M.Sc. (MATHEMATICS)

MAL-513

MECHANICS

DIRECTORATE OF DISTANCE EDUCATION

GURU JAMBHESHWAR UNIVERSITY OF SCIENCE & TECHNOLOGY

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Lesson: 1

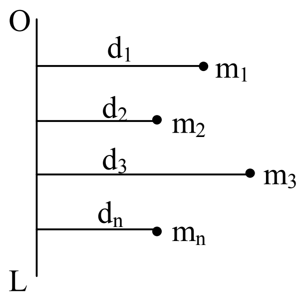
Moment of Inertia-1

1.1 Some definations:-

Inertia:- Inertia of a body is the inability of the body to change by itself its state of rest or state of uniform motion along a straight line.

Inertia of motion:- It is the inability of a body to change by itself its state of motion.

Moment of Inertia:- A quantity that measures the inertia of rotational motion of body is called rotational inertia or moment of inertia of body. M.I. is rotational analogue of mass in linear motion. We shall denote it by I. Let there are n particles of masses m_i, then moment of inertia of the system is



$$I = m_1 d_1^2 + m_2 d_2^2 + \dots + m_n d_n^2$$

$$= \sum_{i=1}^{n} m_i d_i^2$$

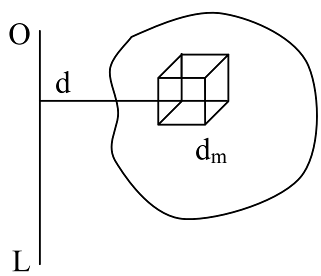
$$\therefore \quad I = \Sigma\, md^2$$

where d_i are the ⊥ distances of particles from the axis.

(i) M.I. in three dimension: - Let us consider a three dimensional body of volume V. Let OL be axis of rotation. Consider an infinitesimal small element of mass d_m, then

mass of small element $d_m = \rho dv$

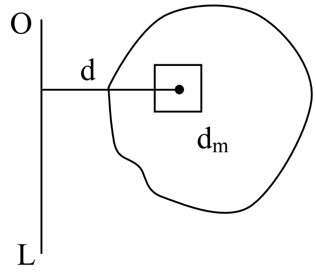
where dv = volume of infinitesimal small element and ρ is the density of material. Then moment of inertia of body is



$$I = \iiint_V d_m d^2$$

or $$I = \iiint_V \rho d^2 \, dv$$

(ii) M.I. in two dimension



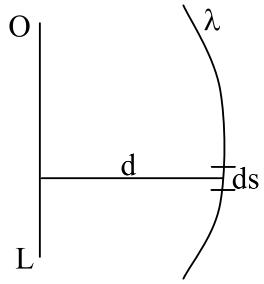
Here mass of small element $d_m = \rho dS$

and moment of inertia is $I = \iint_S d_m d^2$

or $I = \iint_S \rho d^2 \, dS$

where dS = surface area of small element

(iii) M.I. in one dimension



Consider a body (a line or curve) in one dimension. Consider a small element of length ds and mass d_m. Then mass of small element,

$d_m = \rho ds$

M.I. of small element = $d_m d^2$

$\therefore$ M.I. of body $I = \int_s d_m d^2$

or $I = \int_s \rho d^2 \, ds$

Radius of gyration:- Radius of gyration of a body about a given axis is the ⊥ distance of point P from the axis where its whole mass of body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass. This distance is represented by K. When K is radius of gyration,

$$I' = I$$

$$\Rightarrow \quad MK^2 = m(r_1^2 + r_2^2 + ... + r_n^2)$$

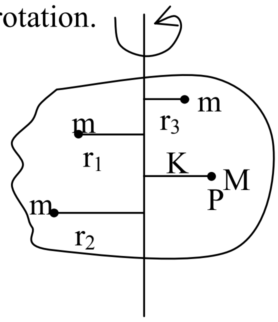
$$= \frac{mn(r_1^2 + r_2^2 + ... + r_n^2)}{n}$$

$$\Rightarrow \quad MK^2 = \frac{M(r_1^2 + r_2^2 + ... + r_n^2)}{n}$$

$$\Rightarrow \quad K = \sqrt{\frac{r_1^2 + r_2^2 + ... + r_n^2}{n}}$$

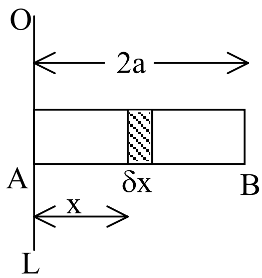
where n is the number of particles of the body, each of mass 'm' and $r_1, r_2 \ldots r_n$ be the perpendicular distances of these particles from axis of rotation.

Where M = m × n = total mass of body.



Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

Example:-**M.I. of a uniform rod of length '2a' about an axis passing through one end and ⊥ to the rod:-**



Let M = mass of rod of length 2a.

OL = axis of rotation passing through one end A and ⊥ to rod.

∴ Mass per unit length of rod = $\frac{M}{2a}$

Consider a small element of breadth δx at a distance 'x' from end A.

∴ Mass of this small element = $\frac{M}{2a}\delta x$

∴ M.I. of small element about axis OL or AL = $\frac{M}{2a}\delta x\, x^2$

∴ M.I. of rod about OL = $\int_0^{2a} \frac{M}{2a} x^2\, dx$

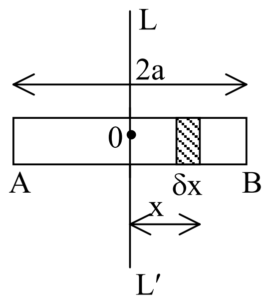
∴ $I = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$

$$I = \frac{M}{2a}\frac{8a}{3} = \frac{4}{3}Ma^2$$

$$\Rightarrow \boxed{I_{OL} = \frac{4}{3}\ Ma^2}$$

Example:- M.I. of a rod about an axis passing through mid-point and ⊥ to rod

Here LL′ is the axis of rotation passing through mid-point '0' of rod having length 2a.



Consider a small element of breadth δx at a distance 'x' from mid-point of rod O.

$\therefore$ Mass of this small element = $\frac{M}{2a}\delta x$

$\therefore$ M.I. of small element about LL′ = $\frac{M}{2a}\delta x.\ x^2$

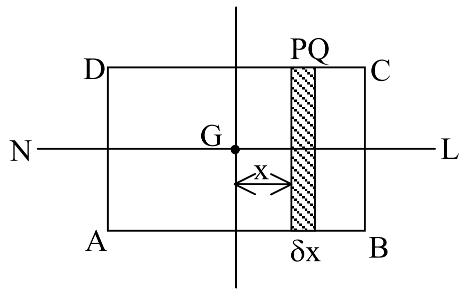
$\therefore$ M.I. of rod about LL′ = $\frac{M}{2a}\int_{-a}^{a} x^2 dx$

$$\therefore\ I_{LL'} = \frac{2M}{2a}\int_0^a x^2 dx = \frac{M}{a}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{M}{3a}a^3 = \frac{Ma^2}{3}$$

$$\Rightarrow \boxed{I_{LL'} = \frac{1}{3}Ma^2}$$

Example:-M.I. of a rectangular lamina about an axis (line) passing through centre and parallel to one side



Let ABCD be a rectangular lamina of mass 'M' and NL be the line about which M.I. is to be calculated.

Let AB = 2a, BC = 2b

Mass per unit area of lamina = $\frac{M}{4ab}$

Consider an elementary strip PQ of length (BC = 2b) and breadth δx and at a distance 'x' from G and parallel to AD.

$\therefore$ Mass of elementary strip = $\frac{M}{4ab} . 2b\,\delta x$

$= \frac{M}{2a}\delta x$

M.I. of this strip about NL $= \frac{b^2}{3}(\text{Mass of strip})$

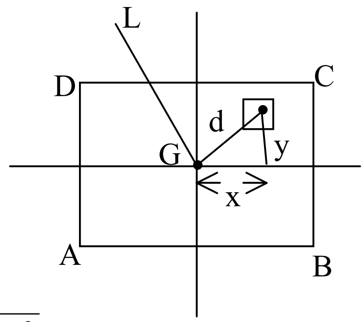
$= \frac{M}{2a}\delta x . \frac{b^2}{3}$

$\therefore$ M.I. of rectangular lamina about NL

$$= \frac{M}{2a}\frac{b^2}{3}\int_{-a}^{a}\delta x$$

$$\Rightarrow \quad I = \frac{M}{2a}\frac{b^2}{3}2a = \frac{Mb^2}{3}$$

Example:- M.I. of rectangular lamina about a line ⊥ to lamina and passing through centre :



Let GL = axis of rotation passing through centre 'G' and ⊥ to lamina ABCD. Consider a small element of surface area $\delta S = \delta x \delta y$

Here ⊥ distance of small element from axis GL, $d = \sqrt{x^2 + y^2}$

$\therefore$ Mass of small element = $\rho \delta x\, \delta y$

M.I. of this small element about GL

$$= \rho\ \delta x\ \delta y\ (x^2 + y^2)$$

$$\therefore \text{ M.I. of lamina} = \int_{-b}^{b}\int_{-a}^{a} \rho\,(x^2 + y^2)dx\ dy$$

$$= 4\rho \int_0^b\int_0^a (x^2 + y^2)\ dx\ dy$$

$$= 4\rho \int_0^b \left(\frac{x^3}{3} + xy^2\right)_0^a dy = 4\rho \int_o^b \left(\frac{a^3}{3} + ay^2\right) dy$$

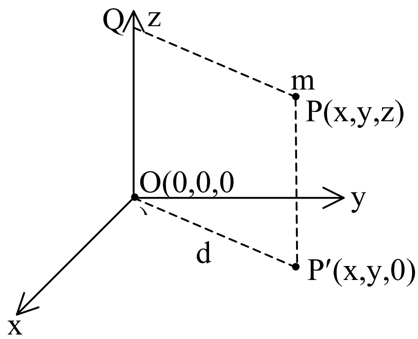
$$= 4\rho \left(\frac{a^3}{3} y + \frac{ay^3}{3}\right)_0^b$$

$$= \frac{4\rho a}{3}(a^2 b + b^3) = \frac{4\rho ab}{3}(a^2 + b^2)$$

$$\therefore\ I = \frac{M}{3}(a^2 + b^2) \qquad [\text{using mass of lamina } M = 4\rho\ ab]$$

1.2 Moments and products of inertia about co-ordinate axes:-

(I) For a particle system



Consider a single particle P of mass 'm' having co-ordinates (x, y, z)

Here $d = \perp$ distance of particle P of mass m from z-axis

$$= PQ = OP' = \sqrt{x^2 + y^2}$$

Therefore, M.I. of particle of mass 'm' about z-axis

$$= md^2 = m\,(x^2 + y^2)$$

$\therefore$ M.I. of **system** of particles about z-axis

$$I_{oz} = \Sigma md^2 = \Sigma m(x^2 + y^2)$$

And Standard notation for M.I about z-axis is C, i.e., $C = \Sigma m\,(x^2 + y^2) = I_{oz}$

Similarly, we can obtain M. I. about x and y-axis which are denoted as under:

About x-axis, $A = \Sigma(y^2 + z^2) = I_{ox}$

About y-axis, $B = \Sigma m\,(z^2 + x^2) = I_{oy}$

Product of Inertia

The quantities

$$D = \Sigma\ myz$$

$$E = \Sigma\ mzx \text{ and}$$

$$F = \Sigma\ mxy$$

are called products of inertia w.r.t. pair of axes (oy, oz), (oz, ox) and (ox, oy) respectively.

For a continuous body:

The M.I. about z-axis, x-axis and y-axis are defined as under

$$C = \iiint_V \rho(x^2 + y^2)\,dx\,dy\,dz$$

$$A = \iiint_V \rho\,(y^2 + z^2)dx\,dy\,dz$$

$$B = \iiint_V \rho\,(z^2 + x^2)dx\,dy\,dz$$

Similarly, the products of inertia w.r.t. pair of axes (oy, oz), (oz, ox) and (ox, oy) respectively are as under

$$D = = \iiint_V \rho\ yz\ dv\ ,E = \iiint_V \rho\ zx\ dV \quad \text{and } F = = \iiint_V \rho\ xy\ dV$$

For laminas in xy plane, we put z = 0, then

$$A = \iint_S \rho\ y^2\ dxdy$$

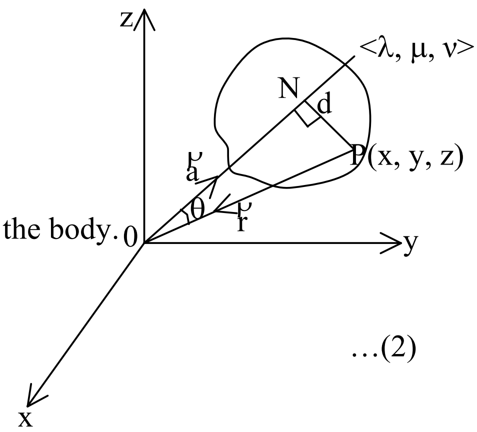
$$B = \iint_S \rho\ x^2\ dx\ dy$$

$$C = \iint_S \rho\ (x^2 + y^2)dxdy$$

$$D = E = 0, \quad F = \iint_S \rho\ xy\ dx\ dy$$

1.3 M.I. of a body about a line (an axis) whose direction cosines are <λ, μ, ν> :-

Let $\hat{a}$ is a unit vector in axis OL whose direction cosines are <λ, μ, ν>.



