity), thus we get

$$\frac{\dot{f}(r)}{R} - \frac{1}{2}v^2 = -\frac{3\mu}{4}\frac{1}{R^{4/3}}$$

$$\Rightarrow 2Rv\frac{dv}{dR} + 3v^2 = -\frac{3\mu}{2}\frac{1}{R^{4/3}} \qquad | \text{ using } (2)$$

To make it exact, we multiply by R^2 so that

$$2R^3 v\frac{dv}{dR} + 3R^2v^2 = -\frac{3\mu}{2}R^{2/3}$$

$$\Rightarrow \frac{d(R^3v^2)}{dR} = -\frac{3m}{2}R^{2/3}$$

Integrating, we get

$$R^3\,v^2 = A - \frac{9m}{10}R^{5/3} \qquad (4)$$

When R = a, $\dot{R} \equiv v = 0$, which gives $A = \frac{9\mu}{10}a^{5/3}$.

Now, we take v = $\dot{R}$< 0 because as the cavity fills, R decreases with time. Thus (4) gives

$$\frac{dR}{dt} = -\left(\frac{9\mu}{10}\right)^{1/2}\left(\frac{a^{5/3} - R^{5/3}}{R^3}\right)^{1/2}$$

Therefore,

$$\left(\frac{9\mu}{10}\right)^{1/2} t = -\int_a^0 \frac{R^{3/2}dR}{(a^{5/3}-R^{5/3})^{1/2}}$$

$= \frac{6a^{5/3}}{5}\int_0^{\pi/2} \sin^2\theta\, d\theta$ | $R^{5/3} = a^{5/3}\sin^2\theta$ i.e., $R = a(\sin\theta)^{6/5}$

$$= \frac{3\pi a^{5/3}}{10}$$

Thus,

$$t = \pi a^{5/3}(10\mu)^{-1/2}.$$

Hence the result.

Example: ***A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subjected to a given constant pressure Π, no other forces being in action on the liquid. The solid sphere suddenly shrinks into a concentric sphere, determine the subsequent motion, and the impulsive action on the sphere.***

Solution. Let v′ be the velocity at a distance r′ from the centre of the sphere at any time t and p be the pressure. The equation of continuity (case of spherical symmetry) is

$$r'^2 v' = f(t) \tag{1}$$

Equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or

$$\frac{\dot f(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \tag{2}$$

Let R, r be the radii of the external and internal boundaries at time t, and V, v be their velocities. These quantities are functions of time t only and

$$V = \dot R,\ v = \dot r$$

Also, $p = 0$, $p = \Pi$ on the internal and the external boundaries respectively. Integrating (2) w.r.t r′ from r′ = r to r′ = R, we get

$$-\dot f(t)\left(\frac{1}{r}-\frac{1}{R}\right) + \frac{1}{2}(v^2 - V^2) = \frac{\Pi}{\rho} \tag{3}$$

But

$$r'^2 v' = f(t) = r^2 v = R^2 V$$

$$\Rightarrow\quad \dot f(t) = 2r\dot r v + r^2\frac{dv}{dt} = 2rv^2 + r^2 v\frac{dv}{dr}$$

Hence (3) becomes

$$-\left(2rv^2 + r^2 v\frac{dv}{dr}\right)\left(\frac{1}{r}-\frac{1}{R}\right) + \frac{v^2}{2}\left(1-\frac{V^2}{v^2}\right) = \frac{\Pi}{\rho}$$

or

$$-\left(2rv^2 + r^2 v\frac{dv}{dr}\right)\left(\frac{1}{r}-\frac{1}{R}\right) + \frac{v^2}{2}\left(1-\frac{r^2}{R^4}\right) = \frac{\Pi}{\rho}$$

Multiplying both sides by $2r^2$ and observing that

$R^3 - r^3 = b^3 - a^3 = c^3$ (say), we obtain

$$-\left(4r^3v^2 + 2r^4v\frac{dv}{dr}\right)\left\{\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right\}$$

$$+ v^2 r^4\left\{\frac{1}{r^2} - \frac{r^2}{(r^3+c^3)^{4/3}}\right\} = \frac{2\Pi}{\rho}r^2$$

or $$\left\{\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right\}\frac{d}{dr}(v^2r^4) - v^2r^4\left\{\frac{1}{r^2} - \frac{r^2}{(r^3+c^3)^{4/3}}\right\} = -\frac{2\Pi}{\rho}r^2$$

Integrating, we get

$$v^2 r^4\left\{\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right\} = \frac{-2\Pi}{3\rho}r^3 + c \tag{4}$$

But $v = 0$, when $r = a$, so $c = \frac{2\Pi a^3}{3\rho}$

Thus, (4) becomes

$$v^2 r^4\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{2\Pi}{3\rho}(a^3 - r^3)$$

i.e. $$v^2\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{2\Pi}{3\rho}\left(\frac{a^3-r^3}{r^4}\right)$$

which gives the expression for the velocity for the subsequent motion.

Now, let P be the impulsive pressure at a distance r' and let r be the radius of the solid sphere, then from the relation.

$$-\nabla P = \rho\bar{q}$$

we find $-\frac{dP}{dr'} = \rho v'$ $\Rightarrow dp = -\rho v'\,dr' = -\rho\frac{r^2v}{r'^2}dr' \mid r'^2v' = r^2v$

Integrating, we get

$$P = \rho v(r^2/r') + c_1$$

Since $P = 0$, when $r' = R$, so $c_1 = -\rho v\frac{r^2}{R}$

Thus $P = \rho vr^2\left(\frac{1}{r'} - \frac{1}{R}\right)$

Putting $r' = r$, we find

$$P = \rho v\; r^2\left(\frac{1}{r} - \frac{1}{R}\right)$$

which gives the impulsive pressure on the surface of the sphere.

The whole impulse on the sphere

$$= 4\pi r^2 P$$

$$= 4\pi\, r^2\, \rho vr^2\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$= 4\pi\, \rho r^3 v(R-r)/R.$$

and the whole momentum destroyed

$$= \int_r^R 4\pi\, r'^2\rho v' dr'$$

$$= 4\pi\rho\int_r^R r^2 v dr' \qquad \mid r'^2 v' = r^2 v$$

$$= 4\pi\rho\, r^2\, v(R-r)$$

Example: ***A sphere of radius a is surrounded by an infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that the pressure at distance r from***

the centre immediately falls to $\pi\left(1-\frac{a}{r}\right)$. ***Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius*** $\frac{a}{2}$, ***the impulsive pressure sustained by the surface of the sphere is*** $\sqrt{7\Pi\rho a^2/6}$.

Solution. Let v′ be the velocity at a distance r′ from the centre of the sphere at any time t and p be the pressure. The equation of continuity (case of spherical symmetry) is

$$\frac{1}{r'^2}\frac{d}{dr'}(r'^2v') = 0 \Rightarrow r'^2v^2 = f(t) \tag{1}$$

Equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = \frac{-1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{| No body forces}$$

or
$$\frac{\dot{f}(t)}{r'^2} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating w.r.t. r′, we get

$$-\frac{\dot{f}(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C$$

Since $r' \to \infty \Rightarrow p = \Pi$, $v' = 0$ so that $C = \Pi/\rho$.

Thus $\frac{-\dot{f}(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho}$ (2)

When, sphere is suddenly annihilated i.e. $r' = a$, $v' = 0$, $p = 0$, then

$$-\frac{\dot{f}(t)}{a} = \Pi/\rho \; i.e.\; \dot{f}(t) = -\frac{\Pi a}{\rho} \tag{3}$$

The velocity v′ vanishes just after annihilation, so from (2) and (3), we get

$$\frac{\Pi a}{\rho r'} = \frac{\Pi - p}{\rho} \Rightarrow \frac{a\Pi}{r'} = \Pi - p$$

Thus, the pressure at the time of annihilation (r′ = r) is

$$\frac{a\Pi}{r} = \Pi - p \Rightarrow p = \Pi\left(1 - \frac{a}{r}\right)$$

which proves the first result.

Now, let P be the impulsive pressure at a distance r′, then from the relation $-\nabla P = \rho\bar{q}$, we get

$$-\frac{dP}{dr'} = \rho v' \quad \Rightarrow dP = -\rho v'\, dr'$$

From the equation of continuity, we have

$$r^2v = r'^2 v' = f(t) \tag{4}$$

So
$$dP = -\rho v\,(r^2/r'^2)\, dr' \tag{5}$$

where r is the radius of the inner surface and v is the velocity there.

Integrating (5), we get

$$P = \rho v\,(r^2/r') + C_1$$

When $r' \to \infty$, P = 0 so that $C_1 = 0$

Thus $P = \rho v(r^2/r')$ (6)

Equation (6) determines the impulsive pressure P at a distance r′. The velocity v at the inner surface of the sphere (p = 0) is obtained from (2) as

$$-\frac{\dot{f}(t)}{r}+\frac{1}{2}v^2=\frac{\Pi}{\rho} \qquad (7)$$

From (4), $\dot{f}(t)=\frac{d}{dt}(r^2v)=r^2\frac{dv}{dt}+v.2r\frac{dr}{dt}=r^2\frac{dv}{dr}\frac{dr}{dt}+2rv^2$

$$\Rightarrow \quad \frac{\dot{f}(t)}{r}=rv\frac{dv}{dr}+2v^2$$

Thus (7) becomes

$$\text{r v}\frac{dv}{dr}+2v^2-\frac{1}{2}v^2=-\frac{\Pi}{\rho}$$

or $$\text{rv}\frac{dv}{dr}+\frac{3}{2}v^2=\frac{-\Pi}{\rho}$$

$$\Rightarrow \quad 2r^3v\frac{dv}{dr}+3v^2r^2=\frac{-2\Pi}{\rho}r^2 \qquad |\text{Multiplying by } r^2$$

$$\Rightarrow \quad \frac{d(r^3v^2)}{dr}=-\frac{2\Pi}{\rho}r^2$$

Integrating, we get

$$r^3v^2=-\frac{2\Pi}{3\rho}r^3+C_2$$

Since r = a, v = 0 so we find $C_2=\frac{2\Pi a^3}{3\rho}$

Therefore, $r^3v^2=\frac{2\Pi}{3\rho}(a^3-r^3)$

The velocity v at the surface of the sphere r = a/2, on which the liquid strikes, is

$$v^2=\frac{2\Pi}{3\rho}\frac{a^3-(a/2)^3}{(a/2)^3}=\frac{14}{3}\frac{\Pi}{\rho}$$

From relation (6), using r = a/2, we get

$$P=\frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}}.\frac{a^2}{r'} \qquad (8)$$

which determines the impulsive pressure at a distance r' from the centre of the sphere. Thus, the impulsive pressure at the surface of the sphere of radius a/2 is given by

$$P=\frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}}\frac{a^2}{a/2}=\sqrt{7\Pi\rho a^2/6}$$

Hence the result

5.6 Check Your Progress:

i) Show that the velocity field $u(x,y)=\frac{A(x^2-y^2)}{(x^2+y^2)^2}$, $v(x,y)=\frac{2Axy}{(x^2+y^2)^2}$, $w=0$ satisfies the equation of motion for inviscid incompressible flow. Determine the pressure associated with this fluid.

[Ans: $p=\frac{A\rho^2}{2(x^2+y^2)^2}$]

ii) Determine the pressure, if the velocity field $q_r, q_\theta=Ar+B, q_z=0$ satisfies the equation of motion $\rho\frac{q_\theta^2}{r}=\frac{dp}{dr}$, where A and B are arbitrary constants.

[Ans: $p=\rho\left\{\frac{1}{2}A^2r^2-\frac{B^2}{2r^2}+2ABlog\,r\right\}+C$]

iii) Prove that the equation of motion is satisfied for an inviscid, incompressible, steady flow with negligible body force whose velocity components are given by

$\boldsymbol{q_r = U\left(1 - \frac{A^3}{r^3}\right) cos\theta, q_\theta = -U\left(1 + \frac{A^3}{2r^3}\right) \sin\theta, q_\phi = 0}$, where A is constant. Find the resultant velocity when $\boldsymbol{r} \to \infty$.

iv) A quantity of liquid occupies a length $\boldsymbol{2l}$ of a straight tube of uniform small bore under the action of a force to a point in the tube varying as the distance from that point. Determine the motion and the pressure.

v) If $\boldsymbol{\widetilde{w}}$ is the impulsive pressure; $\boldsymbol{\phi, \phi'}$ the velocity potential immediately before and after an impulse act, V is the potential of impulses, prove that $\boldsymbol{\widetilde{w} + \rho V + \rho(\phi - \phi') = const.}$, where $\boldsymbol{\rho}$ is the density of the fluid.

5.7 Summary: In this chapter, the impulsive motion equation, Cauchy's pressure equation, Bernoulli's equations, Lagrange’s equation of motion and Euler's equation of motion are derived to explore the issues with fluid flows. The Bernoulli's equation deals with the conservation of a fluid's kinetic, potential, and flow energy as well as their conversion to one another in flow regions where net viscous forces are negligible and other constraining circumstances are present. Also Bernoulli’s theorem is stated and proved.

5.8 Keywords: momentum, impulsive motion, pressure, equation of motion, Bernoulli’s theorem.

5.9 Self-Assessment Test:

SA1: Derive Bernoulli’s equation for unsteady motion of an incompressible fluid and hence derive expression for steady motion.

SA2: If $\boldsymbol{u = \frac{ax-by}{x^2+y^2}, v = \frac{ay+bx}{x^2+y^2}}$ and w=0, investigate the nature of the motion of the liquid and also obtain the velocity potential and pressure at any point P.

SA3: Obtain Bernoulli’s equation for steady irrotational motion of an incompressible fluid.

SA4: Obtain equation of motion under impulsive forces in Cartesian coordinates and prove that impulse satisfies Laplace’s equation.

SA5: The particle velocity for a fluid motion referred to rectangular axes is given by the components

$\boldsymbol{u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, v = 0, w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}}$, where A is constant. Show that this a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube, $\boldsymbol{-a \le x \le a}$, $\boldsymbol{0 \le z \le 2a}$. Also, find the pressure associated with this velocity field.

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CHAPTER-6
IRROTATIONAL MOTION

6.0 Learning Objectives: After reading this chapter, you should be able to understand the boundary condition during flow of the fluid, work with energy equation, circulation, vorticity, vorticity equation, permanence of irrototations, axially symmetric flows and Kelvin Circulation theorem.

6.1 Boundary Conditions: When fluid is in contact with a rigid solid surface (or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact:

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.

Let **n** denote a normal unit vector drawn at the point P of the surface of contact and let **q** denotes the fluid velocity at P. When the rigid surface of contact is at rest, we must have **q.n**=0 at each point of the surface. This expresses the condition that the normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point as shown in figure (a).

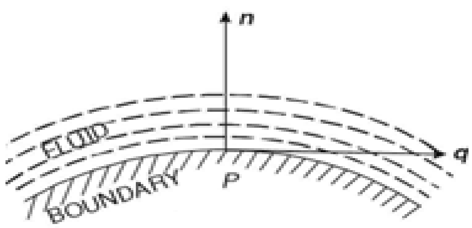


Fig.(a)

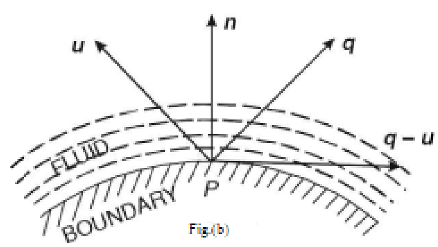


Fig.(b)

Next, let the rigid surface be in motion and let **u** be its velocity at P (figure (b)). Then we must have

$$\boldsymbol{q}.\boldsymbol{n} = \boldsymbol{u}.\boldsymbol{n} \quad \boldsymbol{or} \quad (\boldsymbol{q}-\boldsymbol{u}).\boldsymbol{n} = \boldsymbol{0}$$

Which expresses for the fact that there must be no normal velocity at P between boundary and fluid, i.e., the velocity of the fluid relative to the boundary is tangential to the boundary at its each point.

In particular, if the boundary surface is at rest, then $\boldsymbol{u} = 0$ and the condition becomes

$$\boldsymbol{q} \cdot \boldsymbol{n} = 0 \qquad (2)$$

Another type of boundary condition arrives at a free surface where liquid borders a vacuum e.g. the interface between liquid and air is usually regarded as free surface. For this free surface, pressure p satisfies

$$P = \Pi \qquad (3)$$

where Π denotes the pressure outside the fluid i.e. the atmospheric pressure. Equation (3) is a dynamic boundary condition.

Third type of boundary condition occurs at the boundary between two immiscible ideal fluids in which the velocities are $\boldsymbol{q}_1$ *and* $\boldsymbol{q}_2$ and pressures are p_1 *and* p_2 respectively.

Now, we find the condition that a given surface satisfies to be a boundary surface.

Article : To obtain the differential equation satisfied by boundary surface of a fluid in motion

or

To find the condition that the surface.

$$\mathrm{F}(\boldsymbol{r}, t) = F(x, y, z, t) = 0$$

may represent a boundary surface:

If **q** be the velocity of fluid and **u** be the velocity of the boundary surface at a point P of contact, then

$$(\boldsymbol{q} - \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \Rightarrow \boldsymbol{q}.\boldsymbol{n} = \boldsymbol{u}.\boldsymbol{n} \qquad (1)$$

where $\boldsymbol{q} - \boldsymbol{u}$ is the relative velocity and **n** is a unit vector normal to the surface at P.

The equation of the given surface is

$$\mathrm{F}(\boldsymbol{r}, t) = F(x, y, z, t) = 0 \qquad (2)$$

We know that a unit vector normal to the surface (2) is given by

$$\boldsymbol{n} = \frac{\nabla F}{|\nabla F|}$$

Thus, from (1), we get

$$\boldsymbol{q} \cdot \nabla F = \boldsymbol{u} \cdot \nabla F \qquad (3)$$

since the boundary surface is itself in motion, therefore at time (t + δt), its equation is given by

$$\mathrm{F}(\boldsymbol{r} + \delta\boldsymbol{r}, t + \delta t) = 0. \qquad (4)$$

From (2) and (4), we have

$$\mathrm{F}(\boldsymbol{r} + \delta\boldsymbol{r}, t + \delta t) - F(\boldsymbol{r}, t) = 0$$

i.e. $\mathrm{F}(\boldsymbol{r} + \delta\boldsymbol{r}, t + \delta t) - F(\boldsymbol{r}, t + \delta t) + F(\boldsymbol{r}, t + \delta t) - F(\boldsymbol{r}, t) = 0$

By Taylor's series, we can have

$$(\delta\boldsymbol{r} \cdot \nabla)F(\boldsymbol{r}, t + \delta t) + \delta t \frac{\partial}{\partial t} F\{\boldsymbol{r}, t\} = 0$$

$$\because F(x + \delta x, y + \delta y, z + \delta z) = F(x, y, z) + \delta x \frac{\partial F}{\partial x} + \delta y \frac{\partial F}{\partial y} + \delta z \frac{\partial F}{\partial z} + \ldots + \delta\boldsymbol{r} \cdot \nabla F$$

$$= \mathrm{F(x, y, z)} + \delta\boldsymbol{r} \cdot \nabla F$$

$$\Rightarrow \quad \left(\frac{\delta r}{\delta t} \cdot \nabla\right) F(\boldsymbol{r}, t + \delta t) + \frac{\partial F}{\partial t} = 0$$

Taking limit as δt→0, we get

$$\left(\frac{d\boldsymbol{r}}{dt} . \nabla\right) F + \frac{\partial F}{\partial t} = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial t} + (\boldsymbol{q} . \nabla)F = 0 \quad i.e. \quad \frac{DF}{Dt} = 0 \qquad (5)$$

which is the required condition for any surface F to be a boundary surface

Corollary 1. If **q** = (u, v, w), then the condition (5) becomes

$$\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0$$

In case, the surface is rigid and does not move with time, then $\frac{\partial F}{\partial t} = 0$and the boundary condition is

$$u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0 i.e. (\boldsymbol{q} \cdot \nabla)F = 0$$

Corollary 2. The boundary conditions

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

is a linear equation and its solution gives

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \qquad \frac{D}{Dt}\equiv\frac{d}{dt}\ \ in\ \ Lagrangian\ \ view$$

$\Rightarrow$ $$\frac{dx}{dt}=u,\frac{dy}{dt}=v,\frac{dz}{dt}=w$$

which are the equations of path lines.

Hence once a particle is in contact with the surface, it never leaves the surface.

Corollary 3. From equation (5), we have

$$\boldsymbol{q}\cdot\nabla F=\frac{-\partial F}{\partial t}$$

$\Rightarrow$ $$\boldsymbol{q}\cdot\frac{\nabla F}{|\nabla F|}=\frac{-\partial F/\partial t}{|\nabla F|}$$

$\Rightarrow$ $$\boldsymbol{q}\cdot\boldsymbol{n}=\frac{-\partial F/\partial t}{|\nabla F|}$$

which gives the normal velocity.

Also, from (1), we get

$$\boldsymbol{u}.\boldsymbol{n}=\frac{-\partial F/\partial t}{|\nabla F|} \qquad \because\ \ \boldsymbol{q}.\boldsymbol{n}=\boldsymbol{u}.\boldsymbol{n}$$

which gives the normal velocity of the boundary surface.

Example: ***Obtain the condition for the surface z = f(x, y, t) to be the boundary of a moving fluid***.

Solution. Write $F(x,y,z,t)\ =\ z-f(x,y,t)\ =\ 0$ (1)

By definition $\frac{DF}{Dt}=\frac{\partial F}{\partial t}+(\boldsymbol{q}\cdot\nabla)F=0$

(2)

is the required condition for the surface F$(\boldsymbol{r},t)$ = 0 to be the boundary of a moving fluid.

From (1) and (2), we get

$$\frac{Dz}{Dt}-\frac{\partial f}{\partial x}\frac{Dx}{Dt}-\frac{\partial f}{\partial y}\frac{Dy}{Dt}-\frac{\partial f}{\partial t}\frac{Dt}{Dt}=0$$

$\Rightarrow$ $$\mathrm{w}-\mathrm{u}\frac{\partial f}{\partial x}-v\frac{\partial f}{\partial y}-\frac{\partial f}{\partial t}=0$$

i.e. $$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}-w=0$$

Example: ***Show that the ellipsoid***

$$\boldsymbol{\frac{x^2}{a^2k^2t^{2n}}+kt^4\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=1}$$

***is a possible form of the boundary surface of a liquid*?**

Solution. The surface $F(x,y,z,t)\ =\ 0$ can be a possible boundary surface, if it satisfies the boundary condition.

$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \qquad (1)$$

where u, v, w satisfies the equation of continuity

$$\nabla\cdot\boldsymbol{q}=0 \qquad i.e. \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (2)$$

Here, $F(x, y, z, t) \equiv \frac{x^2}{a^2k^2t^{2n}} + kt^n \left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] - 1 = 0$

$\therefore \frac{\partial F}{\partial t} = -\frac{x^2.2n}{a^2k^2t^{2n+1}} + 4kt^{n-1}\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right]$

$\frac{\partial F}{\partial x} = \frac{2x}{a^2k^2t^{2n}}, \frac{\partial F}{\partial y} = \frac{2kt^4 y}{b^2}, \frac{\partial F}{\partial z} = \frac{2kt^n z}{c^2}.$

Thus, from (1), we get

$$\frac{-x^2}{a^2k^2}\frac{2n}{t^{2n+1}} + nkt^{n-1}\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right]$$

$$+ \frac{2.xu.}{a^2k^2t^{2n}.} + \frac{2kt^n yv}{b^2} + \frac{2kt^n.zw}{c^2} = 0$$

or $$\left(u - \frac{4x}{t}\right)\frac{2x}{a^2k^2t^{2n}} + \left(v + \frac{4y}{2t}\right)\frac{2kyt^4}{b^2} + \left(w + \frac{nz}{2t}\right)\frac{2kzt^n}{c^2} = 0$$

which will hold, if we take

$$\text{u} - \frac{-nx}{t} = 0, v + \frac{ny}{2t} = 0, w + \frac{nz}{2t} = 0$$

i.e. $$\text{u} = \frac{nx}{t}, v = -\frac{ny}{2t}, w = -\frac{nz}{2t} \qquad (3)$$

It will be a justifiable step if equation (2) is satisfied.

i.e. $$\frac{n}{t} + \frac{-n}{2t} + \frac{-n}{2t} = 0.$$

which is true

Hence the given ellipsoid is a possible form of boundary surface of a liquid.

Example: ***Show that the surface***

$$\boldsymbol{\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1}.$$

is a possible form of boundary surface of a fluid. Find also the normal velocity.

Solution. Here, F(x, y, z, t) $\equiv \frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t - 1 = 0$ (1)

$\Rightarrow$ $$\frac{\partial F}{\partial t} = \frac{x^2}{a^2.}(2\tan t)\sec^2 t + \frac{y^2}{b^2.}(2\cot t)(-\operatorname{cosec}^2 t)$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2.}(\tan^2 t), \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2}(\cot^2 t); \quad \frac{\partial F}{\partial z} = 0$$

The condition of boundary surface is

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

Here, it becomes

$$\frac{2x^2}{a^2}(\tan t\sec^2 t) - \frac{2y^2}{b^2}(\cot t\operatorname{cosec}^2 t) + \frac{2xu.}{a^2.}\tan^2 t + \frac{2yv}{b^2}\cot^2 t = 0$$

or $$\frac{2x}{a^2}\tan^2 t\left(u + \frac{x\sec^2 t}{\tan t}\right) + \frac{2y}{b^2}\cot^2 t\left(v - \frac{y\operatorname{cosec}^2 t}{\cot t}\right) = 0$$

which will be satisfied if we take

$$\text{u} = \frac{-x\sec^2 t}{\tan t}, v = \frac{y\operatorname{cosec}^2 t}{\cot t}$$

This will be a justifiable step if the equation of continuity i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is satisfied.

Now, $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{-1}{\sin t\cos t} + \frac{1}{\sin t\cos t} + 0 = 0$$

Hence equation (1) is a possible form of the boundary surface of the liquid.

Now, normal velocity = $\boldsymbol{q.n} = \frac{-\partial F/\partial t}{|\nabla F|}$

$$= \frac{-\partial F/\partial t}{\left[\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2\right]^{1/2}}$$

$$= \frac{-\left[\frac{2x^2}{a^2.} \tan t \sec^2 t - \frac{2y^2}{b^2} \cot t \cos ec^2 t\right]}{\left[\left(\frac{2x}{a^2} \tan^2 t\right)^2+\left(\frac{2y}{b^2} \cot^2 t\right)^2\right]^{1/2}}$$

$$= \frac{a^2y^2 \cot t \cos ec^2 t - b^2x^2 \tan t \sec^2 t}{(b^4x^2 \tan^4 t + a^4y^2 \cot^4 t)^{1/2}}$$

Example: ***Determine the restrictions on f_1, f_2, f_3 if***

$$\boldsymbol{\frac{x^2}{a^2} f_1(t) + \frac{y^2}{b^2} f_2(t) + \frac{z^2}{c^2} f_3(t) = 1}$$

is a possible form of boundary surface of a liquid.

Solution. Here, F $\equiv \frac{x^2}{a^2} f_1(t) + \frac{y^2}{b^2} f_2(t) + \frac{z^2}{c^2} f_3(t) - 1 = 0$ (1)

The boundary surface condition is

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \quad (2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} f_1' + \frac{y^2}{b^2} f_2' + \frac{z^2}{c^2} f_3' + u\frac{2x}{a^2} f_1 + v\frac{2y}{b^2} f_2 + w\frac{2z}{c^2} f_3 = 0$$

$$\Rightarrow \quad \frac{2x}{a^2} f_1\left(u + \frac{xf_1'}{2f_1}\right) + \frac{2y}{b^2} f_2\left(v + \frac{yf_2'}{2f_2}\right) + \frac{2z}{c^2} f_3\left(w + \frac{zf_3'}{2f_3}\right) = 0$$

This equation will be satisfied if we take

$$u = \frac{-xf_1'}{2f_1}, v = \frac{-yf_2'}{2f_2}, w = \frac{-zf_3'}{2f_3}$$

where u, v, w must, satisfy the equation of continuity. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} = 0$

i.e., $-\frac{1}{2}\left[\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}\right] = 0$

Integrating, we get

$$\log f_1 f_2 f_3 = \text{constant}$$

i.e. $\quad f_1 f_2 f_3 = \text{constant}$

which is the required restriction.

Example: ***In the steady motion of homogeneous liquid if the surfaces $f_1 = a_1$ $f_2 = a_2$ define the streamlines, prove that the most general values of the velocity components. u, v, w is***

$$\boldsymbol{F(f_1, f_2)\frac{\partial(f_1,f_2)}{\partial(y,z)}, \quad F(f_1, f_2)\frac{\partial(f_1,f_2)}{\partial(z,x)}, \quad F(f_1, f_2)\frac{\partial(f_1,f_2)}{\partial(x,y)}.}$$

where F is any arbitrary function.

Solution. Here, the motion is given to be steady, therefore streamlines are independent of t i.e., f_1 & f_2 are functions of x, y, z only. Differentiating $f_1 = a_1$, $f_2 = a_2$, we get

$$\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0 \quad (1)$$

$$\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz = 0. \tag{2}$$

Solving these, we get

$$\frac{dx}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{dy}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}} = \frac{dz}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}$$

i.e. $$\frac{dx}{\frac{\partial(f_1,f_2)}{\partial(y,z)}} = \frac{dy}{\frac{\partial(f_1,f_2)}{\partial(z,x)}} = \frac{dz}{\frac{\partial(f_1,f_2)}{\partial(x,y)}} \tag{3}$$

But the differentiating equations of streamline are.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{4}$$

Comparing (3) and (4), we get

$$\frac{u}{\frac{\partial(f_1,f_2)}{\partial(y,z)}} = \frac{v}{\frac{\partial(f_1,f_2)}{\partial(z,x)}} = \frac{w}{\frac{\partial(f_1,f_2)}{\partial(x,y)}} = F \qquad \text{(say)}$$

$\Rightarrow$ $$u = F\frac{\partial(f_1,f_2)}{\partial(y,z)}, v = F\frac{\partial(f_1,f_2)}{\partial(z,x)} w = F\frac{\partial(f_1,f_2)}{\partial(x,y)}.$$

Now, we shall determine the nature of F.

For possible motion, the velocity components must satisfy the equation of continuity namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \left[\frac{\partial F}{\partial x}.\frac{\partial(f_1,f_2)}{\partial(y,z)} + F.\frac{\partial}{\partial x.}\frac{\partial(f_1,f_2)}{\partial(y,z)}\right] + \left[\frac{\partial F}{\partial y}\frac{\partial(f_1,f_2)}{\partial(z,x)} + F.\frac{\partial}{\partial y.}\frac{\partial(f_1,f_2)}{\partial(z,x)}\right]$$
$$+ \left[\frac{\partial F}{\partial z}.\frac{\partial(f_1,f_2)}{\partial(x,y)} + F.\frac{\partial}{\partial z.}\frac{\partial(f_1,f_2)}{\partial(x,y)}\right] = 0 \tag{5}$$

But $\frac{\partial}{\partial x}\frac{\partial(f_1,f_2)}{\partial(y,z)} + \frac{\partial}{\partial y}\frac{\partial(f_1,f_2)}{\partial(z,x)} + \frac{\partial}{\partial z}\frac{\partial(f_1,f_2)}{\partial(x,y)} = 0$ | By the property of Jacobian

$\therefore$ (5) $\Rightarrow \frac{\partial F}{\partial x}\frac{\partial(f_1,f_2)}{\partial(y,z)} + \frac{\partial F}{\partial y}\frac{\partial(f_1,f_2)}{\partial(z,x)} + \frac{\partial F}{\partial z}\frac{\partial(f_1,f_2)}{\partial(x,y)} = 0$

$$\Rightarrow \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} = 0 \, i.e. \frac{\partial(F,f_1,f_2)}{\partial(x,y,z)} = 0$$

But the vanishing Jacobian means that F, f_1, f_2 are not independent.

Therefore, F is a function of f_1 & f_2 i.e., F = F (f_1, f_2)

Hence $u = F(f_1, f_2)\frac{\partial(f_1,f_2)}{\partial(y,z)}, v = F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(z,x)}, w = F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(x,y)}$.

Example: ***Show that all necessary conditions can be satisfied by a velocity potential of the form***

$$\phi = \alpha x^2 + \beta y^2 + \gamma z^2$$

and the boundary surface of the form

$$F \equiv ax^4 + by^4 + cz^4 - X(t) = 0$$

where X(t) is a given function of time and α, β, γ, a, b, c are suitable functions of time.

Solution. Here, the velocity potential is given, therefore the flow is of potential kind. Thus, we have $$\boldsymbol{q} = -\nabla\varphi. = -\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right) = (u, v, w) \tag{1}$$

Also, we know that the necessary condition, that $\bar{q}$ must satisfy, is the equation of continuity.
i.e., $\nabla \cdot \boldsymbol{q} = 0 \quad or \quad \nabla(-\nabla\varphi) = 0$ i.e., $\nabla^2\phi = 0$

i.e., $$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

In the present case it becomes

$$2\alpha + 2\beta + 2\gamma = 0 \Rightarrow \alpha + \beta + \gamma = 0 \tag{2}$$

Now, the boundary surface is

$$F \equiv ax^4 + by^4 + cz^4 - X(t) = 0 \tag{3}$$

and the condition which F must satisfy, is

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \tag{4}$$

Using (1) and (3) in (4), we obtain

$$x^4 a' + y^4 b' + z^4 c' - X'(t) - 2\alpha x(4ax^3) - 2\beta y(4by^3) - 2\gamma z(4cz^3) = 0$$

i.e., $x^4(a'-8\alpha a) + y^4(b'-8\beta b) + z^4(c-8\gamma c) - X'(t) = 0$ (5)

where dashes denote derivative with respect to time t.
Since (3) and (5) both must hold for all points (x, y, z) on the surface, so they are identical.
Comparing these we get

$$\frac{a'-8\alpha a}{a} = \frac{b'-8\beta b}{b} = \frac{c'-8\gamma c}{c} = \frac{X'(t)}{X(t)}$$

These conditions will hold if α, β, γ, a, b, c are some suitable functions of time, where $\alpha + \beta + \gamma = 0$.
Hence ϕ and F = 0 satisfy the necessary condition for velocity potential and boundary surface if α, β, γ, a, b, c are some suitable functions of time.

6.2 Circulation: The flow round a closed curve C is known as circulation and is usually denoted by $\boldsymbol{\Gamma}$. Thus

$$\Gamma = \oint_C \boldsymbol{q.dr}$$

Obviously, when a single-valued velocity potential $\boldsymbol{\phi}$ exists, circulation round C is zero; it being equal to $\boldsymbol{\phi_A - \phi_A}$.

Stokes' theorem : This theorem deals with the concept of rotation in terms of circulation and states as under

*If **q** is the velocity vector point function and S is a surface bounded by a curve C, then*

$$\oint_C \boldsymbol{q.dr} = \int_S \boldsymbol{curl\, q.dS}.\ \boldsymbol{i.e.},\ \ \Gamma = \int_S \boldsymbol{\Omega.n\, dS}.$$

*Where the unit normal vector **n** at any point of S is drawn in the sense in which a right-handed screw would move when rotate in the sense of description of C.*

6.3 Kelvin's Circulation Theorem: The circulation Γ around any material closed contour C moving with the inviscid (non-viscous) fluid is constant for all times, provided that the external forces (body forces) are conservative and derivable from a single valued potential function χ and the density is a function of pressure only.

Proof. The circulation round a closed curve C of fluid particles is defined by

$$\Gamma = \oint_C \boldsymbol{q} \,.\, d\boldsymbol{r},$$

Where **q** is the velocity and **r** is the position vector of a fluid particle at any time t.