Then

$$\hat{a} = \lambda\hat{i} + \mu\hat{j} + \nu\hat{k} \qquad ...(1)$$

Let P(x, y, z) be any point(particle) of mass of the body. Then its position vector $\vec{r}$ is given by

$$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad ...(2)$$

⊥ distance of P from OL,

$$d = PN = OP\sin\theta = |\vec{r}\times\hat{a}| \qquad ...(3)$$

$$\Rightarrow \quad d = (x\hat{i} + y\hat{j} + z\hat{k})\times(\lambda\hat{i} + \mu\hat{j} + \nu\hat{k})$$

$$= |(\nu y - \mu z)\hat{i} + (\lambda z - \nu x)\hat{j} + (\mu x - \lambda y)\hat{k}|$$

$$= \sqrt{(\nu y - \mu z)^2 + (\lambda z - \nu x)^2 + (\mu x - \lambda y)^2}$$

$$\Rightarrow \quad d = \sqrt{\lambda^2(y^2+z^2) + \mu^2(z^2+x^2) + \nu^2(x^2+y^2) - 2\mu\nu yz - 2\lambda\nu xy - 2\lambda\mu xy}$$

Therefore, M.I. of body about an axis whose direction cosine are λ, μ, ν is

$$I_{OL} = \Sigma m\{\lambda^2(y^2+z^2) + \mu^2(z^2+x^2) + \nu^2(x^2+y^2)$$

$$- 2\mu\nu yz - 2\lambda\nu xz - 2\lambda\mu xy\}$$

$$\Rightarrow \quad I_{OL} = A\lambda^2 + B\mu^2 + C\nu^2 - 2\mu\nu D - 2\lambda\nu E - 2\lambda\mu F$$

1.4 Kinetic Energy (K.E.) of a body rotating about O:-

Let axis of rotation be $\overrightarrow{OL}$ through O, then angular velocity about OL is $\vec{w} = w\hat{a}$

Then K.E., $T = \Sigma\frac{1}{2}m\,(\vec{v}.\vec{v})$

$$= \frac{1}{2}\Sigma m|\vec{v}|^2$$

$$\Rightarrow \quad T = \frac{1}{2} = \Sigma m|\hat{a}\times\vec{r}|^2\, w^2 \qquad [\because\ \vec{v} = \vec{w}\times\vec{r} = w\hat{a}\times\vec{r}$$

$$= \frac{1}{2} w^2 \ \Sigma \ md^2 \qquad \text{[using equation (3)]}$$

$$\Rightarrow \qquad T = \frac{1}{2} w^2 I_{OL}$$

This is the required expression for kinetic energy in terms of moment of inertia

1.5 Parallel axis theorem

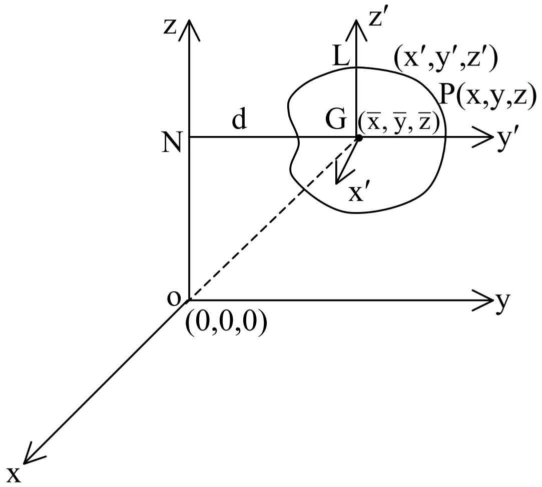
Statement:- For a body of mass 'M', we have

$$C = C' + Md^2$$

where C′ = M. I of body about a line GL through C.G.(centre of mass) and parallel to z-axis

C = M.I. of body about z-axis (i.e. a line parallel to GL) and at a distance 'd' from GL.

Proof :



Let M = Mass of body and P is any point whose co-ordinates w.r.t. oxyz are (x,y,z), G is the centre of mass whose co-ordinates w.r.t. oxyz are ($(\bar{x}, \bar{y}, \bar{z})$).

Let us introduce a new co-ordinate system Gx′y′z′ through G and Co-ordinates of P w.r.t. this system are (x′,y′,z′). Let $\vec{r}_G$ be the position vector of G and r_i position vector of mass m_i w.r.t. oxyz system. Now by definition of centre of mass of body,

$$\vec{r}_G = \frac{\Sigma m_i r_i}{M}$$

when centre of mass cocides with origin at G w.r.t new co-ordinate system G x′y′z′, we have $\vec{r}_G = 0$. Therefor

$$\frac{\Sigma m_i r_i}{M} = 0 \quad \Rightarrow \Sigma m_i r_i = 0$$

$$\Rightarrow \quad \frac{\Sigma m x'}{M} = 0, \quad \frac{\Sigma m y'}{M} = 0, \quad \frac{\Sigma m z'}{M} = 0$$

where

$$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

So, we have

$$\Sigma m x' = \Sigma m y' = \Sigma m z' = 0 \qquad \ldots(1)$$

Now $d^2 = (GN)^2 = (OG)^2 - (ON)^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{z}^2$

$$= \bar{x}^2 + \bar{y}^2 \qquad \ldots(2)$$

Co-ordinates of P w.r.t. (ox, oy, oz) axes is (x, y, z)

Co-ordinates of P w.r.t. (Gx′, Gy′, Gz′) axes is (x′, y′, z′)

Then $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$

Thus, M.I. about z-axis is

$$C = \Sigma m(x^2 + y^2)$$

$$= \Sigma m[(\bar{x} + x')^2 + (\bar{y} + y')^2]$$

$$\Rightarrow \quad C = \Sigma m\,[\bar{x}^2 + x'^2 + 2\bar{x}x' + \bar{y}^2 + y'^2 + 2\bar{y}y']$$

$$= \Sigma m(x'^2 + y'^2) + \Sigma m(\bar{x}^2 + \bar{y}^2) + 2\bar{x}\,\Sigma m x' + 2\bar{y}\,\Sigma m y'$$

$$C = \Sigma m\,(x'^2 + y'^2) + (\bar{x}^2 + \bar{y}^2)\,\Sigma m + 0 \qquad \text{[from (1)]}$$

$$\Rightarrow \quad C = C' + Md^2 \qquad \text{[using (2) and } \Sigma m = M\text{, total mass]}$$

Similarly, M.I. about x and y-axis are given by

$$A = A' + Md^2$$

$$B = B' + Md^2$$

where d is perpendicular distance of P from x and y-axis

For Product of Inertia

Here Product of Inertia w.r.t. pair (ox,oy) is

$F = \Sigma mxy = \Sigma m(\bar{x} + x')(\bar{y} + y')$

$= \Sigma m(\bar{x}\bar{y} + x'\bar{y} + \bar{x}y' + x'y')$

$= \Sigma m\, x'\, y' + \bar{x}\bar{y}\, \Sigma m + \bar{x}\, \Sigma m\, y' + \bar{y}\, \Sigma m\, x'$

$= F' + M\, \bar{x}\bar{y} + 0$ [using (1)]

$\Rightarrow \quad F = F' + M\, \bar{x}\bar{y}$

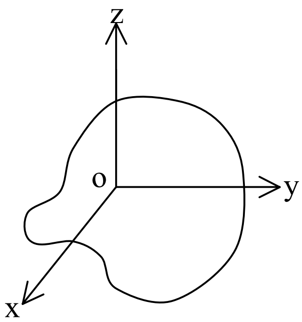
Similarly, for products of Inertia w.r.t. pair (oy, oz) and (oz, ox) respectively, we have $D = D' + M\bar{y}\bar{z}$ and $E = E' + M\bar{z}\bar{x}$.

1.6 Perpendicular axis theorem

(For Two dimensional bodies or mass distribution)

Statement:- The M.I. of a plane mass distribution (lamina) w.r.t. any normal axis is equal to sum of the moments of inertia about any two ⊥ axis in the plane of mass distribution (lamina) and passing through the intersection of the normal with the lamina.

Proof :



Let ox, oy are the axes in the plane of lamina and oz be the normal axis, i.e., xy is the plane of lamina.

Let C is the M.I. about ⊥ axis, i.e., oz axis

Here to prove $C = A + B$

By definition, M.I. of plane lamina about z-axis,

$C = \iint_S \rho\,(x^2 + y^2)\, dS$ [for a continuous body]

$= \iint_S \rho\, x^2\, dS + \iint_S \rho\, y^2\, dS$

$\Rightarrow \quad C = B + A$

For mass distribution,

$C = \Sigma m\,(x^2 + y^2) = \Sigma m\, x^2 + \Sigma m y^2$

$\Rightarrow \quad C = B + A$

For two dimensional body, $D = E = 0$ and $F = \Sigma mxy$.

Converse of perpendicular axis theorem :

Given $\quad C = A + B$

To prove it is a plane lamina.

Proof:- Here $A = \Sigma m(y^2 + z^2)$

$B = \Sigma m(z^2 + x^2)$, $C = \Sigma M(x^2 + y^2)$

Now given $C = A + B$

$\Rightarrow \Sigma(x^2 + y^2) = \Sigma m\,(y^2 + z^2) + \Sigma m(z^2 + x^2)$

$= \Sigma m(y^2 + 2z^2 + x^2)$

$\Rightarrow \Sigma mx^2 + \Sigma my^2 = \Sigma my^2 + 2\Sigma mz^2 + \Sigma mx^2$

$\Rightarrow \quad 2\Sigma mz^2 = 0$

$\Rightarrow \quad \Sigma mz^2 = 0$ for all distribution of mass.

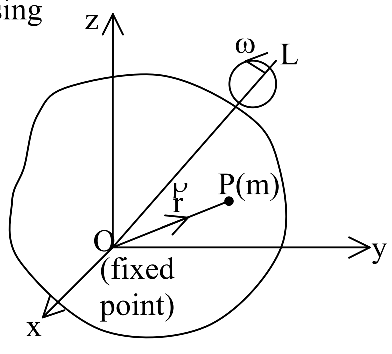
For a single particle of mass 'm',

$mz^2 = 0 \Rightarrow z = 0$ as $m \neq 0$

$\Rightarrow$ It is a plane mass distribution or it is a plane lamina.

1.7 Angular momentum of a rigid body about a fixed point and about a fixed axis:- The turning effect of a particle about the axis of rotation is called angular Momentum.

Let O be the fixed point and OL be an axis passing through the fixed point.



$\vec{w}$ = angular velocity about $\overrightarrow{OL}$

$\vec{r}$ = position vector of P(x, y, z)

$\Rightarrow \quad \vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Also linear velocity of P, $\vec{v} = \vec{w} \times \vec{r}$ …(1)

The angular momentum of body about O is

$$\vec{H} = \Sigma(\vec{r} \times m\vec{v}) = \Sigma[\vec{r} \times m(\vec{w} \times \vec{r})] \quad \text{…(2)}$$

$$\vec{H} = \Sigma m\, [\vec{r} \times (\vec{w} \times \vec{r})]$$

$$= \Sigma m\, [(\vec{r}.\vec{r})\vec{w} - (\vec{r}.\vec{w})\vec{r}]$$

$$[\because\ A \times (B \times C) = (A.\ C)\, B - (A\ .\ B)C]$$

$$= \Sigma m\ r^2\vec{w} - (\vec{r}.\vec{w})\vec{r}]$$

$\Rightarrow$ $$\vec{H} = (\Sigma m r^2)\vec{w} - \Sigma m(\vec{r}.\vec{w})\vec{r} \quad \ldots(3)$$

If $\vec{H} = h_1\,\hat{i} + h_2\,\hat{j} + h_3\,\hat{k}$

$$\vec{w} = w_1\,\hat{i} + w_2\,\hat{j} + w_3\,\hat{k} \quad \ldots(4)$$

Then $\vec{r}.\vec{w} = w_1 x + w_2 y + w_3 z$

$\therefore$ from (3),

$$h_1\ \hat{i} + h_2\,\hat{j} + h_3\,\hat{k} = (\Sigma m\ r^2)(w_1\hat{i} + w_2\,\hat{j} + w_3\,\hat{k}) -$$
$$\Sigma m(w_1x + w_2y + w_3z)\ (x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

Equating coefficients of $\hat{i}$ on both sides,

$$h_1 = \Sigma m\ (x^2 + y^2 + z^2)\ w_1 - \Sigma m(w_1x + w_2y + w_3z)x$$

$$= \Sigma m(y^2 + z^2)w_1 + \Sigma m\ x^2w_1 - \Sigma m w_1\ x^2 - \Sigma m(w_2y + w_3z)x$$

$$= \Sigma m(y^2 + z^2)\ w_1 - (\Sigma m\ xy)\ w_2 - (\Sigma m\ xz)w_3$$

$\therefore$ $$h_1 = Aw_1 - Fw_2 - Ew_3$$

Similarly,

$$h_2 = Bw_2 - Dw_3 - Fw_1$$

$$h_3 = Cw_3 - Ew_1 - Dw_2 \quad \ldots(5)$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Inertia matrix (symmetric 3 × 3 matrix)

1.8 Principal axis and their determination

Definition :- If the axis of rotation $\vec{w}$ is parallel to the angular momentum $\vec{H}$, then the axis is known as principal axis.

If $\vec{w} = |\vec{w}|\hat{a} = w\hat{a}$

$\hat{H} = |\vec{H}|\hat{a} \quad \Rightarrow \quad \vec{H} = n\,\vec{w}$, where n is a constant

$\Rightarrow \quad H = nw$

1.8.1 Theorem:- Prove that in general, there are three principal axes through a point of rigid body.

Proof : For principal axis,

$$\vec{H} = n\vec{w} \Rightarrow H = nw \qquad \ldots(1)$$

Let $\vec{H} = H\hat{a}, \quad \vec{w} = w\hat{a}$

where $\hat{a}$ is a unit vector along principal axis of body through O.

By definition of $\vec{H}$,

$$\vec{H} = \Sigma(\vec{r} \times m\vec{v})$$

$$\Rightarrow \quad \vec{H} = (\Sigma mr^2)\vec{w} - \Sigma m(\vec{r}.\vec{w})\vec{r}$$

Using $\vec{H} = n\vec{w}$, we get

$$n\vec{w} = (\Sigma mr^2)\vec{w} - \Sigma m(\vec{r}.\vec{w})\vec{r}$$

Using $\vec{w} = w\hat{a}$,

$$n\, w\hat{a} = \Sigma mr^2\, w\hat{a} - \Sigma m(\vec{r}.w\hat{a})\vec{r}$$

Cancelling w on both sides & rearranging,

$$(\Sigma mr^2 - n)\hat{a} = \Sigma m(\vec{r}.\hat{a})\vec{r} \qquad \ldots(2)$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \qquad \hat{a} = \lambda\hat{i} + \mu\hat{j} + \nu\hat{k} \qquad \ldots(3)$

where <λ, μ, ν> are direction cosine of principal axis.

$$\Rightarrow \quad (\Sigma mr^2 - n)\,(\lambda\hat{i} + \mu\hat{j} + \nu\hat{k}) = \Sigma m[\lambda x + \mu y + \nu z)\,(x\hat{i} + y\hat{j} + z\hat{k})\,]$$

Equating coefficients of $\hat{i}$ on both sides,

$$\Rightarrow \quad [\Sigma m(x^2 + y^2 + z^2) - n]\,\lambda = \Sigma m(\lambda x^2 + \mu xy + \nu xz)$$

$$\Rightarrow \quad [\Sigma m(y^2 + z^2) - n]\lambda = \Sigma m[\mu xy + \nu xz] \quad \text{[canceling } \Sigma m\,\lambda x^2 \text{ on both sides]}$$

$$\Rightarrow \quad (A - n)\lambda - F\mu - E\nu = 0$$

Similarly $\quad (B - n)\mu - D\nu - F\lambda = 0 \qquad \ldots(4)$

$$(C - n)\nu - E\lambda - D\mu = 0$$

or
$$(A - n)\lambda - F\mu - E\nu = 0$$
$$-F\lambda + (B - n)\mu - D\nu = 0 \qquad \ldots(5)$$
$$-E\lambda - D\mu + (C - n)\nu = 0$$

Equation (5) has a non-zero solution only if

$$\begin{vmatrix} A-n & -F & -E \\ -F & B-n & -D \\ -E & -D & C-n \end{vmatrix} = 0 \qquad \ldots(6)$$

This determinental equation is a cubic in n and it is called characteristic equation of symmetric inertia matrix. This characterstic equation has three roots n_1, n_2, n_3 (say), so n_1, n_2, n_3 are real. Corresponding to $n = (n_1, n_2, n_3)$ (solving equation (5) or (6) for $<\lambda, \mu, \nu>$).

Let the values of (λ, μ, ν) be

$(\lambda_1, \mu_1, \nu_1) \rightarrow n = n_1$

$(\lambda_2, \mu_2, \nu_2) \rightarrow n = n_2$

$(\lambda_3, \mu_3, \nu_3) \rightarrow n = n_3$

These three sets of value determine three principal axes $\hat{a}_1, \hat{a}_2, \hat{a}_3$ given by

$\hat{a}_p = \lambda_p \hat{i} + \mu_p \hat{j} + \nu_p \hat{k}$ where p = 1, 2, 3.

1.8.2 Theorem :- Three principal axes through a point of a rigid body are mutually orthogonal.

Proof : Let the three principal axes corresponding to roots n_1, n_2, n_3 of characteristic equation

$$\begin{vmatrix} A-n & -F & -E \\ -F & B-n & -D \\ -E & -D & C-n \end{vmatrix} = 0$$

be $\hat{a}_1, \hat{a}_2, \hat{a}_3$.

Let $\hat{n}_1, \hat{n}_2, \hat{n}_3$ are all different.

Then from equation,

$$(\Sigma m\, r^2 - n)\ \hat{a} = \Sigma m(\vec{r}.\hat{a})\vec{r}$$

We have

$$(\Sigma mr^2 - n_1)\,\hat{a}_1 = \Sigma m(\vec{r}.\vec{a}_1)\vec{r} \qquad \ldots(1)$$

$$(\Sigma mr^2 - n_2)\ \hat{a}_2 = \Sigma m(\vec{r}.\hat{a}_2)\vec{r} \qquad \ldots(2)$$

$$(\Sigma mr^2 - n_3)\ \hat{a}_3 = \Sigma m(\vec{r}.\hat{a}_3)\vec{r} \qquad \ldots(3)$$

Multiply scalarly equation (1) by $\hat{a}_2$ and equation (2) with $\hat{a}_1$ and then substracts, we get

$$(n_1 - n_2)\,\hat{a}_1.\hat{a}_2 = 0$$

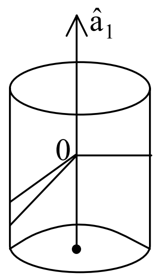
$\Rightarrow \quad \hat{a}_1.\hat{a}_2 = 0$ as $n_1 \neq n_2$

Similarly $\hat{a}_2, \hat{a}_3 = 0$ and $\hat{a}_3.\hat{a}_1 = 0$

$\Rightarrow \quad \hat{a}_1, \hat{a}_2, \hat{a}_3$ are mutually orthogonal.

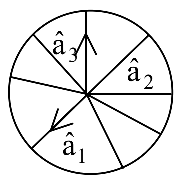
Remarks :- (i) If $n_1 \neq n_2 \neq n_3$ then there are exactly three mutually $\perp$ axis through 0.

(ii) If $n_2 = n_3$ (i.e. two characteristic roots are equal). There is one principal axis corresponding to n_1 through 0. Then every line through 0 & $\perp$ to this $\hat{a}_1$ is a principal axis. Infinite set of principal axis with the condition that $\hat{a}_1$ is fixed.



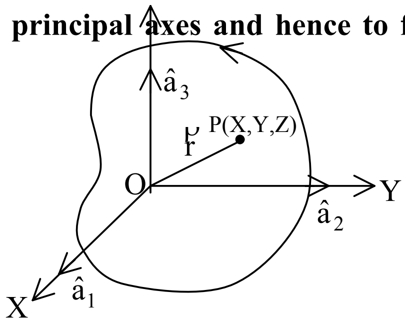
(iii) If $n_1 = n_2 = n_3$, then

Any three mutually $\perp$ axes through O(centre of sphere) are principal axes.



Z

1.9 Moments and products of Inertia about principal axes and hence to find angular momentum of body.



Let $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are the principal axes.

Let us take co-ordinates axes along the principal axes.

$$\vec{r} = \overrightarrow{OP} = X\hat{a}_1 + Y\hat{a}_2 + Z\hat{a}_3$$

$$\therefore \quad r^2 = X^2 + Y^2 + Z^2$$

From equation

$$(\Sigma mr^2 - n)\ \hat{a} = \Sigma m(\vec{r}.\hat{a})\vec{r}$$

We have

$$(\Sigma mr^2 - n_1)\ \hat{a}_1 = \Sigma m(\vec{r}.\hat{a}_2)\vec{r} \qquad \ldots(1)$$

$$(\Sigma mr^2 - n_2)\ \hat{a}_2 = \Sigma m(\vec{r}.\hat{a}_2)\vec{r} \qquad \ldots(2)$$

$$(\Sigma mr^2 - n_3)\ \hat{a}_3 = \Sigma m(\vec{r}.\hat{a}_3)\vec{r} \qquad \ldots(3)$$

From (1),

$$(\Sigma mr^2 - n_1)\hat{a}_1 = \Sigma m[(X\hat{a}_1 + Y\hat{a}_2 + Z\hat{a}_3).\hat{a}_1[X\hat{a}_1 + Y\hat{a}_2 + Z\hat{a}_3]$$

$$= \Sigma m\times(X\hat{a}_1 + Y\hat{a}_2 + Z\hat{a}_3)$$

Equating coefficients of $\hat{a}_1\ \hat{a}_2, \hat{a}_3$,

$$\Sigma m(X^2 + Y^2 + Z^2) - n_1 = \Sigma mX^2$$

$$0 = \Sigma mXY$$

$$0 = \Sigma mXZ \qquad \ldots(4)$$

or $\quad n_1 = \Sigma m(Y^2 + Z^2) = A^*$

& $\quad F^* = 0, \quad E^* = 0$

Similarly from (2) & (3), we get

$$n_2 = B^*,\ D^* = 0,\ F^* = 0$$

& $\quad n_3 = C^*, E^* = 0, D^* = 0$

where A*, B*, C* are M.I. and D*, E*, F* are product of Inertia about principal axes.

Inertia matrix for principal axes through O is

$$\begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix} = \begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & C^* \end{pmatrix}$$

Definition: Three mutually ⊥ lines through any point of a body which are such that the product of inertia about them vanishes are known as **principal axes**.

Expression for angular momentum ($\vec{H}$)

Here $D^* = E^* = F^* = 0$, then from equation,

$$h_1 = Aw_1 - Fw_2 - Ew_3$$

we have

$$h_1 = A^*w_1 - F^*w_2 - E^* w_3$$

$$\Rightarrow \quad h_1 = A^*w_1 \qquad [\because F^* = E^* = 0]$$

Similarly $h_2 = B^* w_2$, $h_3 = C^* w_3$

$$\therefore \quad \vec{H} = h_1 \hat{a}_1 + h_2 \hat{a}_2 + h_3 \hat{a}_3$$

$$= A^* w_1 \hat{a}_1 + B^* w_2 \hat{a}_2 + C^* w_3 \hat{a}_3$$

where (w_1, w_2, w_3) are components of angular velocity about $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$.

A*, B*, C* are also called principal moments of inertia.