Time derivative of Γ following the motion of fluid is

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_C \boldsymbol{q}.\boldsymbol{dr} = \oint_C \frac{d}{dt}(\boldsymbol{q}\cdot d\boldsymbol{r})$$

$$= \oint_C \left[\frac{d\boldsymbol{q}}{dt}\cdot d\boldsymbol{r} + \boldsymbol{q}\cdot\frac{d}{dt}(d\boldsymbol{r})\right]$$

$$= \oint_C \left[\frac{d\boldsymbol{q}}{dt}\cdot d\boldsymbol{r} + \boldsymbol{q}\cdot d\boldsymbol{q}\right] \qquad (1) \qquad \Big|\because \frac{d}{dt}(d\boldsymbol{r}) = d\left(\frac{d\boldsymbol{r}}{dt}\right) = d\boldsymbol{q}$$

Since the system of forces is conservative; therefore F=–∇χ, where χ is a potential function'

Euler's equation of motion is

$$\nabla p \;=\; -\nabla\chi - \frac{1}{\rho}\;\; \nabla p \qquad (2)$$

Multiplying each term of (2) scalarly by d**r**, we get

$$d\boldsymbol{r}\cdot\frac{d\boldsymbol{q}}{dt} = -d\boldsymbol{r}\cdot\nabla\chi - \frac{1}{\rho}d\boldsymbol{r}.\nabla \mathrm{p}$$

i.e. $$\frac{d\boldsymbol{q}}{dt}.\,d\boldsymbol{r} = -d\chi - \frac{dp}{\rho} \qquad \Big|\because d\boldsymbol{r}.\nabla \equiv d \qquad (3)$$

Thus from (1), we get

$$\frac{d\Gamma}{dt} = \oint_C \left(-d\chi - \frac{dp}{\rho} + \boldsymbol{q}\cdot d\boldsymbol{q}\right)$$

$$= \oint_C \left[d\left(\frac{1}{2}\boldsymbol{q}^2 - \chi\right) - \frac{1}{\rho}dp\right]$$

$$= \oint_C d\left(\frac{1}{2}q^2 - \chi\right) - \oint_C \frac{1}{\rho}dp$$

$$= \left[\frac{1}{2}\boldsymbol{q}^2 - \Omega\right]_A^A - \oint_C \frac{dp}{\rho}$$

$$= 0 - \oint_C \frac{dp}{\rho} \qquad (4)$$

where A is any point on the closed contour C. Now, if density is a function of pressure only, then the integral $\oint_C \frac{dp}{\rho}$ vanishes and hence we get

$$\frac{d\Gamma}{dt} = 0 \quad \Rightarrow \; \Gamma = \text{constant for all time}$$

Some Consequences of Circulation Theorem

Corollary. I. In a closed-circuit C of fluid particles moving under the same conditions as in the theorem,

$$\int_S curl\boldsymbol{q}.\,dS = \int_S \omega.\,dS = \text{constant} \qquad (5)$$

where S is any open surface, whose sum is C. To establish (5), we note that, by Stock's theorem,

$$\int_S curl\boldsymbol{q}.\,dS = \oint_C \boldsymbol{q}.\,\boldsymbol{dr} = \Gamma = \text{constant}$$

This shows that the product of the cross-section and angular velocity as any point on a vortex filament is constant all along the vortex filament and for all times.

Corollary II. Under the conditions of the theorem, vortex lines move with the fluid.

Proof. Let C be any closed curve drawn on the surface of a vortex tube. Let S be the portion of the vortex tube rimmed by C. By definition vortex lines lie on S. Thus

$$0 = \int_S curl\boldsymbol{q}.dS = \oint_C \boldsymbol{q}.d\boldsymbol{r} \qquad | \because \text{ on surface circulation is zero}$$

Let C be a material curve and S be a material surface, then

$$\frac{d}{dt}\int_S (\boldsymbol{n}.curl\boldsymbol{q})dS = \int_S \frac{D}{Dt}(\boldsymbol{n}.curl\boldsymbol{q})dS = 0$$

Thus $\boldsymbol{n}.curl\boldsymbol{q}$ remains zero, so that S remains a surface composed of vortex lines. Consequently, vortex lines and tubes move with the fluid i.e. vortex filaments are composed of the same fluid particles. This explains why smoke rings maintain their forms for long periods of time.

Corollary III. Permanence of irrotational motion:

Under the conditions of the theorem, if the flow is irrotational in a material region of the fluid at some particular time (e.g. t = 0 or t = t_0), the flow is always irrotational in that material region thereafter.

i.e. If the motion of an ideal fluid is once irrotational it remains irrotational for ever afterwards provided the external forces are conservative and density ρ is a function of pressure p only.

Proof. Suppose that at some instant (t = t_0), the fluid on the material surface S is irrotational

Then, $curl\ \boldsymbol{q} = \boldsymbol{\omega} = 0$ (1)

for all points of S.

Let C be the boundary of surface S, then

$$\Gamma = \oint_C q.dr = \int_S (n.curl\boldsymbol{q})dS = \int_S (\boldsymbol{n}.\omega)ds = 0 \qquad | \text{ using (1)}$$

But by Kelvin's circulation theorem, Γ is constant for all times. Hence circulation Γ is zero for all subsequent times. At any later time,

$$\int_S n.\omega dS = 0$$

If we now take S to be non-zero infinitesimal element, say ΔS, then

$\boldsymbol{n}.\omega\ \Delta S = 0 \Rightarrow \omega = 0$ at all points of S for all times and the motion is irrotational permanently. This proves the permanency of irrotational motion.

Remarks 1) The above three corollaries are properties of vortex filaments.

2) The Kelvin's theorem is true whether the motion be rotational or irrotational In case of irrotational motion, $\boldsymbol{\omega} = 0$ and thus $\Gamma = 0$.

3) From the results of the theorem, we conclude that vortex filaments must either form closed curves or have their ends on the bounding surface of the fluid. A vortex in an ideal fluid is therefore permanent.

6.4 Vorticity Equation (Helmholtz Theorem): *If the external forces are conservative and density is a function of pressure p only, then*

$$\frac{\boldsymbol{D\Omega}}{\boldsymbol{dt}} = (\boldsymbol{\Omega}.\boldsymbol{\nabla}\,)\boldsymbol{q}$$

Proof: Euler's equation of motion for an ideal fluid under the action of a conservative body force with potential χ per unit mass is

$$\frac{D\boldsymbol{q}}{Dt} = \frac{\partial \boldsymbol{q}}{\partial t} + \nabla\left(\frac{1}{2}\boldsymbol{q}^2\right) - \boldsymbol{q} \times \Omega = -\nabla\chi - \frac{1}{\rho}\nabla p \qquad (1)$$

where the vorticity $\Omega = curl\boldsymbol{q} = \nabla \times \boldsymbol{q}$. If the fluid has constant density, then taking curl of equation (1), we get

$$\nabla \times \frac{\partial \boldsymbol{q}}{\partial t} + \nabla \times \left[\nabla\left(\frac{1}{2}\boldsymbol{q}^2\right)\right] - \nabla \times (\boldsymbol{q} \times \Omega) = \nabla \times \left(-\nabla\chi - \frac{1}{\rho}\nabla p\right)$$

$$\Rightarrow \qquad \nabla \times \frac{\partial \boldsymbol{q}}{\partial t} - \nabla \times (\boldsymbol{q} \times \Omega) = 0$$

$$\frac{\partial \boldsymbol{q}}{\partial t} - (\nabla \times \boldsymbol{q}) - \nabla \times (\boldsymbol{q} \times \Omega) = 0$$

$$\Rightarrow \qquad \frac{\partial \Omega}{\partial t} = \nabla \times (\boldsymbol{q} \times \Omega)$$

$$= (\Omega \cdot \nabla)\boldsymbol{q} - (\boldsymbol{q} \cdot \nabla)\Omega$$

$$\Rightarrow \qquad \frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q} \cdot \nabla)\Omega = (\Omega \cdot \nabla)\boldsymbol{q}$$

i.e. $$\frac{D\Omega}{Dt} = (\Omega \cdot \nabla)\boldsymbol{q} \qquad (2)$$

which is the required vorticity equation.

Equation (2) is called Helmholtz's vorticity equation.

The vorticity equation can also be written as

$$\frac{\boldsymbol{D}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{\Omega}}{\boldsymbol{\rho}}\right) = \left(\frac{\boldsymbol{\Omega}}{\boldsymbol{\rho}}.\boldsymbol{\nabla}\right)\boldsymbol{q} \qquad \textbf{\textit{(3)}}$$

For two-dimensional motion, the vorticity vector Ω is perpendicular to the velocity vector **q** and the R.H.S. of (2) is identically zero. Thus, for two dimensional motion of an ideal fluid, vorticity is constant.

In the case, when body force is not conservative, equation (2) becomes

$$\frac{D\Omega}{Dt} = (\Omega \cdot \nabla)\boldsymbol{q} + curlF$$

where **F** is body force per unit mass.

If we write $\boldsymbol{\Omega} = \xi i + \eta j + \zeta k$, $\boldsymbol{q} = u\boldsymbol{i} + \boldsymbol{v}j + w\boldsymbol{k}$ then the Cartesian form of (3) is

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\xi}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\eta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\zeta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w$$

Remark: *For $\rho = constant$, was originally given by Stoke and Helmholtz and later on extended to the above form by Nanson.*

6.5 Intrinsic or Elastic Strain Energy:

It is the energy stored in the fluid by virtue of compression and is analogous to the one stored in a stretched string. *Intrinsic energy E per unit mass measures the work done by unit mass of the fluid against external pressure, as it passes, under the supposed relation between p and ρ, from its actual state to some standard state in which the pressure and density are p_0 and ρ_0.*

For the incompressible fluid, E=0.

Since the work done in changing the shape of any volume V to V_0 is $\int_V^{V_0} p dv$ and $\rho = mv$, we may set

$$Work\ done = \int_V^{V_0} p\, dv = \int_\rho^{\rho_0} p\, d(m/\rho)$$

Hence $\qquad E = \int_\rho^{\rho_0} p\, d(1/\rho) \qquad$ since m=1.

The total intrinsic energy of a fluid body is often called internal energy and obviously given by $\int_V \rho\, E\, dv$.

6.6 Energy Equation: The rate of change of total energy (kinetic, potential and Intrinsic) of any portion of a compressible inviscid fluid, as it moves about, equals the rate at which the work is being done by the pressure on the boundary. The potential due to external forces is supposed to be independent of time.

Proof: Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dv be volume element surrounding P. Let **q(r,**t) be the velocity of P. Then, the Euler's equation of motion is

$$\frac{d\boldsymbol{q}}{dt} = F - \left(\frac{1}{\rho}\right)\nabla p \qquad (1)$$

Let the external forces be conservative so that there exists a force potential χ which is independent of time. Thus $F = -\nabla\chi \quad and \quad \frac{\partial\chi}{\partial t} = 0$

Using the above results and then multiplying scalarly both sides of (1) with $\rho\, \boldsymbol{q}$, we get

$$\rho\left(\boldsymbol{q}.\frac{dq}{dt}\right) = -\rho\, \boldsymbol{q}.\boldsymbol{\nabla}\chi - \boldsymbol{q}.\nabla p$$

Or $$\rho\frac{d}{dt}\left(\frac{1}{2}\boldsymbol{q}^2\right) + \rho\, \boldsymbol{q}.\boldsymbol{\nabla}\chi = -\boldsymbol{q}.\nabla p$$

Since $$\frac{d\chi}{dt} = \frac{\partial\chi}{\partial t} + (\boldsymbol{q}.\nabla)\chi = (\boldsymbol{q}.\nabla)\chi \quad as\ \frac{\partial\chi}{\partial t} = 0$$

by hypothesis, the above may be rewritten as

$$\rho \frac{d}{dt}\left(\frac{1}{2}q^2 + \chi\right) = -\boldsymbol{q}.\nabla p = -\nabla.(p\boldsymbol{q}) + p\,\nabla.\mathbf{q} \tag{2}$$

Integrating both sides of (2) we get

$$\frac{d}{dt}\left\{\int_V (\frac{1}{2}q^2 + \chi\Big)\rho dv\right\} = -\int_V \nabla.(p\boldsymbol{q})dv + \int_V p\,(\nabla.\mathbf{q}\,)\,dv \tag{3}$$

the left-hand side being valid as $\frac{d(\rho dv)}{dt} = 0$ [since the elementary mass remains invariant throughout the motion] by continuity condition. By virtue of divergence theorem and the equation of continuity the right side of (3) may be simplified to yield

$$\frac{d}{dt}\left\{\int_V (\frac{1}{2}q^2 + \chi\Big)\rho dv\right\} = \int_S \boldsymbol{n}.(p\boldsymbol{q})ds - \int_V \frac{p}{\rho}\frac{d\rho}{dt}dv \tag{4}$$

Where **n** is unit inward normal. Now, by definitions

$$T = \int_V \frac{1}{2}\rho \boldsymbol{q}^2 dv; \quad W = \int_V \rho\chi\, dv; \quad I = \int_V \rho E\, dv \tag{5}$$

are the kinetic, potential and intrinsic(internal) energies respectively, then (4) may be written as

$$\frac{d}{dt}(T+W) = \int_S (\boldsymbol{n}.\boldsymbol{q})p dS - \frac{dI}{dt} \tag{6}$$

$\because$

$\frac{dI}{dt} = \int_V \frac{dE}{dt}\,\rho dv$ [by (5) and $\frac{d}{dt}(\rho v) = 0$

$$= \int_V \frac{dE}{d\rho}\,\frac{d\rho}{dt}\,\rho\, dv$$

$= \int_V \frac{p}{\rho}\frac{d\rho}{dt}\,dv$ [As $E = \int_\rho^{\rho_0} p\, d\left(\frac{1}{\rho}\right) = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\, d\rho \Rightarrow \frac{dE}{d\rho} = \frac{p}{\rho^2}$

Also, the work done by the fluid pressure on an element ds being $p\,ds\,n\,d\boldsymbol{r}$ and the rate at which this is being done is $p\,ds\,n.\boldsymbol{q}.\left(\boldsymbol{q} = \frac{dr}{dt}\right)$, it follows that for the space of volume V, the rate at which work is being done by the fluid pressure is $\int_S (\boldsymbol{n},\boldsymbol{q})\,p\,ds = R(say)$. Thus (6) may be put as

$$\frac{d}{dt}(T+W+I) = R \tag{7}$$

The statement embodied in (7) is what we were interested in and is often quoted as *the Volume Integral form of Bernoulli's equation.*

Energy equation for incompressible fluids:

Since I=0 for incompressible fluids, (7) reduces to

$$\frac{d}{dt}(T+W) = R \tag{8}$$

The energy equation is stated as follows: The rate of increase of energy in the system is equal to the rate at which work is done on the system.

Example: ***An infinite mass of fluid is acted on by a force $\mu/r^{\frac{3}{2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{\frac{1}{2}} c^{\frac{5}{4}}$.***

Solution: At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2 v' = r^2 v \tag{1}$$

When the radius of cavity is r, then

Kinetic energy= $\int_r^\infty \frac{1}{2}\,(4\pi\, r'^2\, \rho dr').\, v'^2$ $\qquad$ $[K.E = \frac{1}{2} \times mass \times (velocity)^2$

$= 2\pi\, \rho\, r^4\, v^2\, \int_r^\infty \frac{dr'}{r'^2}$ $\qquad$ [using (1)

$= 2\pi\, \rho r^3 v^2.$

The initial kinetic energy is zero.

Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{\left(\frac{3}{2}\right)}}$$

So that $V = \frac{2\mu}{r'^{\left(\frac{1}{2}\right)}}$

$\therefore\, work\ done = \int_r^c V\, dm,\quad dm$ being the elementary mass

$= \int_r^c \left(\frac{2\mu}{r'^{\left(\frac{1}{2}\right)}}\right).4\,\pi\, r'^2\, dr'\rho = 8\,\mu\,\rho\, \int_r^c r'^{\left(\frac{3}{2}\right)}\, dr' = \frac{16}{5}\, \pi\mu\, \rho\, (c^{\frac{5}{2}} - r^{\frac{5}{2}})$

We now use energy equation, namely $\qquad$ *Increase in kinetic energy=work done*

This $\Rightarrow 2\pi\, \rho r^3 v^2 - 0 = \frac{16}{5}\, \pi\mu\, \rho (c^{\frac{5}{2}} - r^{\frac{5}{2}})$

$$\therefore\quad \mathrm{v} = \frac{\mathrm{dr}}{\mathrm{dt}} = -\frac{\left(\frac{8\mu}{5}\right)^{\frac{1}{2}}\left(c^{\frac{5}{2}} - r^{\frac{5}{2}}\right)^{\frac{1}{2}}}{r^{\frac{3}{2}}} \tag{2}$$

Wherein negative sign is taken because r decreases as t increases.

Let T be the time of filling up the cavity. Then (2) gives

$\int_0^T dt = -\left(\frac{5}{8\mu}\right)^{\frac{1}{2}}\, \int_c^0 \frac{r^{\frac{3}{2}}\, dr}{\sqrt{\left(c^{\frac{5}{2}} - r^{\frac{5}{2}}\right)}}$ $\quad$ or

$$T = \left(\frac{5}{8\mu}\right)^{\frac{1}{2}} \int_0^c \frac{r^{\frac{3}{2}}\, dr}{\sqrt{\left(c^{\frac{5}{2}} - r^{\frac{5}{2}}\right)}}$$

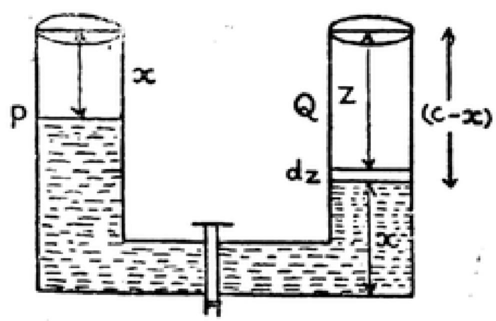
Put $r^{\frac{5}{2}} = c^{\frac{5}{2}} \sin^2\theta$ so that $\left(\frac{5}{2}\right) \times r^{\frac{3}{2}}\, dr = 2c^{\frac{5}{2}} \sin\theta \cos\theta\, d\theta$

$$\therefore T = \left(\frac{5}{8\mu}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{4}{5}\, c^{\frac{5}{4}} \sin\theta\, d\theta = \left(\frac{2}{5\mu}\right)^{\frac{1}{2}} c^{\frac{5}{4}}$$

Example: ***Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled one with water, and the other with air of such a density as to support a column h of water, h being less than c. If a communication be opened between them at their bases, the height x, to which the water rises, is given by the equation***

$$\boldsymbol{cx - x^2 + ch \log\{\frac{c-x}{x})\} = 0}$$

Solution: Let P, Q be two cylinders containing water and air respectively and k be the cross-section of each cylinder before and after the communication is set up, the air and water are at rest. Thus the initial and final kinetic energies are zero. The intrinsic energies also vanish, because of incompressibility.



The potential energy due to position of water in the cylinder P is

$W_a = \int_0^c g\rho\, kz\, dz = \frac{1}{2}\, g\rho k c^2$ (1)

[Here we have considered the density ρ of water simply to support a column h of water]

Let the height x of water rises in cylinder Q then the height (c-x) of water will remain in the cylinder P after communication is opened. The final potential energy is

$$W_b = \int_0^{c-x} g\rho kz\, dz + \int_0^x g\rho kz\, dz$$

$$W_b = \frac{1}{2} g\, \rho j\, [(c-x)^2 + x^2] \tag{2}$$

Loss in potential energy of work done by gravity is

$W_a - W_b = \frac{1}{2} g\, \rho k\, [c^2 - (c-x)^2 - x^2] = g\rho k x\, (c-x)$ (3)

Let p be the pressure when the water rises to a height z, then

$$g\rho h\, k\, c = pk(c-z) \Longrightarrow\ p = g\, \rho hc/(c-z)$$

Since the air has been compressed so the work done in compressing this air in the cylinder Q is

$= -g\rho hck \int_0^x \frac{dz}{c-z} = g\, \rho hck \log\frac{c-x}{c}$ (4)

Total work done=Change in K.E

Hence $\qquad g\, \rho k x\, (c-x) + g\rho hck \log\left\{\frac{c-x}{c}\right\} = 0$

Or $\qquad g\, \rho\, k[cx - x^2 + ch \log\left\{\frac{c-x}{c}\right) = 0$

Or $\qquad cx - x^2 + ch \log\frac{c-x}{c} = 0.$ Proved

6.7 Green's Theorem: *If $\boldsymbol{\phi_1 and\ \phi_2}$ are two continuously differentiable scalar point functions such that $\boldsymbol{\nabla\phi_1\ and\ \nabla\phi_2}$ are also continuously differentiable and S denotes a closed surface bounding any singly-connected region of space, then*

$$\int_V (\nabla\, \boldsymbol{\phi_1}.\nabla\boldsymbol{\phi_2})\boldsymbol{dv} = -\int_V \boldsymbol{\phi_1}\, \nabla^2\, \boldsymbol{\phi_2}\, \boldsymbol{dv} - \int_S \boldsymbol{\phi_1} \frac{\partial \boldsymbol{\phi_2}}{\partial \boldsymbol{n}}\, \boldsymbol{ds}$$
$$= -\int_V \boldsymbol{\phi_2}\, \nabla^2\, \boldsymbol{\phi_1}\, \boldsymbol{dv} - \int_S \boldsymbol{\phi_2} \frac{\partial \boldsymbol{\phi_1}}{\partial \boldsymbol{n}}\, \boldsymbol{ds}$$

Where V is the region enclosed by S and $\boldsymbol{\delta n}$ an element of the normal at any point on the boundary drawn into the region considered.

Some hydrodynamical applications of Green's Theorem:

(1) If $\boldsymbol{\phi_2}$ is constant (=k say). Then $\boldsymbol{\nabla^2\phi_2 = 0} = \frac{\partial\phi_2}{\partial n}$ ***everywhere***. If $\boldsymbol{\phi}$ be the velocity potential of a liquid motion with S, then by Green's theorem, we get

$$\int_S \boldsymbol{k}\ \left(\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{n}}\right) \boldsymbol{dS} = \boldsymbol{0} \qquad \boldsymbol{or} \qquad \int_S \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{n}} \boldsymbol{dS} = \boldsymbol{0}$$

Since $\frac{\partial\phi}{\partial n}$ is the normal velocity outwards, $\frac{\partial\phi}{\partial n}\boldsymbol{dS}$ represents the flow across dS per unit time. Then the above result represents that the total flow of liquid into any closed region at any instant is zero. i.e., the quantity of a liquid inside S remains constant.

(2) **Kinetic Energy of finite liquid:**
The kinetic energy is given by

$$\boldsymbol{T} = \int_V \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{\rho q^2 dv}$$

taken throughout the volume V occupied by the fluid. For irrotational motion $\boldsymbol{q} = -\nabla\boldsymbol{\phi}, \nabla^2\boldsymbol{\phi} = \mathbf{0}$.

Therefore $\boldsymbol{T} = \frac{1}{2}\,\boldsymbol{\rho} \int \nabla\boldsymbol{\phi}.\nabla\boldsymbol{\phi}\, \boldsymbol{dv} = -\frac{1}{2}\int_S \boldsymbol{\phi} \left(\frac{\partial\phi}{\partial n}\right) \boldsymbol{dS}$. |by Green's theorem

taken over the bounding surface of the liquid, *dn* denoting an element of inward-drawn normal.

Physical Interpretation: We know that if **q** is the velocity and $\boldsymbol{\rho}$ the density of the liquid, then K.E. of the liquid within S is

$$\boldsymbol{T} = \int_V \frac{1}{2}\,\boldsymbol{\rho}\, \boldsymbol{q}^2 \boldsymbol{dv} = -\frac{1}{2}\boldsymbol{\rho} \left(\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{n}}\right) \boldsymbol{dS}$$

Since $\boldsymbol{\rho\phi}$ is the impulsive pressure and $-(\frac{\partial\phi}{\partial n})$ the inward velocity, it follows that the K.E. set up by impulses, in a system starting from rest, is the sum of the products of each impulse and half the velocity of its point of application. It also follows that the K.E. of a given mass of liquid moving irrotationally in a simply connected region depends on the motion of its boundaries. Clearly the surface integral

$$-\frac{1}{2}\boldsymbol{\rho} \int_S \boldsymbol{\phi} \left(\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{n}}\right) \boldsymbol{dS}$$

Represents the work done by the impulsive pressure in starting the motion from rest.

(3) If the boundaries are at rest, it follows that $\frac{\partial\phi}{\partial n} = \mathbf{0}$, so that

$$\int_V \frac{1}{2}\,\boldsymbol{\rho}\, \boldsymbol{q}^2 \boldsymbol{dv} = \mathbf{0}, \text{ i.e., } \boldsymbol{q} = \mathbf{0} \text{ at every point}$$

Hence, if the boundaries are fixed, irrotational motion is impossible in a closed simply-connected region.

6.8 Kelvin's Minimum Energy Theorem. The kinetic energy of irrotational motion of a liquid occupying a finite simply connected region is less than that of any other motion of the liquid which is consistent with the same normal velocity of the boundary.

Proof. Let T_1 be the K.E. of the actual motion, $\boldsymbol{q_1}$ be the fluid velocity and ϕ be the velocity potential of the given irrotational motion. Let V be the region occupied by the fluid and S be the surface of this region, then

$$T_1 = \frac{\rho}{2}\int_\tau q_1{}^2 dv = \frac{\rho}{2}\int_\tau (-\nabla\phi)^2\, dv$$

$$= \frac{\rho}{2}\int_S \varphi \frac{\partial\varphi}{\partial n} dS \qquad (1)$$

Let T_2 be the K.E. and $\boldsymbol{q_2}$ be the velocity of any other motion of the fluid consistent with the same normal velocity of the boundary S (or consistent with the same kinetic boundary condition)

For both the motions, the continuity equation is satisfied i.e.

$$\nabla \cdot q_1 = 0 = \nabla . q_2 \tag{2}$$

The boundaries have the same normal velocity

i.e. $\quad q_1 . \boldsymbol{n} = q_2 \cdot \boldsymbol{n}$

i.e. $\quad (q_2 - q_1) \cdot \boldsymbol{n} = 0 \qquad (3)$

Now, let us consider

$$
\begin{aligned}
T_2 - T_1 &= \frac{1}{2}\rho \int_v (q_2^2 - q_1^2)\, dv \\
&= \frac{1}{2}\rho \int_v [2q_2 \cdot (q_2 - q_1) + (q_2 - q_1)^2] dv \\
&= \frac{\rho}{2}\int_v 2q_1 \cdot (q_2 - q_1) dv + \frac{\rho}{2}\int_v (q_2 - q_1)^2 dv \\
&= -\rho \int_v \nabla\phi . (\mathrm{q}_2 - \mathrm{q}_1) \mathrm{dv} + \frac{\rho}{2}\int_v (q_2 - q_1)^2 dv \qquad (4)
\end{aligned}
$$

From vector calculus, we have

$$\nabla\cdot[\phi(q_2 - q_1)] = \nabla\phi\cdot (q_2 - q_1) + \phi\, \nabla\cdot(q_2 - q_1)$$

i.e. $\quad \nabla\phi\cdot (q_2 - q_1) = \nabla\cdot[\phi(q_2 - q_1)] - \phi\nabla\cdot(q_2 - q_1)$

Therefore, from (4), we find

$$
\begin{aligned}
T_2 - T_1 &= -\rho\int_v \nabla\cdot[\phi(q_2 - q_1)] \mathrm{dv} + \rho\int_v \phi\nabla\cdot(q_2 - q_1)\, \mathrm{dv} \\
&\quad + \frac{\rho}{2}\int_v (q_2 - q_1)^2\, \mathrm{dv} \\
&= -\rho \int_S \phi(q_2 - q_1) \cdot \mathbf{n}\, \mathrm{dS} + \rho \int_v \phi\nabla\cdot(q_2 - q_1)\, \mathrm{dv} \\
&\quad + \frac{\rho}{2}\int_v (q_2 - q_1)^2 \mathrm{dv} \qquad \text{|By Gauss theorem} \\
&= \frac{\rho}{2}\int_v (q_2 - q_1)^2 \mathrm{dv} \qquad \text{| using (2) \& (3)} \\
&> 0
\end{aligned}
$$

$$\Rightarrow \quad T_2 > T_1$$

Thus, the irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary (but for which vortices are present inside)

Hence the theorem.

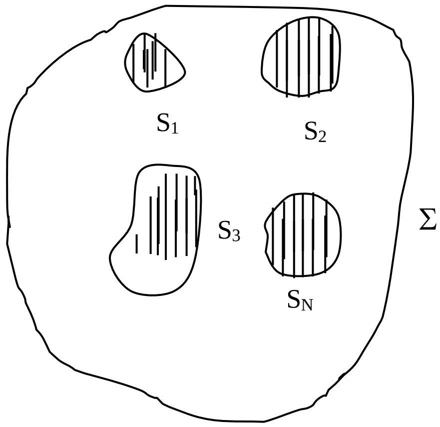
6.9 Kinetic Energy of Infinite Liquid. Theorem: An infinite liquid is in irrotational motion which is at rest at infinity and is bounded internally by solid surface (s)S. Show that the K.E. of the moving fluid is

$$T = \frac{1}{2}\rho \int_S \phi \frac{\partial\phi}{\partial n} dS$$

where S = $S_1 + S_2 + \ldots S_N$ denotes the sum of the inner boundaries $S_1, S_2, \ldots, S_N$ and **n** is normal to S drawn out of the fluid on each boundary.

Proof. Let Σ be a large surface enclosing the surface (s) S and v be the region bounded by S internally and by Σ externally.

Using the result of K.E. for finite liquids, we find that the K.E. T_1 for finite region v is given by



$$T_1 = \frac{\rho}{2}\int_S \varphi \frac{\partial \varphi}{\partial n} dS + \frac{\rho}{2}\int_\Sigma \varphi \frac{\partial \varphi}{\partial n} dS \qquad (1)$$

Now, div $\mathbf{q} = \nabla^2\phi = 0$ throughout v and the divergence theorem accordingly gives

dS = 0

$$\Rightarrow \quad \int_{SU\Sigma} \boldsymbol{n}\cdot\nabla\phi dS = 0 \Rightarrow \int_{SU\Sigma}\frac{\partial \phi}{\partial n} dS = 0$$

$$\Rightarrow \quad \int_S \frac{\partial \varphi}{\partial n} dS + \int_\Sigma \frac{\partial \varphi}{\partial n} dS = 0 \qquad (2)$$

Since the surface S is solid, there is no flow across it, hence $\int_S \frac{\partial \varphi}{\partial n} dS = 0$ (3)

Therefore, from (2), we get

$$\int_\Sigma \frac{\partial \varphi}{\partial n} dS = 0 \qquad (4)$$

For the surface Σ, as Σ goes to infinity, the liquid is at rest

$$\Rightarrow \quad \mathbf{q} = 0 \Rightarrow \nabla\phi = 0 \Rightarrow \phi = \text{constant} = C \text{ (say)} \qquad (5)$$

Hence, as Σ goes to ∞, the K.E. of the liquid is

$$T_1 \to T = \frac{\rho}{2}\int_S \varphi \frac{\partial \varphi}{\partial n} dS + \frac{\rho}{2} c \int_\Sigma \frac{\partial \varphi}{\partial n} dS \qquad \text{| Using (5)}$$

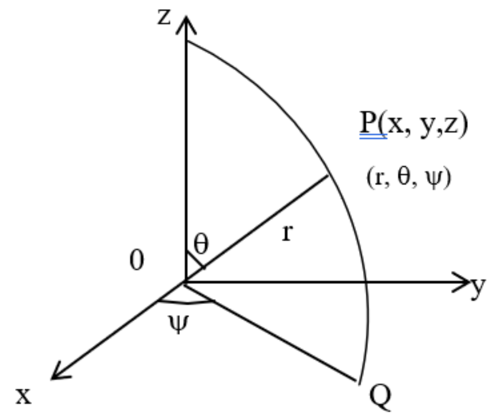
$$= \frac{\rho}{2}\int_S \varphi \frac{\partial \varphi}{\partial n} dS \qquad \text{| Using (4)}$$

Hence the result

Remark. We note that the K.E. for finite and infinite liquid has the same expression.

6.10 Axially Symmetric Flows

A potential flow which is axially symmetric about the axis $\theta = 0, \pi$ (i.e., z-axis is taken as the axis of symmetry) has the property that at any point P, all the scalar and vector quantities associated with the flow are independent of azimuthal angle ψ such that $\frac{\partial}{\partial \psi} \equiv 0$, where (r, θ, ψ) are spherical polar co-ordinates.



The equation of continuity div **q**= 0 for steady flow of an incompressible fluid becomes.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, q_\theta) = 0 \qquad (1)$$

For irrotational motion **q**= –∇ϕ, where ϕ is velocity potential and thus

$$q_r = -\frac{\partial\varphi}{\partial r},\; q_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}$$

From equation (1), we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) = 0 \qquad (2)$$

Let a solution of (2) in separable variables r, θ has the form

$$\phi = -R(r)\,\Theta(\theta) \qquad (3)$$

Using (3) in (2), we get

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}(R\Theta)\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(R\Theta)\right] = 0$$

$$\Rightarrow \quad \Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$$

$$\Rightarrow \quad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \qquad (4)$$

The L.H.S. of (4) is a function of r only while the R.H.S. is a function of θ only. The equation can therefore be satisfied if and only if either side is a constant, say n(n+1) and thus we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1) \qquad (5)$$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta\sin\theta = 0 \qquad (6)$$

To solve (5), we put

$$R = r^m \Rightarrow \frac{dR}{dr} = mr^{m-1}$$

Thus (5) $\Rightarrow \frac{1}{r^m}\frac{d}{dr}(r^2 m r^{m-1}) = n(n+1)$

$$\Rightarrow m\frac{d}{dr}(r^{m+1}) = r^m n(n+1)$$

$$\Rightarrow m(m+1)\,r^m = r^m\, n(n+1)$$

$$\Rightarrow (m^2 + m - n^2 - n) = 0$$

$\Rightarrow$ (m−n) (m+n+1) = 0

$\Rightarrow$ m = n or m = −(n+1)

Therefore, solution of (5) can be written as

$$R(r) = A_n r^n + B_n r^{-(n+1)} \tag{7}$$

To solve (6), we put

$$\cos\theta = \mu$$

$$\Rightarrow \frac{d}{d\theta} \equiv \frac{d\mu}{d\theta}\frac{d}{d\mu} \equiv -\sin\theta \frac{d}{d\mu}$$

Therefore, equation (6) becomes.

$$-\sin\theta \frac{d}{d\mu}\left[\sin\theta\,(-\sin\theta)\frac{d\Theta}{d\mu}\right] \quad + n\,(n+1)\,\Theta \sin\theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left(\sin^2\theta \frac{d\Theta}{d\mu}\right) + n(n+1)\Theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left[(1-\cos^2\theta)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0 \tag{8}$$

Equation (8) is a Legendre's Equation and possesses a solution known as Legendre Function of the first kind $P_n(\mu)$

Therefore,

$$\Theta = P_n(\mu)$$

Hence the general solution of (3) is of the form

$$\begin{aligned}\phi(r,\theta) &= -R(r)\,\Theta\,(\theta)\\ &= -[A_n r^n + B_n r^{-(n+1)}]\,P_n(\cos\theta)\end{aligned} \tag{9}$$

(Complete solution is the sum of all such solutions i.e. $\sum_{n=0}^{\infty}$ …….)

6.10.1. Uniform Flow. Consider the flow which corresponds to a potential given by (9) with

$$A_n = U\delta_{in},\ B_n = 0, \quad (n = 0, 1, 2, \ldots\ldots)$$

where U is a constant, δ_{ij} is Knonecker delta $= 1\ for\ i = j$ and $= 0\ for\ i \neq j$

Since $P_1(\cos\theta) = \cos\theta$, equation (9) becomes

$$\phi(r,\theta) = -Ur\cos\theta \equiv -Uz \qquad |\ z = r\cos\theta$$

Thus

$$\mathbf{q} = -\nabla\phi = -\frac{\partial\phi}{\partial z}\mathbf{k} = U\,\mathbf{k}$$

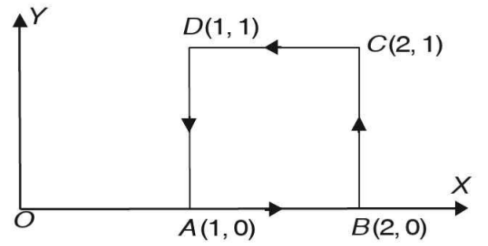
which is a uniform streaming motion of the fluid with speed U along the direction of z-axis or the axis θ = 0.

Example: ***A velocity field is given by q=*** $\boldsymbol{\frac{-iy+jx}{x^2+y^2}}$***.Determine whether the flow is irrotational. Calculate the circulation round (a) a square with its corners at (1,0), (2,0), (2,1), (1,1); (b) a unit circle with centre at the origin.***

Solution: we have $\boldsymbol{curl\, q} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = k\left\{\frac{(y^2-x^2)+(x^2-y^2)}{(x^2+y^2)^2}\right\} = 0$

Hence **curl q**=0 everywhere except at the origin. Thus the flow is irrotational. It has a singularity at the origin where the velocity becomes infinite.

(a) Draw a square in the Cartesian plane as A(1,0),B(2,0),C(2,1),D(1,1).



Then circulation around the square ABCD is given by

$$\Gamma = \int \boldsymbol{q}\,.dr = \int_A^B \boldsymbol{q}\,.dr + \int_B^C \boldsymbol{q}\,.dr + \int_C^D \boldsymbol{q}\,.dr + \int_D^A \boldsymbol{q}\,.dr$$

Along AB(i.e. x-axis), y=0 so dy=0 and hence $\boldsymbol{dr} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} = dx\boldsymbol{i}$

Therefore $\int_A^B \boldsymbol{q}\,.dr = \int_{x=1}^{x=2}\left(\frac{\boldsymbol{-iy+jx}}{\boldsymbol{x^2+y^2}}\right).(\,dx\boldsymbol{i}) = \int_{\boldsymbol{x=1}}^{\boldsymbol{x=2}}\left(\frac{1}{x}j\right).(dx\boldsymbol{i}) = \boldsymbol{0}$

Along BC(parallel to y-axis), x=2 so dx=0 and hence $\boldsymbol{dr} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} = dy\,\boldsymbol{j}$

Therefore $\int_B^C \boldsymbol{q}\,.dr = \int_{y=0}^{y=1}\left(\frac{\boldsymbol{-iy+jx}}{\boldsymbol{x^2+y^2}}\right).dy\boldsymbol{j} = \int_0^1 \frac{2dy}{y^2+4} = 2.\frac{1}{2}\left[\tan^{-1}\frac{y}{2}\right]_0^2 = \tan^{-1}\frac{1}{2}$

Similarly, $\int_C^D \boldsymbol{q}\,.dr = \tan^{-1}2 - \tan^{-1}1$

$\int_D^A \boldsymbol{q}\,.dr = -\tan^{-1}1$

Therefore, $\Gamma = \tan^{-1}\frac{1}{2} + \tan^{-1}2 - \tan^{-1}1 - \tan^{-1}1 = \cot^{-1}2 + \tan^{-1}2 - \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2} - 2\frac{\pi}{4} = 0$

[as $\cot^{-1}2 + \tan^{-1}2 = \frac{\pi}{2}$]

Since $curl\; q$ is zero everywhere inside the square path, we could have got the same results directly from Stoke's theorem.

(b) To obtain circulation around the unit circle with its centre at the origin, we use polar coordinates for convenience.

Let $x = r\cos\theta, y = r\sin\theta$, we have $\boldsymbol{q} = \frac{-r\sin\theta\, i + r\cos\theta\, j}{r^2} = -\frac{\sin\theta}{r}i + \frac{\cos\theta}{r}j$

and $\boldsymbol{q} = ui + vj$ so that $u = -\frac{\sin\theta}{r}\quad and\; v = \frac{\cos\theta}{r}$

Therefore, $q_r = u\cos\theta + v\sin\theta\;\; and\; q_\theta = -u\sin\theta + v\cos\theta = \frac{1}{r}$

Thus, $\Gamma = \int \boldsymbol{q}.dr = \int_0^{2\pi} r.\frac{1}{r}\,d\theta = 2\pi$

Example: ***Liquid of density ρ is flowing in two dimensions between the oval curves $r_1r_2 = a^2$, $r_1r_2 = b^2$ where r_1 and r_2 are the distances measured from two fixed points if the motion is***

irrotational and quantity q per unit time across any line joining the bounding curves, then the kinetic energy is $\boldsymbol{\pi\, \rho q^2/log(b/a)}$***.***

Solution: The two-dimensional irrotational motion occurs in a doubly connected region. The equation to the curves are

$$r_1 r_2 = a^2 \quad and\; r_1 r_2 = b^2$$

Let the complex potential w is of the form

$$w = iA\log[(z - z_1)(z - z_2)] = \phi + i\,\psi$$

or $$\phi + i\psi = iAlog\left(r_1 e^{i\theta_1}\, r_2 e^{i\theta_2}\right) = iA\log\left\{r_1 r_2\, e^{i(\theta_1 - \theta_2)}\right\}$$

as $$z - z_1 = r_1 e^{i\theta_1}\; and\; z - z_2 = r_2 e^{i\theta_2}$$

Separating real and imaginary parts, we have

$$\phi = -A(\theta_1 + \theta_2), \psi = A\log r_1 r_2$$

Now $$q = \psi_b - \psi_a = A\log b^2 - A\log a^2 = 2A\log\left(\frac{b}{a}\right)$$

or $$A = q/\left[2\log\left(\frac{b}{a}\right)\right]$$

since the region is doubly connected, the circulation k is given by $k = A(2\pi + 2\pi) = 4\pi A$

Hence the kinetic energy of a cyclic irrotational becomes

$$T = -\frac{1}{2}\rho \int \phi \frac{\partial\phi}{\partial n}\, dS - \frac{1}{2}\,\rho k \frac{\partial\phi}{\partial n}\, dS = -2\pi A\,\rho \int_A^B d\psi$$

The second integral vanishes on a rigid boundary

$$T = 2\pi A\,\rho(\psi_b - \psi_a) = \pi\,\rho q^2/\log\left(\frac{b}{a}\right)$$

6.11 Check Your Progress:

i) Find the circulation about the square enclosed by the lines $x = \pm 2, y = \pm 2$ for the flow $u = x + y, v = x^2 - y$. [Ans=0]

ii) Show that if $\phi = -(ax^2 + by^2 + cz^2)/2, V = -(lx^2 + my^2 + cn^2)/2$ where a, b,c; l,m,n are functions of time and a+b+c=0,irrotational motion is possible with a free surface of equi-pressure if

$(l + a^2 + \dot{a})e^{2\int a\, dt}; \left(m + b^2 + \dot{b}\right)e^{2\int b dt}; (n + c^2 + \dot{c})e^{2\int c dt}$ are constants.

iii) A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid satisfying the condition for the continued existence of velocity potential ϕ under conservative forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is

$$-\rho \iint \frac{\partial\phi}{\partial t} \cdot \frac{\partial\phi}{\partial n}\, dS$$

where ρ is the density and ∂n an element of normal to dS drawn into the fluid.

iv) Show that in the motion of fluid in two dimensions if the co-ordinates (x,y) of an element at any time can be expressed in terms of initial coordinates (a,b) and the time, the motion is irrotational if $\frac{\partial(\dot{x},x)}{\partial(a,b)} + \frac{\partial(\dot{y},y)}{\partial(a,b)} = 0$.

v) In irrotational motion in two dimensions, prove that $\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = \boldsymbol{q}\, \nabla^2 \boldsymbol{q}$.

6.12 Summary: In this chapter boundary conditions are discussed in detail which play an important role during the study of flow of fluid. Irrotational and circulation of flow are also defined in this chapter. The expressions for energy equation, Kinetic energy of liquid, Kelvin circulation theorem are derived and also discussed Axially symmetric flows.

6.13 Keywords: Boundary surface, Circulation, energy equation, kinetic energy, irrotational, axially symmetric flows.

6.14 Self -Assessment Test:

SA1: Show that under certain conditions the motion of frictionless fluid if once irrotational, will always be so, is true also when each particle is acted upon by a resistance varying as the velocity.

SA2: Obtain Cauchy's integral using circulation theorem.

SA3: Deduce from the principle that the kinetic energy set up is a minimum that, if a mass of incompressible liquid be given at rest, completely filling a closed vessel of any shape and if any motion of the liquid be produced suddenly by giving arbitrarily prescribed normal velocities at all points of its bounding surface subject to the condition of constant volume, the motion produced is irrotational.

SA4: Prove that under certain conditions, to be stated, the motion of a fluid if once irrotational, is always irrotational afterwards.

SA5: Show that in an irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

SA6: Show that the rate per unit of time at which work is done by the internal pressure between the parts of a compressible fluid is

$$\int \mathrm{p}(\nabla.\mathrm{q})\mathrm{dv}$$

where p is the pressure, and q the velocity at any point, and the integration extends through the volume of the fluid.

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CHAPTER-7

MOTION OF SPHERE

7.0 Learning Objectives: After studying this chapter, you should be able to study the motion of a sphere through the liquid at rest at infinity, obtain the lines of flow relative to the sphere when liquid streaming past a fixed sphere, determine the pressure on the surface when sphere moving through infinite liquid, calculate the force acting on the sphere due to the presence of the fluid.

7.1 Motion of a Sphere: To study irrotational motion in three-dimensions with a particular reference to the motion of a sphere. We shall consider certain special forms of solution of the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1}$$

which, in spherical polar coordinates (r,θ,ψ), reduces to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}=0 \tag{2}$$

When there is symmetry about a line (say z-axis), ϕ is independent of ψ and hence (2) reduces to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0 \tag{3}$$

In the case of motion of sphere the velocity potential is known to have the form $f(r)\cos\theta$. Substituting $\phi=f(r)\cos\theta$ in (3), we have

$$\left(\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}\right)\cos\theta-\frac{f(r)}{r^2}\cos\theta-\frac{\cos\theta}{r^2}\,f(r)=0$$

Or
$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}-\frac{2f}{r^2}=0$$

Or
$$r^2\left(\frac{d^2f}{dr^2}\right)+2r\frac{df}{dr}-2f=0$$

which is homogeneous differential equation. As usual, its solution is $f(r)=Ar+\frac{B}{r^2}$. Hence a solution of (3) of the form $f(r)\cos\theta$ may be taken as

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \tag{4}$$

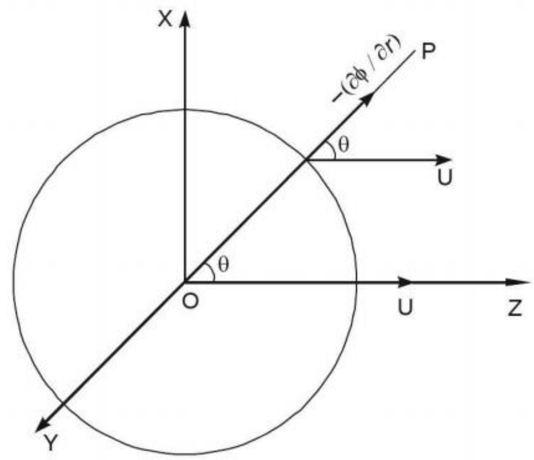
7.2 Motion of a Sphere through a Liquid at rest at infinity:

Take origin at the Centre of the sphere and the axis of z in the direction of motion. Let the sphere move with velocity U along the z-axis. To determine the velocity potential ϕ that will satisfy the given boundary conditions, we have the following considerations:

(i) ϕ satisfies the Laplace's equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0, \tag{1}$$

wherein we have used the fact that there is symmetry of flow about z-axis.



(ii) Boundary condition at the surface of the sphere $r = a$, namely

Normal velocity at any point of the sphere=velocity of the liquid at that point in that direction

i.e., $-\left(\frac{\partial \phi}{\partial r}\right) = U\cos\theta, when\ r = a$ (2)

(iii) Since the liquid is at rest at infinity, we must have

$$-\frac{\partial \phi}{\partial r} = 0,\ at\ r = \infty \tag{3}$$

The above considerations (i) and (ii) suggest that ϕ must be of the form $f(r)\cos\theta$ and hence it may be assumed as

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{4}$$

From (4), $$-\frac{\partial \phi}{\partial r} = -\left(A - \frac{2B}{r^3}\right)\cos\theta \tag{5}$$

Putting $r = \infty$ in (5) and using (3), we get

$0 = A\cos\theta$ so that A=0 (6)

Putting r=a in (5) and using (2) and (6), we get

$$U\cos\theta = \left(\frac{2B}{a^3}\right)\cos\theta$$

So that $$B = \frac{Ua^3}{2} \tag{7}$$

Thus, $$\phi = \frac{1}{2}U\frac{a^3\cos\theta}{r^2} \tag{8}$$

which determines the velocity potential for the flow.

(a) Streamline flow: We now determine the equations of lines (streamlines) of flow. The differential equation of the lines of flow at the instant the centre of sphere is passing through the origin is given by

$$\frac{dr}{\frac{\partial \phi}{\partial r}} = \frac{r\,d\theta}{\frac{\partial \phi}{r\partial \theta}}$$

Or $\dfrac{dr}{\left(\frac{Ua^3}{r^3}\right)\cos\theta} = \dfrac{r\,d\theta}{\left(\frac{Ua^3}{2r^3}\right)\sin\theta}$ |using (8)

Or $\frac{1}{r}\,dr = 2\cot\theta\,d\theta$ (9)

Integrating (9), $\log r = 2\log\sin\theta + \log c$ or $r = c\sin^2\theta$

which is the equation of the lines of flow.

7.3 Liquid Streaming past a fixed Sphere: *For a liquid streaming past a fixed sphere, obtain the lines of flow relative to the sphere.*

Let the sphere be at rest and let the liquid flow past the sphere with velocity U in the negative direction of z-axis. This motion may be deduced by reducing the sphere to rest by superposing a velocity $-U$ parallel to z-axis both to the sphere and liquid and we must add to the velocity potential a term $Urcos\theta$ to account for the additional velocity, then we have

$$\phi = \frac{Ua^3}{2r^2}\,cos\theta + Ur\cos\theta = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad (1)$$

| For $-\frac{\partial\phi}{\partial z} = -U, \phi = Uz = U\,r\cos\theta$

This is the velocity potential when the liquid is streaming past a fixed sphere. The stream lines are given by

$$\frac{dr}{-\frac{\partial\phi}{\partial r}} = \frac{rd\theta}{-\left(\frac{1}{r}\right)\partial\phi/\partial\theta}$$

or $$\frac{dr}{\frac{Ua^3\cos\theta}{r^3} - U\cos\theta} = \frac{rd\theta}{-\left(\frac{Ua^3}{2r^3}+U\right)\sin\theta}$$

$$\left(\frac{2r^3+a^3}{a^3-r^3}\right)\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\,d\theta$$

or $$-\frac{2\cos\theta}{\sin\theta}\,d\theta = \left[\frac{3r^2}{r^3-a^3} - \frac{1}{r}\right]dr$$

Integrating, $-2\log\sin\theta = \log(r^3 - a^3) - \log r - \log c$

or $rc = \sin^2\theta\,(r^3 - a^3)$ (2)

The lines of flow relative to the sphere are given by (2).

Example: ***Show that when a sphere of radius a moves with uniform velocity U through a perfect incompressible infinite fluid, the acceleration of a particle of the fluid at (r,0) is*** $\boldsymbol{3U^2\left(\frac{a^3}{r^4} - \frac{a^6}{r^7}\right)}$.

Solution: Superimpose a velocity $-U$ both to the sphere and the liquid. This reduces the sphere to rest and the velocity potential of the flow is given by

$$\phi = U(r + a^3/2r^2)\cos\theta \qquad (1)$$

$$\therefore \quad \dot{r} = -\frac{\partial\phi}{\partial r} = -U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \qquad (2)$$

and $$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \qquad (3)$$

Again from (2), we have

$$\ddot{r} = U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\dot{\theta} - U\frac{3a^3}{r^4}\dot{r}\cos\theta = U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\dot{\theta} + \frac{3a^3}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta \qquad (4)$$

Clearly, for a point (r,0) the velocity is only along the direction of r and hence the acceleration will also be only along r.
Thus, the required acceleration $=\ddot{r}$ only at (r,0)

$$= \frac{3a^3}{r^4}\, U^2\left(1-\frac{a^3}{r^3}\right) = 3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right), \text{ using (4) and noting that } \theta = \dot{\theta} = 0$$

Example*: An infinite homogeneous liquid is flowing steadily past a rigid boundary consisting partly of the horizontal plane y=0 and partly of a hemispherical boss $x^2+y^2+z^2=a^2$ with irrotational motion which tends, at a great distance from the origin to uniform velocity U parallel to the axis of z. Find the velocity potential and the surface of equal pressure.*

Solution: *The velocity potential of the motion of a liquid streaming past a fixed sphere with velocity U in the negative direction of z-axis is given by*

$$\boldsymbol{\phi = U\left(r+\frac{a^3}{2r^2}\right)\cos\theta} \qquad \textbf{(1)}$$

Solution: Let y-axis be taken in vertical upward direction. Then the motion under consideration is such that velocity perpendicular to the plane y=0(i.e. xz-plane) vanishes. Hence y=0 may be taken as a stream surface. Also the hemisphere above y=0 is also a stream surface. Accordingly, for the hemispherical boss $x^2+y^2+z^2=a^2$ on y=0, the velocity potential is given by (1).
Since U is uniform and the hemisphere is at rest, the motion is steady. Hence by Bernoulli's theorem, the pressure at any point is given by

$$\boldsymbol{\frac{p}{\rho}+\frac{q^2}{2}=c} \qquad \textbf{(2)}$$

Hence the surface of equal pressure are given by putting p=constant in (2). Therefore, these are given by $q^2 = constant.(as\ \rho\ is\ constant)$, i.e., by

$$\boldsymbol{\left(-\frac{d\phi}{dr}\right)^2+\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = const.}$$

$$\left[U\left(1-\frac{a^3}{r^3}\right)\cos\theta\right]^2 + \left[\frac{U}{r}\left(r+\frac{a^3}{2r^2}\right)\sin\theta\right]^2 = \text{constant}$$

or
$$\left(1-\frac{a^3}{r^3}\right)\cos^2\boldsymbol{\theta}+\left(1+\frac{a^3}{2r^3}\right)\sin^2\boldsymbol{\theta} =$$
$constant, as\ U\ is\ a\ constant.$

7.4 Equation of Motion of a Sphere:

Let a sphere of radius '*a*' advance with velocity U in an infinite mass of liquid at rest at infinity. The velocity potential and stream function are given by

$$\phi = \frac{\frac{1}{2}Ua^3\cos\theta}{r^2}; \quad \Psi = -\frac{1}{2}\,U\frac{a^3\sin\theta}{r} \qquad (1)$$

The Kinetic energy of the fluid is given by

$$T_f = -\frac{1}{2}\rho\iint\phi\,\frac{\partial\phi}{\partial n}\,dS \qquad (2)$$

$$= -\frac{1}{2}\rho\int_0^{2\pi}\int_0^{\pi}\left(\frac{\frac{1}{2}Ua^3\cos\theta}{r^2}\right)\left(-\frac{Ua^3\cos\theta}{r^3}\right)a^2\sin\theta\,d\theta dz$$

$$= \frac{U^2}{4}\rho a^3 \int_0^{\pi} \cos^2\theta \sin\theta . \int_0^{2\pi} dz \quad \text{at r=a}$$

$$= \frac{1}{3}\pi\rho a^3 U^2 = \frac{1}{4}M'U^2 \text{ ; where } \quad M' = \frac{4}{3}\pi a^3 \rho \tag{3}$$

Here M' is the mass of the liquid displaced by the sphere. If M is the mass of the sphere, the total kinetic energy of the fluid and the sphere is

$$T = \frac{1}{2}(M + M')U^2 \tag{4}$$

Let R be the external force per unit mass in the direction of motion of the sphere. Let us use the result that the rate of doing work is equal to the rate of increase in K.E.

Thus $\quad$ $$\text{RU} = = \frac{1}{2}\frac{d}{dt}\left[\left(M + \frac{M'}{2}\right)U^2(t)\right]$$

$$= \left(M + \frac{M'}{2}\right)U\frac{dU}{dt}$$

$$\Rightarrow \quad \text{M}\frac{dU}{dt} = R - \frac{1}{2}M'\frac{dU}{dt} \tag{5}$$

If the liquid is not there, then M′ = 0 and the equation of motion of the sphere is

$$\text{M}\frac{dU}{dt} = R \tag{6}$$

Comparing equation (5) & (6), we note that the presence of the liquid offers a resistance of the amount $\frac{1}{2}M'\frac{dU}{dt}$ to the motion of the sphere

Let R′ be the external force per unit mass on the sphere when there is no liquid, then

MR = external force on the sphere in the presence of the liquid.

= MR′ – M′R′ = (M –M′) R′

Since, $\quad$ $\text{M} = \frac{4\pi\sigma a^3}{3}, M' = \frac{4\pi\rho a^3}{3}$

$$\therefore \quad \text{R} = \left(\frac{\sigma-\rho}{\sigma}\right)R' \tag{7}$$

From equations (5) & (7), we find

$$\text{M}\frac{dU}{dt} = \left(\frac{\sigma-\rho}{\sigma}\right)R' - \frac{1}{2}M'\frac{dU}{dt}$$

or $\quad$ $$\left(M + \frac{M'}{2}\right)\frac{dU}{dt} = \left(\frac{\sigma-\rho}{\sigma}\right)R' = \left(\frac{M-M'}{M}\right)R'$$

$$\therefore \quad \text{M}\frac{dU}{dt} = \left(\frac{M-M'}{M+\frac{M'}{2}}\right)R' = \left(\frac{\sigma-\rho}{\sigma+\frac{1}{2}}\right)R' \tag{8}$$

This is the required equation of motion of a sphere in a liquid at rest at infinity.

From equations (6) & (8), we note that the effect of the presence of the liquid reduces the external forces in the ration σ –ρ : σ + $\frac{\rho}{2}$.

Note: Sometimes the above ratio is expressed as s-1: s+1/2, where $s = \sigma/\rho$ is the specific gravity of the sphere compared with the liquid.

7.5 Pressure Distribution on a Sphere:

At a point on a sphere moving through an infinite liquid the pressure is given by the formula

$$\frac{p-p_0}{\rho} = \frac{1}{2} a\, f \cos\theta_1 + \frac{1}{8}\, v^2 (9\cos^2\theta - 5)$$

where v is the velocity , f the acceleration of the sphere, and θ, θ_1 are the angles between the radii and the direction of v , f respectively, and p_0 is the pressure at infinity.

Let the coordinates of the centre C of the moving sphere referred to fixed axes be (x_0, y_0, z_0) and

let $$\dot{x_0} = U,\ \dot{y_0} = V, \dot{z_0} = W \tag{1}$$

Let (x, y, z) be the coordinates of any point P in the liquid.

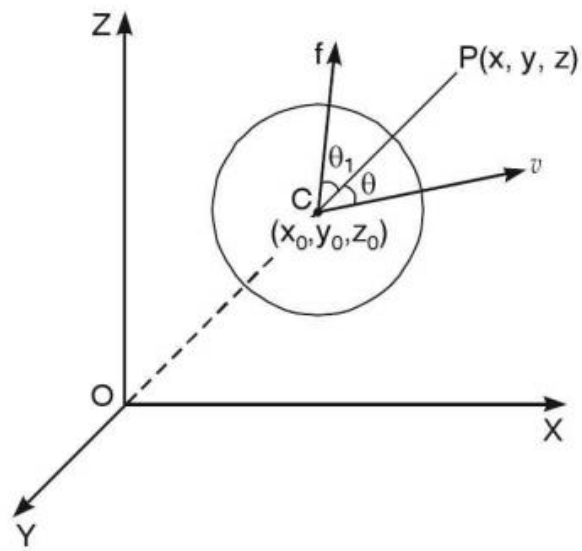
Let θ, θ_1 be the angles between CP and the directions of *v* ,*f* respectively.

Let CP=r. Then, we have

$$r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \tag{2}$$

Let $l, m\ n$ be the direction cosines of CP, then

$$l = \frac{x-x_0}{r}, \qquad m = \frac{y-y_0}{r}, \qquad n = \frac{z-z_0}{r} \tag{3}$$



Also $$v^2 = U^2 + V^2 + W^2 \tag{4}$$

$$v\cos\theta = resolved\ part\ of\ v\ along\ CP = Ul + Vm + Wn$$

$$= U\frac{x-x_0}{r} + V\frac{y-y_0}{r} + W\frac{z-z_0}{r} \tag{5}$$

$$f\cos\theta_1 = resolved\ part\ of\ f\ along\ CP = \dot{U}l + \dot{V}m + \dot{W}n$$

$$=\dot{U}\frac{x-x_0}{r} + \dot{V}\frac{y-y_0}{r} + \dot{W}\frac{z-z_0}{r} \tag{6}$$

The velocity potential at a fixed point of space (x, y, z) is given by

$$\phi = \frac{a^3}{2r^2}\ v\cos\theta$$

$$\phi = \frac{a^3}{2r^2}\ [U\frac{x-x_0}{r} + V\frac{y-y_0}{r} + W\frac{z-z_0}{r}] \quad (7)$$

From (2), $2r\frac{\partial r}{\partial x} = 2(x-x_0)$ so that $\frac{\partial r}{\partial x} = \frac{x-x_0}{r}$ (8)

Differentiating (7) partially with respect to x, we get

$$\frac{\partial \phi}{\partial x} = \frac{\frac{1}{2}a^3 U}{r^3} - \frac{3a^3}{2r^4}\frac{\partial r}{\partial x}\ [U\ (x-x_0) + V(y-y_0) + W(z-z_0)]$$

$$= \frac{1}{2}\frac{a^3 U}{r^3} - \frac{3a^3}{2r^4}\ (x-x_0)v\cos\theta$$ [by (5) and (8)

Similarly, differentiating (7) partially with respect to y and z , we get

$$\frac{\partial \phi}{\partial y} = \frac{1}{2}\frac{a^3 V}{r^3} - \frac{3a^3}{2r^4}\ (y-y_0)v\cos\theta\ ; \quad \frac{\partial \phi}{\partial z} = \frac{1}{2}\frac{a^3 W}{r^3} - \frac{3a^3}{2r^4}\ (z-z_0)v\cos\theta$$

$$\therefore q^2 = \left(-\frac{\partial \phi}{\partial x}\right)^2 + \left(-\frac{\partial \phi}{\partial y}\right)^2 + \left(-\frac{\partial \phi}{\partial z}\right)^2$$

$$= \left(\frac{a^6 v^2}{4r^6}\right)(a + 3\cos^2\theta) \quad (9)$$

From (2), $r\frac{dr}{dt} = (x-x_0)\dot{x_0} - (y-y_0)\dot{y_0} - (z-z_0)\dot{z_0}$

$$= -U(x-x_0) - V(y-y_0) - W(z-z_0) \quad (10)$$

Differentiating (7) partially with respect to t and using (5) and (6) , we get

$$\frac{\partial \phi}{\partial t} = \frac{a^3}{2r^3}\ (fr\ cos\ \theta_1 - v^2) + 3\frac{a^3}{2r^5}(r^2 v^2\cos^2\theta)$$

Thus, $\frac{\partial \phi}{\partial t} = \left(\frac{a^3}{2r^3}\right)(fr\cos\theta_1 - v^2 + 3v^2\cos^2\theta)$ (11)

Let P the potential function due to external forces. Then the pressure at any point to the liquid is given by Bernoulli's equation, namely,

$$\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + P = f(t) \quad (12)$$

At infinity $r = \infty, p = p_0\ and\ so\ \frac{\partial \phi}{\partial t} = 0\ and\ q = 0$ from (11). Hence (12) gives

$F(t) = \frac{p_0}{\rho} + P.$

So (12) reduces to

$$\frac{p-p_0}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2}\ q^2 = \frac{a^3}{2r^3}\ (fr\cos\theta_1 - v^2 + 3v^2\cos^2\theta\ ) - \frac{1}{8}\frac{a^6 v^2}{r^6}\ (1 + 3\cos^2\theta)$$

$$\therefore\ \frac{p-p_0}{\rho}\ a^3 f\frac{1}{2r^2}\cos\theta_1 - \frac{a^3 v^2}{8r^6}\ (4r^3 + a^3) + \frac{3a^3}{8r^6}v^2(4r^3 - a^3)\cos^2\theta$$

(13)

Putting r=a in (13), pressure at any point on the surface of the sphere is given by

$$\frac{p-p_0}{\rho} = \frac{1}{2}\ af\cos\theta_1 + \frac{1}{8}v^2(9\cos^2\theta - 5) \quad (14)$$

Corollary 1. When sphere moves uniformly, i.e., when f=0, pressure at point on the surface of the sphere r=a is given by (putting f=0 in (14))

$\frac{p-p_0}{\rho} = \frac{1}{8} v^2 (9\cos^2\theta - 5)$ (15)

or $\frac{p-p_0}{\rho} = \frac{1}{8} v^2 [\frac{9}{2}(1+\cos 2\theta) - 5] = \frac{1}{16} v^2 (9\cos 2\theta - 1)$ (16)

Corollary 2. Resultant thrust when there is no acceleration

From (15) ; $p = p_0 + \frac{1}{8}\rho v^2 (9\cos^2\theta - 5)$

So the resultant thrust on the sphere

$$= -\int p\cos\theta\, ds = -\int_0^\pi p\cos\theta\, d\theta . 2\pi a\sin\theta$$

$$-2\pi a^2 \int_0^\pi [p_0 + \frac{1}{8} v^2 \rho (9\cos^2\theta - 5)] \sin\theta \cos\theta\, d\theta = 0$$

Which is in conformity with D'Alembert's Paradox.

Corollary 3. Resultant thrust when there is acceleration.

When f is not zero, the resultant thrust due to that part will be

$$-\int_0^\pi \frac{\rho 1}{2} a f\cos\theta_1 .\ 2\pi a\sin\theta_1 . a\, d\theta_1 = -\pi a^3 \rho f \int_0^\pi \cos^2\theta_1 \sin\theta_1\, d\theta_1 = -\frac{2}{3}\pi a^3 f\rho - \frac{1}{2}M' f$$

Where $M' = \frac{4}{3}\pi a^3 \rho = mass\ of\ the\ liquid\ displaced.$

7.6 Drag Force on a Sphere: *Show that the fluid pressure exerts a force* $\frac{1}{2}M'\dot{U}$ *opposing the motion where M'=mass of the liquid displaced by the sphere*

Or

A sphere moves in a fluid at rest at infinity. Calculate the force acting on the sphere due to the presence of the fluid.

The velocity potential for the resulting motion is given by

$$\phi = \frac{Ua^3}{2r^2}\cos\theta \quad (1)$$

Where U is the velocity of motion of a sphere in a fluid at rest at infinity.

The Kinetic energy T of the liquid on the surface of the sphere is given by

$$T = \frac{1}{4}M'U^2$$

The drag force on the sphere may be obtained by integrating the resolved component of pressure force over the surface of the sphere.

This result may also be obtained by equating the rate of change of K.E. of the fluid to the work done by the fluid forces. Let F denote drag force, then

$$\boldsymbol{\frac{dT}{dt} = \frac{Workdone}{time} = \frac{Force \times distance}{time} = Force \times velocity = FU}$$

or $FU = \frac{d}{dt}\left(\frac{1}{4}M'U^2\right) = \frac{M'}{2} U\dot{U}$

or $F = \frac{1}{2}M'\dot{U}$

Note: If U= constant, then drag force is zero.

Example: ***Prove that a sphere projected in a liquid under gravity describes a parabola of latus rectum***

$$\left[\boldsymbol{\frac{2\sigma+\rho}{\sigma-\rho}}\right]\left(\boldsymbol{\frac{U^2}{g}}\right),$$

***where σ and ρ** are the densities of sphere and liquid and U is the horizontal velocity.*

Or Discuss motion of sphere under gravity.

Solution: In considering the motion of a sphere, we consider its virtual mass equal to $M+\frac{1}{2}M'$, where M and M' are masses of sphere and liquid displaced by sphere. By Newton's second law of motion

$$\left(M+\frac{1}{2}M'\right)\dot{U}=Mg-M'g$$

$$\frac{4}{3}\pi a^3\left(\sigma+\frac{1}{2}\rho\right)\dot{U}=\frac{4}{3}\pi a^3(\sigma-\rho)g$$

$$acceleration=\dot{U}=\frac{\sigma-\rho}{\sigma-\frac{1}{2}\rho}g=\frac{2(\sigma-\rho)}{2\sigma+\rho}g$$

Length of lactus rectum $=2(Horizontal\ velocity)^2/acceleration$

$$=\frac{2U^2}{\dot{U}}=2U^2\frac{2\sigma+\rho}{2(\sigma-\rho)g}=\frac{U^2}{g}\frac{2\sigma+\rho}{\sigma-\rho}.$$

Example: ***An infinite ocean of an incompressible liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is disturbed by the sphere, and the pressure in the liquid at a great distance from the obstacle is*** **Π. *Show that the thrust on that half of the sphere on which the liquid impinges is*** $\boldsymbol{\pi a^2\left[\Pi-\frac{\rho U^2}{16}\right)}$.

Solution: For the velocity potential ϕ of the liquid,

$$\phi=\frac{Ua^3}{2r^2}\cos\theta+Ur\cos\theta$$

Since the motion is steady and there are no extraneous forces

$$\frac{p}{\rho}=C-\frac{1}{2}q^2$$

At a great distance $q=U\ and\ p=\Pi, so\ C=\frac{\Pi}{\rho}+\frac{1}{2}U^2$

Hence, $\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{2}U^2-\frac{1}{2}q^2$

Now, at r=a, $q^2=\left(-\frac{\partial\phi}{\partial r}\right)^2+\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$

Therefore, $p=\Pi+\frac{1}{2}\rho U^2-\frac{1}{2}\rho\frac{9}{4}U^2\sin\theta$

The liquid impinges on half of the sphere, hence thrust on that half

$$=\int_0^{\frac{\pi}{2}}\rho\cos\theta\ (2\pi a\sin\theta)\ a\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} [\,\Pi + \tfrac{1}{2}\,\rho U\,(1 - \tfrac{9}{4}\sin^2\theta)\,]\,2\pi\, a^2 \sin\theta\cos\theta\, d\theta$$
$$= \pi a^2 \left(\Pi - \frac{\rho U^2}{16}\right).$$

***Example:* A sphere whose radius at time t is $a + b\,\cos nt$, is held in a stream of liquid of density ρ, whose velocity at a great distance is U. Prove that the resultant thrust on the sphere is $2\pi\,\rho(a + b\,\cos\,nt)^2\,nb\,\sin nt$.**

Solution: We first consider the liquid to be at rest at infinity and take note of the throbbing of the sphere. Since the motion is symmetrical about the centre O of the sphere, the velocity v at a distance r from O is entirely radial and depends only on r and t.

Let $R = a + b\cos nt$; so that the velocity at any point on the sphere is $-nb sin\, nt$. The equation of continuity, therefore, is

$$r^2 v = R^2\,(-nb\sin nt)$$

This gives
$$v = -\frac{\partial\phi_0}{\partial r} = -\frac{R^2}{r^2}\,nb\sin nt$$

Whence
$$\phi_0 = -\frac{R^2}{r}\,nb\sin nt \qquad (1)$$

As the velocity potential ϕ_1 of a liquid streaming with velocity U past a fixed sphere of radius R is $U\cos\theta\left[r + \frac{R^3}{2r^2}\right]$, the motion of the liquid under conditions of the problem is obtained by superposing $\phi_0\ on\ \phi_1$. Thus

$$\phi = \phi_0 + \phi_1 = Ur\cos\theta + \frac{UR^3}{2r^2}\cos\theta - \frac{R^2}{r}\,nb\sin nt$$

At r = R
$$\begin{cases} \dfrac{\partial\phi}{\partial r} = nb\sin nt \\ \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{3}{2}\,U\sin\theta \end{cases}$$

Since $\dot{R} = -nb\sin nt$, we also have , when r=R

$$\frac{\partial\phi}{\partial t} = [-\frac{3}{2}\,U\cos\theta\,.\,nb\sin nt + 2(nb\sin nt)^2 - R\,n^2\,b\cos nt\,]\,,$$

Also,
$$q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = \frac{9}{4}\,U^2\sin^2\theta\, n^2 b^2\sin^2 nt.$$

We now make use of the pressure equation, viz,

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2 + C$$

Which gives, on substituting the values of $\frac{\partial\phi}{\partial t}$ $and\ q,$

$$\left(\frac{p}{\rho}\right) = -\left(\frac{3U}{2}\right)\,nb\cos\theta\sin nt + 2(nb\sin nt\,)^2$$
$$-\,n^2 b\,R\cos nt - \left(9\frac{U^2}{8}\right)\sin^2\theta - \frac{1}{2}n^2 b^2\sin^2 nt + C$$

The resultant thrust on the sphere acts along the initial line by virtue of symmetry, and is of magnitude F, where

$$F = \int_0^{\frac{\pi}{2}} p\cos\theta\; 2\,\pi\, R\sin\theta\,.\,R\,d\theta$$

Measured in the sense of the stream,

or $$F = 2\pi R^2 \int_0^{\pi} \rho \sin\theta . \cos\theta \left(-\frac{3U}{2}\, nb \cos\theta \sin nt\right) d\theta \; + 0$$

$$= -3\pi\rho\, UR^2 nb \sin nt \,.\frac{2}{3}$$

Thus, $$F = -2\pi\rho\, U(a + b\cos nt)^2\, nb \sin nt.$$

7.7 Check Your Progress:

i). Prove that when the sphere is motion with uniform velocity U, the pressure on the part of its surface where the radius makes an angle θ with direction of motion is increased on account of the motion by the amount $\frac{\rho U^2}{16}(9\, cos\, \theta - 1)$,where ρ is the density of the liquid.

ii) A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity u along the x-axis. If the pressure at infinity is zero, prove that at a point x in advance of the centre

$$p = \frac{1}{2}\rho\, a^3 \left[\frac{u}{x^2} + u^2\left(\frac{2}{x^2} - \frac{a^3}{x^6}\right)\right]$$

iii) A solid sphere moves through quiescent frictionless liquid whose boundaries are at a distance from it great compared with its radius. Prove that at each instant the motion in the liquid depends only on the position and velocity of the sphere at that instant. Prove that the liquid steams pas the sides of the sphere with half the velocity of the sphere.

iv) A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity U along x-axis. If the pressure is zero, prove that at a point x in advance of the centre

$$P = \frac{1}{2}\rho a^3 \left[\frac{U}{x^2} + U^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right]$$

v) Prove that the thrust on the half of the sphere on which the liquid impinges is $\pi a^2(\Pi - \frac{1}{16}\rho U^2)$, where Π is the pressure at infinity, U the undisturbed velocity of the liquid and ρ the density.