1.10 Momental Ellipsoid:-

We know that M.I., I_{OL} of a body about the line whose d.c.'s are $<\lambda, \mu, \nu>$ is

$$I_{OL} = I = A\lambda^2 + B\mu^2 + C\nu^2 - 2D\mu\nu - 2E\nu\lambda - 2F\lambda\mu \qquad \ldots(1)$$

Let P(x,y, z) be any point on OL and OP = R, then $\vec{R} = R(\lambda\hat{i} + \mu\hat{j} + \nu\hat{k}) = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \quad \lambda = \frac{x}{R}, \quad \mu = \frac{y}{R}, \quad \nu = \frac{z}{R} \quad \ldots(2)$$

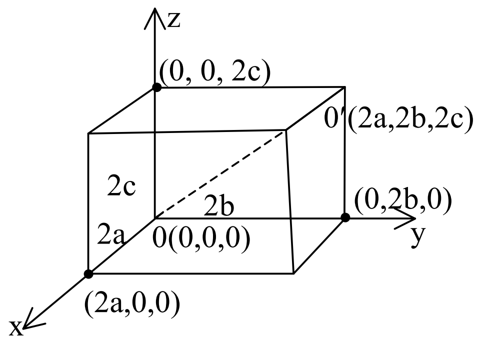
Now let P moves in such a way that IR^2 remains constant, then from (1), (2), we get

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = IR^2 = \text{constant}$$

Since coefficients of x^2, y^2, z^2 i.e. A, B, C all are positive, this equation represents an ellipsoid known as momental ellipsoid.

Example:- A uniform solid rectangular block is of mass 'm' and dimension 2a × 2b × 2c. Find the equation of the momental ellipsoid for a corner 'O' of the block, referred to the edges through O as co-ordinates axes and hence determine M.I. about OO′ where O′ is the point diagonally opposite to O.

Solution :



Taking x, y, z axes along the edges of lengths 2a, 2b, 2c, we obtain

$$A = \iiint_V \rho(y^2 + z^2)dv$$

$$= \int_0^{2a}\int_0^{2b}\int_0^{2c} \rho\,(y^2 + z^2)dz\,dy\,dx$$

$$= \int_0^{2a}\int_0^{2b} \rho\left(y^2 z + \frac{z^3}{3}\right)_0^{2c} dy\,dx$$

$$= \rho\int_0^{2a}\int_0^{2b}\left(y^2\,2c + \frac{1}{3}8c^3\right)dy\,dx$$

$$= \rho.\ 2c \int_0^{2a}\int_0^{2b} \left(y^2 + \frac{4}{3}c^2 \right) \ dy\ dx$$

$$= \rho.\ 2c \int_0^{2a} \left(\frac{y^3}{3} + \frac{4}{3}c^2 y \right)_0^{2b} \ dx$$

$$= \frac{\rho 2c}{3} \int_0^{2a} (8b^3 + 4c^2\ 2b)\ dx$$

$$= \rho \frac{2c}{3} 8b \int_0^{2a} (b^2 + c^2) dx$$

$$= \rho \frac{16bc}{3} (b^2 + c^2) 2a = (8abc\ \rho)\ \frac{4}{3} (b^2 + c^2)$$

$\Rightarrow$ $$A = \frac{4M}{3} (b^2 + c^2)$$

Similarly $B = \dfrac{4M}{3}(c^2 + a^2),\ \ C = \dfrac{4M}{3}(a^2 + b^2)$

$$D = \iiint_V \rho yz\ dV = \rho \int_0^{2a}\int_0^{2b}\int_0^{2c} yz\ \ dz\ dy\ dx$$

$$= \pi \int_0^{2a}\int_0^{2b} y \left[\frac{z^2}{2} \right]_0^{2c} \ dy\ dx$$

$$= \frac{\rho}{2} \int_0^{2a}\int_0^{2b} y(4c^2)\ dy\ dx$$

$$= 2c^2\rho \int_0^{2a}\int_0^{2b} y\ dy\ dx = 2c^2\rho \int_0^{2a} \left[\frac{y^2}{2} \right]_0^{2b} \ dx$$

$$= c^2\ \rho \int_0^{2a} 4b^2\ \ dx = 4b^2c^2\ \rho \int_0^{2a} dx$$

$\Rightarrow$ $$D = 4b^2c^2\rho.\ 2a = (8abc\ \rho)bc = M\ bc$$

Similarly $E = Mca$, $\quad F = Mab$

Using these in standard equation of momental ellipsoid, we get

$$\frac{4M}{3}[(b^2 + c^2)x^2 + (c^2 + a^2)\,y^2 + (a^2 + b^2)z^2]$$

$$- 2M[bcyz + cazx + ab\,xy] = IR^2 \qquad \ldots(1)$$

which is required equation of momental ellipsoid.

To find M.I. about OO′ :-

using x = 2a, y = 2b, z = 2c as O′(2a, 2b, 2c)

and $R^2 = 4(a^2 + b^2 + c^2)$

from (1),

$$I_{OO'} = \frac{\frac{4M}{3}[(b^2+c^2)4a^2+(c^2+a^2)4b^2+(a^2+b^2)4c^2]-8M(b^2c^2+c^2a^2+a^2b^2)}{4(a^2+b^2+c^2)}$$

$$\Rightarrow\ I_{OO'} = \frac{8M}{3}\left[\frac{2(2a^2b^2+2b^2c^2+2c^2a^2)-3(a^2c^2+a^2b^2+b^2c^2)}{4(a^2+b^2+c^2)}\right]$$

$$\Rightarrow\ I_{OO'} = \frac{2M}{3}\frac{(b^2c^2+c^2a^2+a^2b^2)}{(a^2+b^2+c^2)}$$

Lesson-2 Moment of Inertia-2

2.1 Equimomental Systems:- Two systems are said to be equimomental if they have equal M.I. about every line in space.

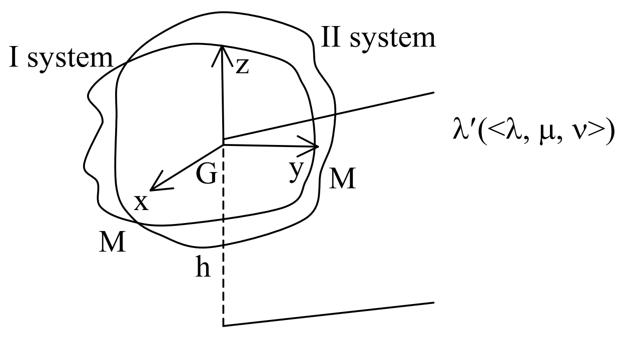
2.1.1 Theorem:- The necessary and sufficient condition for two systems to be equimomental are :

(i) They have same total mass.

(ii) They have same centroid.

(iii) They have same principal axes.

Proof:- Part A : The condition (i) to (iii) are sufficient. Here we assume that if (i) to (iii) hold, we shall prove that two systems are equimomental. Let M be the total mass of each system.



Let G be the common centroid of both the system. Let A*, B*, C* be the principal M.I. about principal axes through G for both the systems. Let λ be any line in space with d.c. <λ, μ, ν>. We draw a line λ′ similar to λ passing through G. Let h = ⊥ distance of G fromλ.

M.I. about λ′ for both the system is

$$I_{\lambda'} = A^*\lambda^2 + B^*\mu^2 + C^*\nu^2$$

[Θ Product of inertia about principal axes i.e. D* = E* = F* = 0]

So by parallel axes theorem, the M.I. of both the system about λ is

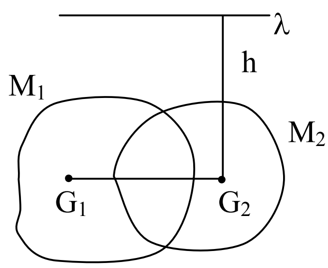
$$I_\lambda = I_{\lambda'} + Mh^2$$

$\Rightarrow \quad I_\lambda = A^*\lambda^2 + B^*\mu^2 + C^*\nu^2 + Mh^2$

Hence both the system have same M.I. about any line of space. So they are equimomental.

Part B:- The conditions are necessary. Here we assume that the two systems are equimomental and derive condition (i) to (iii). Let M_1 and M_2 be the total masses of the two systems respectively and G_1 & G_2 are their centroid respectively.

Condition (i)



Since the systems are equimomental i.e. they have same M.I., 'I' (say) about line G_1G_2(in particular). Let λ be the line in space which is parallel to G_1 G_2 at a distance h. Then by parallel axes theorem, M.I. of Ist system about $\lambda = I + M_1h^2$ and M.I. of IInd system about $\lambda = I + M_2h^2$.

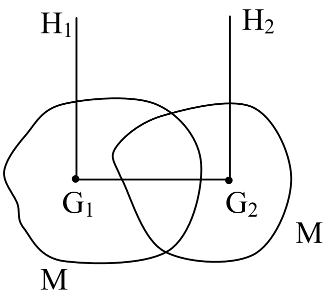
Since the two systems are equimomental, therefore we have,

$$I + M_1h^2 = I + M_2h^2$$

$\Rightarrow \quad M_1 = M_2 = M$ (say)

This implies that both the systems have same total mass.

Condition (ii)



Let G_1H_1 and G_2H_2 be two parallel lines each being $\perp$ to $G_1\, G_2$. Let I* be the M.I. of either system about a line G_1H_1 and $\perp$ to G_1G_2 (through G_1)

Using parallel axes theorem,

M.I. of Ist system about $G_2H_2 = I^* + M\,(G_1G_2)^2$

M.I. of IInd system about $G_2H_2 = I^* - M\,(G_1G_2)^2$

As the systems are equimomental, therefore

$$I^* + M\,(G_1G_2)^2 = I^*\, M(G_1G_2)^2$$

$\Rightarrow \quad (G_1G_2)^2 = 0$ as $M \neq 0$

$\Rightarrow \quad G_1 = G_2 = G$ (say)

$\Rightarrow \quad$ Both the systems have same centroid.

Condition (iii):- Since the two systems are equimomental, they have the same M.I. about every line through their common centroid. Hence they have same principal axes and principal moments of inertia.

2.2 Coplanar distribution:-

2.2.1Theorem:- (i) Show that for a two dimensional mass distribution (lamina), one of the principal axes at O is inclined at an angle θ to the x-axis through O

such that $\quad \tan 2\theta = \dfrac{2F}{B-A}$

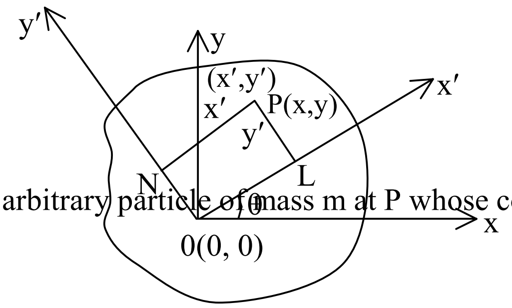
where A, B, F have their usual meanings.

(ii) Show that maximum and minimum values of M.I. at O are attained along principal axes.

OR

Theorem: - For a 2-D mass distribution (lamina), the value of maximum and minimum M.I. about lines passing through a point O are attained through principal axes at O.

Proof :-



Let us consider an arbitrary particle of mass m at P whose coordinates w. r.t. axes through O are (x,y), then for mass distribution, we have

M.I. about x-axis i.e. $A = \Sigma my^2$

M.I. about y-axis i.e. $B = \Sigma mx^2$ …(1)

and Product of inertia $F = \Sigma mxy$

We take another set of ⊥ axes ox′, oy′ such that ox′ is inclined at an angle θ with x-axis.