7.8 Summary: In this chapter we have discussed the theory of irrotational motion in three dimensions with the motion of sphere. The equation of motion of sphere in an infinite mass of liquid at rest at infinity has been derived. For a liquid streaming past a fixed sphere, the lines of flow relative to the sphere is obtained. Also when a sphere moves through an infinite liquid, the pressure distribution on formula is derived.

7.9 Keywords: irrotational, motion of sphere, infinite liquid, velocity potential, pressure distribution, thrust, drag force.

7.10 Self-Assessment Test:

SA1: Find the velocity potential when a sphere of radius a is moving with velocity U in he liquid at rest at infinity.

SA2: A sphere of radius a is moving with velocity U through an infinite liquid at rest at infinity. If p_0 be the pressure at infinity, show that the pressure at any point of the surface of the sphere, the radius to which point makes an angle θ with the direction of motion is given by

$$p = p_0 + \frac{1}{2}\,\rho U^2 \left[1 - \left(\frac{9}{4}\right)\sin^2\theta\,\right].$$

SA3: A sphere moves through a liquid at rest at infinity with a uniform velocity; prove that the equation to the lines of flow is

$$r = a\sin^2\theta$$

SA4: A solid sphere of radius a move along a straight line in an ideal liquid of density ρ, which is moving irrotationally and is at rest at infinity. Show that the magnitude of the resultant thrust of the liquid on the sphere at any instant is $\frac{2}{3}\,\pi a^3\rho f$, where f is the instantaneous value of the acceleration of the sphere.

SA5: For a solid sphere moving under gravity in an infinite liquid prove that the effect of the liquid is to reduce the acceleration due to gravity in the ratio

$$(s-1):(s+\frac{1}{2})$$

s being the specific gravity of the sphere compared with the liquid.

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CHAPTER-8

STOKE'S STREAM FUNCTION

8.0 Learning Objectives: After reading this chapter, you should be able to understand the motion in two dimensions, define Lagrange's stream function, understand three-dimensional hydrodynamical singularities: source, sinks and doublets in three-dimensions their images in infinite plane and spherical surface and derive the velocity potential functions for these singularities, define and understand the importance of Stoke's Stream function.

8.1 Motion in Two-dimensions: When the lines of motion are all parallel to a fixed plane (say xy-plane) and velocity at the corresponding points of all planes parallel to that of xoy-plane has the same magnitude and direction, the motion is said to be two-dimensional i.e., there is no velocity potential parallel to the z-axis. Obviously in this case w=0 and u, v are functions of x,y and t only.

In the figure, let xoy be the fixed plane and x'o'y' a plane parallel to it. If from a point P in the xOy plane a normal is drawn to meet x'O'y' in P', P' is called the point corresponding to P. Let **q** be the velocity at P making an angle θ with Ox; then the velocity at P' will also be **q** making an angle θ with Ox'. This velocity **q** shall be a function of x, y and time, but not of z.

For convenience, let us consider the fluid in two-dimensional motion to be confined between two hypothetical parallel planes at unit distance apart. The plane of reference, the plane xOy is taken midway and parallel to these planes. Therefore, any closed curve drawn in the plane xOy will be a cross-section of a cylindrical surface of unit length. Accordingly, when we speak of the flow across a curve in the xOy plane, we really mean the flow across unit length of the cylinder whose cross-section on the xOy plane is that curve.

8.2 Lagrange Stream Function $\boldsymbol{\psi}$: In two-dimensional motion of incompressible fluid, the velocity **q** is a function of x, y,t but not of z, so that the differential equation of the stream lines is given by

$$\frac{dx}{u}=\frac{dy}{v} \quad \text{or} \quad vdx-udy=0 \tag{1}$$

The equation of continuity is

$$\nabla.\boldsymbol{q}=0 \text{ i.e.,} \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

But (2) is the condition that the differential equation (1) should be exact; it follows that (1) must be a perfect differential, $d\psi$ (say).

Thus

$$vdx-udy=d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=0$$

So that
$$u=-\frac{\partial \psi}{\partial y};v=\frac{\partial \psi}{\partial x}.$$

We call the function ψ the Lagrange stream function or current function.

Obviously the stream lines are given by the solution of (1), i.e., $\psi = constant$. Thus the stream function is constant along a stream line. It is clear from the foregoing considerations that the existence of stream function is merely a consequence of the continuity and incompressibility of the fluid. The current function always exists in all types of two-dimensional motion whether rotation or irrotational.

Physical meaning of Lagrange stream function: Consider a curve I in the xy-plane. If the tangent at any point P of the element ds makes an angle θ with x-axis, the direction cosines of the normal there at (directed from right to left) shall be $(-\sin\theta, \cos\theta, 0)$. The flow, Q, across the curve I from right to left is

$$Q = \int_t \rho\, \boldsymbol{q}.n\, ds = \rho \int_t [(\frac{\partial\psi}{\partial y}\sin\theta) + \frac{\partial\psi}{\partial x}\cos\theta\,]ds$$

$$\left\{q = k \times \nabla\psi = \left[-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}, 0\right]\right\}$$

$$= \rho \int_t \left(\frac{\partial\psi}{\partial y}\, dy + \frac{\partial\psi}{\partial x}\, dx\right) = \rho\ \int_t d\psi = (\psi_2 - \psi_1)\rho$$

Where ψ_1, ψ_2 are the values at the initial and final points of the curve. Thus, the difference between the values of stream function at any two points of a curve equals the flow across that curve.

Corollary: Let AB be an infinitesimal arc of a curve whose length is δs. Then flow across it is $Q = \rho\boldsymbol{q}\delta s$ as well as $Q = \rho(\psi_2 - \psi_1) = \rho d\psi$. Thus, $\boldsymbol{q} = \frac{d\psi}{ds}$, the velocity in terms of the steam function.

8.3 Irrotational Motion in Two Dimensions:

When the motion is irrotational, we know that

$$\xi = \frac{1}{2}\left\{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right\} = 0, ;\ \eta = \frac{1}{2}\left\{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right\} = 0; \zeta = \frac{1}{2}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = 0$$

In two-dimensional motion, first two equations are automatically zero. Substituting for u and v in terms of stream function, the third equation becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Which shows that the stream function satisfies Laplace equations.

Also, in this case, since velocity potential exists, we have

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y};\ \ v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

Hence the equation of continuity, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ gives

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Which shows that ϕ also satisfies Laplace equation.

It can be easily seen that

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}=-uv+uv=0$$

Hence the family of curves $\phi(x,y)=constant$ and $\psi(x,y)=constant$ cut orthogonally at all their points of intersection.

8.4 Complex Potential and Velocity: The relation $w=\phi+i\psi\ where\ \phi$ is the velocity potential and ψ is the stream function of a two-dimensional irrotational motion of a perfect fluid, is known as the complex potential of the fluid motion.

Now since $\quad \frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \quad$ and $\quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$

Known as Cauchy-Riemann equations, it follows that w is an analytic function of $z=x+iy$ in a region where $\phi\ and\ \psi$ are single valued functions.

Conversely, if w is analytic, its real and imaginary parts($\phi\ and\ \psi$) give the velocity potential and stream function for a possible two dimensional irrotational fluid motion.

Again, differentiating the relation

$$w=\phi+i\psi=f(z)$$

With respect to x, we get

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(z)\ \frac{\partial z}{\partial x}$$

Or $\quad -u+iv=f'(z)$

Or $\quad -u+iv=\frac{dw}{dz}$

Hence $\left|\frac{dw}{dz}\right|=|-u+iv|=\sqrt{u^2+v^2},$

Thus $\left|\frac{dw}{dz}\right|$ represents the velocity at any point.

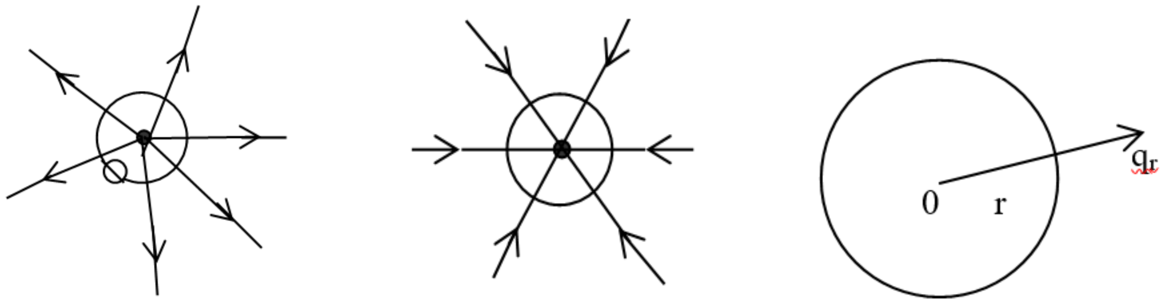
The points where velocity is zero are called stagnation points. Thus, for stagnation points $\frac{dw}{dz}=0.$

8.5 Sources, Sinks and Doublets (Three-dimensional Hydrodynamical Singularities)

8.5.1 Source: An outward symmetrical radial flow of fluid in all directions is termed as a three-dimensional source or a **point source** or a **simple source.**

Thus, a source is a point at which fluid is continuously created and distributed e.g. an expanding bubble of gas pushing away the surrounding fluid. If the volume of fluid per unit time which is emitted from a simple source at 0 is constant and equal to 4πm, then m is termed as strength of the source.

8.5.2 Sink: A negative source is called a **sink**. At such points, the fluid is constantly moving radically inwards from all directions. Thus, a simple sink of strength m is a simple source of strength –m.



8.6 Velocity Potential due to a Simple Source of Strength m. Let there be a source of strength m at a point O the centre, we draw a sphere of radius r around O. The flow across the sphere per unit volume is given by

$$\int_S \mathbf{q}.\mathbf{n}\,dS$$

In case of a source there is only the radial velocity i.e. $\overline{q}$ has only radial component q_r .

Therefore, the flow is

$$= \int_S q_r \, dS \qquad | \mathbf{q}.\mathbf{n} = q_r, \text{ since } \mathbf{q} \textit{ and } \mathbf{n} \text{ have same directions i.e. radial direction.}$$

$$= q_r \, (4\pi \, r^2).$$

Thus, we get

$$4\pi m = q_r \, (4\pi \, r^2)$$

$$\Rightarrow \qquad q_r = \frac{m}{r^2} = -\frac{\partial}{\partial r}\left(\frac{m}{r}\right) \tag{1}$$

It is observed that curl $\mathbf{q} = 0$ (except at r = 0), therefore for irrotational flow,

$$q_r = -\frac{\partial \phi}{\partial r} \qquad | \mathbf{q} = -\nabla \phi \tag{2}$$

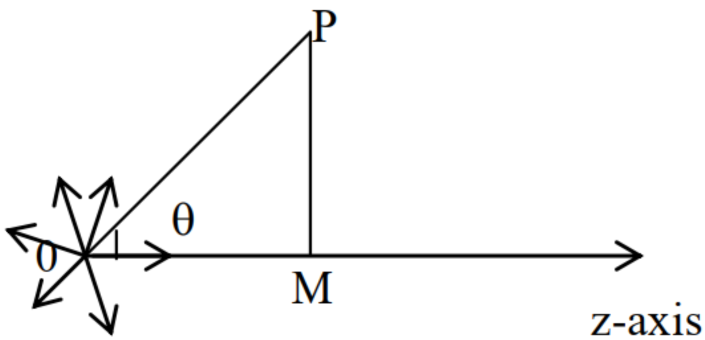
From (1) & (2), we find

$$\phi = \frac{m}{r}$$

which is the required expression for the velocity potential for a source.

Remarks. (i) For a simple sink of strength m, the velocity potential is $\phi = -\frac{m}{r}$

(ii) A source or sink implies the creation or annihilation of liquid at a point. Both are points at which the velocity potential and stream function for two dimensional case become infinite and therefore, they require special analysis.



8.7 A simple Source in Uniform Stream. Let us consider a simple source of strength m at 0 in a uniform stream having undisturbed velocity $U\hat{k}$, $\hat{k}$ be the unit vector along z-axis which is taken as the axis of symmetry of the flow.

We shall find the velocity potential at any point P (z, θ, ψ). From P, draw perpendicular on OZ. Let OP = r, $\angle POZ = \theta$; OM = z

We observe that the velocity potential of the uniform stream in the absence of source is

$$\mathbf{q} = -\nabla\phi \quad \Rightarrow U\mathbf{k} = -\frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = -U \Rightarrow \phi = -Uz$$

$$\phi_1 = -Uz = -Ur\cos\theta \tag{1}$$

and the velocity potential of the simple source is

$$\phi_2 = \frac{m}{r} \tag{2}$$

Thus, the velocity potential of the combination is

$$\phi = \phi_1 + \phi_2 = -Ur\cos\theta + \frac{m}{r}$$

$$= -\left(Ur\cos\theta - \frac{m}{r}\right) \tag{3}$$

From here, the velocity components at $P(r, \theta, \psi)$ are

$$q_r = -\frac{\partial \phi}{\partial r} = U\cos\theta + \frac{m}{r^2}$$

$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta \qquad \left| \begin{array}{l} 0 \le \theta \le \pi \\ 0 \le \psi \le 2\pi \\ \dfrac{\partial}{\partial \psi} \equiv 0 \end{array} \right.$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \psi} = 0$$

The stagnation points $(\mathbf{q} = 0)$ are given by $U\cos\theta + \dfrac{m}{r^2} = 0,\ \sin\theta = 0 \quad \Rightarrow \theta = 0 \text{ or } \pi$

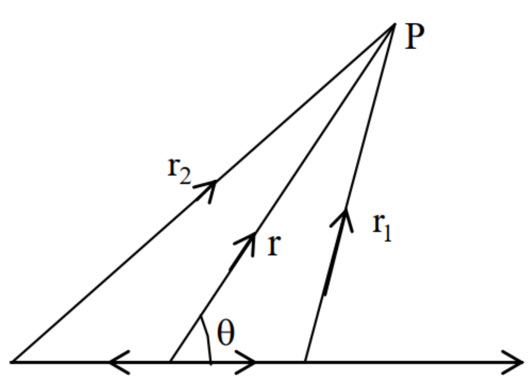
But $\theta = 0$ gives r to be imaginary $\Rightarrow \theta = \pi$ and $r = \sqrt{\dfrac{m}{U}}$

Thus there is only one stagnation point $\left(\sqrt{\dfrac{m}{U}}, \pi, 0\right)$

8.8. Doublet (Dipole). The combination of a source and a sink of equal strength, at a small distance apart, is called a doublet.

8.8.1 Velocity Potential of Doublet.

Suppose that there is a simple source of strength m at O_1 and a simple sink of strength m at O_2. Origin O is taken as the mid-point. of O_1 O_2. It is also assumed that there is no other source or sink. Let P be a fixed point within the fluid and



$$\boldsymbol{OP} = \boldsymbol{r}, \boldsymbol{O_1P} = \boldsymbol{r_1}, \boldsymbol{O_2P} = \boldsymbol{r_2}, \angle POO_1 = \theta$$

$$\boldsymbol{OO_1} = \boldsymbol{h}, \boldsymbol{OO_2} = -\boldsymbol{h}, h = |\boldsymbol{h}|$$

The velocity potential at P due to the combination of source and sink at O_1 and O_2 is

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} = \frac{mr_2 - mr_1}{r_1 r_2}$$

$$= \frac{m(r_2 - r_1)}{r_1 r_2} = \frac{m(r_2^2 - r_1^2)}{r_1 r_2 (r_1 + r_2)}$$

$$= \frac{m(\mathbf{r}_2 - \mathbf{r}_1).(\mathbf{r}_2 + \mathbf{r}_1)}{r_1 r_2 (r_1 + r_2)}$$

But $\quad \mathbf{r}_2 - \mathbf{r}_1 = 2\mathbf{h} \, and \; r_2 + \mathbf{r}_1 = 2\mathbf{r}$

$$\left| \begin{aligned} \mathbf{r}_2 &= \mathbf{r}\mathbf{h} + \mathbf{r} \\ \mathbf{r}_1 &= -\mathbf{r}h + \mathbf{r} \end{aligned} \right.$$

Thus $\quad \phi = \dfrac{m(2\mathbf{h}).(2\mathbf{r})}{r\, r_2 (r_1 + r_2)} = \dfrac{4m\mathbf{h}.\mathbf{r}}{r_1 r_2 (r_1 + r_2)}$

$$= \frac{2\mu.\mathbf{r}}{r_1 r_2 (r_1 + r_2)}, \text{ where } \boldsymbol{\mu} = 2m\mathbf{h} \tag{1}$$

In equation (1), let us first keep $\boldsymbol{\mu}$ a finite constant and non-zero vector, so that $\mu = |\boldsymbol{\mu}|$ is a finite constant and non-zero scalar. Let $\boldsymbol{h} \rightarrow \mathbf{0} \; along \; \boldsymbol{O_1O}$.

Then $m \rightarrow \infty$ in such a way that $\bar{\mu}$ remains the same finite non-zero constant vector. In that case, both r_1, $r_2 \rightarrow r$ and thus under this limiting process, (1) results in

$$\phi = 2.\boldsymbol{\mu}.\frac{\boldsymbol{r}}{r^3} = \frac{\mu r \cos\theta}{r^3} = \frac{\mu \cos\theta}{r^2} \tag{2}$$

The limiting source sink combination obtained at 0 when we keep the direction of **h** fixed but let $h \rightarrow 0$ and $m \rightarrow \infty$ with $\mu = 2mh$ remaining a finite non-zero constant, is called a three-dimensional doublet (or dipole). The scalar quantity μ is called the moment or strength of the doublet. The vector quantity $\boldsymbol{\mu} = \mu\hat{\mu}$ is called the vector moment of the doublet & $\hat{\mu}$ (unit vector from 0_2 to 0_1) determines the direction of the axis of the doublet from sink to source.

From (2), the velocity components are given by

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{2\mu\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{r^3}$$

$$q_\psi = 0$$

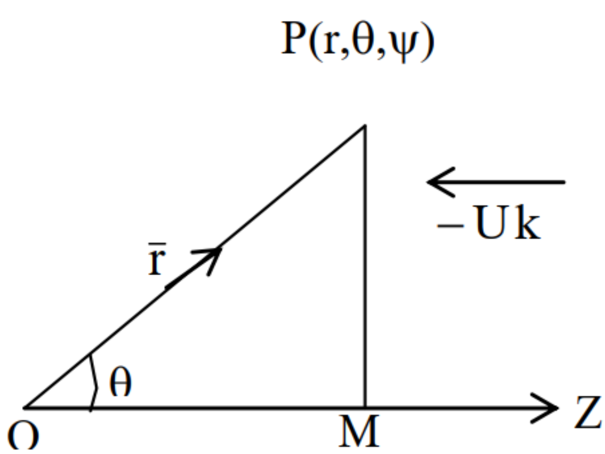
The streamlines due to the doublet are given by

$$\frac{dr}{\frac{2\mu\cos\theta}{r^3}} = \frac{rd\theta}{\frac{\mu\sin\theta}{r^3}} = \frac{r\sin\theta d\psi}{0}$$

$\Rightarrow$ $\quad d\psi = 0 \quad \Rightarrow \psi = \text{constant and } \frac{dr}{r} = 2\cot\theta\, d\theta$

$\Rightarrow$ $\quad r = A\sin^2\theta$

8.8.2 Doublet in a Uniform Stream. Let there be a doublet of vector moment $\boldsymbol{\mu}=\mu\,\mathbf{k}$ at O in a uniform stream whose velocity in the absence of the doublet is $-U\mathbf{k}$ (U = constant).



Let P be a point in the fluid having spherical polar co-ordinates (r, θ, ψ), the direction **OZ** of the doublet's axis being the line θ = 0. We shall find the resultant velocity potential due to the combination of the uniform stream and the doublet. We know that the velocity potential due to the uniform stream is

$$\phi_1 = U\,z = Ur\cos\theta \qquad (1)$$

and the velocity potential due to a doublet at O, is

$$\phi_2 = \frac{\mu\cos\theta}{r^2} \qquad (2)$$

Thus, the resultant velocity potential at P. due to the combination, is

$$\phi = \phi_1 + \phi_2 = (Ur + \mu \bar{r}^{2}) \cos\theta$$

From here, the velocity component are

$$q_r = -\frac{\partial \phi}{\partial r} = -\left(U - \frac{2\mu}{r^3}\right)\cos\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \left(U + \frac{\mu}{r^3}\right)\sin\theta$$

$$q_\psi = \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \psi} = 0$$

Stagnation points are determined by solving.

$$\left(U - \frac{2\mu}{r^3}\right)\cos\theta = 0, \left(U + \frac{\mu}{r^3}\right)\sin\theta = 0 \qquad |\,\mathbf{q} = 0$$

which are satisfied when $\sin\theta = 0$ and $r = \left(\frac{2\mu}{U}\right)^{1/3}$

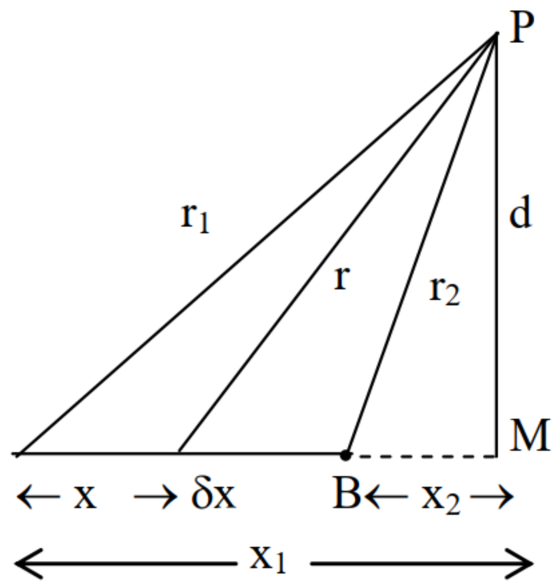
Thus, we have the two stagnation points.

$$\left(\left(\frac{2\mu}{U}\right)^{1/3}, 0\right) \text{ and } \left(\left(\frac{2\mu}{U}\right)^{1/3}, \pi\right)$$

which lie on the axis of symmetry.

If we write r = a i.e., $a = \left(\frac{2\mu}{U}\right)^{1/3}$ i.e. $\mu = \frac{1}{2} U a^3$, then for the region $r \geq a$, we obtain the same velocity potential as for a uniform flow past a fixed impermeable sphere of radius a and centre 0. Thus, for $r \geq a$, the effect of the sphere is that of a doublet of strength $\mu = \frac{1}{2} Ua^3$ situated at its centre, its axis pointing upstream. So, the sphere can be represented by a suitably chosen singularity at its centre.

8.9 Line Distribution of Sources. Let us consider a uniform line source AB of strength m per unit length. This means that the elemental section of AB at a distance. x from A and of length δx is a point source of strength $m\delta x$.



Let P be a point in the fluid at a distance r from this element, then the velocity potential at P due to the point source is $\dfrac{m\delta x}{r}$.

The total velocity potential at P due to the entire line distribution AB (= $2l$) is

$$\phi = m\int_0^{2l}\frac{dx}{r} \qquad (1)$$

Let AM = x_1, BM = x_2, where AM is the orthogonal projection of AP on AB. Also, let PM = d, AP = r_1, BP = r_2. Since $r^2 = (x_1 - x)^2 + d^2 = (x_1 - x)^2 + r_1^2 - x_1^2$, therefore from (1), we get

$$\phi = m\int_0^{2l}\frac{dx}{\sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}}$$

$$= m\left[\frac{\log\left\{(x_1 - x) + \sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}\right\}}{-1}\right]_0^{2l} \qquad \left|\ \because \int_\alpha^\beta \frac{1}{\sqrt{x^2 + a^2}}dx = \left[\log\left(x + \sqrt{x^2 + a^2}\right)\right]_\alpha^\beta\right.$$

$$= m\left[\log\left\{(x_1 - x) + \sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}\right\}\right]_{2l}^{0}$$

$$= m\left[\log(x_1 + r_1) - \log\left\{x_2 + \sqrt{x_2^2 + r_1^2 - x_1^2}\right\}\right] \qquad \left|\ \because x_1 - 2l = x_1 - AB = x_2\right.$$

$$= m\log\left(\frac{x_1 + r_1}{x_2 + r_2}\right), \text{ where } r_1^2 - x_1^2 = d^2 = r_2^2 - x_2^2.$$

Again, the relation $r_1^2 - x_1^2 = r_2^2 - x_2^2$

$$\Rightarrow \qquad \frac{r_1 + x_1}{r_2 + x_2} = \frac{r_2 - x_2}{r_1 - x_1} = \frac{r_1 + r_2 + x_1 - x_2}{r_1 + r_2 + x_2 - x_1}$$

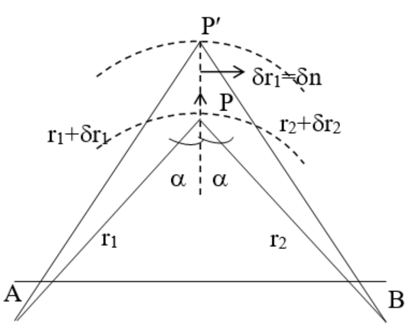
$$= \frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}$$

Thus, $$\phi = m \log\left(\frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}\right)$$

$$= m \log\left(\frac{a + l}{a - l}\right) \qquad (2)$$

where 2a is the length of major axis of the ellipsoid of revolution through P having A and B as foci since for such an ellipsoid $r_1 + r_2$ = constant. It follows from here that the equipotential surfaces ϕ = constant are precisely the family of confocal ellipsoid $r_1 + r_2 = 2a$ obtained when a is allowed to vary.

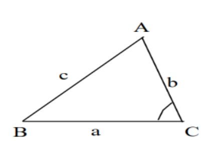
Expression for Velocity: The velocity at P is given by $\mathbf{q} = -\nabla\phi = -\left(\frac{\partial \phi}{\partial n}\right)\mathbf{n}$ (3)



Let P be any point on the ellipsoid specified by parameter a and P′ the neighboring point on the ellipsoid specified by parameter a + δa, wherein $\mathbf{PP}' = \delta n\, \mathbf{n}$

Thus $$\mathbf{q} = -m\frac{\partial}{\partial \mathbf{n}}\left[\log\frac{a+l}{a-l}\right]\mathbf{n} = -m\left[\frac{1}{a+l} - \frac{1}{a-l}\right]\frac{\partial a}{\partial n}\mathbf{n} = \frac{2lm}{a^2 - l^2}\frac{\partial a}{\partial n}\mathbf{n} \qquad (4)$$

The normal at P to the a-surface bisects the angle 2α between the focal radii AP, BP.



Now,

$$(r_1 + \delta r_1)^2 = r_1^2 + (\delta n)^2 - 2r_1\,\delta n \cos(180-\alpha)$$

$$= r_1^2 + (\delta n)^2 + 2r_1\,\delta n \cos\alpha$$

$$\left|\begin{array}{l} \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} \\ \Rightarrow c^2 = a^2 + b^2 - 2ab\cos C \end{array}\right.$$

$$\Rightarrow \qquad 2r_1\,\delta r_1 = 2r_1\,\delta n \cos\alpha + (\delta n)^2 - (\delta r_1)^2$$

$$\Rightarrow \qquad \delta r_1 = \delta n \cos\alpha \qquad \left|\ (\delta r_1)^2 = (\delta n)^2\right.$$

$$\Rightarrow \qquad \frac{\partial r_1}{\partial n} = \cos\alpha$$

Similarly, $\dfrac{\partial r_2}{\partial n} = \cos\alpha$

Since, $2a = r_1 + r_2$

$$\Rightarrow \quad 2\frac{\partial a}{\partial n} = \frac{\partial r_1}{\partial n} + \frac{\partial r_2}{\partial n}. = \cos\alpha + \cos\alpha = 2\cos\alpha$$

$$\Rightarrow \quad \frac{\partial a}{\partial n} = \cos\alpha$$

and thus, from equation (4), the velocity of fluid at P is given by $\mathbf{q} = \left[\dfrac{2lm\cos\alpha}{a^2 - l^2}\right]\mathbf{n}$

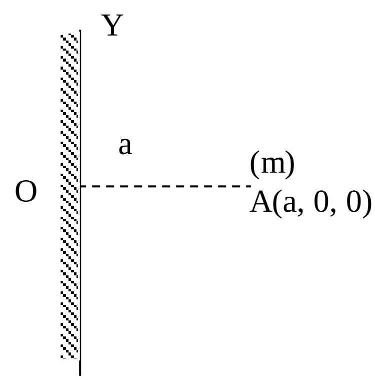
8.10 Hydrodynamical Images for Three Dimensional Flows

Let us consider a fluid containing a distribution of sources, sinks and doublets. If a surface S can be drawn in the fluid across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface S may be said to be images of one another w.r.t. to the surface. Further, if the surface S be considered as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unaltered.

8.10.1. Images in a Rigid Impermeable Infinite Plane

(i) Image of a source in a plane: consider a simple source of strength m situated at A(a, 0, 0) at a distance a from an infinite plane YY′.

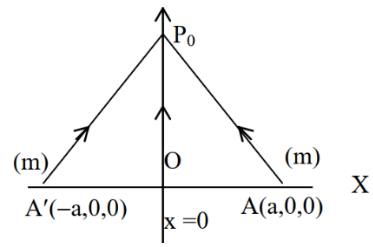
We shall show that the appropriate image system for this is an equal source of strength m at A′(–a, 0, 0), the reflection of A in the plane.



To prove this, we consider two equal sources f strength m at A(a, 0, 0) & A′ (–a, 0, 0) with no rigid boundary. Let P_0 be any point on the plane YY′. Then the fluid velocity at P_0 due to the two sources is

Y′

$$\mathbf{q} = \frac{m}{(AP_0)^3}\mathbf{AP}_0 + \frac{m}{(A'P_0)^3}\mathbf{A'P}_0 \qquad \left| \; \mathbf{q} = \frac{m}{r^2}\mathbf{r} = \frac{m}{r^3}\mathbf{r} \right.$$



$$\Rightarrow \qquad \mathbf{q} = \frac{m}{(AP_0)^3}\ (\boldsymbol{AP_0} + \boldsymbol{A'P_0})$$

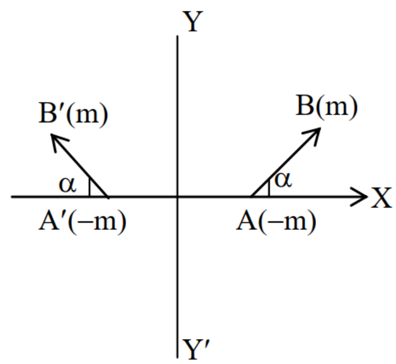
$$= \frac{m}{(AP_0)^3}(2\boldsymbol{OP_0}) = \frac{2m}{(AP_0)^3}(\mathbf{OP}_0) \qquad \left| \begin{array}{l} \because \mathbf{AP}_0 + \mathrm{A'P}_0 \\ = (\mathbf{AO} + OP_0) + (\mathbf{A'}O + \mathbf{O}P_0) \\ = 2OP_0 \end{array} \right.$$

This shows that at any point P_0 of the plane YY′, the fluid flows tangentially to the plane x = 0 and so there is no transport of fluid across this plane O

Let ϕ denotes the velocity potential then, at all points P_0 on the plane YY′, the normal component of velocity is zero

$\Rightarrow \dfrac{\partial\phi}{\partial n} = 0$. Hence, the image of a source at A in the rigid plane YY′ is a source at A′, as required.

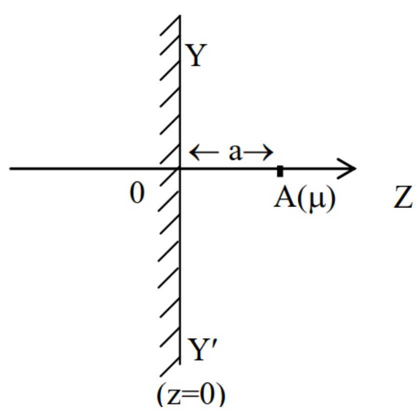
(ii) Image of Doublet in a Plane: Consider a pair of sources –m at A and m at B, taken close together and on one side of the rigid plane YY'. The image system is –m at A', m at B′, where A′ & B′ are respectively the reflections of A and B in the plane YY′. In the limiting case, when B→A along BA in such a way as to form a doublet at A, we find that the image of a doublet in an infinite impermeable rigid plane is a doublet of equal strength and symmetrically disposed to the other w.r.t the plane.

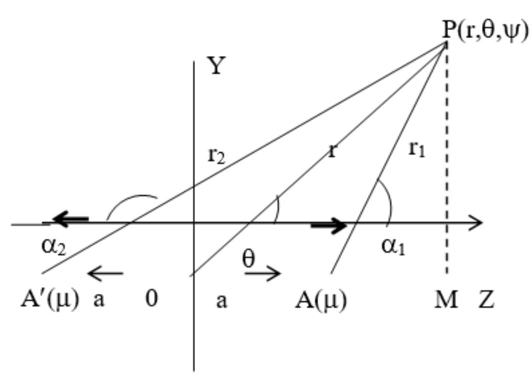


Example: ***A three-dimensional doublet of strength μ whose axis is in the direction $\overline{\text{OZ}}$ is distant a from the rigid plane z = 0 which is the sole boundary of liquid of constant density ρ, infinite in extent. If p_∞ be the pressure at ∞, show that the pressure on the plane is least at a distance $\frac{a\sqrt{5}}{2}$ from the doublet***

Solution. Let there be a doublet of strength μ at the point A with OA = a and YY′ (i.e. z = 0) be the infinite plane. Then the image system is an equal doublet of strength μ at A′, the reflection of A in the plane z = 0, and the axis along **ZO**. The line OZ is taken as the initial line $\theta = 0$ and plane z = 0 is $\theta = \pi/2$. so that $P(r, \theta, \psi)$ is confined to the region $0 \le \theta \le \pi/2$. Let AP = r_1, A′P = r_2 and α_1, α_2 be the angles which these lines make with the axis of the doublets as shown in the figure.

Then, the velocity potential at P is





$$\phi = \frac{\mu\cos\alpha_1}{r_1^2} + \frac{\mu\cos\alpha_2}{r_2^2} \tag{1}$$

where $$\left.\begin{aligned} r_1^2 &= r^2 + a^2 - 2ra\cos\theta \\ r_2^2 &= r^2 + a^2 + 2ra\cos\theta \end{aligned}\right\} \tag{2}$$

(By cosine formulae in Δ POA, POA′)

But $$\cos\alpha_1 = \frac{AM}{r_1} = \frac{OM - OA}{r_1} = \frac{r\cos\theta - a}{r_1}$$

and $$\cos(180 - \alpha_2) = \frac{A'M}{r_2} = \frac{A'O + OM}{r_2} = \frac{a + r\cos\theta}{r_2}$$

$\Rightarrow$ $$\cos\alpha_2 = -\frac{(a + r\cos\theta)}{r_2}$$

Using these relations in (1), we get

$$\phi = \frac{\mu}{r_1^2}\left(\frac{r\cos\theta - a}{r_1}\right) + \frac{\mu}{r_2^2}\left[\frac{-(a + r\cos\theta)}{r_2}\right]$$

$$= \mu\left[\frac{r\cos\theta - a}{r_1^3} - \frac{r\cos\theta + a}{r_2^3}\right] \tag{3}$$

Further from (2), we have

$$2\,r_1\,\frac{\partial r_1}{\partial r} = 2r - 2a\cos\theta \quad \Rightarrow \frac{\partial r_1}{\partial r} = \frac{r - a\cos\theta}{r_1}$$

Similarly, $$\frac{\partial r_2}{\partial r} = \frac{r + a\cos\theta}{r_2}, \frac{\partial r_1}{\partial \theta} = \frac{ra\sin\theta}{r_1}$$

$$\frac{\partial r_2}{\partial \theta} = -\frac{ra\sin\theta}{r_2}.$$

Thus from (3), the velocity components are given by

$$q_r = -\frac{\partial\phi}{\partial r} = \mu\left[\frac{\cos\theta}{r_2^3} - 3\left(\frac{\partial r_2}{\partial r}\right)\frac{1}{r_2^4}(r\cos\theta + a) - \frac{\cos\theta}{r_1^3} + 3\left(\frac{\partial r_1}{\partial r}\right)\frac{1}{r_1^4}(r\cos\theta - a)\right]$$

$$= \mu\left[\frac{\cos\theta}{r_2^3} - 3\frac{(r + a\cos\theta)(r\cos\theta + a)}{r_2^5} - \frac{\cos\theta}{r_1^3} + \frac{3(r - a\cos\theta)(r\cos\theta - a)}{r_1^5}\right]$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu}{r}\left[\frac{r\sin\theta}{r_1^3} + 3\frac{(r\cos\theta - a)\left(\frac{\partial r_1}{\partial\theta}\right)}{r_1^4} - \frac{r\sin\theta}{r_2^3} - 3\frac{(r\cos\theta + a)\left(\frac{\partial r_2}{\partial\theta}\right)}{r_2^4}\right]$$

$$= \frac{\mu}{r}\left[\frac{r\sin\theta}{r_1^3} + \frac{3ra\sin\theta(r\cos\theta - a)}{r_1^5} - \frac{r\sin\theta}{r_2^3} + 3\frac{ra\sin\theta(r\cos\theta + a)}{r_2^5}\right]$$

$$q_\psi = 0$$

When the point P lies on the plane YY′ or $\theta = \pi/2$, we have $r_1^2 = r_2^2 = r^2 + a^2$ and so at $(r, \pi/2, \psi)$, the velocity components are

$$q_r = -6\mu\, ra/(r^2 + a^2)^{5/2},\ q_\theta = 0,\ q_\psi = 0\ .$$

Along the streamline through this point, Bernoulli's equation is

$$\frac{p}{\rho} + \frac{1}{2}\mathbf{q}^2 = const = \frac{p_\infty}{\rho},$$

where $\mathbf{q} = 0$ at infinity.

Thus, the pressure at any point on the plane YY′ is given by

$$p = p_\infty - \frac{1}{2}\rho\left[36\mu^2 a^2 r^2 / (r^2 + a^2)^5\right]$$

i.e. $$p(r) = p_\infty - \frac{18\rho\mu^2 a^2 r^2}{(r^2 + a^2)^5}$$

Now,

$$p'(r) = \frac{dp}{dr} = 36\rho\mu^2 a^2 r(4r^2 - a^2)/(r^2 + a^2)^6$$

which gives $p'(r) = 0$ when $r = \frac{1}{2}a$

Also

$$p'\left(\frac{a}{2} -\right) < 0,\ p'\left(\frac{a}{2} +\right) > 0$$

i.e. $p'(r)$ changes sign from negative to positive when r passes through $\frac{a}{2}$

$\Rightarrow$ p is minimum at $r = \frac{a}{2}$ $\theta = \pi/2$

i.e. at the point $P_0\left(\frac{a}{2}, \pi/2, \psi\right)$

The distance P_0A is given by

$$\sqrt{\left(\frac{a}{2}\right)^2 + a^2} = \frac{\sqrt{5}}{2}a$$

Hence p is least at a distance $\frac{\sqrt{5}}{2}a$ from the doublet and the minimum value is

$$p_{min.} = p_\infty - \frac{9}{2}\rho\mu^2\left(\frac{4}{5}\right)^5 \frac{1}{a^6}$$

8.11 Images in Impermeable Spherical Surfaces. We have already studied the effect of placing a solid impermeable sphere in a uniform stream of incompressible fluid, taking the case of axial symmetry. Here, we discuss the disturbance produced when a sphere is placed in more general flow.

We shall make use of Weiss's Sphere Theorem which states as follows:

Let $\phi(r, \theta, \psi)$ be the velocity potential at a point P having spherical polar co-ordinates (r, θ, ψ) in an incompressible fluid having irrotational motion and no rigid boundaries. Also suppose that ϕ has no singularities within the region $r \le a$. Then if a solid impermeable sphere of radius a is introduced into the flow with its centre at the origin of co-ordinates, the new velocity potential at P in the fluid is

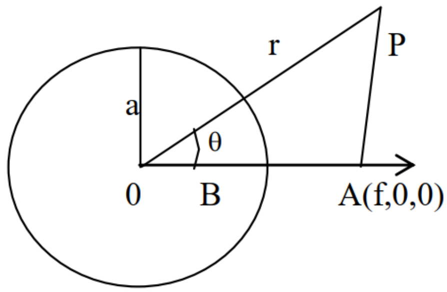
$$\phi(r, \theta, \psi) + \frac{a}{r}\phi\left(\frac{a^2}{r}, \theta, \psi\right) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta, \psi)\, dR, \ (r > a)$$

where r and $\frac{a^2}{r}$ are the inverse points w.r.t the sphere of radius a."

Here, the last two terms refer to perturbation potential due to the presence of the sphere.

(i) **Image of a Source in a Sphere:** Suppose a source of strength m is situated at point A at a distance f(> a) from the centre of the sphere of radius a.

Let B be the inverse point of A w.r.t. the sphere, then $OB = a^2/f$



The velocity potential at P(r, θ, ψ) in the fluid due to a simple source of strength m at A(f, 0, 0) is

$$\phi(r, \theta) = \frac{m}{AP}$$

From Δ OAP, $\cos\theta = \dfrac{(OP)^2 + (OA)^2 - (AP)^2}{2(OP)(OA)} = \dfrac{r^2 + f^2 - (AP)^2}{2rf}$

$$\Rightarrow \qquad AP = \sqrt{r^2 + f^2 - 2rf\cos\theta}$$

Thus, the velocity potential is

$$\phi(r, \theta) = m(r^2 + f^2 - 2rf\cos\theta)^{-1/2} \qquad (1)$$

Introducing a solid sphere in the region $r \le a$, where $a < f$, we obtain on using Weiss's sphere theorem, a perturbation potential

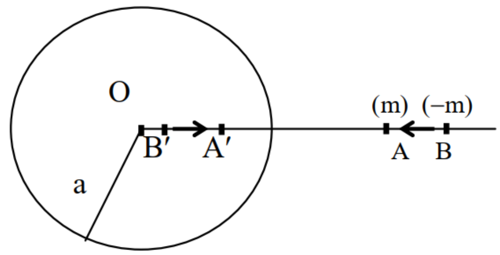
$$\frac{a}{r}\phi\left(\frac{a^2}{r}, \theta\right) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta)\, dR$$

i.e. $$\frac{am}{r}\left[\frac{a^4}{r^2} + f^2 - 2\frac{a^2}{r} f\cos\theta\right]^{-1/2} - \frac{m}{a}\int_0^{a^2/r} [R^2 + f^2 - 2Rf\cos\theta]^{-1/2}\, dR$$

i.e. $$\frac{(ma/f)}{\sqrt{r^2 - 2r(a^2/f)\cos\theta + (a^2/f)^2}} - \frac{m}{a}\int_0^{a^2/r} \frac{dR}{\sqrt{R^2 - 2Rf\cos\theta + f^2}}$$

This shows that the image system of a point source of strength m placed at distance f(> a) from the centre of solid sphere consists of a source of strength $\dfrac{ma}{f}$ at the inverse point $\dfrac{a^2}{f}$ in the sphere, together with a continuous line distribution of sinks of uniform strength $\dfrac{m}{a}$ per unit length extending from the centre to the inverse point.

(ii) Image of a doublet in a sphere when the axis of the doublet passes through the centre of the sphere: Let us consider a doublet AB with its axis **BA** pointing towards the centre 0 of a sphere of radius a. Let OA = f, OB = f + δf. Let A′, B′ be the inverse points of A & B in the sphere so that



$OA' = a^2/f$, $OB' = a^2/(f+\delta f)$.

At A, B we associate simple sources of strengths m and −m so that the strength of the doublet is $\mu = m\delta f$, where μ is to remain a finite non-zero constant as $m \to \infty$ and $\delta f \to 0$ simultaneously.

$$B'A' = OA' - OB' = \frac{a^2}{f} - \frac{a^2}{f+\delta f} = \frac{a^2}{f} - \frac{a^2}{f}\left(1+\frac{\delta f}{f}\right)^{-1}$$

$$= \frac{a^2}{f} - \frac{a^2}{f} + \frac{a^2}{f}\frac{\delta f}{f} \text{ to the first order}$$

$$= \frac{a^2}{f^2}\delta f \text{ to the first order}$$

Now, from the case of "Image of source in a sphere", the image of m at A consists of $\frac{ma}{f}$ at A′ together with a continuous line distribution from O to A′ of sinks of strength $\frac{m}{a}$ per unit length and the image of −m at B consists of $\frac{-ma}{(f+\delta f)}$ at B′ together with a continuous line distribution from O to B′ of sources of strength $\frac{m}{a}$ per unit length.

The line distribution of sinks and sources from 0 to B′ cancel each other leaving behind a line distribution of sinks of strength $\frac{m}{a}$ per unit length from B′ to A′ i.e. sink of strength $\frac{m}{a}B'A' = \frac{m}{a}\left(\frac{a^2}{f^2}\delta f\right) = \frac{a}{f^2}(m\delta f) = \frac{\mu a}{f^2}$ at B′. The source at B′ is of strength

$$\frac{-ma}{f+\delta f}=\frac{-ma}{f}\left(1+\frac{\delta f}{f}\right)^{-1}=-\frac{ma}{f}\left(1-\frac{\delta f}{f}\right) \quad , \qquad \text{to the first order terms}$$

$$=\frac{-ma}{f}+\frac{ma}{f^2}\delta f=\frac{-ma}{f}+\frac{\mu a}{f^2}$$

which is equivalent to a sink $\frac{ma}{f}$ at B' and a source $\frac{\mu a}{f^2}$ at B′.

As there is already a sink $\frac{\mu a}{f^2}$ at B′, therefore source and sink at B′ neutralize. Finally, we are left with source $\frac{ma}{f}$ at A′ and a sink. $\frac{ma}{f}$ at B′. Thus, to the first order, we obtain a doublet at A′ of strength

$$\frac{ma}{f}\ (B'A')=\frac{ma}{f}\,\frac{a^2}{f^2}\delta f$$

$$=\frac{ma^3}{f^3}\delta f=\frac{\mu a^3}{f^3}.$$

Hence in the limiting case as $\delta f\to 0$, $m\to\infty$, we obtain a doublet at A of strength μ with its axis towards O, together with a doublet at the inverse point A′ of strength $\frac{\mu a^3}{f^3}$ with its axis away from O.

8.12 Stoke's Stream Function (Stream Function for an Axi-Symmetric Flow): If the streamlines in all the planes passing through a given axis are the same, the fluid motion is said to be axi-symmetric. We have already considered such flow for irrotational motion in spherical polar co-ordinates. (r, θ, ψ) in which the line $\theta = 0$ is the axis of symmetry.

Suppose the z-axis be taken as axis of symmetry, then $q_\theta = 0$ and the fluid motion is the same in every plane θ = constant (meridian plane) and suppose that a point P in the fluid may be specified by cylindrical polar co-ordinates (r, θ, z). Thus, all the quantities associated with the flow are independent of θ. The equation of continuity in cylindrical co-ordinates, becomes

$$\frac{\partial}{\partial r}(rq_r)+\frac{\partial}{\partial z}(rq_z)=0$$

i.e. $$\frac{\partial}{\partial r}(rq_r)=-\frac{\partial}{\partial z}(rq_z) \tag{1}$$

This is the condition of exactness of the differential equation

$$rq_r dz - r\,q_z\,dr = 0 \tag{2}$$

This means that (2) is an exact differential equation and let L.H.S. be an exact differential dΨ(say)

Therefore,

$$rq_r\, dz - rq_z\, dr = d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial z} dz$$

which gives

$$\frac{\partial \Psi}{\partial r} = -rq_z, \frac{\partial \Psi}{\partial z} = rq_r \tag{3}$$

The function Ψ in (3) is called **Stoke's stream function.**

The equation of streamlines in the meridian plane θ = constant at a fixed time t is

$$\frac{dr}{q_r} = \frac{dz}{q_z}$$

$$\Rightarrow \qquad q_z\, dr = q_r\, dz$$

Using (3), we get

$$-\frac{1}{r}\frac{\partial \Psi}{\partial r} dr = \frac{1}{r}\frac{\partial \Psi}{\partial z} dz$$

$$\Rightarrow \qquad \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial z} dz = 0$$

$$\Rightarrow \qquad d\Psi = 0$$

$$\Rightarrow \qquad \Psi = \text{constant} = C$$

which represent the streamlines.

Property of Stoke's Function:

2π times the difference of the values of Stoke's stream function at two points in the same meridian plane is equal to the flow across the angular surface obtained by the revolution around the axis of curve joint the points.

Proof: Let dS be an element of the curve and θ is its inclination to the axis, then outward flow across the surface of revolution is equal to

$$Q = \rho \int_S \boldsymbol{q}.\boldsymbol{n}\, dS = \rho \int_S (q_r\, n_r + q_z n_z) dS$$

where **n** is the outward normal to the surface S, i.e., directed away from the z-axis. Since

$$n_r dS = r\, d\theta dz, \qquad n_z dS = -r\, d\theta\, dr$$

Therefore, $Q = \rho \int_0^{2\pi} d\theta \int_A^B (q_r\, r\, dz - q_z r\, dr) = 2\pi\, \rho\, \int_A^B \left(\frac{\partial\Psi}{\partial z} dz + \frac{\partial\Psi}{\partial r} dr\right)$

$$= 2\pi\rho \int_A^B d\Psi = 2\pi\rho(\Psi_B - \Psi_A).$$

8.12.1 Stoke's Stream Function in Spherical Polar Co-ordinates (r, θ ψ) : We consider the axi-symmetric motion in r, θ plane such that $q_\psi = 0$. The equation of continuity in spherical polar co-ordinates becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(r \sin\theta q_\theta) = 0$$

i.e. $$\frac{\partial}{\partial r}(r^2 \sin\theta q_r) = \frac{\partial}{\partial\theta}(-r \sin\theta q_\theta) \qquad (1)$$

This is condition of exactness for the different equation

$$r \sin\theta\, q_\theta\, dr - r^2 \sin\theta\, q_r\, d\theta = 0 \qquad (2)$$

Thus the expression on L.H.S. of (2) is equal to an exact differential function Ψ such that

$$r \sin\theta\, q_\theta\, dr - q_r\, r^2 \sin\theta\, d\theta = d\Psi = \frac{\partial\Psi}{\partial r} dr + \frac{\partial\Psi}{\partial\theta} d\theta$$

$$\Rightarrow \quad \frac{\partial\Psi}{\partial r} = q_\theta r \sin\theta, \frac{\partial\Psi}{\partial\theta} = -q_r\, r^2 \sin\theta.$$

Remark. In the above cases, the motion need not be irrotational i.e. velocity potential may not exist. In case of irrotational motion, it can easily be shown that the velocity potential ϕ and the Stoke's stream function Ψ do not satisfy C–R equations due to the fact that Ψ is not harmonic.

8.12.2 Stoke's Stream Function for a Uniform Stream: Let a uniform stream with velocity U be in the direction of z-axis such that $\mathbf{q} = U\,\mathbf{k}$. Then, from the relations

$$q_z = -\frac{\partial\phi}{\partial z} = -\frac{1}{r}\frac{\partial\Psi}{\partial r}, \quad q_r = -\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial z}$$

we get $$U = -\frac{1}{r}\frac{\partial\Psi}{\partial r}, 0 = \frac{1}{r}\frac{\partial\Psi}{\partial z}$$

$$\Rightarrow \quad \frac{\partial\Psi}{\partial r} = -Ur, \frac{\partial\Psi}{\partial z} = 0$$

$\Rightarrow \quad \Psi = -U\,\dfrac{r^2}{2}$, where the constant of integration is found to be zero.

In spherical polar co-ordinates we have

$$\Psi = -\frac{U}{2}(r \sin\theta)^2 = -\frac{U}{2} r^2 \sin^2\theta.$$

8.12.3. Stoke's Stream Function for a Simple Source at origin: In case of simple source

$$\mathbf{q} = f(r)\,\hat{r}$$

But we have already calculated that for a source of strength m at origin.

$$\mathbf{q} = \frac{m}{r^2}\hat{r}(r > 0) \text{ in spherical polar co-ordinates.}$$

i.e. $$(q_r, q_\theta) = \frac{m}{r^2}\hat{r} \qquad (1)$$

Also, we know that in spherical polar co-ordinates,

$$q_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta},\ q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \qquad (2)$$

From (1) & (2), we get

$$\frac{m}{r^2} = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, \frac{\partial\Psi}{\partial r} = 0$$

$$\Rightarrow \qquad \frac{\partial\Psi}{\partial\theta} = -m\sin\theta, \frac{\partial\Psi}{\partial r} = 0$$

$$\Rightarrow \qquad \Psi = m\cos\theta\,.$$

A constant may be added to this solution and this is usually done to make $\Psi = 0$ along the axis of symmetry $\theta = 0$. In such case,

$$\Psi = m(\cos\theta - 1)$$

For a sink of strength m at origin, the Stoke's stream function is

$$\Psi = m(1 - \cos\theta)$$

8.12.4 Stoke's Stream Function for a Doublet at origin: We assume that the flow is due to only a doublet at origin 0 of strength μ. Taking the axis $\theta = 0$ of the system of spherical co-ordinates to coincide with the axis of the doublet, we find that the velocity potential at $P(r, \theta, \psi)$ is

$$\phi = \frac{\mu\cos\theta}{r^2} \ (r > 0) \qquad (1)$$

$$\Rightarrow \qquad q_r = -\frac{\partial\phi}{\partial r} = \frac{2\mu\cos\theta}{r^3}, q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{r^3} \quad q_\psi = 0 \qquad (2)$$

But the relations between the velocity components and the Stoke's stream function Ψ are

$$q_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta},\ q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \qquad (3)$$

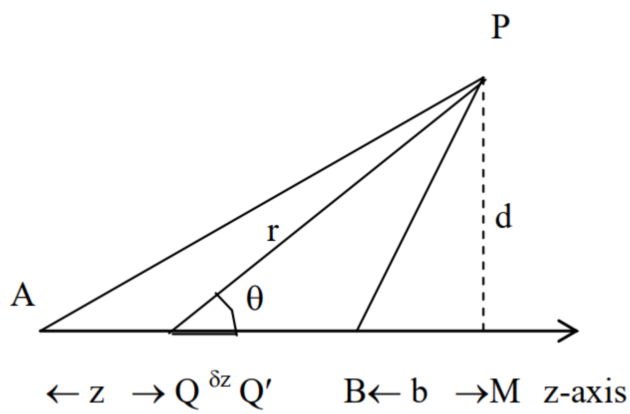
From (2) and (3), we get

$$\frac{\partial\Psi}{\partial\theta} = -\frac{2\mu\sin\theta\cos\theta}{r}, \frac{\partial\Psi}{\partial r} = \frac{\mu\sin^2\theta}{r^2}$$

Integrating, we get

$$\Psi = \frac{-\mu\sin^2\theta}{r}$$

8.12.5 Stoke's Stream Function due to a Uniform Line Source: Let a uniform line source of fluid extends along the streamline segment AB of length l. Consider an element QQ′ of length δz at a distance z (= AQ) from A. Thus, we have a simple source of strength m δz, where m is the constant source strength per unit length of the distribution along AB.



Let QP = r, $\angle PQB$ = Q, PM = d

The Stoke's stream function $\delta\Psi$ at P for the simple source of strength $m\delta z$ at Q is $m\delta z(\cos\theta - 1)$. Then, the value of the Stoke's stream function Ψ at P due to entire line source AB is given by

$$\Psi = m\int_0^l (\cos\theta - 1)dz = m\int_0^l \cos\theta dz - m\int_0^l dz$$

$$= m\int_0^l \frac{l+b-z}{\sqrt{d^2+(l+b-z)^2}}dz - ml$$

In ΔPQM, $\cos\theta = \frac{QM}{PQ} = \frac{QB+BM}{PQ}$

$$= \frac{l-z+b}{r} = \frac{l-z+b}{\sqrt{d^2+(l+b-z)^2}}$$

Putting $l + b - z = x \Rightarrow dz = -dx$

When $z = 0, x = l + b,$

when $z = l, x = b$

Therefore,

$$\Psi = m \int_{l+b}^{b} \frac{x(-dx)}{\sqrt{d^2 + x^2}} - ml$$

or $$\Psi = \frac{m}{2} \int_{b}^{l+b} (d^2 + x^2)^{-1/2} (2x)\, dx - ml$$

$$= \frac{m}{2} \left[\frac{\sqrt{d^2 + x^2}}{1/2} \right]_{b}^{l+b} - ml$$

$$= m\left[\sqrt{d^2 + (l+b)^2} - \sqrt{d^2 + b^2}\right] - ml$$

$$= m[AP - BP] - mAB$$

$$= m[AP - BP - AB]\ .$$

As p is the only variable point, the simpler form m (AP–BP) can be taken for evaluating velocity components at P. The stream surfaces are

$$\Psi = \text{constant i.e. } AP - BP = \text{constant.}$$

These are confocal hyperboloids of revolution about AB, with A and B as foci.

We have shown earlier that the equipotential were confocal ellipsoids of revolution about AB with the same foci. Also, it is well known result that two families of confocal intersect orthogonally.

8.12.6 Stoke's Stream Function for a Doublet in a Uniform Stream: Let a doublet of vector moment $\mu\mathbf{k}$ is situated at origin 0 in a uniform stream whose undisturbed velocity is $-U\mathbf{k}$.

In spherical polar co-ordinates (r, θ, ψ), the Stoke's stream functions for each separate distribution are

$$\Psi_1 = \frac{1}{2} Ur^2 \sin^2\theta \qquad \text{(for uniform stream, } \mathbf{q} = -U\mathbf{k}\text{)}$$

$$\Psi_2 = -\frac{\mu}{r} \sin^2\theta \qquad \text{(for doublet at origin)}$$

Hence the stream function for the combination is

$$\Psi (r, \theta) = \left(\frac{1}{2} Ur^2 - \mu / r\right) \sin^2\theta$$

The equation of the stream surfaces are $\Psi(r, \theta)$ = constant.

In particular, the stream surfaces for which $\Psi = 0$ are given by

$$\left(\frac{1}{2} Ur^2 - \mu / r\right) \sin^2\theta = 0$$

$\Rightarrow$ $\sin\theta = 0$ or $\frac{1}{2}Ur^2 - \frac{\mu}{r} = 0$

$\Rightarrow$ $\theta = 0, \pi$ i.e. the z-axis or $r = \left(\frac{2\mu}{U}\right)^{1/3}$, the surface of the sphere with centre 0 and radius $\left(\frac{2\mu}{U}\right)^{1/3}$

Example: ***A point source of strength Ua^2 is introduced at 0 in a uniform stream whose undisturbed velocity is Ui. Show that over the surface of revolution $r \sin \theta = 2a \cos \frac{\theta}{2}$ there is no flow, the system of spherical polar co-ordinates being used with $\theta = 0$ taken as x-axis. If a rigid surface of revolution having above equation is introduced into the flow of velocity Ui after removal of the point source, explain why the two models are hydrodynamically equivalent for corresponding points in the field of flow?***

Solution. Stoke's stream function for uniform stream is $-\frac{1}{2}Ur^2 \sin^2\theta$ and for a point source at 0, it is $Ua^2(\cos\theta - 1)$. Thus, the total stream function for the combination is

$$\Psi(r, \theta) = -\frac{1}{2}Ur^2 \sin^2\theta + Ua^2(\cos\theta - 1)$$

$$\bar{q} = \frac{m}{a^2}\hat{i} \Rightarrow U\hat{i} = \frac{m}{a^2}\hat{i}$$

$$\Rightarrow m = Ua^2$$

$$= U\left[a^2(\cos\theta - 1) - \frac{r^2}{2}\sin^2\theta\right]$$

The stream surfaces are given by

$$\Psi = \text{constant}$$

i.e. $$U\left[a^2(\cos\theta - 1) - \frac{r^2}{2}\sin^2\theta\right] = \text{constant}$$

i.e. $$a^2(1 - \cos\theta) + \frac{r^2}{2}\sin^2\theta = \text{constant}$$

i.e. $$a^2\, 2\sin^2\frac{\theta}{2} + \frac{r^2}{2}\sin^2\theta = \text{constant}$$

i.e. $$2a^2\left(1 - \cos^2\frac{\theta}{2}\right) + \frac{r^2}{2}\sin^2\theta = \text{constant}$$

i.e. $$-2a\cos^2\frac{\theta}{2}+\frac{r^2}{2}\sin^2\theta = \text{constant.}$$

In particular, taking the constant on R.H.S. to be zero, the corresponding stream surface is

$$r^2 \sin^2\theta = 4a^2 \cos^2\frac{\theta}{2}$$

i.e. $$r \sin\theta = 2a \cos\frac{\theta}{2} \qquad (1)$$

which is the required surface of revolution.

We note that equation (1) is satisfied by

(i) $\cos\frac{\theta}{2} = 0$ i.e. $\theta = \pi$

(ii) $r = a \operatorname{cosec}\frac{\theta}{2}$

As no flow takes place over the surface of revolution $r = a \operatorname{cosec}\frac{\theta}{2}$, we may introduce a rigid boundary over the surface, excluding the fluid and source within its interior. Then the hydrodynamical image of the external flow $U\hat{i}$ in the surface is the point source Ua^2 at 0.

Hence the two models are hydrodynamically equivalent for corresponding points in the field of flow.

Also, we have

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r},\ q_\psi = 0$$

$$\Rightarrow \qquad q_r = U\left(\cos\theta+\frac{a^2}{r^2}\right)$$

$$q_\theta = -U\sin\theta,\ q_\psi = 0$$

$\therefore$ Fluid speed q at point P is

$$q = \sqrt{q_r^2+q_\theta^2+q_\psi^2}$$

$$= U\sqrt{1+2\left(\frac{a}{r}\right)^2\cos\theta+\left(\frac{a}{r}\right)^4}$$

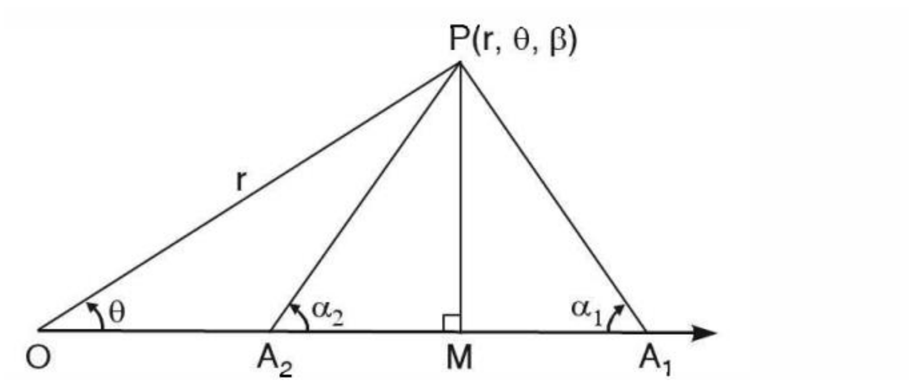
On the surface of revolution $r = a \operatorname{cosec}\frac{\theta}{2}$, the fluid velocity is tangential to the surface and

$$q = \frac{U}{2}\sqrt{5 + 2\cos\theta - 3\cos^2\theta}$$

Stagnation point on the surface is $\cos\theta = -1 \Rightarrow \theta = \pi$.

Example: ***Doublets of strengths μ_1 and μ_2 are situated at points A_1, A_2 whose Cartesian coordinates are $(0, 0, c_1)$, $(0, 0, c_2)$, their axes being directed towards and away from the origin respectively. Find the condition that there is no transport of fluid over the surface of the sphere $x^2 + y^2 + z^2 = c_1c_2$.***

Solution:



Let the spherical polar coordinates of P be (r, θ, β) with initial line OA_2A_1. Let the axis of the doublet at A_1 *and* A_2 make angles α_1 *and* α_2 with A_1P, A_2P . Then the velocity potential ϕ at P is

$$\phi = \frac{\mu_2 \cos\alpha_2}{A_2P^2} + \frac{\mu_1 \cos\alpha_1}{A_1P^2} \qquad (1)$$

Given $OA_1 = c_1 \; ad \; OA_2 = c_2$.Then from figure,

$$A_1P = (r^2 - 2rc_1\cos\theta + c_1)^2 \text{ and } A_2P = (r^2 - 2rc_2\cos\theta + c_2^2)^2 \qquad (2)$$

$$\cos\alpha_1 = \frac{MA_1}{A_1P} = \frac{OA_1 - OM}{A_1P} = (c_1 - r\cos\theta)\,/\,A_1P \qquad (3)$$

$$\text{Similarly, } \cos\alpha_2 = \left(\frac{r\cos\theta - c_2}{A_2P}\right) \qquad (4)$$

Where M is the foot of the perpendicular drawn from P on OA_1

Using (3) and (4), (1) becomes

$$\phi = \frac{\mu_2(r\cos\theta - c_2)}{A_2P^3} + \frac{\mu_1(c_1 - r\cos\theta)}{A_1P^3} \qquad (5)$$

From (2) and (5)

$\phi = \phi(r,\theta) = \mu_2\,(r\cos\theta - c_2)\,(r^2 - 2rc_2\cos\theta + c_2^2)^{-\frac{3}{2}} + \mu_1\,(c_1 - r\cos\theta)\,(r^2 - 2rc_1\cos\theta + c_1^2)^{-\frac{3}{2}}$ (6)

When there is no transport of fluid over the sphere $x^2 + y^2 + z^2 = (\sqrt{c_1c_2})^2$, we have

$\frac{\partial\phi}{\partial r} = 0\ \ when\, r = \sqrt{c_1c_2}$ (7)

Using (7) and (6), we have

$\mu_2\, c_2^{-\frac{3}{2}} = \mu_1 c_1^{-\frac{3}{2}}$ or $\frac{\mu_2}{\mu_1} = \left(\frac{c_2}{c_1}\right)^{\frac{3}{2}}$ which is the required condition.

Example: ***Discuss the motion for which Stoke's stream function is given by*** $\boldsymbol{\Psi = \frac{1}{2}V\,[a^4r^{-2}\cos\theta - r^2]\sin^2\theta}$***, where r is the distance from a fixed point and*** $\boldsymbol{\theta}$ ***is the angle, this distance makes with a fixed direction.***

Solution: Given $\Psi = \frac{1}{2}V\,[a^4r^{-2}\cos\theta - r^2]\sin^2\theta$ (1)

Evidently, Ψ is a sum of two terms. Here liquid flows with velocity V parallel to the x-axis in presence of a fixed solid of revolution

$$\Psi = \frac{1}{2}V\frac{a^4}{r^2}\cos\theta\sin^2\theta$$

is the stream function for a solid which is moving with velocity parallel to the negative direction of x-axis. In this case boundary condition is

$$\Psi = \frac{1}{2}Vr^2\sin^2\theta + const.$$

On boundary

$$\frac{1}{2}Vr^2\sin^2\theta + const. = \frac{\frac{1}{2}Va^4}{r^2}\cos\theta\sin^2\theta$$

This implies $const. = 0, r^2\sin^2\theta = \frac{a^4}{r^2}\cos\theta\sin^2\theta$

Or $r^4 = a^4\cos\theta$

It follows that the given stream function gives the motion of a liquid flowing past a solid $r^4 = a^4\cos\theta$, moving with velocity V along x-axis.

8.13 Check Your Progress:

i) Prove that the velocity potential at a point P due to a uniform finite line source AB of strength m per unit length is of the form $\phi = m \log f$, where $f = \frac{r_2+x_2}{r_1+x_1} = \frac{r_1-x_1}{r_2-x_2} = \frac{a-l}{a+l}$, where $AB = 2l, PA = r_1, PB = r_2, NA = x_1, NB = x_2$, N being the foot of perpendicular from P on line AB, and 2a the length of major axis of the spheroid through P having A and B as foci.

ii) If AB be a uniform line source, and A, B equal sinks of such strength that there is no total gain or loss of fluid, show that the stream function (Stoke's stream function) at any point is

$\Psi = C[(r_1 - r_2)^2 - c^2]\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$, where c is the length of AB and r_1, r_2 are the distances of the points considered from A and B.

iii) A and B are a simple source and sink of strengths m and m' respectively in an infinite liquid. Show that the equation of streamlines is $m \cos\theta - m' \cos\theta' = constant$, where θ and θ' are the angles which AP, BP make with AB. P being any point. Prove also that if $m > m'$, the cone defined by the equation $\cos\theta = 1 - \left(\frac{2m'}{m}\right)$ divides the streamlines issuing from A into two sets, one extending to infinity and the other terminating at B.

iv) Find the Stoke's stream function where fluid motion is due to a source of strength (flux $4\pi m$) at a fixed point A and a translation of the fluid of velocity U. Explain how this solution can be used to deduce the motion of fluid past a blunt nosed cylindrical body whose diameter is ultimately 4a, where $a^2 = m/U$.

v) A solid of revolution is moving along its axis in an infinite liquid; show that the kinetic energy of the liquid is $-\frac{1}{2}\pi\rho\int \frac{\Psi}{\omega}\frac{\partial\Psi}{\partial n}\,ds$, where Ψ is the Stoke's function, ω the distance of a point from the axis and the integral is taken round a meridian curve of the solid. Hence obtain the kinetic energy of infinite liquid due to the motion of a sphere through it with velocity V.

8.14 Summary: In this chapter, we have defined Lagrange's stream function and its physical significance, obtained the velocity potential function for three-dimensional source, sink and doublet and discussed the image of source and doublets in plane and spherical surfaces. In this chapter we have also defined Stoke's Stream function and obtained the values of stream function for source and doublets in uniform flow of fluid.

8.15 Keywords: velocity potential, three-dimensional, source, doublets, Lagrange's stream function, images, Stoke's stream function, axi-symmetric flow.

8.16 Self-Assessment Test:

SA1: Find value of Stoke' stream function in case of a simple source on the axis of x and a uniform line source along the axis.

SA2: What is Stoke's stream function? Prove that 2π times the difference of its values at two points in the same meridian plane is equal to the flow across the angular surface obtained by the revolution round the axis of a curve joining the two points.

SA3: Show that the image with regard to a sphere of a doublet whose axis passes through the centre is a doublet at the inverse point.

SA4: Verify that $\Psi = (Ar^{-2}\cos\theta + Br^2)\sin^2\theta$, is a possible form of Stoke's stream function, find the corresponding velocity potential.

SA5: Show that a uniform stream of velocity U can be obtained as the limit $a \to \infty$ of the field due to a source of strength $2\pi\, a^2 U$ at (-a, 0, 0) and a sink of strength $-2\pi\, a^2 U$ at (a,0,0).

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CHAPTER-9

CONFORMAL TRANSFORMATION

9.0 Learning Objectives: After reading this chapter, you should be able to understand the transformation from one-plane to another plane and particularly conformal transformation, the images using complex potential theory, know the application to fluid mechanics, derive Blasius theorem, circle theorem, understand Joukovski transformation and aerofoils.

9.1 Transformation:

The equations $\xi = \xi(x, y),\ \eta = \eta(x.y)$ defines a map or transformation from xy-plane to $\xi\eta$ −plane. By means of these equations a domain or curve of xy-plane is mapped on the corresponding curve of $\xi\eta - plane$.

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \xi}{\partial x} . \frac{\partial \eta}{\partial y} - \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}$$

The determinant J is called Jacobian of the transformation. Let $t = f(z) = \xi + i\,\eta$ be analytic function of z so that Cauchy Riemann equations

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}, \qquad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}$$

are satisfied.