Then equation of line ox′ is given by

$$y = x \tan\theta$$

$$\Rightarrow \quad y \cos\theta - x \sin\theta = 0 \qquad \ldots(2)$$

Changing θ to $\theta + \frac{\pi}{2}$, equation of oy′ is

$$-y \sin\theta - x \cos\theta = 0$$

$$\Rightarrow \quad y \sin\theta + x \cos\theta = 0 \qquad \ldots(3)$$

Let P(x′, y′) be co-ordinates of P relative to new system of axes ox′, oy′, then

PL = y′ = length of ⊥ from P on ox′

$$= \frac{y\cos\theta - x\sin\theta}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$= y\cos\theta - x\sin\theta \qquad \ldots(4)$$

Similarly x′ = PN = length of ⊥ from P on oy′

$$= \frac{y\sin\theta + x\cos\theta}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$= y\sin\theta + x\cos\theta \qquad \ldots(5)$$

Therefore,

M.I. of mass distribution (lamina) about ox′ is

$$I_{ox'} = \Sigma my'^2 = \Sigma m(y\cos\theta - x\sin\theta)^2$$

$$= \Sigma\, m(y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta)$$

$$= \cos^2\theta\, \Sigma\, my^2 + \sin^2\theta\, \Sigma\, mx^2 - 2\sin\theta\cos\theta\, \Sigma mxy$$

$$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta \qquad \ldots(6)$$

Similarly M.I. of mass distribution (lamina) about oy′is given by

$$I_{oy'} = A\cos^2\left(\frac{\pi}{2}+\theta\right) + B\sin^2\left(\frac{\pi}{2}+\theta\right) - F\sin 2\left(\frac{\pi}{2}+\theta\right)$$

$$= A\sin^2\theta + B\cos^2\theta + F\sin 2\theta \qquad \ldots(7)$$

Product of inertia w.r.t pair of axes (ox′, oy′),

$$I_{x'y'} = \Sigma m x' y'$$

$$= \Sigma m(y\sin\theta + x\cos\theta)(y\cos\theta - x\sin\theta)$$

$\Rightarrow$ $$I_{x'y'} = \sin\theta\cos\theta\,\Sigma my^2 - \sin\theta\cos\theta\,\Sigma\, mx^2 - \sin^2\theta\,\Sigma mxy + \cos^2\theta\,\Sigma mxy$$

$$= A\sin\theta\cos\theta - B\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)F$$

$$= (A-B)\frac{\sin 2\theta}{2} + F\cos 2\theta \qquad \ldots(8)$$

The axes ox′, oy′ will be principal axes if

$$I_{x'y'} = 0$$

Using equation (8),

$$\frac{1}{2}(A-B)\sin 2\theta + F\cos 2\theta = 0$$

$\Rightarrow$ $$\tan 2\theta = \frac{2F}{B-A}$$

$\Rightarrow$ $$\theta = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} \qquad \ldots(9)$$

This determines the direction of principal axes relative to co-ordinates axes. We shall now show that maximum/minimum (extreme) values of $I_{ox'}$, $I_{oy'}$ are obtained when θ is determined from (9),

We rewrite, $I_{ox'}$ and $I_{oy'}$ as

$$I_{ox'} = \frac{1}{2}(A+B) - \frac{1}{2}[(B-A)\cos 2\theta + 2F\sin 2\theta]$$

$$I_{oy'} = \frac{1}{2}(A+B) + \frac{1}{2}[(B-A)\cos 2\theta + 2F\sin 2\theta]$$

For maximum and minimum value of $I_{ox'}$, $I_{oy'}$,

$$\frac{d}{d\theta}(I_{ox'}) = 0 \quad \text{and} \quad \frac{d}{d\theta}(I_{oy'}) = 0$$

i.e. $\frac{d}{d\theta}$ [(B–A) cos 2θ + 2F sin 2θ] = 0

$\Rightarrow$ $-(B-A)2 \sin 2\theta + 4F \cos 2\theta = 0$

$$\Rightarrow \quad \tan 2\theta = \frac{2F}{B-A} \qquad \ldots(11)$$

Similarly $\frac{d}{d\theta}$ [(B–A) cos 2θ + 2F sin 2θ] = 0

$\Rightarrow$ $-(B-A)\, 2 \sin 2\theta + 4F \cos 2\theta = 0$

$$\Rightarrow \quad \tan 2\theta = \frac{2F}{B-A}$$

So extreme values of $I_{ox'}$ and $I_{oy'}$ are attained for θ given by equation (11) already obtained in (9).

Therefore, the greatest and least values of M.I. for mass distribution (lamina) through O are obtained along the principal axes.

The extreme values are obtained as under

We have, $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2F}{B-A}$

$$\Rightarrow \quad \frac{\sin 2\theta}{2F} = \frac{\cos 2\theta}{B-A} = \frac{1}{\sqrt{4F^2 + (B-A)^2}}$$

$$\Rightarrow \quad \sin 2\theta = \frac{2F}{\sqrt{4F^2 + (B-A)^2}}$$

$$\& \cos 2\theta = \frac{B-A}{\sqrt{4F^2 + (B-A)^2}}$$

Now writing

$$I_{ox'} = \frac{1}{2}(A+B) - \frac{1}{2}[(B-A)\cos 2\theta + 2F \sin 2\theta]$$

Using values of cos 2θ and sin 2θ , we obtain the extreme values of $I_{ox'}$ and $I_{oy'}$ as under

$$I_{ox'} = \frac{1}{2}(A+B) - \frac{1}{2}\left[\frac{(B-A)(B-A)}{\sqrt{4F^2+(B-A)^2}} + \frac{4F^2}{\sqrt{4F^2+(B-A)^2}}\right]$$

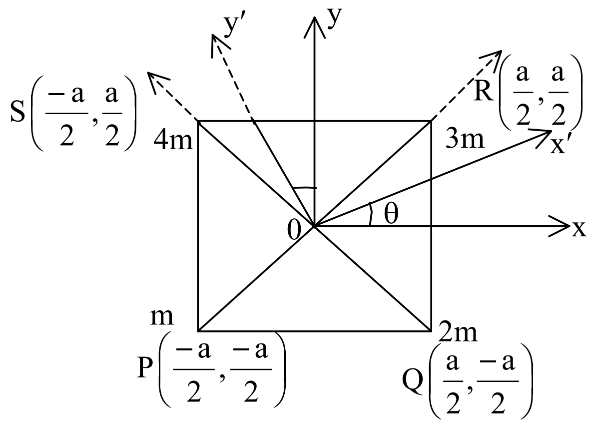
$$= \frac{1}{2}(A+B) - \frac{1}{2}\left[\frac{(B-A)^2+4F^2}{\sqrt{4F^2+(B-A)^2}}\right]$$

$$= \frac{1}{2}(A+B) - \frac{1}{2}[\sqrt{4F^2+(B-A)^2}]$$

Similarly $I_{oy'} = \frac{1}{2}(A+B) + \frac{1}{2}[\sqrt{4F^2+(B-A)^2}]$

Example 1:- A square of side 'a' has particles of masses m, 2m, 3m, 4m at its vertices. Show that the principal M. I. at centre of the square are $2ma^2$, $3ma^2$, $5ma^2$. Also find the directions of principal axes.

Solution :



Taking origin O at the centre of square and axes as shown in the figure, we have

A = M.I. of system of particles about x-axis.

$$= \sum_{i=1}^{4} m_i y_i^2 = m\left(\frac{-a}{2}\right)^2 + 2m\left(\frac{-a}{2}\right)^2 + 3m\left(\frac{a}{2}\right)^2 + 4m\left(\frac{a}{2}\right)^2$$

$\Rightarrow$ $A = \frac{5}{2}ma^2$ ...(1)

$$B = \Sigma m_i x_i^2 = m\left(\frac{-a}{2}\right)^2 + 2m\left(\frac{-a}{2}\right)^2 + 3m\left(\frac{a}{2}\right)^2 + 4m\left(\frac{a}{2}\right)^2$$

$\Rightarrow \quad B = \frac{5}{2} ma^2$ …(2)

$\therefore \quad C = B + A = 5ma^2$

For a two-dimensional mass distribution, D = E = 0 and

$$F = \Sigma m_i x_i y_i = m\left(\frac{-a}{2}\right)\left(\frac{-a}{2}\right) + 2m\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right) + 3m\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + 4m\left(\frac{-a}{2}\right)\left(\frac{a}{2}\right)$$

$$\Rightarrow \quad F = \frac{ma^2}{4} + \left(\frac{2ma^2}{4}\right) + \frac{3ma^2}{4} - \frac{4ma^2}{4} = ma^2 - \frac{3}{2} ma^2$$

$$\Rightarrow \quad F = \frac{-1}{2} ma^2$$

Let ox′, oy′ be the principal axes at O s. t. $\angle x'ox = \theta$.

Then, we have $I_{ox'} = A \cos^2\theta - 2F \sin\theta \cos\theta + B \sin^2\theta$

$I_{oy'} = A \sin^2\theta + 2F \sin\theta \cos\theta + B \cos^2\theta$

and $I_{x'y'} = \frac{1}{2}(A - B) \sin 2\theta + F \cos 2\theta$

Since ox′ & oy′ are principal axes, therefore $I_{x'y'} = 0$

$\Rightarrow \quad \frac{1}{2}(A - B) \sin 2\theta + F \cos 2\theta = 0$ …(3)

$$\Rightarrow \quad \tan 2\theta = \frac{2F}{B - A}$$

(3) $\Rightarrow \cos 2\theta = 0 \qquad [\because A = B = \frac{5}{2} ma^2]$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$\Rightarrow$ Diagonals OR & OS are principal axes.

Therefore, $I_{OR} = \frac{5}{2} ma^2\left(\frac{1}{2}\right) - 2\left(\frac{-ma^2}{2}\right)\left(\frac{1}{2}\right) + \frac{5}{2} ma^2\left(\frac{1}{2}\right)$ [using equation (I)]

$\Rightarrow \quad I_{OR} = 3ma^2$

and $I_{os} = 2ma^2 \qquad \left[\because I_{oy'} = \frac{5}{2} ma^2\left(\frac{1}{2}\right) - 2.\frac{1}{2} ma^2\left(\frac{1}{2}\right) + \frac{5}{2} ma^2\left(\frac{1}{2}\right)\right]$

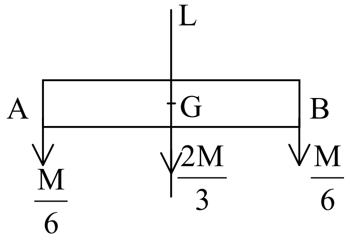
M.I. about z-axis is C = B + A

$\Rightarrow$ $C = I_{OR} + I_{OS} = 3ma^2 + 2ma^2$

$\Rightarrow$ $C = 5ma^2$.

Example 2 :- Show that a uniform rod of mass 'M' is equimomental to three particles situated one at each end of the rod and one at its middle point, the masses of the particle being $\frac{M}{6}, \frac{M}{6}$ and $\frac{2M}{3}$ respectively.

Solution:- Let AB = 2a is the length of rod having mass 'M'.



Let m, M−2m, m are the masses at A, G, B respectively. This system of particles has same centroid and same total mass M. This system of particles has the same M.I. (i.e. each zero) about AB, passing through common centroid 'G'. Therefore, systems are equimomental.

To find m:- we take M.I. of two systems (one system is rod of mass 'M') and other system consists of particles.

M.I. of rod about GL = $\frac{Ma^2}{3}$

M.I. of particles about GL = $ma^2 + 0 + ma^2$

$= 2ma^2$

As systems are equimomental,

$\therefore$ $2ma^2 = \frac{Ma^2}{3}$

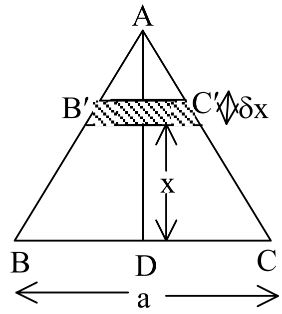
$\Rightarrow$ $m = \frac{M}{6}$

& $M - 2m = M - \frac{M}{3} = \frac{2M}{3}$

So masses of particles at A, G, B are $\frac{M}{6}, \frac{2M}{3}, \frac{M}{6}$ respectively.

Example 3 :- Find equimomental system for a uniform triangular lamina.

Solution:- Let M = Mass of Δ lamina.



Let ⊥ distance of A from BC = h

i.e. AD = h

First find M.I. of Δ lamina ABC about BC.

$M = \frac{1}{2} ah\,\sigma$, where σ = surface density of lamina

⇒ $\sigma = \frac{M}{\left(\frac{ah}{2}\right)}$ (density = Mass/area)

Now $\frac{B'C'}{BC} = \frac{h-x}{h}$

⇒ $B'C' = \frac{a(h-x)}{h}$ = length of strip

Area of strip $B'C' = \frac{a(h-x)}{h}\delta x$

Mass of strip $= \frac{M}{\left(\frac{ah}{2}\right)} . \frac{a(h-x)\delta x}{h}$

$= \frac{2M}{h^2}(h-x)\delta x$

∴ M.I. of strip B′C′ about BC $= \frac{2M}{h^2}(h-x)x^2\,\delta x$

∴ M.I. of Δ lamina ABC about BC

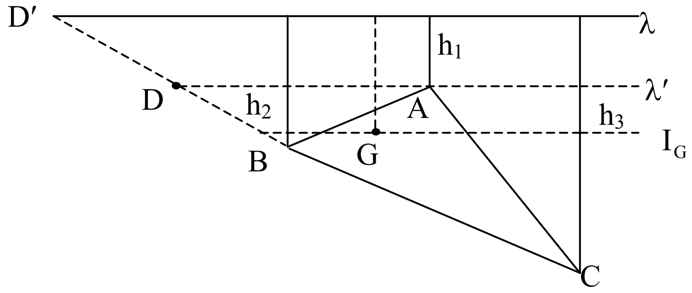
$$= \int_0^h \frac{2M}{h^2}(h-x)\,x^2\,dx$$

$$= \frac{2M}{h^2}\left[\frac{hx^3}{3} - \frac{x^4}{4}\right]_0^h$$

$$= \frac{2M}{h^2}\left(\frac{-h^4}{4} + \frac{h^4}{3}\right) = \frac{2M}{h^2}\frac{h^4}{12}$$

⇒ $I = \frac{1}{6}Mh^2$ …(1)

Now we apply this result to general case of finding M.I. about any line λ in the plane of lamina.