Then $$J = \frac{\partial \xi}{\partial x} . \frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial x}\left(-\frac{\partial \eta}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 = \left|\frac{\partial}{\partial x}(\xi + i\eta)^2\right| = \left|\frac{\partial t}{\partial x}\right|^2 = \left|\frac{dt}{dx}\right|^2$$

$$\because \frac{dt}{dx} = \frac{\partial t}{\partial x}$$

Or $$J = \left|\frac{dt}{dx}\right|^2 = |f'(z)|^2$$

Or $$|f'(z)| = \left|\frac{dt}{dz}\right| = \sqrt{J} = \text{h (say)}$$

where $$h^2 = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 = \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2$$

9.2 Conformal Transformation:

Suppose a bi-uniform mapping of a region of the z-plane on a region of the t-plane is connected by the relation

$$t = f(z) \tag{1}$$

where f(z) be a single-valued differentiable within a closed contour C in the xy-plane(z-plane) and $t = \xi + i\eta$ be another complex variable in $\xi\eta$ −plane(t-plane). By relation corresponding to each point in the z-plane within or on C, there will be a point *t* in the t-plane and points on C or within C will lie on or within a certain contour C’ in t-plane. The necessary condition for existence of such a

mapping of z-plane into t-plane is that $f'(z)$ should never vanish at any point on or within C, or in other words, dt/dz must exist independent of the direction of δz.
Let $z, z_1\ and\ z_2$ be represented by the points A, B and C respectively in the z-plane and let the corresponding values t, t_1 and t_2 be represented by point P, Q and R in t-plane.
Then

$$\frac{t_1-t}{z_1-z}=\frac{f(z_1)-f(z)}{z_1-z}; \qquad \frac{t_2-t}{z_2-z}=\frac{f(z_2)-f(z)}{z_2-z}.$$

Provided that AB and AC are small enough, we have

$$\frac{t_1-t}{z_1-z}=f'(z); \quad \frac{t_2-t}{z_2-z}=f'(z)$$

and consequently $\quad \frac{t_1-t}{z_1-z}=\frac{t_2-t}{z_2-z}=f'(z)=\frac{dt}{dz}$

It follows, by taking arguments and modulus that

$$\frac{AC}{AB}=\frac{PR}{PQ}\ and\ \theta=\psi.$$

Therefore, the triangles ABC and PQR are directly similar.

Also $\quad \left|\frac{t_1-t}{z_1-z}\right|=\left|\frac{f'(z)}{1}\right|=\left|\frac{\frac{dt}{dz}}{1}\right|$, it follows that the linear combinations are in the ratio of 1: $\left|\frac{dt}{dz}\right|$, and the ratio of the corresponding small areas, i.e.

$$\frac{\Delta PQR}{\Delta ABC}=|f'(z)|^2=f'(z)\overline{f'(z)}=\frac{dt}{dz}\frac{\overline{dt}}{dz} \text{ are in the ratio } 1:\frac{dt}{dz}.$$

Thus the mapping given by (1) is such that an infinitesimal triangle in one plane maps into a directly similar infinitesimal triangle in the other plane, preserving the angles and the similarity of the corresponding infinitesimal triangles.
Since small elements of area are unaltered in shape, the transformation is said to be Conformal.
The factor $\left|\frac{dt}{dz}\right|$ is often referred to as the linear combination.
By a proper choice of transformation, motion with a complicated boundary can be deduced from that with a simpler boundary. An extensive use is made of several sets of transformations and applied successfully to potential flow in two-dimensions. Thus a problem which stands unsolved in one physical configuration (say z-plane) may be solved into another configuration (say t-plane) by some suitable transformation. The problem thus may be regarded not as that of finding a direct solution, but of finding a proper transformation into a configuration which admits of an immediate solution. Though not always applicable, it is the most reliable method to derive exact solution.

9.3 Applications to Fluid Mechanics:

Conformal Invariance of hydrodynamical singularities

An essential feature of conformal mapping is that the vanishing of the Laplacian ($\nabla^2=0$) remains invariant under transformation. Thus a harmonic function remains harmonic after conformal transformation to a new coordinate plane. This may be ascertained for transformation $t=f(z)$. Thus $\phi(x,y)$ which in terms of ξ,η may be written as $\Phi(\xi,\eta)$, then

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}=\left|\frac{dz}{dt}\right|^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

So that if dz/dt is not infinite then

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0;\Rightarrow\frac{\partial^2\Phi}{\partial \xi^2}+\frac{\partial^2\Phi}{\partial \eta^2}=0$$

Thus, if $\phi(x,y)$ is a general potential function in the z-plane, then $\Phi(\xi,\eta)$ is necessarily a general potential function in the t-plane. In other words, (velocity) potential functions transform into (velocity) potential functions. Furthermore, any curve or boundary i.e. stream line in the z-plane along which ϕ is constant is mapped into a new curve or boundary i.e. stream line in the t-plane along which Φ is constant.

We now discuss the retention of some important hydrodynamic singularities on conformal transformation:

(I) **Source.** Let there be a source of strength m at the point $P(z_0)$ in the z-plane and enclose it by a small curve C. Let $P'(t_0)$ be the corresponding point in t-plane and let C' be the corresponding curve in the t-plane, then C' must enclose P'; because the domain D in the z-plane is mapped in one-to-one function onto D' in the t-plane under analytic function $t=f(z)$ transformation.

The flow across C by the definition of a source is $2\pi m\rho$. The flow across C is also given in terms of the stream function by $-\rho\int d\psi,$ and since each point on C' corresponds to one and only one point on C, we must have, $-\rho\int_C d\psi=-\rho\int_{C'}d\psi$ taken in the same sense. This means that the flow across C' is $2\pi m\rho$ and this will be the same for any small closed curve surrounding $P'(t_0)$. Thus there must be a source m at $P'(t_0)$.Hence, we can say that in a conformal transformation a source is transformed into an equal source. If C' encircles P' only once, the source will be of strength m. If C' encloses P' n times when C encircles P once, the strength of the source at P will be (m/n).

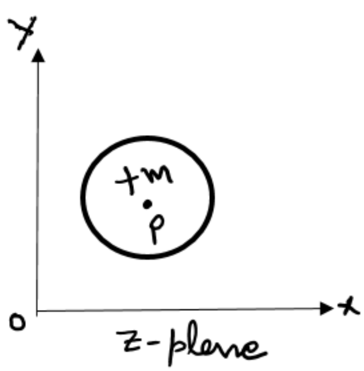
Example: Consider the motion $t=z^{\frac{1}{n}}$ (1)

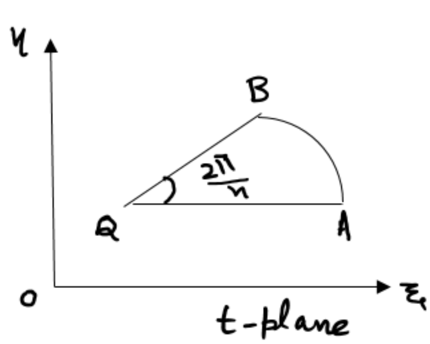
Consider a source of strength +m at P. Let $(r.\theta)$ be the polar coordinates of the branch point P in z-plane and (R,ϕ) be the coordinates of the corresponding point Q in t-plane.

From (1), we have

$$Re^{i\phi}=\left(r\,e^{i\theta}\right)^{\frac{1}{n}}=r^{\frac{1}{n}}\,e^{\left(\frac{\theta}{n}\right)i}$$

$$\Rightarrow\qquad R=r^{\frac{1}{n}}\quad and\ \ \phi=\frac{\theta}{n}.$$



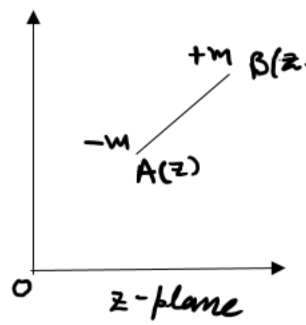


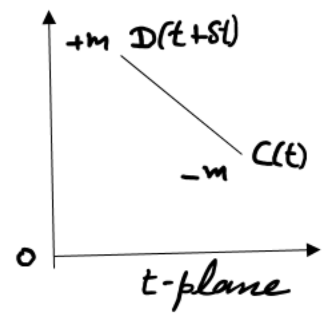
It follows that a complete circle round P in the z-plane corresponds to an arc AB($= 2\pi/n$) in the t-plane. Since the flow across the circle is equal to that across the arc AB.

$$2\pi m = (2\pi/n)m' \Rightarrow \mathrm{m}m' = nm$$

Therefore, corresponding to a source of strength +m at P there will be a source of strength +m'(=nm) at Q.

(II) **Doublet:** Let the doublet of strength μ at the point A(z) in the z-plane be obtained by the combination of source –m at A and +m at B, so that (Lim m.AB)=μ; as $\delta z \to 0, m \to \infty$.





On transforming, we get a source m at D and a sink –m at C in the t-plane. If AB is small enough, magnification gives

$\frac{CD}{AB} = \left|\frac{dt}{dz}\right|$ so that m.CD=m.AB dt/dz|.

Proceeding to the limit, the result is $\mu' = \mu\ \left|\frac{dt}{dz}\right|$; which is the strength of the doublet in t-plane.

If the doublet in the z-plane is inclined at α with the real axis, the doublet in t-plane will be inclined with the real axis at an angle β, where

$$\beta = arg\delta t = \arg\{\left(\frac{dt}{dz}\right)\delta z\} \qquad | \because \frac{dt}{dz}\ \delta z = \delta t$$

$$= \arg\left(\frac{dt}{dz}\right) + \arg\delta z = \arg\left(\frac{dt}{dz}\right) + \alpha$$

Thus, a doublet of strength μ and inclined at α with real axis in the z-plane transforms conformally into a doublet of strength $\mu'\left(= \mu\ \left|\frac{dt}{dz}\right|\right)$ and inclination $\beta(= \alpha + \arg(dt/dz))$ with real axis in the t-plane.

(III) Vortex filament: Let there be a vortex at $P(z_0)$ of strength K in the z-plane and let $P'(t_0)$ be the corresponding point in the t-plane; the connecting mapping being $t = f(z)$. Let C be any small closed curve surrounding $P(z_0)$ and C' the corresponding small closed curve surrounding $P'(t_0)$.

The circulation round C, by definition, is $-\int_C d\phi = -[\phi]_C = K$.

Since each point on C' corresponds to one and only one point on C, we must have $-\int_C d\phi = -\int_{C'} d\phi$ and thus, circulation around any small closed curve C' surrounding $P'(t_0)$ is K. Therefore, there must be a vortex of strength K at $P'(t_0)$. Hence a vortex is transformed into an equal vortex at the corresponding point.

These vortices however do not necessarily continue to move so as to occupy corresponding points; if however we know the motion of one, the motion of the other is usually deduced by a device due to Routh'.

(iv) Kinetic energy: Let $P(z_0)$ be a point inside a small ΔABC in the z-plane. Let $P'(t_0)$ be the corresponding point to P inside the corresponding small $\Delta A'B'C'$ in the t-plane. The Kinetic energy of the fluid in the two triangles is respectively

$$\left.\begin{aligned} T_1 = \tfrac{1}{2}\rho q^2\, \Delta ABC = \tfrac{1}{2}\,\rho\,\left|\tfrac{dw}{dz}\right|^2 \Delta ABC \\ T_2 = \tfrac{1}{2}\rho q'^2\, \Delta A'B'C' = \tfrac{1}{2}\,\rho\,\left|\tfrac{dw}{dt}\right|^2 \Delta A'B'C' \end{aligned}\right\} \qquad (1)$$

where w is the complex potential for the fluid motion. Now

$$\Delta' = \left|\frac{dt}{dz}\right|^2 \Delta \ \text{ and } \left|\frac{dw}{dz}\right| = \left|\frac{dw}{dt}\right|\left|\frac{dt}{dz}\right|,$$

By conformality, we have from (1) that $T_1 = T_2$.

Notes: (1) The complex potential $w = \phi + i\,\psi$, of a flow is invariant under a conformal mapping; because ϕ and ψ are both harmonic and hence conformally invariant.

The complex potential, $w = \phi + i\psi$, performs a conformal mapping onto the w-plane, where $\psi - lines$ and $\phi - lines$ are respectively horizontal and vertical lines.

(2) The retention of the character of the hydrodynamic singularities during transformations is of considerable importance in solving certain problems. In some transformations mathematical singularities appear from the transformation. Physically, these correspond to stagnation points and are not termed hydrodynamic singularities.

Example: ***Use the method of transformation to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \pi/3$, the solution is***

$$\boldsymbol{\phi + i\psi = -m\, log\,\{(z^3 - z_0^3)(z^3 - z_0'^3)\}},$$

where $z_0 = x_0 + iy_0$ and $z_0' = x_0 - iy_0$.

Solution: Changing the motion from z-plane to t-plane by the transformation $t = z^3$,where $t = Re^{i\phi}$ and $z = re^{i\theta}$.

or $$Re^{i\phi} = r^3\, e^{i\theta} \Rightarrow R = r^3 \ and\ \phi = 3\,\theta.$$

Thus, the boundaries $\theta = 0$ and $\theta = \pi/3$ in z-plane transform to $\phi = 0\ and\ \phi = \pi$ in t-plane.

A source of strength +m is placed at the point z_0 in z-plane bounded by the line $\theta = 0\ and\ \theta = \pi/3$ corresponds by transformation to the points $t_0 = z_0^3$ bounded by the real axis $\phi = 0\ and\ \phi = \pi$ in t-plane. The image system consists of (i) a source of strength +m at $t_0 = z_0^3$ (ii) a source of strength +m at $t_0' = z_0'^3$.

Thus the complex potential becomes

$$w = -mlog(t - z_0^3) - m\log(t - z_0'^3)$$

or $$w = -m\log(z^3 - z_0^3) - mlog\,(z^3 - {z_0'}^3)$$

or $$\phi + i\psi = -m\log\{(z^3 - z_0^3)(z^3 - z_0'^3)$$

Example: ***An area A is bounded by that port of the x-axis for which x>a and by that branch of $x^2 - y^2 = a^2$ which is in the positive quadrant. There is a two-dimensional unit source at (a,0) which sends out liquid uniformly in all direction. Show by means of the transformation $w = log(z^2 - a^2)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbolas. Determine the stream lines corresponding to $\psi = 0, \pi/4$, and $\pi/2$. If ρ_1 and ρ_2 are distance of a point P within the fluid from the point $(\pm a, 0)$. Show that the velocity of the fluid at P is measured by 2OP/ $\rho_1\rho_2$, O being the origin.***

Solution: The complex potential is given by $w = \phi + i\psi = \log(z^2 - a^2)$

or $\phi + i\psi = \log\{(x+iy)^2 - a^2\} = \log\{(x^2 - y^2 - a^2) + 2ixy\}$

or $\psi = \tan^{-1}\frac{2xy}{x^2-y^2-a^2}$ $or \tan\psi = \frac{2xy}{x^2-y^2-a^2}$

The stream lines are given by $\psi = constant$.

i.e., $\frac{2xy}{x^2-y^2-a^2} = constant = k(say)$

if k=0, the stream lines will be x=0 and y=0.

If k is infinite, the stream lines will be $x^2 - y^2 = a^2$.

Thus the liquid flows in the area A bounded by $x = 0, y = 0$ and $x^2 - y^2 = a^2$, $i.e.,$ the portion of a rectangular hyperbola in the positive quadrant.

Again the complex potential w can be written as

$$w = \log(z - a) + \log(z + a),$$

Which shows that the image of a unit source at the point(a,0) consists of a unit source at the point $(-a, 0)$ with regard to y-axis.

The velocity of the fluid at any point is $q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z-a} + \frac{1}{z+a}\right| = \frac{|2z|}{|z-a||z+a|} = \frac{2OP}{\rho_1\rho_2}$

The stream lines corresponding to $\psi = 0$ and $\psi = \frac{\pi}{2}$ are x=0, y=0 and $x^2 - y^2 = a^2$.

Also, the steam lines corresponding to $\psi = \pi/4$ are

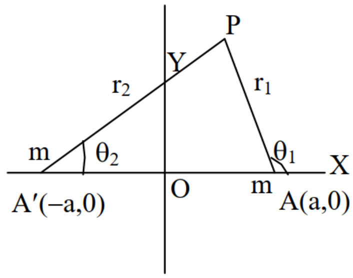
$$\frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{4} = 1$$
$$x^2 - y^2 - a^2 = 2xy.$$

9.4 Images in Two Dimensions: If a surface S can be drawn in a moving fluid in such a way that there is no transport of fluid across that surface then any system of sources, sinks and doublets on one side of the surface is said to be the image system of sources, sinks and doublets on the other side with regard to the surface S. The fluid flows tangentially to the surface.

As there is no flow across the surface s, the surface S is necessarily a streamline. If we introduce a rigid boundary in place of the surface then the fluid motion will remain unaltered and the fluid velocity at any point, normal to the rigid boundary must vanish.

To discuss the images in two dimensions, we use complex potential.

9.4.1. Image of a Line Source in a Plane. Without loss of generality we take the rigid impermeable plane to be x = 0 and perpendicular to the plane of flow (xy-plane). Thus we are to determine the image of a line source of strength m per unit length at A(a, 0) w.r.t. the streamline OY. Let us place a line source per unit length at A′(–a, 0).



The complex potential of strength at a point P due to the system of line sources, is given by

$$\boldsymbol{w = -m\log z - a) - m\log(z + a) = -m\log\left(r_1 e^{i\theta_1}\right) - m\log\left(r_2 e^{i\theta_2}\right)}$$

$$\boldsymbol{= -m\log\{r_1 r_2 e^{i(\theta_1+\theta_2)}\} = -m\log r_1 r_2 - i\,m(\theta_1 + \theta_2)}$$

$$\boldsymbol{\phi + i\psi = -m\log r_1 r_2 - i\,m(\theta_1 + \theta_2)}$$

$$\boldsymbol{\psi = -m\,(\theta_1 + \theta_2)}$$

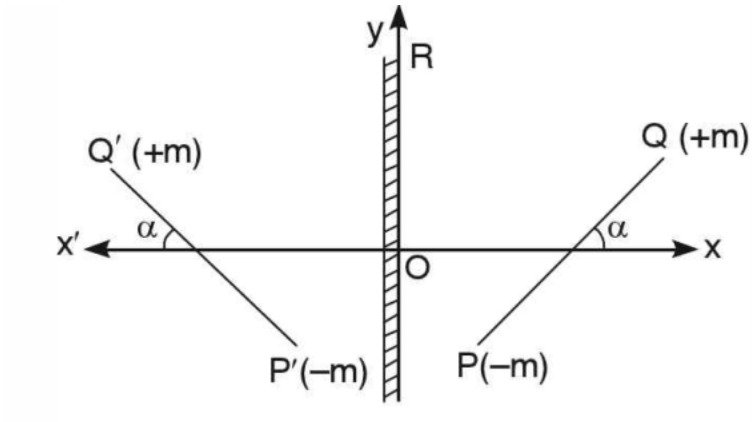
If P lies on y-axis, then PA = PB $\Rightarrow \angle PAB = \angle PBA$

$$\boldsymbol{i.e.\ \pi - \theta_1 = \theta_2 \Rightarrow \theta_1 + \theta_2 = \pi}$$

Thus $\psi = -m\pi = \text{constant}$

which shows that y-axis is a streamline. Hence the image of a line source of strength m per unit length at A(a, 0) is a source of strength m per unit length at A′(–a, 0). In other words, image of a line source w.r.t. a plane (a stream line) is a line source of equal strength situated on opposite side of the plane (stream line) at an equal distance.

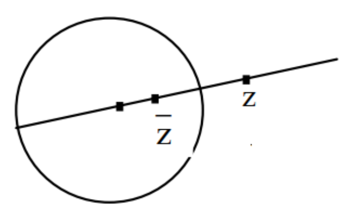
9.4.2 Image of a Line Doublet in a Plane. Let us consider the rigid impermeable plane to be x = 0 and perpendicular to the plane of flow (xy-plane). Thus, we are to determine the image of a line doublet w.r.t.



the stream line OY. Let there be line sources at the points P and Q, taken very close together, of strengths −m and m per unit length. Their respective images in OY are −m at P′, m at Q′, where P′, Q′ are the reflections of P, Q in OY. The line PQ makes angle α with OX. Thus P'Q' makes angle $(\pi-\alpha)$ with OX. In the limiting case, as $m\to\infty$, $PQ\to 0$, we have equal line doublets at P and P′ with their axes inclined at α, $(\pi-\alpha)$ to OX. Hence, either of the line doublet is the hydrodynamical image of the other in the infinite rigid impermeable plane (stream line) x = 0

9.5 Milne-Thomson Circle Theorem: Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities within the circle $|z| = a$. Then on introducing the solid circular cylinder $|z| = a$, with impermeable boundary, into the flow, the new complex potential for the fluid outside the cylinder is given by

$$w = f(z) + \bar{f}\,(a^2/z),\ |z| \geq a$$



$$z\bar{z} = a^2$$

Proof. Let C be he **cross-section** of the cylinder with equation $|z| = 1$.

Therefore, on the circle C, $|z| = a \Rightarrow z\,\bar{z} = a^2 \Rightarrow \bar{z} = a^2/z$

where $\bar{z}$ is the image of the point z w.r.t. the circle. If z is outside the circle, then $\bar{z} = a^2/z$ is inside the circle. Further, all the singularities of f(z) lie outside C and the singularities of $f(a^2/z)$ and therefore those of $\bar{f}\,(a^2/z)$ lie inside C. Therefore $\bar{f}(a^2/z)$ introduces no singularity outside the cylinder. Thus, the functions f(z) and f(z) + $\bar{f}$ (a^2/z) both have the same singularities outside C. Therefore the conditions satisfied by f(z) in the absence of the cylinder are satisfied by f(z) + $\bar{f}$ (a^2/z) in the presence of the cylinder. Further, the complex potential, after insertion of the cylinder |z| = a, is

$$w = f(z) + \bar{f}\,(a^2/z) = f(z) + \bar{f}\,(\bar{z})$$

$$= f(z) + \overline{f(z)}$$

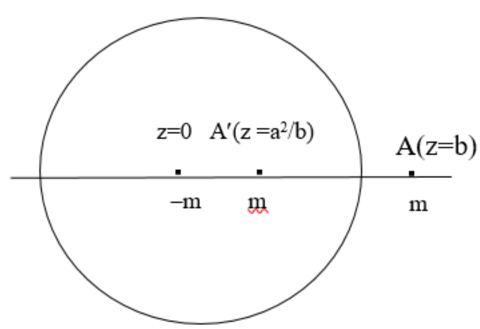
= a purely real quantity

But we know that $w = \phi + i\psi$. It follows that $\psi = 0$

This proves that the circular cylinder |z| = a is a streamline i.e. C is a streamline. Therefore, the new complex potential justifies the fluid motion and hence the circle theorem.

Remark: *The Milne-Thomson circle theorem provides a conventional method for finding the image system of a given two dimensional systems which lies outside a circular boundary. For, if w=f(z) represents the given system in the presence of the circular boundary |z|=a, then* $\boldsymbol{w = f(a^2/z)}$ *represents the image system.*

(i) Image of source with respect to circle of radius a: Consider a source of strength +m at z=b so that the complex potential due to this source is



Let a circular cylinder |z|=a (where a<d) be inserted, then by circle theorem, the complex potential is given by

$$\phi + i\psi = w = -m \log (z-b) - m \log((a^2/z)-b)$$

$$= -m \log (z-b) - m \log\left(-\frac{b}{z}\right)\left(z - \frac{a^2}{b}\right)$$

$$= -m \log (z-b) - m \log (z-a^2/b) + m \log z + \text{constant} \qquad (1)$$

Ignoring the constant term, we observe from (1) that the complex potential represents a line source +m at z = b, another line source +m at the inverse point z = a^2/b and an equal line sink –m at the centre of the circle. Thus the image of a line source of strength m per unit length at z = b in a cylinder is an equal line source at the inverse point $z = a^2/b$ together with an equal line sink $-m$ at the centre $z = 0$ of the circle.

(ii) Image of a Line Doublet relative to the Circle. Let there be a line doublet of strength μ per unit length at the point $z = d$, its axis being inclined at an angle α with the x-axis. The line doublet is assumed to be perpendicular to the plane of flow i.e. parallel to the axis of cylinder. The complex potential in the absence of the cylinder, is

$$\frac{\mu e^{ie}}{z-d}$$

When the cylinder $|z| = a$ is inserted, the complex potential, by circle theorem, becomes

$$w = \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu e^{-i\alpha}}{(a^2/z)-d}$$

$$= \frac{\mu e^{i\alpha}}{z-d} - \frac{\mu e^{-i\alpha} z}{d\left(z - a^2/d\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu z\, e^{i(\pi-\alpha)}}{d\left(z - a^2/d\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu e^{i(\pi-\alpha)}}{d} + \frac{\mu a^2}{d^2}\frac{e^{i(\pi-\alpha)}}{z - a^2/d} \qquad (1)$$

If the constant term (second term) in (1) is neglected, then the complex potential in (1) is due to a line doublet of strength μ per unit length at $z = d$, inclined at an angle α with x-axis and another line doublet of strength $\frac{\mu a^2}{d^2}$ per unit length at the inverse point $z = a^2/d$ inclined at an angle $\pi-\alpha$ with x-axis.

Thus the image of a line doublet of strength μ per unit length $z = d$ inclined at angle α with x-axis is a line doublet of strength $\frac{\mu a^2}{d^2}$ per unit length at the inverse point a^2/d which is inclined at an angle $\pi-\alpha$ with x-axis.

9.6 Drag Force and Lift Force of Immersed Bodies:

A body of arbitrary shape and orientation when immersed in a fluid stream experiences forces and moments from the flow. Choose axis $\boldsymbol{e_1}$ parallel to the free stream and positive down-stream. The force on the body in direction $\boldsymbol{e_1}$ is called Drag and moment about $\boldsymbol{e_1}$ the rolling moment. The drag is essentially a flow loss and must be overcome if the body is to move against the stream.

A force perpendicular to $\boldsymbol{e_1}, \boldsymbol{e.g.}$, such as the one bearing the weight of the body is called Lift (direction $\boldsymbol{e_2}$. The moment about the lift axis $\boldsymbol{e_2}$ is called yaw. The third component $\boldsymbol{e_3}(= \boldsymbol{e_2} \times \boldsymbol{e_1})$ along which there is neither a loss nor a gain is termed the side force and the moment about $\boldsymbol{e_3}$ is called a pitching moment.

For bodies with symmetry about lift-drag axis, the problem reduces to a 2-D case : two forces (lift and drag) and the pitching moment.

9.7 Blasius Theorem

In a steady two-dimensional irrotational flow given by the complex potential w = f(z), if the pressure forces on the fixed cylindrical surface C are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then neglecting the external forces,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

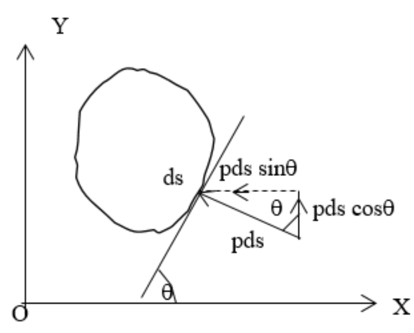
$$M = \text{Real part of } \left[-\frac{\rho}{2}\int_C z\left(\frac{dw}{dz}\right)^2 dz\right]$$

where ρ is the density of the fluid

Proof. Let ds be an element of arc at a point P(x, y) and the tangent at p makes an angle θ with the x-axis. The pressure at P(x, y) is pds, p is the pressure per unit length. pds acts along the inward normal to the cylindrical surface and its components along the co-ordinate axes are

$dX = pds \cos(\theta + \boldsymbol{\pi/2}) = -pds \sin\theta$,

$dY = pds \sin(\boldsymbol{\theta + \pi/2}) = pds \cos\theta$



The pressure at the element ds is

$dF = dX + idY = -p \sin\theta \, ds + ip \cos\theta \, ds$

pds

$$= ip\,(\cos\theta + i\sin\theta)\,ds \qquad \left|\begin{array}{l} \quad pds\,\sin\theta \text{ along negative } x-\text{axis} \\ \Rightarrow -pds\,\sin\theta \text{ along positive } x-\text{axis} \end{array}\right.$$

$$= ip\left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds \qquad \left|\cos\theta = \frac{dx}{ds},\quad \sin\theta = \frac{dy}{ds}\right.$$

$$= ip\,(dx + idy) = ip\,dz \tag{1}$$

The pressure equation, in the absence of external forces, is

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant}$$

or $$p = -\frac{1}{2}\rho q^2 + k \tag{2}$$

Further $$\frac{dw}{dz} = -u + iv = -q\cos\theta + iq\sin\theta$$

$$= -q\,(\cos\theta - i\sin\theta) = -q\,e^{-i\theta} \tag{3}$$

and $$dz = dx + idy = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds = (\cos\theta + i\sin\theta)\,ds = e^{i\theta}\,ds \tag{4}$$

The pressure on the cylinder is obtained by integrating (1). Therefore,

$$F = X + iY = \int_C ipdz = \int_C i\,(k - 1/2\,\rho q^2)\,dz$$

$$= -\frac{i\rho}{2}\int_C q^2\,dz \qquad \left|\ \because \int_C dz = 0\right.$$

$$= -\frac{i\rho}{2}\int_C q^2\,e^{i\theta}ds$$

From here ;

$$X - iY = \frac{i\rho}{2}\int_C q^2\,e^{-i\theta}\,ds$$

$$= \frac{i\rho}{2}\int_C (q^2\,e^{-2i\theta})\,e^{i\theta}\,ds$$

$$= \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz \qquad \left|\ \text{using (3) \& (4)}\right.$$

The moment M is given by

$$M = \int_C \left|\bar{r} \times d\bar{F}\right| = \int_C [(pds\,\sin\theta)\,y + (pds\,\cos\theta)\,x]$$

$$= \int_C \left[p\left(\frac{dy}{ds}\right) y\,ds + p\left(\frac{dx}{ds}\right) x\,ds \right] \qquad \left| \begin{array}{l} \vec{r} \times d\vec{F} \\ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ -p\,ds\sin\theta & p\,ds\cos\theta & \end{vmatrix} \end{array} \right.$$

$$= \int_C \ p(x\,dx + ydy]$$

$$= \int_C \left(k - \frac{1}{2}\rho q^2\right) \ (xdx + ydy)$$

$$= k\int_C d\left[\frac{1}{2}(x^2 + y^2)\right] - \frac{\rho}{2}\int_C \ q^2\,(xdx + ydy)$$

$$= -\frac{\rho}{2}\int_C \ q^2(xdx + ydy) \qquad \mid \because \ 1^{st} \text{ integral vanishes.}$$

$$= \frac{-\rho}{2}\int_C \ q^2\,(x\cos\theta + y\sin\theta)\,ds \qquad \left| \begin{array}{l} dx = \cos\theta ds \\ dy = \sin\theta ds \end{array} \right.$$

$$= \text{Real part of } \left[\frac{-\rho}{2}\int_C q^2(x + iy)(\cos\theta - i\sin\theta)ds\right]$$

$$= \text{Real part of } \left[\frac{-\rho}{2}\int_C q^2 z e^{-i\theta} ds\right]$$

$$= \text{real part of} \left[\frac{-e}{2}\int_C z(q^2 e^{-2i\theta})e^{i\theta}ds\right]$$

$$= \text{real part of } \left[-\frac{e}{2}\int_C z\left(\frac{dW}{dz}\right)^2 dz\right].$$

Hence the theorem.

Example: ***What arrangement of sources and sinks will give rise to the function W = log*** $\left(z - \frac{a^2}{z}\right)$***? Also prove that two of the streamlines are a circle r = a and x = 0.***

Solution. We have $w = \log\left(z - \frac{a^2}{z}\right) = \log\left(\frac{z^2 - a^2}{z}\right)$

i.e. $\phi + i\psi = \log(z^2 - a^2) - \log z$

$$= \log(z-a) + \log(z+a) - \log z \qquad (1)$$

This represents a line source at $z = 0$ and two line sinks at $z = \pm a$, each of strength unity per unit length. We can write

$$\phi + i\psi = \log(x-a+iy) + \log(x+a+iy) - \log(x+iy)$$

$$\Rightarrow \quad \psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left(\frac{\dfrac{y}{x-a}+\dfrac{y}{x+a}}{1-\dfrac{y^2}{x^2-a^2}}\right) - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right) - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left[\left(\frac{x^2+y^2+a^2}{x^2+y^2-a^2}\right)\frac{y}{x}\right] \qquad (2)$$

Since ψ = constant is the equation of the streamlines, therefore equations for streamlines are

$$y(x^2+y^2+a^2) = (x^2+y^2-a^2)x \tan\alpha$$

where α is a constant.

In particular, if we take $\alpha = \pi/2$, then we get the streamlines as

$$(x^2+y^2-a^2)\,x = 0$$

i.e. $x^2+y^2-a^2 = 0$, $x = 0$

i.e. $x^2+y^2 = a^2$ $\quad x = 0$

i.e. $r = a$, $x = 0$. Hence the result.

Example: ***Find the stream function of the two dimensional motion due to two equal sources of strength m at a distance 2a apart and an equal sink of strength 2m between them. Determine the stream lines. Also find the fluid speed at any point.***

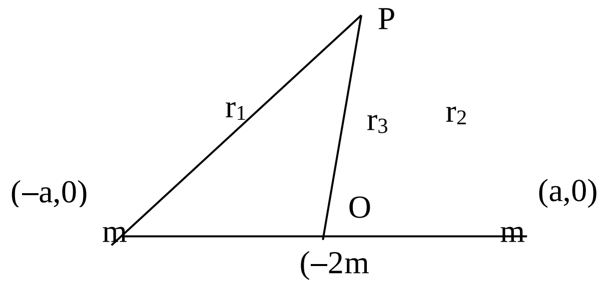
Solution: The complex potential of the given system is

$$W = -m\log(z-a) - m\log(z+a) + 2m\log z \qquad (1)$$

$$= m\log z^2 - m\log(z^2-a^2)$$

$$= m[\log z^2 - \log(z^2-a^2)]$$

i.e., $\phi + i\psi = m[\log(x^2-y^2+2ixy) - \log(x^2-y^2-a^2+2ixy)]$



$$\Rightarrow \quad \psi = m\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right) - \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)\right]$$

$$= m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2-(x^2-y^2)a^2}\right] \qquad (2)$$

Streamlines are given by ψ = constant.

i.e. $\psi = \text{constant} = m\tan^{-1}\left(\frac{-2}{\lambda}\right)$, say (3)

Therefore, from (2) & (3), we obtain

$$\frac{2}{\lambda} = \frac{2a^2xy}{(x^2+y^2)^2-(x^2-y^2)a^2}$$

$$\Rightarrow \qquad (x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$$

where $\boldsymbol{\lambda}$ is a variable parameter.

Now, the fluid velocity is given by

$$u - iv = -\frac{dw}{dz} = -\left(\frac{2m}{z} - \frac{m}{z-a} - \frac{m}{z+a}\right)$$

$$= \frac{2ma^2}{z(z-a)(z+a)}$$

$\therefore$ The fluid speed is given by

$$q = |u - iv| = \left|-\frac{dw}{dz}\right| = \left|\frac{2ma^2}{z(z-a)(z+a)}\right|$$

$$= \frac{2ma^2}{|z||z-a||z+a|}$$

$$= \frac{2ma^2}{r_1 r_2 r_3}, \text{ where } |z| = r_3, \ |z-a| = r_2, |z+a| = r_1$$

9.8 The Joukowski Transformation: The transformation

$$t = z + \left(\frac{c^2}{4z}\right),$$

Is among the most important transformations of two-dimensional fluid motion and utilized to obtain the fluid streaming past a fixed elliptic cylinder from that of a circular cylinder. Obviously, when |t| is large nearly, so that the corresponding distant parts of the two planes are unaltered. Thus a uniform stream at infinity in the w-plane corresponds to that, of the same strength and direction in z-plane. By considering the inverse transformation

$$z = \frac{1}{2}(t \pm \sqrt{(t^2 - c^2)})$$

or confining our attention to the positive square root viz

$$z = \frac{1}{2}(t + \sqrt{(t^2 - c^2)}) \quad (1)$$

It can be easily shown that the region outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

will map into the region outside the circle $|t| = \frac{1}{2}(a + b)$.

9.9 Analysis of Joukowski's Transformation:

The Joukowski transformation

$$t = z + (a^2/z) \quad (1)$$

is one of the most important transformations used for the purpose of investigating two-dimensional transformation used for the purpose of investigating two dimensional fluid motion. We shall explain it here in some detail:

Let us denote corresponding points in the z-plane and t-plane by the same letter without and with dashes. We commence observations categorically.