Let h_1, h_2, h_3 are length of $\perp$ drawn from corners (or points) A, B, C respectively of ΔABC. s. t. $h_1 < h_2 < h_3$

We extend BC to meet a point 'D' on line λ'. We draw a line λ' through A and parallel to λ.

Distances of C and B from λ' are $h_3 - h_1$, $h_2 - h_1$

Let M_1 is the mass of ΔACD and M_2 is the mass of ΔABD

This $\Rightarrow$ $\quad M = M_1 - M_2$

and $$\frac{M_1}{M_2} = \frac{\sigma\frac{1}{2}AD(h_3 - h_1)}{\sigma.\frac{1}{2}AD(h_2 - h_1)}$$

$$\Rightarrow \quad \frac{M_1}{M_2} = \frac{h_3 - h_1}{h_2 - h_1}$$

$$\Rightarrow \quad \frac{M_1}{h_3 - h_1} = \frac{M_2}{h_2 - h_1} = \frac{M_1 - M_2 = M}{h_3 - h_2}$$

$$\Rightarrow \quad M_1 = \frac{M(h_3 - h_1)}{h_3 - h_2}, \quad M_2 = \frac{M(h_2 - h_1)}{h_3 - h_2} \qquad \ldots(2)$$

We denote I_λ as the M. I. of ΔABC about λ and $I_{\lambda'}$ as the M.I. of ΔABC about λ' and I_G as the M.I. of ΔABC about a line parallel to λ or λ' through centre of mass (G) of ΔABC. So then

$I_{\lambda'}$ = M.I. of ΔACD – M.I. of ΔABD

$= \frac{1}{6} M_1(h_3 - h_1)^2 - \frac{1}{6} M_2(h_2 - h)^2$

$= \frac{M}{6}\left[\frac{(h_3 - h_1)^3 - (h_2 - h_1)^3}{h_3 - h_2}\right]$ [using equation (2)]

$= \frac{M}{6}\left[\frac{(h_3 - h_2)}{(h_3 - h_2)}\right] [(h_3 - h_1)^2 + (h_2 - h_1)^2 + (h_3 - h_1)(h_2 - h_1)]$

$[\because a^3 - b^3 = (a - b)(a^2 + b^2 + ab)]$

$= \frac{M}{6}[h_3^2 + h_1^2 - 2h_1h_3 + h_2^2 + h_1^2 - 2h_1h_2 + h_2h_3 - h_1h_2 - h_1h_3 + h_1^2]$

$= \frac{M}{6}[3h_1^2 + h_2^2 + h_3^2 + h_2h_3 - 3h_3h_1 - 3h_1h_2]$ …(3)

Now $\perp$ distance of G from $\lambda = \frac{(h_1 + h_2 + h_3)}{3}$ …(4)

and $\perp$ distance of G from $\lambda' = \left(\frac{h_1 + h_2 + h_3}{3} - h_1\right)$ …(5)

Using parallel axes theorem,

$I_\lambda = I_G + \frac{M}{9}(h_1 + h_2 + h_3)^2$ …(6)

and $I_{\lambda'} = I_G + \frac{M}{9}(h_2 + h_3 - 2h_1)^2$ …(7)

$I_\lambda = \frac{M}{6}(3h_1^2 + h_2^2 + h_3 + h_2h_3 - 3h_1h_3 - 3h_1h_2 + 4h - 2c + h_1h_3)$

(7) $\Rightarrow$ $I_G = I_{\lambda'} - \frac{M}{9}(h_2 + h_3 - 2h_1)^2$ …(8)

Put equation (8) in (6),

$I_\lambda = I_{\lambda'} + \frac{M}{9}(h_1 + h_2 + h_3)^2 - \frac{M}{9}(h_2 + h_3 - 2h_1)^2$

$= \frac{M}{6}[3h_1^2 + h_2^2 + h_3^2 + h_2 h_3 - 3h_3h_1 - 3h_1h_2]$

$- \frac{M}{9}(h_2 + h_3 - 2h_1)^2 + \frac{M}{9}(h_1 + h_2 + h_3)^2$

$$= \frac{M}{6}[3h_1^2 + h_2^2 + h_3^2 + h_2h_3 - 3h_1 h_3 - 3h_1h_2] + \frac{M}{9}[h_1^2 + h_2^2 + h_3^2 + 2h_1h_2$$

$$+ 2h_2h_3 + 2h_3h_1 - h_2^2 - h_3^2 - 4h_1^2 - 2h_2h_3 + 4h_1h_3 + 4h_1h_2]$$

$$\Rightarrow \quad I_\lambda = \frac{M}{6}[h_1^2 + h_2^2 + h_3^2 + h_1 h_2 + h_2 h_3 + h_1 h_3]$$

$$\Rightarrow \quad I_\lambda = \frac{M}{3}\left[\left(\frac{h_1 + h_2}{2}\right)^2 + \left(\frac{h_2 + h_3}{2}\right)^2 + \left(\frac{h_3 + h_1}{2}\right)^2\right]$$

= M. I. of mass $\frac{M}{3}$ placed at mid-point of A and B +

M.I. of M.I. of mass $\frac{M}{3}$ placed at mid-point of B and C +

M.I. of mass $\frac{M}{3}$ placed at mid-point of C and A.

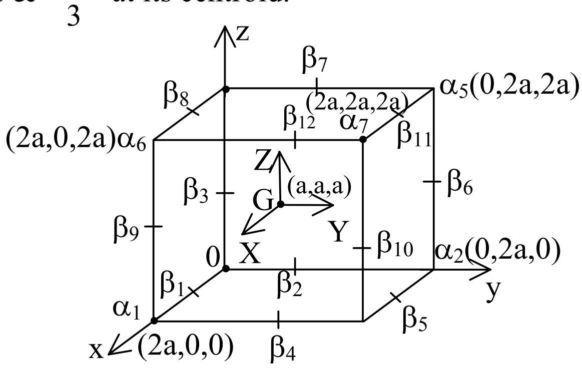
i.e. which is same as M.I. of equal particles of masses $\frac{M}{3}$ at the mid-points of sides of ΔABC.

Example 4:- Find equimomental system for a uniform solid cuboid. OR
Show that a uniform solid cuboid of mass 'M' is equimomental with

(i) Masses $\frac{M}{24}$ at the mid-points of its edges & $\frac{M}{2}$ at its centre.

(ii) Masses $\frac{M}{24}$ at its corners & $\frac{2M}{3}$ at its centroid.

Solution:-



Let length of edge of cuboid = 2a

Coordinates of mid-point of edges of cuboid are

$\beta_1 = (a, 0, 0)$, $\beta_2 = (0, a, 0)$, $\beta_3 = (0, 0, a)$, $\beta_4 = (2a, a, 0)$, $\beta_5 = (a, 2a, 0)$

$\beta_6 = (0, 2a, a)$, $\beta_7 =)0, a, 2a)$, $\beta_8 = (a, 0, 2a)$, $\beta_9 = (2a, 0, a)$, $\beta_{10} = (0, 2a, a)$

$\beta_{11} = (a, 2a, 2a)$, $\beta_{12} = (2a, a, 2a)$

Let G be centroid & ρ is the density of cuboid, then $M = \rho V = \rho(2a)^3 = 8\rho a^3$, …(1)

Now we find M.I. and Product of Inertia of cuboid about co-ordinates axes.

Therefore, A = M.I. of cuboid about x-axis

$$= \iiint_v \rho(y^2 + z^2)dv = \rho\int_0^{2a}\int_0^{2a}\int_0^{2a}(y^2 + z^2)dx\,dy\,dz$$

$$= \frac{8}{3}a^2(8\rho a^3) = \frac{8}{3}Ma^2 \qquad \text{[using (1)]}$$

Similarly B = M. I. of cuboid about y-axis = $\frac{8}{3}Ma^2$

C = M.I. of cuboid about z-axis = $\frac{8}{3}Ma^2$

Now D = product of inertia of cuboid w.r.t. pair (oy, oz)

$$\Rightarrow \quad D = \int_0^{2a}\int_0^{2a}\int_0^{2a}\rho\, yz\,dx\,dy\,dz = (da^3\rho)\,a^2$$

$$\Rightarrow \quad D = Ma^2$$

Similarly E= F = Ma^2

(i) Now consider a system of particles in which 12 particles each of mass $\frac{M}{24}$ are situated at mid-point of edges.

i.e. at β_i (i = 1 to 12) and a particle of mass $\frac{M}{2}$ at G.

Total mass of this system = $12\left(\frac{M}{24}\right) + \frac{M}{2}$

$$= \frac{M}{2} + \frac{M}{2} = M$$

$\Rightarrow$ The two systems have same mass. Also the centroid of these particles at β_i and G is the point G itself which is centroid of cuboid.

$\Rightarrow$ the two systems have same centroid.

Let A' = M.I. of system of particles at (β_i, G) about x-axis.

$$= \Sigma m(y^2 + z^2) + \frac{M}{2}(2a^2)$$

$$= \frac{M}{24}[0 + a^2 + a^2 + a^2 + 4a^2 + 5a^2 + 5a^2 + 4a^2 + a^2 + 5a^2 + 8a^2 + 5a^2]$$

$$+ \frac{M}{2}(2a^2)$$

$$\Rightarrow \quad A' = \frac{M}{24}(40a^2) + Ma^2 = \frac{64}{24}Ma^2 = \frac{8}{3}Ma^2$$

Similarly B' = M.I. of system of particle about y-axis = $\Sigma m(z^2 + x^2)$

$$\Rightarrow \quad B' = \frac{8}{3}Ma^2$$

Similarly $C' = \frac{8}{3} Ma^2$

Now D' = M.I. of system of particles w.r.t. (oy, oz)

$$= \Sigma myz$$

$$= \frac{M}{24}[0 + 0 + 0 + 0a^2 + 0 + 2a^2 + 2a^2 + 0 + 0 + 2a^2 + 4a^2 + 2a^2] + \frac{M}{2}(a^2)$$

$$\Rightarrow \quad D' = \frac{M}{24}(12a^2) + \frac{M}{2}a^2 = \frac{M}{2}a^2 + \frac{M}{2}a^2 = Ma^2$$

Similarly $E' = F' = Ma^2$

$\Rightarrow$ Both the systems have same M.I. and product of inertia referred to co-ordinate axes through O.

Using parallel axes theorem, both systems (i.e. cuboid & particles) have identical moments and products of inertia referred to parallel axes through common centroid G. So both the systems have same principal axes and principal M.I.

Therefore both the systems are equimomental.

(ii) Now let A'' = M.I. of system of particles at (α_i & G) about x-axis

$$= \frac{M}{24}(0 + 4a^2 + 4a^2 + 4a^2 + 8a^2 + 4a^2 + 8a^2) + \frac{2}{3}M(2a^2)$$

$$= \frac{M}{24}(32\ Ma^2) + \frac{4}{3}Ma^2 = \frac{4}{3}Ma^2 + \frac{4}{3}\ Ma^2$$

$\Rightarrow \qquad A'' = \frac{8}{3}Ma^2$

Similarly $B'' = C'' = \frac{8}{3}Ma^2$

Also D'' = P.I. of system of particles w.r.t. (oy, oz) axes

$$= \frac{M}{24}[0 + 0 + 0 + 0 + 4a^2 + 0 + 4a^2] + \frac{2M}{3}(a^2)$$

$$= \frac{8}{24}Ma^2 + \frac{2M}{3}a^2 = \frac{1}{3}Ma^2 + \frac{2}{3}Ma^2$$

$\Rightarrow \qquad D'' = Ma^2$

Similarly $E'' = F'' = Ma^2$

$\Rightarrow$ Both the systems have same M.I. and product of inertia referred to co-ordinate axes through O.