(i) For very large values of z, we have $t \neq z$ nearly so that the distant parts of the z and t-planes remain unaffected. Thus, a uniform stream (wind) at infinity in z-plane will appear as the same uniform stream(wind) at infinity in t-plane.

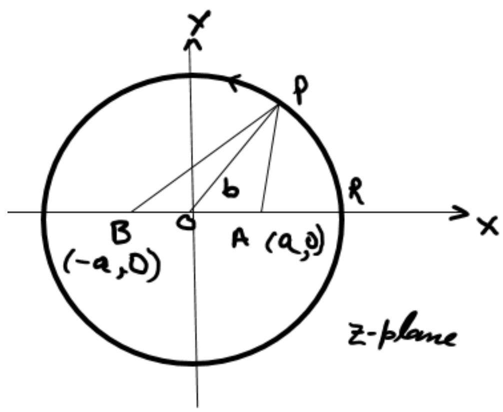
(ii) Since $\frac{dt}{dz} = \mathbf{1 - a^2/z^2}$, the points $\boldsymbol{z \pm a}$ and therefore the corresponding points $\boldsymbol{t = \pm 2a}$ are such where the transformation ceases to be conformal. Accordingly, we avoid mapping the region of which these critical (singular) points are interior; though these may appear on the boundary.

(iii) we now apply Joukowski's transformation to |z|=b, where b>a.

From (i) , $\boldsymbol{t \pm 2a = (z \pm a)^2/z}$, so that

$$\boldsymbol{|t+2a| + |t-2a| = [|z+a| + |z-a|]/|z|}$$

$$\boldsymbol{= (AP^2 + BP^2)/b}$$



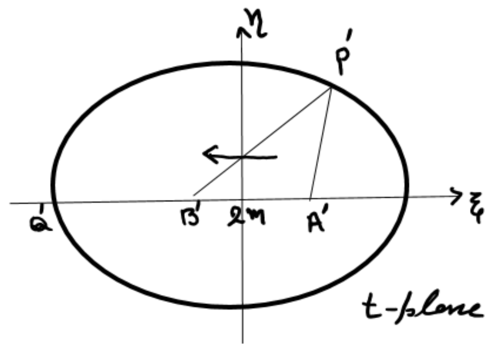
By Median-theorem(Geometry)

$$|t+2a| + |t-2a| = \frac{2(b^2+a^2)}{b} = constant$$

Thus as P describes the circle |z|=b, in the z-plane, P' describes an ellipse with the points $t = \pm 2a$ as foci, in the w-plane. The lengths of the major and minor axes of the ellipse are respectively

$$a_1 = \frac{2(b^2+a^2)}{b}, b_1 = \frac{2(b^2-a^2)}{b}.$$

Thus if b increases indefinitely, the ellipse expands indefinitely. Therefore, the transformation (1) maps circles in z-plane whose centre is at the origin into confocal ellipses in the t-plane.



This could follow from (1) directly, for

$$\xi = z\left(1+\frac{a^2}{x^2+y^2}\right);\ \eta = y\left(1-\frac{a^2}{x^2+y^2}\right) \qquad (2)$$

transforms the concentric circles $x^2+y^2=b^2$ into

$$\frac{\xi^2}{b^2+a^2}+\frac{\eta^2}{b^2-a^2}=\frac{1}{b^2}$$

which are confocal ellipses whose foci are $(\pm 2a, 0)$.

(iv) Consider the circle |z|=a. Then (2) implies

$$\xi = 2x = 2a\cos\theta, \eta = 0. But -1 \le \cos\theta \le 1$$

Hence $$-2a \le \xi \le 2a, \eta = 0.$$

Thus we see that the map transforms the circle |z|=a in z-plane to the line A'B' of t-plane where A'(2a,0), B'(-2a,0) are the points of ξ −axis is t-plane.

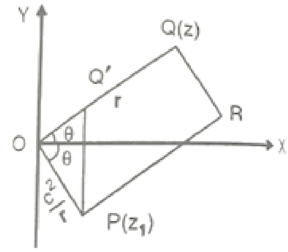
(v) The transformation can be expressed as $z^2 - tz + a^2 = 0$ whence $z = \frac{1}{2}[t \pm (t^2-4a^2)^{\frac{1}{2}}]$

It can be shown that $z = \frac{1}{2}[t + (t^2-4a^2)^{\frac{1}{2}}]$transforms the domain |z|≥a onto the whole t-plane and $z = \frac{1}{2}[t - (t^2-4a^2)^{\frac{1}{2}}]$ transforms the domain $|z| \le a$ onto the whole t-plane.

(vi) The Joukowski transformation as equivalent to the successive transformations

$$\zeta = z + t \ \ where\ t = \frac{c^2}{z}$$

Or $$w = r\,e^{i\theta} + (c^2/\mathrm{r})e^{-i\theta},\ \ z = re^{i\theta}$$



Draw a perpendicular PQ' to the real axis to meat OQ at Q'. i.e., OQ'=OP=c^2/r

Therefore, OQ.OQ'=$r\left(\frac{c^2}{r}\right) = c^2$

Thus, Q and Q' are inverse points with regard to O. The position of the point P is obtained by reflecting OQ', where Q' is the inverse point, in the real axis. Again the position of the point R(ζ) is obtained by the points Q and P by completing the parallelogram OPRQ. Thus the transformation is completed by the fourth vertex R which is represented by ζ. The point O will be made to describe a circle whereas the point Q' will then describe the inverse of this circle. The point P will therefore describe a circle obtained by reflecting the locus of Q' in the real axis.

9.10 Aerofoil: The aerofoil has a profile of fish type. It is used in modern aero planes. Such an aerofoil has a blunt leading edge and a sharp trailing edge. The projection of the profile on the double tangent is the chord. The ratio of the span to the chord is the aspect ratio. The locus of the point midway between the points in which an ordinate perpendicular to the chord meets the profile is known as the camber line of a profile. The camber is the ratio of the maximum ordinate of the camber line to the chord.

The theory of the flow round such an aerofoil is made on the following assumptions.

a) The air behaves as an incompressible inviscid fluid.
b) The aerofoil is a cylinder whose cross-section is a curve of the fish type.
c) The flow is two-dimensional irrotational cyclic motion.

The assumptions are simply approximation to the actual state of affairs. The profiles obtained by conformal transformation of a circle by the simple Joukowski transformation make good wing shapes, and that the lift can be determined from the known flow with respect to a circular cylinder.

Joukowski Aerofoil:

Consider a circle in the z-plane touching at B with AB as radius. Consider a small circle with BA as diameter. Let another large circle touch this circle at B(-a,0). Take a point P(z) on the large circle. Let P' be its inverse point with respect to the circle AB. Again let P'' be the reflection of P' with respect to x-axis then the coordinate of P'' is $\frac{a^2}{z}$.

If we draw a parallelogram with OP and OP'' as adjacent sides and if OQ be the diagonal of this parallelogram, then coordinates of Q will be $t = z + \frac{a^2}{z}$.

The locus of point Q(t) is a fish-shaped contour which is called Joukowski aerofoil on account of its resemblance with the section of an aeroplane wing. The terminal points of the aerofoil are called trailing edge and leading edge respectively as shown in figure.

9.11 Joukowski's Hypothesis:

We have $t = z + \frac{a^2}{z}\ and \frac{dt}{dz} = 1 - a^2/z^2$. If q' be the velocity at any point p'(t) corresponding to the velocity q and P(z) in z-plane, then

$$q = \left|\frac{dw}{dz}\right|,\ q' = \left|\frac{dw}{dt}\right|$$

Also
$$q' = \left|\frac{dw}{dz}\ \frac{dz}{dt}\right| = q\left|\frac{dz}{dt}\right| = q\left|\left(1 - \frac{a^2}{z^2}\right)^{-1}\right|$$

Evidently $q' = \infty\ if\ z = \pm a.$ But $z = \pm a$ corresponds to $t = \pm 2a.$ The point z=a is inside the circle and its transform t=2a is also inside the aerofoil and consequently need not concern us. Thus velocity is infinite at t=-2a, the trailing edge of the aerofoil. Hence in order to avoid infinite velocity at t=-2a, the velocity at B(z=-a) is taken to be zero, i.e., B is taken as stagnation point of the flow in z-plane. This is known as Joukowski hypothesis.

9.12 Kutta-Joukowski Theorem: ***When a cylinder (or an aerofoil) of any shape is placed in a uniform stream (wind) of speed U, the resultant thrust on the cylinder is a lift of magnitude $k\rho U$ per unit length and is at right angles to the stream(wind), where k is the circulation around the cylinder.***

Proof: Since at a great distance from the cylinder, the flow reduces to that of a uniform stream, the complex potential will approximately be of the form

$$w = Uz\, e^{-i\alpha} z + \frac{ik}{2\pi} logz\ + \frac{A}{z} + \cdots$$

Where A is a constant and α is the angle of attack of the stream (wind). Here the first term gives the velocity of the wind at infinity and the second refers to the circulation k round the cylinder.

Now,
$$\frac{dw}{dz} = U\, e^{-i\alpha} + \frac{ik}{2\pi}\,\frac{.1}{z} - \frac{A}{z^2} \ldots$$

Therefore,
$$\left(\frac{dw}{dz}\right)^2 = \left(U\, e^{-i\alpha} + \frac{ik}{2\pi}\,\frac{.1}{z} - \frac{A}{z^2} \ldots\right)^2$$

If (X,Y) be the components of the force (pressure thrust) acting on the surface c of the cylinder, then by Blasius theorem,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = \int_C \left(U\, e^{-i\alpha} + \frac{ik}{2\pi}\,\frac{.1}{z} - \frac{A}{z^2} \ldots\right)^2 dz$$
$$= \int_C [U^2\, e^{-2i\alpha} + ik\frac{U}{\pi} e^{-i\alpha}\frac{1}{z} + O\left(\frac{1}{z^2}\right) \ldots]\, dz$$

This integrand has a pole at z=0. Residue at z=0 is the coefficient of $\frac{1}{z}$ in the expansion of the integrand which is equal to $\frac{ik}{\pi U} e^{-i\alpha}$.

By Cauchy residue theorem,

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \frac{ik}{\pi}\; U\, e^{-i\,\alpha} = -2k\; U\, e^{-i\alpha}$$

Therefore, $\quad X - iY = \frac{i\rho}{2}\left(-2k\; U\; e^{-i\alpha}\right) = -ik\;\rho U\; e^{-i\alpha} = -ik\rho U\;(\cos\alpha - i\sin\alpha)$

So that $\quad X = k\rho U\sin\alpha\;\;;\;\; Y = k\rho\; U\cos\alpha$

Force(L) =$(X^2 + Y^2) = k\rho\; U.$

Thus the force $k\rho\; U$(maximum lift) acts at right angles to the stream. This force is called lift force.

Corollary: Moment about the origin is given by the Blasius formula

$$M + iN = -\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz$$

where M is the moment about the origin of the pressure thrusts on the cylinder(aerofoil) and N is the imaginary part of the integral.

Thus $\quad M = Real\; part\; of\;\; 2\pi\; i\;\rho\; U\; e^{-i\alpha}\; A$

Since the result contains A(constant), it follows that the couple depends on the form of the cylinder.

Example: ***Find the complex potential for liquid streaming, without circulation, past the circular cylinder|=(a+b)/2, the undisturbed streaming being uniform cross flow at incidence*** $\boldsymbol{\alpha}$. ***Show that the Joukowski transformation*** $\boldsymbol{\zeta = (\xi + i\eta) = z + [\frac{a^2-b^2}{4z}]}$

(1)

determine the flow past the elliptic cylinder $\boldsymbol{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1}$ ***. If |z| is large, show that***

$\boldsymbol{w = U\; e^{-i\alpha}\;\zeta + [U(a+b)(b\cos\alpha + i\sin\alpha)/2\zeta]}$ ***(2)***

Hence find the couple exerted per unit length on the cylinder.

Solution: The complex potential of uniform cross flow is $Uze^{-i\alpha}$. When |z|=(a+b)/2 is inserted, use of Circle theorem yields

$$w = \; Uze^{-i\alpha} + U\, e^{i\,\alpha}(a+b)^2/4z \tag{3}$$

Any point on the cylinder $|z| = \frac{a+b}{2}$ is $z = (a+b)e^{i\,\theta}/2$. Consider map (1) on this circle

$$\zeta = \left[\frac{(a+b)e^{i\theta}}{2}\right] + \left[\frac{(a-b)e^{-i\,\theta}}{2}\right] = \frac{1}{2}\; a\left(e^{i\theta} + e^{-i\theta}\right) + \frac{1}{2}b\left(e^{i\theta} - e^{-i\theta}\right)$$

i.e., $\quad \xi + i\,\eta = a\cos\theta + i\; b\sin\theta.$

Thus $\quad \xi = a\cos\theta; \eta = b\sin\theta \quad ; \quad \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1. (circle\; transforms\; to\; ellipse)$

Now $\quad \frac{dw}{dz} = \frac{dw}{d\zeta}\,.\frac{d\zeta}{dz} = \frac{dw}{d\zeta}\left[1 - \frac{a^2-b^2}{4z^2}\right]$

Thus, if |z| is large, the flow pattern is unaltered $\left(\frac{dw}{dz} = \frac{dw}{d\zeta}\right)$. We invert the map (1) to get

$$z^2 - z\,\zeta\;(a^2 - b^2)/4=0 \text{ or } z = \zeta - [(a^2 - b^2/4z] \tag{i}$$

So $\quad z = \zeta - \left(\frac{a^2-b^2}{4}\right)\left(\zeta - \frac{a^2-b^2}{4\zeta}\right) \quad$ |using (i)

$$= \zeta - \frac{a^2-b^2}{4\zeta} = \zeta + (b^2 - a^2)/4\zeta \qquad \text{(ii)}$$

From equation (ii) and equation (3)

$$w = Ue^{-i\alpha}\left[\zeta + \frac{b^2-a^2}{4\zeta}\right] + U\frac{(a+b)^2e^{i\alpha}}{4}.\frac{1}{\zeta}\left[1 + \frac{b^2-a^2}{4\zeta^2}\right]^{-1}$$

$$= Ue^{-i\alpha}\zeta + \frac{U(a+b)}{2\zeta}(b\cos\alpha + i\,a\sin\alpha) + O\left(\frac{1}{\zeta^2}\right)$$

For large $\zeta, O\left(\frac{1}{\zeta^2}\right)$ is neglected and the above yields result (2).

If C is any curve surrounding the cylinder, then couple M is given by

$$M = \frac{1}{2}\rho\, Re\oint_C \zeta\ \left(\frac{dw}{d\zeta}\right)^2 d\zeta \qquad (4)$$

Now the coefficient of $\frac{1}{\zeta}\ in\ \zeta\left(\frac{dw}{d\zeta}\right)^2 = coefficient\ of\ \frac{1}{\zeta^2}\ \ in\ \left(\frac{dw}{d\zeta}\right)^2$.

Here $\left(\frac{dw}{d\zeta}\right)^2 = [\,Ue^{-i\alpha} - \frac{U(a+b)}{2\zeta^2}(b\cos\alpha + i\,a\sin\alpha)]^2$

By Cauchy Residue theorem, Equation (4) gives

$$M = -\frac{1}{2}\rho\, U^2(a+b)Re\{(2\pi i)e^{-i\alpha}(b\cos\alpha + i\,\mathrm{a}\sin\alpha)$$

$$= \pi\,\rho\,U^2(a^2 - b^2)\sin\alpha\cos\alpha.$$

9.13 Check Your Progress:

i) A source of fluid situated in space of two dimensions, is of such strength that $2\pi\rho m$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of the source is $2\pi\,\rho m^2a^2/r(r^2 - a^2)$, where a is the radius of the disc and r is the distance of the source from its centre. In what direction is the disc urged by the pressure?

ii) Between the fixed boundaries $\theta = \frac{\pi}{6}\, and\ \theta = -\frac{\pi}{6}$, there is a two-dimensional liquid motion due to a source at the point $(r = c, \theta = \alpha)$ and a sink at the origin, absorbing, water at the same rate as the source produces it. Find the stream function, and show that one of the steam lines is a part of the curve $r^3\sin 3\alpha = c^3\sin 3\,\theta$.

iii) Find the lines of flow in the two-dimensional fluid motion given by

$$\phi + i\,\psi = -\left(\frac{1}{2}\right)n(x+iy)^2e^{2int}.$$

Prove or verify that the paths of the particles of the fluid (in polar coordinates) may be obtained by eliminating t from the equations.

$$r\cos(nt+\theta) - x_0 = r\sin(nt+\theta) - y_0 = nt(x_0 - y_0)$$

iv) Show that the velocity potential $\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$ gives a possible motion. Determine the form of streamlines and curves of equal speed.

v) Explain the derivation of a Joukowski aerofoil by the transformation $t = z + \sum_{r=1}^{n}\frac{a_r}{z^r}$ applied to centre z_0 and radius a. Obtain the lift formula $L = 4\pi\,\rho\,a\,U^2\sin(\alpha+\beta)$ and show that the moment about the point $t = z_0$ is $M = 2\pi\,\rho b^2\,U^2 \sin 2(\alpha+\gamma)$, $where\ \alpha$ is the angle of attack and β, b, γ constants of transformation.

9.14 Summary: In this chapter we have discussed the conformal transformation which has wide applications to fluid mechanics. The images in two dimensions using transformation have been discussed. Drag Force and Lift Force of Immersed Bodies have also been defined in this chapter. Milne-Circle theorem and Blasius theorem have been derived by using Euler's equation. In the last Joukovski transformation, its analysis and Joukovski aerofoils have been discussed. Also, Kutta-Joukovski theorem has been derived.

9.15 Keywords: Conformal, transformation, images, Blasius, Circle theorem, Joukovski transformation, aerofoils.

9.16 Self-Assessment Test:

SA1: An aerofoil section is obtained by the mapping $\zeta = z + [(a-b)^2/z]$ applied to the circle |z-b|=a, where b is positive real and b<a. Find the value of circulation $\Gamma = 2\pi K$ needed to make the velocity finite at the trailing edge when the current impinges at an angle α to the axis of the aerofoil. Using the value of K obtained, find the force exerted by the current on the aerofoil.

SA2: A long infinite circular cylinder of radius a is placed in uniform stream $-U\boldsymbol{i}$ and fluid surrounding the cylinder is given a circulation $2\pi K$.Discuss the geography of stagnation points. Show also, that the cylinder experiences an uplifting force of magnitude $2\pi\rho K\,U$ per unit length.

SA3: State and prove Blasius theorem for an open curve AB. Find the force on the quadrant $0 \le \theta \le \pi/2$ of the circle |z|=a placed in uniform streaming U**i** when liquid pressure at infinity is p_∞.

SA4: In the conformal representation of the two-dimensional motion of a fluid, prove that if a source exists in one fluid, there will be a source at corresponding point of the other fluid.
SA5: Establish the theorem of Kutta and Joukowski.

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CHAPTER-10

MOTION OF CYLINDERS AND VORTEX MOTION

10.0 Learning Objectives: After reading this chapter, one should be able to understand the two-dimensional irrotational motion produced by motion of circular, co-axial and elliptic cylinders, understand the vortex motion, permanence of vorticity and can be able to obtain the complex potential for the rectilinear vortices.

10.1 Two-dimensional Irrotational Motion Produced by Motion of Cylinders:

Here, we discuss two-dimensional irrotational motion produced by the motion of cylinders in an infinite mass of liquid at rest at infinity (the local fluid moves with the cylinder). The cylinders move at right angles to their generators which are taken parallel to z-axis. Thus, we get the xy-plane as the plane of flow. For the sake of simplicity, we take the cylinders of unit length. For such motion, the stream function ψ or velocity potential ϕ) is determined in the light of the following conditions.

(i) ψ satisfies Laplace equation i.e., $\nabla^2\psi = 0$ at every point of the liquid.

(ii) Since the liquid is at rest at infinity, so

$$\frac{\partial\psi}{\partial x} = 0 \text{ and } \frac{\partial\psi}{\partial y} = 0 \text{ at infinity.}$$

(iii) Along any fixed boundary, the normal component of velocity must be zero so that $\frac{\partial\psi}{\partial s} = 0$ i.e.

ψ = constant, which means that the boundary must coincide with a streamline.

(iv) On the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to normal component of velocity of the cylinder.

Further, we observe that the two-dimensional solution of the Laplace equation $\nabla^2\psi = 0$, in polar co-ordinates (r, θ), is

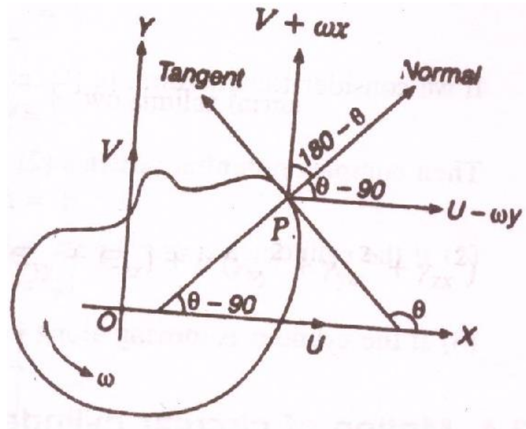
$$\psi = A_n r^n \cos n\theta + B_n r^n \sin \theta$$

where n is any integer, A_n and B_n being constants. Also, all the observations made for ψ, are valid for velocity potential ϕ, where ϕ and ψ satisfy C–R equations.

10.2 General Motion of Cylinder:

Let O be the cross-section of any cylinder be moving perpendicular to its generators which are moving with linear velocities U and V along x,y-axis with angular velocity $\boldsymbol{\omega}$.

Let P(x,y) be on the boundary of the cylinder. Let the tangent PT at P makes an angle $\boldsymbol{\theta}$ with the x-axis. PN is normal at P. We have $\boldsymbol{x = r\cos\theta, y = r\sin\theta}$.



Linear velocity along x-axis= dx/dt

or $\frac{dx}{dt} = \frac{d}{dt}(r\cos\theta) = -r\sin\theta\frac{d\theta}{dt} = -r\sin\theta\,\omega$ since $\omega = \frac{d\theta}{dt}$

Therefore, $\frac{dx}{dt} = -y\,\omega$

Linear velocity along y-axis $= \frac{dy}{dt} = \frac{d}{dt}(r\sin\theta) = r\cos\theta\frac{d\theta}{dt} = x\omega$

Velocity components at P along x-axis and y-axis are $U + \frac{dx}{dt}, V + \frac{dy}{dt}$ $i.e.\ U - y\,\omega, V = \omega x$

But normal component of velocity of boundary =normal component of velocity of liquid

$$-\frac{\partial\psi}{\partial s} = (U - y\,\omega)\cos(90 - \theta) + (V + x\omega)\cos(180 - \theta)$$

or $$-\frac{\partial\psi}{\partial s} = U - y\omega\sin\theta - (V + x\,\omega)\cos\theta$$

$$= (U - y\,\omega)\frac{dy}{ds} - (V + x\,\omega)\frac{dx}{ds}$$

or $$d\psi = (-U + y\omega)dy + (V + x\,\omega)dx$$

Integrating $$\psi = -Uy + \frac{y^2}{2}\omega + Vx + \frac{x^2}{2}\omega + C$$

$$\psi = (Vx - Uy) + \frac{\omega}{2}(x^2 + y^2) + C$$

Where C is constant of integration.

This is the required expression for the general motion of the cylinder.

Note: If the motion is pure rotation then linear velocities are equal to zero, therefore

$$\psi = \frac{\omega}{2}(x^2 + y^2) + C$$

But $z\,\bar{z} = x^2 + y^2$

Therefore, $$\psi = \frac{\omega}{2}z\,\bar{z} + C$$

If we consider the equation of the cross-section of cylinder as boundary of the form

$$z\,\bar{z} = f(z) + f(\bar{z})$$

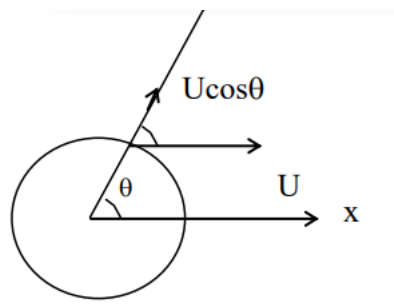
The complex potential satisfies $\boldsymbol{\psi}$ is

$$W = I\,\omega f(z)$$

10.3. Motion of a Circular Cylinder. Let us consider a circular cylinder of radius a moving with velocity U along x-axis in an infinite mass of liquid at rest at infinity. The velocity potential ϕ which is the solution of $\nabla^2\phi = 0$, must satisfy the following conditions.

(i) $\left(-\dfrac{\partial\phi}{\partial r}\right)_{r=a} = U\cos\theta$

(ii) $-\dfrac{\partial\phi}{\partial r}$ and $-\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} \to 0$ as $r\to\infty$



A suitable form of ϕ is

$$\phi\,(r,\theta) = \left(Ar + \frac{B}{r}\right)\cos\theta \qquad (1)$$

$$\Rightarrow \qquad -\frac{\partial\phi}{\partial r} = \left(-A + \frac{B}{r^2}\right)\cos\theta \qquad (2)$$

Applying conditions (i) and (ii) in (2), we get

$$\left(-A + \frac{B}{a^2}\right)\cos\theta = U\cos\theta, (-A + 0.B) = 0 \text{ for all } \theta.$$

$$\Rightarrow \quad -A + \frac{B}{a^2} = U,\ A = 0$$

$$\Rightarrow \qquad A = 0,\ B = U\,a^2$$

Thus $$\phi(r,\theta) = \frac{Ua^2}{r}\cos\theta \qquad (3)$$

The second condition of (ii) is evidently satisfied by ϕ in (3)

But $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{(C–R equation)}$$

so, $$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{Ua^2}{r^2}\cos\theta$$

i.e. $$\frac{\partial\psi}{\partial\theta} = -\frac{Ua^2}{r}\cos\theta$$

Neglecting constant of integration, we get

$$\psi = -\frac{Ua^2}{r}\sin\theta \tag{4}$$

Thus $$W = \phi + i\psi = \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{Ua^2}{re^{i\theta}} = \frac{Ua^2}{z}$$

which gives the complex potential for the flow.

Remarks. (i) For the case of 'Uniform flow past a fixed circular cylinder', using circle theorem, we have obtained the complex potential as

$$W = f(z) + f(a^2/z)$$

$$= Uz + U\frac{a^2}{z}$$

where the cylinder moves with velocity U along positive direction of x-axis. If we give a velocity U to the complete system, along the positive direction of x-axis, then the stream comes to rest and the cylinder moves with velocity U in x-direction.

Thus, we get

$$W = Uz + U\frac{a^2}{z} - Uz = \frac{Ua^2}{z}$$

(ii) Similarly, if we impose a velocity U in the negative direction of x-axis to the complete system in the present problem, then the cylinder comes to rest and the liquid flows past the fixed cylinder with velocity U in negative x-axis direction and thus we get

$$W = \frac{Ua^2}{z} + Uz.$$

(iii) If we put $Ua^2 = \mu$, then we get

$$W = \frac{\mu}{z} = \frac{\mu e^{i\theta}}{z - 0}$$

which shows that the complex potential due to a circular cylinder with velocity U along x-axis in an infinite mass of liquid is the same as the complex potential due to a line doublet of strength $\mu = Ua^2$ pre unit length situated at the centre with its axis along x-axis.

Example: ***A circular cylinder of radius a is moving in the fluid with velocity U along the axis of x. Show that the motion produced by the cylinder in a mass of fluid at rest at infinity is given by the complex potential***

$$W = \phi + i\psi = \frac{Ua^2}{z - Ut}$$

Find the magnitude and direction of the velocity in the fluid and deduce that for a marked particle of fluid whose polar co-ordinates are (r, θ) referred to the centre of the cylinder as origin,

$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{U}{r}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right) \text{ and } \left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

Solution. The cylinder is given to be moving along x-axis. At time t, it has moved through a distance Ut. Taking z = Ut as the origin, the complex potential is

$$W = \phi + i\psi = \frac{Ua^2}{z - Ut}$$

Therefore
$$-\frac{dW}{dz} = \frac{Ua^2}{(z - Ut)^2} = \frac{Ua^2}{r^2}e^{-2i\theta}, \quad z - Ut = re^{i\theta}$$

i.e.
$$u - iv = \frac{Ua^2}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$\Rightarrow \quad u = \frac{Ua^2}{r^2}\cos 2\theta, \quad v = \frac{Ua^2}{r^2}\sin 2\theta$$

Therefore,
$$q = \sqrt{u^2 + v^2} = \frac{Ua^2}{r^2}$$

The direction of velocity is $\tan\alpha = \frac{v}{u} = \tan 2\theta \Rightarrow \alpha = 2\theta$

When the cylinder is fixed and its centre is at 0, then

$$W = Uz + \frac{Ua^2}{z} = U(x + iy) + \frac{Ua^2}{r^2}(x - iy)$$

i.e.
$$\phi + i\psi = Ur(\cos\theta + i\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$\Rightarrow \quad \phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r}, \quad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

The streamlines are given by ψ = constant

$$\Rightarrow \qquad \left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

Further,

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{Ua^2}{r^2}\cos\theta \qquad | \bar{q} = -\nabla\phi$$

$$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta + \frac{Ua^2\sin\theta}{r^2}$$

$$\Rightarrow \qquad \frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U\cos\theta}{r} + \frac{Ua^2\cos\theta}{r^3} + \frac{iU\cos\theta}{r} + i\frac{Ua^2\sin\theta}{r^3}$$

$$= \frac{U}{r}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right)$$

Hence the result.

10.4. Equation of Motion of a Circular Cylinder (Equation of Motion of Circular Cylinder without Circulation). Let a circular cylinder of radius a move with a uniform velocity U along x-axis in a liquid at rest at infinity. The complex potential for the resulting motion, is $\phi + i\psi = W = \frac{Ua^2}{z}$, where origin is taken at the centre of the cylinder.

Thus, $$\phi = \frac{Ua^2}{r}\cos\theta, \quad \psi = -\frac{Ua^2}{r}\sin\theta$$

so $$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = -U\cos\theta$$

Let T_1 be the K.E. of the liquid on the boundary of the cylinder and T_2 that of the cylinder. Let σ and ρ be the densities of material of the cylinder and the liquid respectively. Then

$$T_1 = -\frac{\rho}{2}\int_C \phi\frac{\partial\phi}{\partial n}ds$$

$$= -\frac{\rho}{2}\int_0^{2\pi}\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} a\,d\theta, \quad s = a\theta \;\Rightarrow ds = a\,d\theta \qquad | \; l = r\theta$$

$$= \frac{\rho}{2}\int_0^{2\pi}\left(\frac{Ua^2}{a}\cos\theta\right)(U\cos\theta)\,a\,d\theta$$

$$= \frac{\rho U^2a^2}{2}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{\pi\rho U^2a^2}{2} = (\pi a^2\rho)\frac{U^2}{2} = M'\frac{U^2}{2},$$

where $M' = \pi a^2\rho$ = mass of the liquid displaced by the cylinder of unit length.

K.E. of the cylinder, $T_2 = \frac{1}{2} MU^2$, $M = \pi a^2\sigma$

Thus, total K.E. of the liquid and cylinder is

$$T = T_1 + T_2 = \frac{1}{2}(M + M')\, U^2 \tag{1}$$

Let R be the external force on the cylinder in the direction of motion. We use the fact that rate of change of total energy is equal to the rate at which work is being done by external forces at the boundary.

$$\therefore \qquad RU = \frac{1}{2}\frac{d}{dt}(M + M')\, U^2 \qquad \left| \; \frac{\text{work done}}{\text{time}} = \frac{\text{force. distance}}{\text{time}} = \text{force. velocity} \right.$$

$$= \frac{M + M'}{2} 2U \frac{dU}{dt}$$

$$= (M + M')\, U \frac{dU}{dt}$$

$$\Rightarrow \qquad M\frac{dU}{dt} = R - M'\frac{dU}{dt} \tag{2}$$

Equation (2) is the equation of motion of the cylinder. This shows that the presence of liquid offers resistance (drag force) to the motion of the cylinder, since if there is no liquid, then $M' = 0$ and we get

$$M\frac{dU}{dt} = R \tag{3}$$

Now, if $\frac{R}{M}$ = external force on the cylinder per unit mass be constant and conservative, then by the energy equation, we get

$$\frac{1}{2}(M + {}')\, U^2 - (M - \boldsymbol{M'})\frac{R}{M} r = \text{constant} \tag{4}$$

where r is the distance moved by the cylinder in the direction of R. Diff. (4) w.r.t. t, we get

$$(M + M')\, U\frac{dU}{dt} - (M - M')\frac{R}{M} U = 0$$

or $$M\frac{dU}{dt} = \frac{M - M'}{M + M'} R = \frac{\pi\sigma a^2 - \pi\rho a^2}{\pi\sigma a^2 + \pi\rho a^2} R$$

i.e. $$M\frac{dU}{dt} = \frac{\sigma - \rho}{\sigma + \rho} R \tag{5}$$

which gives another form of equation of motion

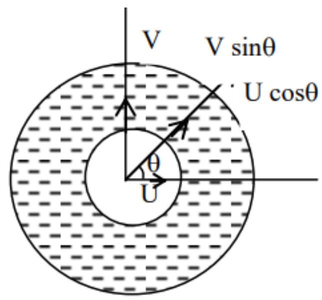
If U = (u, v) and R = (X, Y), then

$$M\frac{du}{dt} = \frac{\sigma-\rho}{\sigma+\rho}X, \ M\frac{dv}{dt} = \frac{\sigma-\rho}{\sigma+\rho}Y \tag{6}$$

Are the equations of motion of the cylinder in Cartesian co-ordinates. Comparing (3) and (5), it can be said that the effect of the presence of the liquid is to reduce external forces in the ratio

$$\sigma - \rho : \sigma + \rho.$$

10.5. Motion of two co-axial cylinders. Let us consider two co-axial cylinders of radii a and b (a < b). The space between them is filled with liquid of density ρ. Let the cylinders move parallel to themselves in directions at right angles with velocities U and V respectively, as shown in the figure



The boundary conditions for the velocity potential ϕ which is the solution of $\nabla^2\phi = 0$, are $(\bar{q} = -\nabla\phi)$

(i) $$-\frac{\partial\phi}{\partial r} = U\cos\theta, \ r = a \tag{1}$$

(ii) $$-\frac{\partial\phi}{\partial r} = V\sin\theta, \ r = b \tag{2}$$

A suitable form of velocity potential is

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \tag{3}$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta \tag{4}$$

Using (1) & (2) in (4), we get

$$-U\cos\theta = \left(A - \frac{B}{a^2}\right)\cos\theta + \left(C - \frac{D}{a^2}\right)\sin\theta$$

$$-V\sin\theta = \left(A - \frac{B}{b^2}\right)\cos\theta + \left(C - \frac{D}{b^2}\right)\sin\theta$$

Comparing co-efficient of $\cos\theta$ and $\sin\theta$, we get

$$A - \frac{B}{a^2} = -U, \quad C - \frac{D}{a^2} = 0$$

$$A - \frac{B}{b^2} = 0, \quad C - \frac{D}{b^2} = -V$$

Solving these equations, we obtain

$$A = -\frac{Ua^2}{a^2 - b^2}, \; B = \frac{-Ua^2b^2}{a^2 - b^2}, \; C = \frac{Vb^2}{a^2 - b^2}, D = \frac{Va^2b^2}{a^2 - b^2}$$

Thus, (3) becomes

$$\phi = -\frac{Ua^2}{a^2 - b^2}\left(r + \frac{b^2}{r}\right)\cos\theta + \frac{Vb^2}{a^2 - b^2}\left(r + \frac{a^2}{r}\right)\sin\theta$$

$$= \frac{Ua^2}{b^2 - a^2}\left(r + \frac{b^2}{r}\right)\cos\theta - \frac{Vb^2}{b^2 - a^2}\left(r + \frac{a^2}{r}\right)\sin\theta \qquad (5)$$

The expression for ψ can be obtained from

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

i.e. $$\frac{\partial\psi}{\partial\theta} = r\frac{\partial\phi}{\partial r}$$

$$= \frac{Ua^2}{b^2 - a^2}\left(r - \frac{b^2}{r}\right)\cos\theta - \frac{Vb^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\sin\theta$$

Integrating and neglecting the constant of integration, we get

$$\psi = \frac{Ua^2}{b^2 - a^2}\left(r - \frac{b^2}{r}\right)\sin\theta + \frac{Vb^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\cos\theta \qquad (6)$$

It should be noted that the values of ϕ and ψ given by (5) and (6), hold only at the instant when the cylinders are on starting i.e. the initial motion.