Using parallel axes theorem, both systems (i.e. cuboid & particles) have identical moments and products of inertia referred to parallel axes through common centroid G. So both the systems have same principal axes and principal M.I.

Therefore both the systems are equimomental.

Self Assessment Questions

1. Find Principal direction at one corner of a rectangular lamina of dimension 2a and 2b.
2. Find equimomental system for a parallelogram or parallelogram is equimomental with particles of masses M/6 at mid-points of sides of $||^{gm}$ & $\frac{M}{3}$ at the intersection of diagonals.

Lesson-3 Generalized co-ordinates and Lagrange's Equations

3.1 Some definations

3.1.1 Generalized Co-ordinates

A dynamical system is a system which consists of particles. It may also include rigid bodies. A Rigid body is that body in which distance between two points remains invariant. Considering a system of N particles of masses $m_1, m_2, \ldots m_N$ or m_i $(1 \le i \le N)$. Let (x, y, z) be the co-ordinates of any particle of the system referred to rectangular axes. Let position of each particle is specified by n independent variables $q_1, q_2, \ldots q_n$ at time t. That is

$$x = x(q_1, q_2, \ldots q_n; t)$$

$$y = y(q_1, q_2, \ldots q_n; t)$$

$$z = z(q_1, q_2, \ldots q_n; t)$$

The independent variables q_j are called as "**generalized co-ordinates**" of the system. Here we use '·' to denote total differentiation w.r.t. time.

$$\therefore \quad \dot{q}_1 = \frac{dq_1}{dt}, \quad \dot{q}_j = \frac{dq_j}{dt} \quad \text{etc. } (j = 1,2,3\ldots\ldots n)$$

The n quantities $\dot{q}_j = \frac{dq_j}{dt}$ are called "**generalized velocities**".

3.1.2 Holomonic system :- If the n generalized co-ordinate $(q_1, q_2, \ldots q_n)$ of a given dynamical system are such that we can change only one of them say q_1 to $(q_1 + \delta q_1)$ without making any changes in the remaining (n−1) co-ordinates, the system is said to be **Holonomic** otherwise it is said to be "**Non-Holonomic**" system.

3.1.3 Result :- Let system consisits of N particles of masses m_i $(1 \le i \le N)$ & q_j (j = 1 to n) are generalized coordinates.

Let $\vec{r}_i$ be the position vector of particle of mass m_i at time t. Then

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n; t) \qquad \ldots(1)$$

Then $\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt}$

$$\Rightarrow \quad \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \vec{r}_i}{\partial t}$$

$$\Rightarrow \quad \dot{\vec{r}}_i = \dot{q}_1\frac{\partial \vec{r}_i}{\partial q_1} + \dot{q}_2\frac{\partial \vec{r}_i}{\partial q_2} + \ldots + \dot{q}_n\frac{\partial \vec{r}_i}{\partial q_n} + \frac{\partial \vec{r}_i}{\partial t}$$

We regard $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$, t as independent variables. So,

$$\boxed{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}}$$

3.1.4 Virtual displacement:- Suppose the particles of a dynamical system undergo a small instantaneous displacement independent of time, consistent with the constraint of the system and such that all internal and external forces remain unchanged in magnitude & direction during the displacement.

3.1.5 Virtual Work & Generalised forces:- Consider a dynamical system consisting of N particles of masses m_i $(1 \le i \le N)$. Let m_i is the mass of ith particle with position vector $\vec{r}_i$ at time t, it undergo a virtual displacement to position $\vec{r}_i + \delta\vec{r}_i$.

Let $\vec{F}_i$ = External forces acting on m_i

$\vec{F}_i'$ = Internal forces acting on m_i

Therefore, virtual work done on m_i during the displacement $\delta\vec{r}_i$ is

$$(\vec{F}_i + \vec{F}_i').\delta\vec{r}_i$$

$\therefore$ Total work done on all particles of system is,

$$\delta W = \sum_{i=1}^{N}(\vec{F}_i + \vec{F}_i').\delta\vec{r}_i$$

$$= \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i + \sum_{i=1}^{N} \vec{F}_i' \cdot \delta\vec{r}_i$$

where δW is called virtual work function. If internal foces do not work in virtual displacement,

$$\text{then } \sum_{i=1}^{N} \vec{F}_i' \cdot \delta\vec{r}_i = 0$$

so $$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i$$

Let X_i, Y_i, Z_i are the components of $\vec{F}_i$ and δx_i, δy_i, δz_i are the components of $\delta\vec{r}_i$

i.e. $\vec{F}_i = (X_i, Y_i, Z_i)$ and $\delta\vec{r}_i = (\delta x_i, \delta y_i, \delta z_i)$

Then $\delta W = \sum_{i=1}^{N} (X_i \delta x_i + Y_i\, \delta y_i + Z_i \delta z_i)$. If the system is Holonomic ,i.e., the co-ordinate q_j changes to $q_j + \delta q_j$ without making any change in other $(n-1)$ co-ordinate.

Let this virtual displacement take effect & suppose the corresponding work done on the dynamical system to be $Q_j\, \delta q_j$, then $Q_j\, \delta q_j = \sum_{i=1}^{N} \vec{F}_i . \delta\vec{r}_i$

Now, if we make similar variations in each of generalized co-ordinate q_j, then

$$\delta W = \sum_{j=1}^{n} Q_j\, \delta q_j = \sum_{i=1}^{N} \vec{F}_i . \delta\vec{r}_i$$

Here Q_j are known as Generalised forces and δq_j are known as generalised virtual displacements.

3.2 Constraints of Motion:- When the motion of a system is restricted in some way, constraints are said to have been introduced.

Constraints

- Holonomic
  Co-ordinates are related by equations $f(\vec{r}_1, \vec{r}_2 ... \vec{r}_n, t) = 0$
- Non-Holonomic
  Co-ordinates through inequalities

Holonomic

Scleronomic
1. Time independent i.e. derivative w.r.t t is zero
2. Independent of velocities $(\dot{x}, \dot{y}, \dot{z})$

Rheonomic
Time dependent i.e. derivative w.r.t. t is non-zero.

Constraints depends explicitly on time

Example of Holonomic constraints

1. $(\vec{r}_i - \vec{r}_j)^2 =$ constant

2. $f(\vec{r}_1, \ldots, \vec{r}_n, t) = 0$

Example of non-Holonomic constraints

Motion of particle on the surface of sphere. Constraints of motion is $(r^2 - a^2) \geq 0$ where a is radius of sphere.

3.3 Lagrange's equations for a Holonomic dynamical system:- Lagrange's equations for a Holonomic dynamical system specified by n-generalised co-ordinates q_j (j = 1, 2, 3….. n) are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$

where T = K.E. of system at time t and Q_j = generalized forces.

Consider a dynamical system consisting of N particles. Let m_i, $\vec{r}_i$ be the mass, position vector of ith particle at time t and undergoes a virtual displacement to position $\vec{r}_i + \delta\vec{r}_i$.

Let $\vec{F}_i$ = External fore acting on m_i

$\vec{F}_i'$ = Internal force acting on m_i

Then equation of motion of ith particle of mass m_i is

$$\vec{F}_i + \vec{F}_i' = m_i \ddot{\vec{r}}_i \qquad \ldots(1)$$

The total K.E. of the system is,

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} \qquad \ldots(2)$$

Now $$\frac{d}{dt}\left[\frac{\partial \vec{r}_i}{\partial q_j}\right] = \left(\frac{\partial}{\partial t} + \sum_{k=1}^{n} \dot{q}_k \frac{\partial}{\partial q_k}\right)\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \qquad \ldots(3)$$

$$\left[\because \quad \frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial t} + \sum_{k=1}^{n} \dot{q}_k \frac{\partial \vec{r}_i}{\partial q_k} \quad \Rightarrow \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^{n} \dot{q}_k \frac{\partial \vec{r}_i}{\partial q_k}\right]$$

$$(3) \Rightarrow \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) + \sum_{k=1}^{n} \dot{q}_k \frac{\partial}{\partial q_k}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial \vec{r}_i}{\partial t}\right) + \sum_{k=1}^{n} \dot{q}_k \frac{\partial}{\partial q_j}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$$

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial \vec{r}_i}{\partial t}\right) + \frac{\partial}{\partial q_j}\left[\sum_{k=1}^{n} \dot{q}_k \frac{\partial \vec{r}_i}{\partial q_k}\right] \qquad [\because \dot{q}_k \text{ are independent of } q_j]$$

$$= \frac{\partial}{\partial q_j}\left[\frac{\partial}{\partial t} + \sum_{k=1}^{n} \dot{q}_k \frac{\partial}{\partial q_k}\right]\vec{r}_i$$

$$= \frac{\partial}{\partial q_j}\left[\frac{d\vec{r}_i}{dt}\right] = \frac{\partial}{\partial q_j}(\dot{\vec{r}}_i)$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \qquad \ldots(4)$$

Also we know that

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \qquad \ldots(5)$$

Consider

$$\frac{d}{dt}\left[\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right] = \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$= \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \qquad \text{[using (4)]}$$

$$\Rightarrow \quad \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left[\dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right] - \left(\dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right) \qquad \text{[using (5)]}$$

$$= \frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(\dot{\vec{r}}_i)^2\right] - \frac{1}{2}\left[\frac{\partial}{\partial q_j}(\dot{\vec{r}}_i^{\,2})\right]$$

Multiplying both sides by m_i & taking summation over i = 1 to N.

$$\Rightarrow \quad \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\Sigma m_i \dot{\vec{r}}_i^{\,2}\right)$$

$$\Rightarrow \quad \sum_{i=1}^{N}(F_i + \vec{F}_i')\frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j} \qquad \text{[using (1) \& (2)]} \qquad \ldots(6)$$

Also we have the relation,

$$\delta W = \sum_{j=1}^{n} Q_j \ \delta q_j = \delta q_j = \sum_{i=1}^{N} \vec{F}_i \partial \vec{r}_i = \sum_{i=1}^{N} [\vec{F}_i + \vec{F}_i']\delta \vec{r}_i \qquad \ldots(7)$$

Since the system is Holonomic, we regard all generalized co-ordinates except q_j as constant. Then, (7) gives

$$Q_j \ \delta q_j = \sum_{i=1}^{N}(\vec{F}_i + \vec{F}_i')\delta \vec{r}_i \qquad \ldots(8)$$

$$\Rightarrow \quad Q_j = \sum_{i=1}^{N}(\vec{F}_i + \vec{F}_i')\frac{\delta \vec{r}_i}{\delta q_j}$$

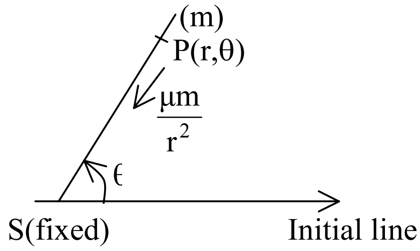
$$\Rightarrow \quad Q_j = \sum_{i=1}^{N}(\vec{F}_i + \vec{F}_i')\frac{\partial \vec{r}_i}{\partial q_j} \qquad \ldots(9)$$

from (6) & (9), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots n$$

This is a system of n equations known as Lagrange equations.

3.4 Example:- Planetary Motion :-



P → Planet

S → Sun(fixed)

Let (r, θ) be the polar co-ordinates of P w.r.t. S at time t.