Corollary. If the cylinders move in the same direction, then the boundary conditions are

(i) $-\frac{\partial\phi}{\partial r} = U\cos\theta, \; r = a$

(ii) $-\frac{\partial\phi}{\partial r} = V\cos\theta, \; r = b$

Using these conditions in (4), comparing co-efficients of $\cos\theta$ and $\sin\theta$ and then solving the resulting equations, we get

$$A = \frac{Ua^2 - Vb^2}{b^2 - a^2},\ B = \frac{-UVa^2b^2}{b^2 - a^2},\ C = 0, D = 0$$

So, $$\phi = \frac{1}{b^2 - a^2}\left[\left(Ua^2 - Vb^2\right)r - \frac{UVa^2b^2}{r}\right]\cos\theta$$

and $$\psi = \frac{1}{b^2 - a^2}\left[\left(Ua^2 - Vb^2\right)r + \frac{UVa^2b^2}{r}\right]\sin\theta$$

Example: ***An infinite cylinder of radius a and density σ is surrounded by a fixed concentric cylinder of radius b and the intervening space is filled with liquid of density ρ. Prove that the impulse per unit length necessary to start the inner cylinder with velocity V is***

$$\frac{\pi a^2}{b^2 - a^2}[(\sigma+\rho)\, b^2 - (\sigma-\rho)\, a^2]V$$

Suppose that V is taken along the x-axis.

Solution. Let the velocity potential be

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \tag{1}$$

The boundary conditions are $(\bar{q} = -\nabla\phi)$

(i) $-\frac{\partial\phi}{\partial r} = V\cos\theta,\ r = a$

(ii) $-\frac{\partial\phi}{\partial r} = 0,\ r = b$

Applying these conditions in (1) and then comparing co-efficient of cosθ and sinθ, we get

$$A - \frac{B}{a^2} = -V,\quad C - \frac{D}{a^2} = 0$$

$$A - \frac{B}{b^2} = 0,\quad C - \frac{D}{b^2} = 0$$

Solving for A, B, C, D, we obtain

$$A = \frac{Va^2}{b^2 - a^2},\ B = \frac{Va^2b^2}{b^2 - a^2},\ C = D = 0$$

Thus, the potential (1) is

$$\phi = \frac{1}{b^2 - a^2}\left(Va^2r + \frac{Va^2b^2}{r}\right)\cos\theta$$

Now, the impulsive pressure at a point on r = a (along x-axis), is

$$P = (\rho\,\phi)_{r=a} = \frac{\rho V a^2}{b^2 - a^2}\left(r + \frac{b^2}{r}\right)\cos\,|_{r=a}$$

$$= \frac{\rho V a}{b^2 - a^2}(a^2 + b^2)\cos\theta$$

The impulsive pressure on the mole cylinder is

$$\int_0^{2\pi} \frac{\rho V a}{b^2 - a^2}\ (a^2 + b^2)\cos\theta.\ a\cos\theta\ d\theta$$

$$= -\pi\ a^2\rho\left(\frac{b^2 + a^2}{b^2 - a^2}\right)V$$

Now, change in momentum = the sum of impulsive forces

Therefore, $\pi\ a^2\sigma\ (V-0) = I - \pi a^2\rho\left(\frac{b^2 + a^2}{b^2 - a^2}\right)V$

$\Rightarrow$ $I = \pi a^2\sigma\ V + \pi a^2\rho\left(\frac{b^2 + a^2}{b^2 - a^2}\right)V$

Thus, impulse due to external forces, is

$$I = \frac{\pi a^2 V}{b^2 - a^2}[\ \sigma\ (b^2 - a^2) + \rho(b^2 + a^2)]$$

$$= \frac{\pi a^2 V}{b^2 - a^2}\ [(\sigma + \rho)\ b^2 - (\sigma - \rho)\ a^2]$$

Hence the result.

10.6 Elliptic Coordinates: Let $z = \cos h\,\zeta, where\ z = x + iy, \zeta = \xi + i\eta$

Then $x + iy = c\ cos\ \ (\xi + i\eta) = c(\cosh\xi\cos\eta + i\sinh\xi\sin\eta)$

So that $x = c\cosh\xi\cos\eta;\ \ y = c\sinh\xi\sin\eta$ (1)

Obviously,

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1 \tag{2}$$

And

$$\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1 \tag{3}$$

Thus ξ=const. and $\eta = const.$ represent confocal ellipses and hyperbolas respectively and the distance between the foci in each case is 2c.

If a, b are the semi-axes of the ellipse (2), we have for $\xi = \xi_0$

$$a = c\cosh\xi_0\,, b = c\sinh\xi_0 \; so\; that\; a^2 - b^2 = c^2$$

Also $$a + b = c(\cosh\xi_0 + \sinh\xi_0) = c\,e^{\xi_0};$$

$$a - b = c\ \cosh\xi_0 - \sinh\xi_0) = c\,e^{-\xi_0};$$

so that $e^{2\xi_0} = (a+b)/(a-b)$ or $\xi_0 = \frac{1}{2}\log\left[\frac{a+b}{a-b}\right]$

The parameters ξ, η are called the elliptic co-ordinates.

10.7 Motion of an elliptic cylinder:

(i) *To determine the* ϕ *and* ψ *when an elliptic cylinder moves in an infinite liquid with velocity U parallel to major axis of the section.*

We know stream function for any cylinder moving with velocities U and V parallel to axes and rotating with an angular velocity ω; is given by

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C \qquad (1)$$

As elliptic cylinder moves along major axis i.e., x-axis, then $V = 0, \omega = 0$, then (1) becomes

$$\psi = -Uy + c \qquad (2)$$

Now let the cross-section of the cylinder be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Or $$\frac{x^2}{c^2 cosh^2\alpha} + \frac{y^2}{c^2\sinh^2\alpha} = 1$$

where $a = c\cosh\alpha\,, b = c\sinh\alpha, \quad \alpha = \xi$

The elliptic coordinates are $x = c\cosh\alpha\cos\eta,\ y = c\sinh\alpha\sin\eta$

So (1) will contain $\sin\eta$. Also since the velocity is to vanish at infinity ($\xi = \infty$), ψ must be of the form.

$$e^{-\xi}\sin\eta$$

We can take complex potential of the form

$$\phi + i\,\psi = A\,e^{-(\xi + i\,\eta)} = A\,e^{-\xi}\,e^{-i\eta} = \mathrm{A\,e^{-\xi}}\,(\cos\eta - i\sin\eta) \qquad (3)$$

Equating imaginary part on both sides

$$\psi = -A\,e^{-\xi}\sin\eta \qquad (4)$$

At the boundary $\xi = \alpha$, we have

$$\xi = -Uc\sinh\alpha\sin\eta$$

Therefore, $-A\,e^{-\alpha}\sin\eta = -Uc\sinh\alpha\sin\eta + c$

We get

$$c=0 \text{ and } A\, e^{-\alpha} = Uc \sinh\alpha$$

$$\Rightarrow \qquad A = Uc\, e^{\alpha} \sinh\alpha$$

Putting these values in (3), we get

$$\psi = -Uc\, e^{\alpha-\xi} \sinh\alpha \sin\eta \tag{5}$$

is the stream function when the elliptic cylinder $\xi = \alpha$ is moving parallel to its major axis with velocity U.

Put $b = c\sinh\alpha\ in\ (5)$ we get

$$\psi = -Ub\, e^{\alpha}\, e^{-\xi} \sin\eta$$

$$= -Ub \sqrt{\frac{a+b}{a-b}}\, e^{-\xi} \sin\eta$$

As $$e^{\alpha} = \sqrt{\frac{a+b}{a-b}}$$

Similarly, from (2) equating real part

$$\phi = A\, e^{-\xi}\cos\eta = Ub\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}\cos\eta$$

The complex potential is given by

$$w = \phi + i\,\psi = Ub\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}(\cos\eta - i\sin\eta)$$

$$w = Ub\sqrt{\frac{a+b}{a-b}}\, e^{-\xi} e^{-i\eta} = Ub\sqrt{\frac{a+b}{a-b}}\, e^{-(\xi+i\,\eta)}$$

(ii) ***To determine the velocity function and the steam function when an elliptic cylinder moves in an infinite liquid with velocity V parallel to the axial plane through the minor axis of a cross-section.***

We know stream function ψ for any cylinder moving with velocities U and V parallel to axes and rotating with angular velocity ω, is given by

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C \tag{1}$$

As elliptical cylinder moves along minor axis, i.e., y-axis, then $U = 0, \omega = 0$, put in (1)

$$\psi = Vx + c \tag{2}$$

Now let the cross-section of the cylinder be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or
$$\frac{x^2}{c^2 cosh^2\alpha}+\frac{y^2}{c^2\sinh^2\alpha}=1$$

where $a = c\cosh\alpha\,, b = c\sinh\alpha, \quad \alpha = \xi$

The elliptic coordinates are $x = c\cosh\alpha\cos\eta,\ y = c\sinh\alpha\sin\eta$

So (1) will contain $\sin\eta$. Also since the velocity is to vanish at infinity $(\xi = \infty)$, ψ must be of the form.

$$e^{-\xi}\cos\ \eta$$

We assume
$$\psi = A\,e^{-\xi}\cos\eta \tag{3}$$

From (2) and (3) equating ψ at $\xi = \alpha$ we have

$$Vc\cosh\alpha\cos\eta + c = Ae^{-\alpha}\cos\eta$$

We get $0 = cA = Vce^{\alpha}\cos\alpha$

Putting these values in (3) we get

$$\psi = Vc\ e^{\alpha}\cosh\alpha\ e^{-\xi}\cos\eta$$

$$= Va\ \sqrt{\frac{a+b}{a-b}}\ e^{-\xi}\cos\eta$$

As $a = c\cosh\alpha$ and $e^{a} = \sqrt{\frac{a+b}{a-b}}$

Also, that
$$\phi = Va\ \sqrt{\frac{a+b}{a-b}}\ e^{-\xi}\sin\eta$$

The complex potential is given by

$$w = \phi + i\,\psi = iVa\ \sqrt{\frac{a+b}{a-b}}\ e^{-(\xi+i\,\eta)}$$

(iii) ***To determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in the direction making an angle θ with the major axis of the cross-section of the cylinder.***

The components of v along the axis of coordinates are

$$U = v\cos\theta \quad and\ V = u\sin\theta$$

Let $\omega_1\ and\ \omega_2$ be the complex potentials corresponding to the motion of the cylinder with velocity U and V along x and y-axis, we have

$$\omega_1 = Ub\sqrt{\frac{a+b}{a-b}}\; e^{-(\xi+i\,\eta)} = b\,v\cos\theta\sqrt{\frac{a+b}{a-b}}\; e^{-(\xi+i\eta)}$$

$$\omega_2 = iVa\sqrt{\frac{a+b}{a-b}}\; e^{-(\xi+i\,\eta)} = ia\,v\sin\theta\sqrt{\frac{a+b}{a-b}}\; e^{-(\xi+i\eta)}$$

Hence the complex potential due to velocity v is given by

$$\omega = \omega_1 + \omega_2 = v\sqrt{\frac{a+b}{a-b}}\; e^{-(\xi+i\eta)}\,(b\cos\theta + i\,a\sin\theta)$$

$$= v\sqrt{\tfrac{a+b}{a-b}}\; e^{-(\xi+i\eta)}\{\,c\sinh\alpha\cos\theta + ic\cosh\alpha\sin\theta\}$$

$$\omega = c\,v\sqrt{\tfrac{a+b}{a-b}}\,e^{-\zeta}\sinh(\alpha+i\,\theta) = v(a+b)e^{-\zeta}\sinh(\alpha+i\,\theta) \quad [\text{as } c^2 = a^2 - b^2$$

10.8 Kinetic energy of Elliptic Cylinder:

i) When the elliptic cylinder $\xi = \alpha$ moves with velocity U parallel to x-axis; we have

$\phi = Ub\sqrt{\tfrac{a+b}{a-b}}\; e^{-\alpha}\cos\eta = Ub\cos\eta\; e^{a}\, e^{-\alpha} = Ub\cos\eta$ $\qquad$ $[e^{\alpha} = \left(\tfrac{a+b}{a-b}\right)^{\frac{1}{2}}$

and $\psi = -Ub\left(\tfrac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\alpha}\sin\eta = -Ub\sin\eta$

We know kinetic energy of T of the liquid on the boundary of the cylinder

$$T = -\frac{1}{2}\,\rho\int \phi d\psi$$

$$= \tfrac{1}{2}\rho\, U^2 b^2 \int_0^{2\,\pi}\cos^2\eta\; d\eta$$ [as $\eta\ varies\ from\ 0\ to\ 2\pi$

$$=\tfrac{1}{2}\,\rho U^2 b^2 \pi$$

ii) When the elliptic cylinder $\xi = \alpha$ moves with velocity V to y-axis

$$\boldsymbol{\phi = Va\sqrt{\frac{a+b}{a-b}}\; e^{-\alpha}\sin\eta\,;\psi = Va\sqrt{\frac{a+b}{a-b}}\; e^{-\alpha}\cos\eta}$$

Therefore, $T=-\frac{1}{2}\rho\int\phi\,d\psi$$=-\frac{1}{2}\rho\int Va\cos\eta\, Va\cos\eta\, d\eta$

$$=-\frac{1}{2}\rho V^2 a^2\int\cos^2\eta\, d\eta$$

$$=\frac{1}{2}\rho V^2 a^2\pi$$

iii) when the elliptic cylinder moves with velocity V on a direction making an angle θ with the major axis

As above $T=\frac{1}{2}\pi V^2\,(b^2\cos^2\theta+a^2\sin^2\theta)$

10.9 Vortex Motion

Introduction: It is known that all possible motion of an ideal fluid can be divided in two classes irrotational and vortex motion. So, for we have confined our attention to the cases involving irrotational motion only. But the most general displacement of a fluid involves rotation such that the rotational vector (vortex vector or vorticity) $\Omega=curl\,\boldsymbol{q}\neq\boldsymbol{0}$. Here we consider the theory of rotational or vortex motion. First of all, we revisit some elementary definitions.

10.10 Definitions:

Vorticity If **q** be the velocity vector of a fluid particle. Then the vector quantity, $\Omega(=$ curl **q**) is called the vorticity vector**.**

Let $\Omega=\xi i+\eta j+\zeta k$ so that (ξ,η,ζ) are the vorticity vector components or the components of the spin. Then we have

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\quad \eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\qquad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

where $\boldsymbol{q}=ui+vj+wk$

If ξ,η,ζ are all zero, the motion is irrotational and velocity function ϕ exists. If ξ,η,ζ are not all zero, the motion is rotational.

In case of two-dimensional motion, we know that w=0 and u & v are function x & y only and hence for two-dimensional case

$$\xi=0,\eta=0,\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

It follows that in two-dimensional motion there can be at the most only one component of spin and its axis is perpendicular to the plane of the motion.

Vortex Line: A vortex line is a curve drawn in the fluid such that the, tangent to it is in the direction of the vorticity vector at that point at that instant.

An element of arc length **dr** along the vortex line is tangent to the vorticity vector.

So, the equation of vortex line is

$$\boldsymbol{\Omega}\times\boldsymbol{dr}=\boldsymbol{0}$$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \xi & \eta & \zeta \\ dx & dy & dz \end{vmatrix} = \mathbf{0}$$

$$(\eta\, dz - \zeta dy)i - (\xi dz - \zeta dx)j + (\xi dy - \eta dx)k = 0$$

or $$\eta\, dz - \zeta dy = 0;\ \xi dz - \zeta dx = 0;\ \xi dy - \eta dx = 0$$

Putting in combine form, we have

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

Vortex Tube: The vortex line drawn through each point of a closed curve enclose a tubular space in the fluid which is called the vortex tube. A vortex filament or simply a vortex is a vortex tube of infinitesimal cross-section.

Circulation: If C is a closed curve, then **circulation** about C is given by

$$\Gamma = \oint_C \mathbf{q}.d\mathbf{r} = \int_S \mathbf{n}.curl\,\mathbf{q}\,dS = \int_S \mathbf{n}.\Omega dS = \int_S \Omega.\,dS$$

The quantity $|\mathbf{n}\cdot\Omega|\delta S$ us called the strength of the vortex tube. A vortex tube with a unit strength is called a unit vortex tube.

Circular Vortex: The section of a cylindrical vortex tube whose cross-section is a circle of radius a, by the plane of motion is a circle and the liquid inside such a tube is said to form a circular vortex.

If w is the angular velocity and πa^2 the cross-sectional area of the vortex tube, then circulation

$$\Gamma = \oint_C \mathbf{q}\cdot d\,r = \int_S \ curl\ \mathbf{q}\cdot\mathbf{n}dS = \int_S \ curl\,\mathbf{q}\cdot dS$$
$$= \text{w} \int_S dS = w\pi a^2 = K(say)$$

This product of the cross-section and angular velocity at any point of the vortex tube is constant along the vortex and is known as the strength of the circular vortex.

10.11 Properties of the Vortex:

Every vortex satisfies the following fundamental properties:

1. **Every vortex is always composed of the same elements of fluid.**

Consider an element of fluid whose coordinates are (x, y, z) at any instant t. Let (a, b, c) are the initial coordinates, then

$$\frac{da}{\xi_0} = \frac{db}{\eta_0} = \frac{dc}{\zeta_0} = \frac{dz}{\omega_0} = \mu\ (say)$$

Since $$dx = \frac{dx}{da}da + \frac{dy}{db}db + \frac{dz}{dc}dc$$

or $$dx = \mu\left(\xi_0\frac{dx}{da} + \eta_0\frac{dy}{db} + \zeta_0\frac{dz}{dc}\right) = \rho_0\,\xi\,\frac{ds_0}{\rho\omega_0}$$

Hence $\left(\frac{\rho_0 ds}{\omega_0}\right) = \left(\frac{\rho ds}{\omega}\right) = \epsilon$ [by Cauchy integral

Let (u, v, w) be the component velocities at (x, y, z), and $(u + du, v + dv, w + dw)$ be the velocities at a neighboring point $(x + dx, y + dy, z + dz)$ on the same vortex line

Since $\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{ds}{\omega} = \frac{\epsilon}{\rho}$

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}\ dz$$

or
$$du = \frac{\epsilon}{\rho}\left(\xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}\right) = \epsilon\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)$$

The quantity du is the rate at which the projection of the element ds on the x-axis is increasing in length. The projection is equal to $\epsilon\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)$, the line ds still forms part of a vortex line.

2. The product of the angular velocity of any vortex into its cross-section is constant with respect to the time, and is the same throughout its length.

Let S_1 and S_2 be cross sectional areas at the end points P and Q of a vortex filament. Let **n** be unit outward normal vector on S. Then, by Gauss Divergence theorem

$$\int_S W.n\ dS = \int_V \nabla.W\ dV = \int \nabla.\left(\frac{1}{2}\ \nabla \times \boldsymbol{q}\right)\ dV$$
$$= \frac{1}{2}\int_v (div\ curl\ \boldsymbol{q})dV = 0\ as\ div\ curl = 0$$

Therefore,
$$\int_S W.n\ dS = 0$$

Where S is the closed surface enclosing volume of V of the fluid in the vortex tube

$$or\ \int_{S_1} W.n\ dS + \int_{S_2} W.n\ dS + \int_{walls} W.n\ dS = 0$$

But on the walls of the tube, W is along the tube and so $\int_{walls} W.n\ dS = 0\ as\ W.n = 0$ on the walls.

Hence $\int_{S_1} W.n\ dS + \int_{S_2} W.n\ dS = 0$

or $\int_{S_1} W.n_1\ dS = \int_{S_2} W.n_2\ dS$

where $n_1 and\ n_2$ are unit outward normal on the surfaces S_1 and S_2 drawn in the same sense. This implies

$$\int_S W.n\ dS = constant$$

This proves the required result.

10.12 Kelvin's proof of permanence: Permanence of irrotational motion:

Under the conditions of the Kelvin's Circulation theorem [Art 6.3 Chapter 6], if the flow is irrotational in a material region of the fluid at some particular time (e.g. t = 0 or t = t_0), the flow is always irrotational in that material region thereafter.

i.e. If the motion of an ideal fluid is once irrotational it remains irrotational for ever afterwards provided the external forces are conservative and density ρ is a function of pressure p only.

Proof. Suppose that at some instant ($t = t_0$), the fluid on the material surface S is irrotational

Then, $curl\ \boldsymbol{q} = \boldsymbol{\omega} = 0$ (1)

for all points of S.

Let C be the boundary of surface S, then

$$\Gamma = \oint_C q.dr = \int_S (n.curl\boldsymbol{q})dS = \int_S (\boldsymbol{n}.\omega)ds = 0 \qquad | \text{ using (1)}$$

But by Kelvin's circulation theorem, Γ is constant for all times. Hence circulation Γ is zero for all subsequent times. At any later time,

$$\int_S n.\omega dS = 0$$

If we now take S to be non-zero infinitesimal element, say ΔS, then

$\boldsymbol{n}.\omega\ \Delta S = 0 \Rightarrow \omega = 0$ at all points of S for all times and the motion is irrotational permanently. This proves the permanency of irrotational motion.

Example: ***Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are*** $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\mu}\ (\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}, \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{y}}, \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{z}})$ ***where*** $\boldsymbol{\mu}, \boldsymbol{\phi}$ ***are function of x,y,z,t.***

Solution: Stream lines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1}$$

and vortex lines are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \tag{2}$$

Stream lines (1) and vortex line (2) will be at right angles if

$$u\xi + v\eta + w\zeta = 0 \tag{3}$$

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4}$$

Using (4), (3) we obtain

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Which is the necessary and sufficient condition in order that $u\ dx + v\ dy + w\ dz$ may be perfect differential. So, we may write

$$u\ dx + v\ dy + w\ dz = \mu\ d\phi = \mu(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial \phi}{\partial z}dz)$$

Therefore, $u = \mu\frac{\partial \phi}{\partial x},\ v = \mu\frac{\partial \phi}{\partial y},\ w = \mu\frac{\partial \phi}{\partial z}$

Example: Assuming that in an infinite unbounded mass of incompressible fluid, the circulation in any closed circuit is independent of time, show that the angular velocity of any element of the

fluid moving rotationally varies as the length of the element measured in the direction of the axis of rotation.

Solution: Let ω be the angular velocity of the element ds and let σ be the its area of cross-section. Then circulation in any closed circuit surrounding this element is $2\omega\sigma$. Let the element ds form a part of the vortex filament s so that the circulation is constant all along this element. Furthermore, since circulation assumed to be independent of time, we get

$$2\omega\sigma = constant$$

Again, since the liquid is incompressible

$$\sigma\, ds = constant$$

After dividing, we have

$$\frac{\omega}{ds} = constant$$

which proves the result.

10.13 General Theory of Vortex Motion:

The vorticity is defined as $\Omega = curl\ \boldsymbol{q}$ **(1)**

In physical applications one is often concerned with the problem of expressing the vorticity field in terms of the velocity field. To obtain **q(r**, t) in terms of Ω we need to invert the equation (1). We do this as follows:

We consider flow regions in which the fluid motion is due to vortices only i.e., there are no sources, sinks etc. motion due to which is essentially irrotational. Assuming the fluid to be incompressible, the equation of continuity is

$$div\ \boldsymbol{q} = 0 \qquad (2)$$

Since divergence of curl of a vector vanishes identically, we can write

$$\boldsymbol{q} = curl\ \boldsymbol{A} \qquad (3)$$

A is called the vector potential for velocity. Also since for any scalar function ϕ, curl(grad ϕ) $=$ 0, **A** is indeterminate to within the gradient of scalar function. To make A unique we stipulate

$$\text{div}\ \mathbf{A}=0 \qquad (4)$$

From (3), curl curl A=curl **q**

$$-\boldsymbol{\nabla}^{\mathbf{2}}\, \boldsymbol{A} + \nabla(\nabla.A) = \Omega$$

Using (4) this gives

$$\nabla^2 A = -\Omega$$

This is Poisson's equation for A whose solution can be expressed as

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{4\pi}\iiint \frac{\Omega'(r',t)}{|r-r'|}\, dV'$$

Where $\boldsymbol{\Omega}'(\boldsymbol{r}'.t)$ is the vorticity at a point r' of the vortex tube, and dV' is the volume element of the vortex tube around the point r', the integration above extends over the whole vortex tubes.

The velocity field at a point P(r) as given by (3) is

$$\boldsymbol{q}(\boldsymbol{r},t) = curl\,\frac{1}{4\pi}\iiint \frac{\Omega^{'}(r^{'},t)}{|r-r^{'}|}\,dV'$$

10.14 Vorticity in Two-dimensions: For an incompressible fluid in the xy-plane, we have

$$\mathbf{q} = (u, v, 0), \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$$

$$\therefore \qquad \Omega = \nabla \times \mathbf{q} = (0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

$$= \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

which shows that in two-dimensional flow, the vorticity vector is perpendicular to the plane of flow.

Also, $\qquad \Omega = |\,\Omega\,| = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

Thus $\qquad \Omega = \mathbf{k}\,\Omega$

Now, for this case, the Helmholtz's vorticity equation

$$\frac{d\Omega}{dt} = (\Omega \cdot \nabla)\mathbf{q} \text{ gives}$$

$$\frac{d\Omega}{dt} = 0 \;\Rightarrow\; \Omega = \text{constant}$$

i.e. Ω = constant.

which shows that in the two-dimensional motion of an incompressible fluid, the vorticity of any particle remains constant.

Here, we may regard Ω as a vortex strength per unit area.

Also, in terms of stream function, we have

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

$$\therefore \qquad \Omega = \mathbf{k}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \mathbf{k}\,\nabla^2 \psi$$

i.e. $\qquad \Omega = \nabla^2\psi$

This gives vorticity in terms of the stream function.

10.15 Rectilinear or columnar Vortex Filament: The strength k of circular vortex is given by

$$k = w\pi a^2.$$

If we let $a \to 0$ and $\omega \to \infty$ such that the above product remains constant, we get a rectilinear vortex filament and represent it by a point in the plane of motion. Such vortex filament may be regarded as straight gravitating rod of fluid lying perpendicular to the plane of flow. It is also termed as a uniform line vortex. The strength of a vortex filament is positive when the circulation round it is anticlockwise and negative when clockwise.

In vortex motion the curvature of the stream lines introduces the action of centrifugal force which must be counter balanced by a pressure gradient in the fluid.

Different Types of Vortices: We may divide vortices into the following four types

1. Forced vortex in which the fluid rotates as a rigid body with constant angular velocity.

2. Free cylindrical vortex for which the fluid moves along streamlines which are concentric circles in horizontal planes and there is no variation of total energy with radius.

3. Free spiral vortex which is a combination the free cylindrical vortex and a source (radial flow)

4. Compound vortex in which the fluid rotates as a forced vortex at the centre and as a free vortex outside.

10.16 Complex Potential for Circulation about a Circular Cylinder (Circular vortex) :

In case of a doubly connected region, the possibility of cyclic motion does exist and as such we proceed to explain it presently in the case of circle.

If the circulation in a closed circuit is $2\pi k$, then k is called the strength of the circulation.

Let s consider the complex potential

$$w = \phi = i\psi = ik\log z \qquad (1)$$

On the circular cylinder $|z| = a, z = a\,e^{i\theta}$

Thus, $w = ik \log (a\,e^{i\theta}) = ik (\log a + i\theta)$

i.e. $\qquad \phi + i\psi - k\theta + ik \log a$

$\Rightarrow \qquad \phi = -k\theta,\ \psi = k \log a =$ constant.

This shows that the circular cylinder is a streamline and thus equation (1) gives the required complex potential for circulation about a circular cylinder.

When the fluid moves once round the cylinder in the positive sense, θ increases by 2π and then

$$\phi_1 = -k(\theta + 2\pi) = -k\theta - 2\pi k$$
$$= \phi - 2\pi k$$

$\therefore$ circulation $= 2\pi k = \phi - \phi_1$

= decrease in ϕ moving once round the circuit.

Hence there is a circulation of amount $2\pi k$ about the cylinder.

Also, $\qquad \dfrac{dw}{dz} = \dfrac{ik}{z}$

$\Rightarrow \qquad q = \left| -\dfrac{dW}{dz} \right| = \dfrac{k}{r}$

i.e. $k = rq$

$\therefore \qquad k = q$ when $r = 1$

Thus k is the sped at unit distance from the origin.

10.17 Complex Potential for Rectilinear Vortex (Line Vortex) Let us consider a cylindrical vortex tube whose cross-section is a circle of radius a; surrounded by infinite mass of liquid. We assume that vorticity over the area of the circle is constant and is zero outside the circle.

Let ψ be the stream function, then

$$\Omega = \nabla^2 \psi\, \mathbf{k}$$

i.e. $\Omega = \nabla^2\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2}$

$$= \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

Since there is a symmetry about the origin ψ is a function of r only and so $\dfrac{\partial^2\psi}{\partial\theta^2} = 0$.

$\therefore$ $\Omega = \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{d\psi}{dr}\right)$, for $r < a$

$= 0$, for $r > a$

i.e. $\dfrac{d}{dr}\left(r\dfrac{d\psi}{dr}\right) = r\,\Omega$ for $r < a$

$= 0$, for $r > a$

Integrating, we find

$r\dfrac{d\psi}{dr} = \Omega\dfrac{r^2}{2} + A$, for $r < a$

$= B$, for $r > a$

We are interested in the fluid motion outside the cylinder $|z| = a$. Therefore, integrating the second of the above result, we get

$\psi = B \log r + C$, for $r > a$.

The constant C may be chosen to be zero. Further, for $r > a$, the vorticity is zero and the fluid motion is irrotational, therefore velocity potential ϕ exists and is related to γ as

$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\partial\psi}{\partial r} = \frac{B}{r}$$

$\Rightarrow$ $\phi = -B\theta + D$

$\Rightarrow$ $\phi = -B\theta$, neglecting D

Let k be the circulation while moving once round the cylinder, then

k = decrease in value of ϕ on describing the circuit once

$= -B[\theta-(\theta+2\pi)] = 2\pi B$

$\Rightarrow$ $B = k/2\pi = K$(say)

Thus, $\phi = -k\theta$ and $\psi = k \log r$

Hence $W = \phi + i\psi = -k\theta + ik \log r$

$= ik(\log r + i\theta)$

$= ik \log z == i\dfrac{k}{2\pi}\log z.$

If the rectilinear vortex is situated at the point $z = z_0$, then by shifting the origin, we get

$W = ik \log(z - z_0)$

If there are vortices of strengths $k_1, k_2, \ldots k_n$ situated at $z_1, z_2, \ldots, z_n$ respectively, then the complex potential is

$$W = ik_1 \log(z-z_1) + ik_2 \log(z-z_2) + \ldots + ik_n \log(z-z_n)$$

Remarks (i) By a vortex, we mean a rectilinear vortex or line vortex.

(ii) $K = k/2\pi$, where k is the strength of a vortex and k that of circulation

(iii) Velocity component of a single vortex

$$\frac{dW}{dz} = -u + iv = \frac{ik}{2\pi(z - z_0)} = \frac{ik}{2\pi re^{i\theta}}$$

$$-u + iv = \frac{ik}{2\pi r} (\cos\theta - i\, sin\theta)$$

$$u = -\frac{k}{2\pi}\frac{y-y_0}{r^2}, v = \frac{k}{2\pi}\frac{x-x_0}{r^2}$$

where $r^2 = (x - x_0)^2 + (y - y_0)^2$

Example: An infinite liquid contains to parallel, equal and opposite rectilinear vortex filaments at a distance 2b. Show that the paths of the fluid particles relative to the vortices can be represented by the equation

$$\boldsymbol{log\left(\frac{(r^2 + b^2 - 2rb\cos\theta)}{(r^2 + b^2 + 2\, r\, b\cos\theta)}\right) + \frac{r\cos\theta}{b} = constant.}$$

O is the middle point of the join which is taken as x-axis.

Solution: Let the vortices of strengths $+k, -k$ be at $A(-b, 0), B(b, 0)$ such that AB is along x-axis. The complex potential due to this vortex pair at P(x,y) is

$$W = \frac{ik}{2\pi}\log(z + b) - \frac{ik}{2\pi}\log(z - b)$$

or $W = \phi + i\,\psi = \frac{ik}{2\pi}[\log(x + b + iy) - \log(x - b + iy)]$

Equating imaginary parts from both sides

$\psi = \frac{k}{4\,\pi}[\log\{(x + b)^2 + y^2\} - \log\{(x - b)^2 + y^2\}]$ (1)

The vortex pair will move along a line parallel to y-axis with velocity

$$\frac{k}{2\pi(AB)} = \frac{k}{2\pi(2b)} = \frac{k}{4\pi b}$$

To reduce the system to rest, we have to superimpose a velocity $\left(\frac{k}{4\pi b}\right)$parallel to y-axis. if ψ' be the stream function due to this disturbance, then

$$-\frac{k}{\pi b} = v = -\frac{\partial\phi'}{\partial y} = \frac{\partial\psi'}{\partial x}, \quad \psi' = -\frac{kx}{4\pi b}$$

The stream lines relative to the vortex system are given by $\psi = const.\, i.e.,$

$$\frac{k}{4\pi}[\log\{(x + b)^2 + y^2\} + \log\{(x - b)^2 + y^2\}] + \frac{x}{b} = constant$$

Changing into polar coordinates,

$$\log\left\{\frac{(x - b)^2 + y^2}{(x + b)^2 + y^2}\right\} + \frac{x}{b} = constant.$$

$$\log\left\{\frac{(r\cos\theta - b)^2 + r^2\sin^2\theta}{(r\cos\theta + b)^2 + r^2\sin^2\theta}\right\} + \frac{r\cos\theta}{b} = constant.$$

10.18 Check Your Progress:

i) Show that when a cylinder moves uniformly in a given straight line in an infinite liquid, the path of any point in the field is given by the equations

$$\frac{dz}{dt} = \frac{Va^2}{(z' - Vt)^2}; \frac{dz'}{dt} = \frac{Va^2}{(z - Vt)^2}$$

Where V is the velocity of cylinder, a its radius, and z, z' are $x + iy, x - iy$ where x, y are the coordinates measured from the starting point of the axis, along and perpendicular to its direction of motion.

ii) An infinite cylinder of radius a and density ρ is surrounded by a fixed concentric cylinder of radius b, and the intervening space is filled with liquid of density ρ. Prove that the impulse per unit length necessary to store the inner cylinder with velocity V us

$$\frac{\pi a^2}{b^2 - a^2} \{(\sigma + \rho)b^2 - (\sigma - \rho)a^2\} \text{ V.}$$

iii) Show that with proper choice of units, the motion of an infinite liquid produced by the motion of an elliptic cylinder parallel to one of its principal axes is given by the complex function $w = e^{-\zeta}, where\ z = 2\cosh\zeta$.

iv) When an infinite liquid contains two parallel equal and opposite rectilinear vortices at a distance 2b, prove that the stream lines relative to the vortices are by the equation

$$\log\left[\frac{x^2 + (y - b)^2}{x^2 + (y + b)^2}\right] + \frac{y}{b} = C$$

The origin being the middle point of the join, which is taken for y-axis.

v) Three parallel rectilinear vortices of same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

10.19 Summary: In this chapter the motion of circular, co-axial and elliptic cylinder in an infinite mass of liquid has been discussed. It is discussed in this chapter about the vortex motion, defined vortex line, vortex tube and vortex filament. The permanent of vorticity is also explained here and proved two important properties of vortex filament. At the end, the complex potential for the rectilinear vortices is derived in this chapter.

10.20 Keywords: Irrotational motion, circular cylinder, co-axial cylinder, elliptic cylinder, vortex motion, vortex filament, rectilinear vortices.

10.21 Self -Assessment Test:

SA1: Find the velocity potential and stream function when a circular cylinder of radius a is moving in an infinite mass of liquid at rest at infinity with velocity U in the direction of x-axis.
SA2: Find kinetic energy when an elliptic cylinder rotates in an infinite mass of liquid at rest at infinity.
SA3: Find the velocity potential and stream function in case of an elliptic cylinder rotating in an infinite mass of liquid at rest at infinity.
SA4: Find velocity potential, stream function and velocity components due to a rectilinear vortex filament.

SA5: Find the necessary and sufficient condition that vortex lines may be at right angles to the stream lines.
SA6: Prove that the product of the cross-section and angular velocity at any point on a vortex filament is constant all along the vortex filament and for all time.

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