Under the action of inverse square law of attraction, force = $\frac{\mu m}{r^2}$

radial velocity = $\dot{r}$

transverse velocity = $r\dot{\theta}$

Here (r, θ) are the generalized co-ordinates of the system and K.E. is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad [\because \ \theta^2 = \dot{r}^2 + r^2\dot{\theta}^2]$$

where r, θ, $\dot{r}, \dot{\theta}$ are independent. As the system is Holomonic, the virtual work function is given by

$$\delta W = \left(\frac{-\mu m}{r^2}\right)\delta r + 0 \quad [\because \delta W = \Sigma Q_j\ \delta q_j = Q_1\ \delta q_1 + Q_2\ \delta q_2 = Q_r\ \delta r + Q_2\ \delta\theta]$$

$$\Rightarrow \qquad Q_r = \frac{-\mu m}{r^2}$$

$$Q_\theta = 0$$

Now $\quad \frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\right]$

$$\Rightarrow \qquad \frac{\partial T}{\partial r} = mr\dot{\theta}^2$$

and $\quad \frac{\partial T}{\partial \theta} = 0$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\,\dot{\theta}$$

Therefore Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \qquad \ldots(1)$$

and $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \qquad \ldots(2)$$

$(1) \Rightarrow$ $$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 = \frac{-\mu m}{r^2} \qquad \ldots(3)$$

and $$m\ddot{r} - mr\dot{\theta}^2 = \frac{-\mu m}{r^2}$$

$(3) \Rightarrow$ $$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\mu m}{r^2}$$

$\Rightarrow$ $$\ddot{r} - r\dot{\theta}^2 = \frac{-\mu}{r^2}$$

and $(4) \Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

3.5 Lagrange equation for a conservative system of forces

Suppose that the forces are conservative & the system is specified by the generalized co-ordinate q_j ($j = 1, 2, \ldots .n$). So we can find a potential function

$V(q_1, q_2, \ldots, q_n)$

such that $\delta W = -\delta V$, where $\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \ldots + \frac{\partial V}{\partial q_n}\delta q_n$

$\Rightarrow$ $$\delta W = -\sum_{j=1}^{n}\left(\frac{\partial V}{\partial q_j}\right)\delta q_j$$

$\Rightarrow$ $$\sum_{j=1}^{n} Q_j\, \delta q_j = -\sum_{j=1}^{n}\left(\frac{\partial V}{\partial q_j}\right)\delta q_j$$

$\Rightarrow$ $$Q_j = \frac{-\partial V}{\partial q_j}$$

Therefor, Lagrange's equation for a conservative holonomic dynamical system becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

Let L = T − V where L = Lagrange's function

or L = K. E. − P. E

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Since V does not depend upon $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$

$$\Rightarrow \quad \frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

3.6 Generalised components of momentum and impulse

Let q_j (j = 1, 2,…n) be generalized co-ordinate at time t for a Holonomic dynamical system. Let T = T($q_1, q_2, …, q_n, \dot{q}_1, \dot{q}_2 .... \dot{q}_n, t$) . Then, the n quantities p_j is defined by

$p_j = \frac{\partial T}{\partial \dot{q}_j}$ are called generalized components of momentum.

We know that Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(p_j) - \frac{\partial T}{\partial q_j} = 0$$

Now T = $\frac{1}{2}mv^2 = \frac{1}{2}m\,\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Then $p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$

Similarly $p_y = m\dot{y}$, $p_z = m\dot{z}$

For generalized forces Q_j (j = 1, 2,….n) for dynamical system, the n quantities J_j defined by

$$\underset{\substack{Q_j\to\infty \\ \tau\to 0}}{\text{Lt}}\left[\int_0^{\tau} Q_j\, dt\right] = J_j \quad \text{(finite)}$$ when limit exists are called **generalised impulses**.

Since $\delta W = \sum_{j=1}^{n} Q_j\ \delta q_j$

$$\therefore \qquad \int_0^{\tau} \delta W\ dt = \sum_{j=1}^{n} \delta q_j \left[\int_0^{\tau} Q_j\ dt\right]$$

$$\therefore \qquad \underset{\substack{Q_j\to\infty \\ \tau\to 0}}{\text{Lt}} \int_0^{\tau} \delta W\ dt = \sum_{j=1}^{n} \delta q_j \left[\underset{\substack{Q_j\to\infty \\ \tau\to 0}}{\text{Lt}} \int_0^{\tau} Q_j\ dt\right]$$

$$\Rightarrow \qquad \delta U = \sum_{j=1}^{n} J_j\ \delta q_j$$

where δU is called impulsive virtual work function.

3.7 Lagrange's equation for Impulsive forces

It states that generalized momentum increment is equal to generalized impulsive force associated with each generalized co-ordinate.

Derivation:- We know that Lagrange's equation for Holonomic system are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$\Rightarrow \qquad \frac{d}{dt}(p_j) - \frac{\partial T}{\partial q_j} = Q_j \qquad \ldots(1)$$

Integrating this equation from t = 0 to t = τ we get

$$(p_j)_{t=\tau} - (p_j)_{t=0} = \int_0^{\tau} \frac{\partial T}{\partial q_j}\, dt + \int_0^{\tau} Q_j\ dt$$

Let $Q_j \to \infty$, $\tau \to 0$ in such a way that

$$\lim_{\substack{Q_j \to \infty \\ \tau \to 0}} \int_0^{\tau} Q_j \, dt = J_j \text{ (finite)} \quad (j = 1, 2, \ldots, n)$$

Further as the co-ordinate q_j do not change suddenly,

$$\underset{\tau \to 0}{\text{Lt}} \int_0^{\tau} \frac{\partial T}{\partial q_j} dt = 0$$

Writing $\Delta p_j = \underset{\tau \to 0}{\text{Lt}} [(p_j)_{t=\tau} - (p_j)_{t=0}]$,

We thus obtain Lagrange's equation in impulsive form

$$\Delta p_j = J_j, \; j = 1, 2, \ldots n$$

3.8 Kinetic energy as a quadratic function of velocities

Let at time t, the position vector of ith particle of mass m_i of a Holonomic system is $\vec{r}_i$, then K.E. is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} \qquad \ldots(1)$$

where N is number of particles. Suppose the system to be Holonomic & specified by n generalized co-ordinates q_j, then $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n, t)$

$$\Rightarrow \quad \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \dot{q}_1 \frac{\partial \vec{r}_i}{\partial q_1} + \dot{q}_2 \frac{\partial \vec{r}_i}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial \vec{r}_i}{\partial q_n} + \frac{\partial \vec{r}_i}{\partial t} \qquad \ldots(2)$$

From (1) & (2),

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\dot{q}_1 \frac{\partial \vec{r}_i}{\partial q_1} + \dot{q}_2 \frac{\partial \vec{r}_i}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial \vec{r}_i}{\partial q_n} + \frac{\partial \vec{r}_i}{\partial t} \right)^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i \left(\dot{q}_1 \frac{\partial \vec{r}_i}{\partial q_1} + \dot{q}_2 \frac{\partial \vec{r}_i}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial \vec{r}_i}{\partial q_n} \right)^2$$

$$+ \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2 + \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial t} \left(\dot{q}_1 \frac{\partial \vec{r}_i}{\partial q_1} + \ldots + \dot{q}_n \frac{\partial \vec{r}_i}{\partial q_n} \right)$$

$$\Rightarrow \quad T = \frac{1}{2}[(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots.)$$

$$+ 2(a_1 \dot{q}_1 + a_2 \dot{q}_2 + \ldots. + a_n \dot{q}_n) + a] \qquad \ldots(3)$$

where $a_{rs} = a_{sr} = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_r}\right)\left(\frac{\partial \vec{r}_i}{\partial q_s}\right)$, $s \geq r$

$$a_r = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_r}\right)\left(\frac{\partial \vec{r}_i}{\partial t}\right)$$

$$a = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$$

equation (3) shows that T is a quadratic function of the generalized velocities.

Special Case :- When time t is explicitly absent, then $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n)$

$$\Rightarrow \qquad \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \dot{q}_1 \frac{\partial \vec{r}_i}{\partial q_1} + \dot{q}_2 \frac{\partial \vec{r}_i}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial \vec{r}_i}{\partial q_n}$$

and $\qquad \frac{\partial \vec{r}_i}{\partial t} = 0$

From (3), we get

$$T = \frac{1}{2}[a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots]$$

$$= \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} a_{rs}\,\dot{q}_r\dot{q}_s$$

Thus the K.E. assumes the form of a Homogeneous quadratic function of the generalized velocities $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$.

In this case, using Euler's theorem for Homogeneous functions

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \ldots + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T$$

$$\Rightarrow \qquad \dot{q}_1 p_1 + \dot{q}_2 p_2 + \ldots + \dot{q}_n p_n = 2T$$

3.9 Donkin's Theorem :-Let a function $F(u_1, u_2, \ldots, u_n)$ have explicit dependence on n independent variables $u_1, u_2 \ldots u_n$. Let the function F be transformed to another function $G = G(v_1, v_2 \ldots v_n)$ expressed in terms of a new set of n independent variables $v_1, v_2, \ldots v_n$ where these new variables are connected to the old variables by a given set of relation

$$V_i = \frac{\partial F}{\partial u_i}, \; i = 1, 2.....n \qquad ...(1)$$

& the form of G is given by

$$G(v_1, v_2.... v_n) = \sum_{i=1}^{n} u_i \, v_i \; - F(u_1, u_2....u_n) \qquad ...(2)$$

then the variables $u_1, u_2... u_n$ satisfy the dual transformation

$$u_i = \frac{\partial G}{\partial v_i} \qquad ...(3)$$

& $$F(u_1, u_2... u_n) = \sum_{i=1}^{n} u_i \, v_i \; - G(v_1, v_2....v_n)$$

This transformation between function F & G and the variables u_i & v_i is called Legendre's dual transformation.

Proof : Since G is given by

$$G(v_1, v_2,...., v_n) = \sum_{k=1}^{n} u_k \, v_k \; - F(u_1, u_2,...., u_n)$$

Then $$\frac{\partial G}{\partial v_i} = \frac{\partial}{\partial v_i}\left[\sum_{k=1}^{n} u_k v_k - F(u_1, u_2 ... u_n)\right]$$

$$= \sum_{k=1}^{n} \frac{\partial u_k}{\partial v_i} \, v_k + \sum_{i=1}^{n} u_k \frac{\partial V_k}{\partial V_i} - \sum_{k=1}^{n} \frac{\partial F}{\partial u_k} \frac{\partial u_k}{dv_i}$$

$$\Rightarrow \quad \frac{\partial G}{\partial v_i} = \sum_{k=1}^{n} \frac{\partial v_k}{\partial v_i} v_k + \sum u_k \, \delta_{ki} - \sum_{k=1}^{n} \frac{\partial H}{\partial u_k} \frac{\partial u_k}{\partial v_i}$$

$$= \sum_{k=1}^{n} \frac{\partial u_k}{\partial v_i} v_k + u_i - \sum_{k=1}^{n} \frac{\partial F}{\partial u_k} \frac{\partial u_k}{\partial v_i}$$

$$= \sum_{k=1}^{n} \frac{\partial u_k}{\partial v_i} \frac{\partial F}{\partial u_k} + u_i - \sum_{k=1}^{n} \frac{\partial F}{\partial u_k} \frac{\partial u_k}{\partial v_i} \qquad \left[\Theta \; (1) \Rightarrow v_k = \frac{\partial F}{\partial u_k}\right]$$

$$= u_i$$

$$\Rightarrow \quad \boxed{\frac{\partial G}{\partial v_i} = u_i}$$

3.10 Extension of Legender's dual transformation

Further suppose that there is a another set of m independent variables α_1, $\alpha_2 \ldots \alpha_n$ present in both F & G.

$\Rightarrow$ $F = F(u_1, u_2 \ldots. u_n, \alpha_1, \alpha_2 \ldots. \alpha_m)$

$G = G(v_1, v_2 \ldots v_n, \alpha_1, \alpha_2 \ldots. \alpha_m)$

then there should be some extra condition for Legendre's dual transformation to be satisfied. These conditions are

$$\frac{\partial F}{\partial \alpha_j} = \frac{-\partial G}{\partial \alpha_j}, \ j = 1, 2 \ldots. m \quad \text{L.H.S.}$$

Consider $G = G(v_1, v_2 \ldots v_n, \alpha_1, \alpha_2 \ldots \alpha_m)$

$$= \sum_{i=1}^{n} u_i \, v_i - F(u_1, u_2 \ldots u_n, \alpha_1, \alpha_2 \ldots \alpha_m) \quad \text{R.H.S.}$$

From L.H.S.

$$\delta G = \sum_{i=1}^{n} \frac{\partial G}{\partial v_i} \delta v_i + \sum_{j=1}^{m} \frac{\partial G}{\partial \alpha_j} \delta \alpha_j \qquad \ldots(1)$$

From R.H.S.,

$$\delta G = \sum_{i=1}^{n} u_i \, \delta v_i + \sum_{i=1}^{n} v_i \, \delta u_i - \sum_{i=1}^{n} \frac{\partial F}{\partial u_i} \delta u_i - \sum_{j=1}^{m} \frac{\partial F}{\partial \alpha_j} \delta \alpha_j \qquad \ldots(2)$$

Equating (1) & (2),

$$\sum_{i=1}^{n} \frac{\partial G}{\partial v_i} \delta v_i + \sum_{j=1}^{m} \frac{\partial G}{\partial \alpha_j} \delta \alpha_j = \sum_{i=1}^{n} u_i \, \delta u_i + \sum_{i=1}^{n} v_i \, \delta u_i - \sum_{i=1}^{n} \frac{\partial F}{\partial u_i} \delta u_i - \sum_{j=1}^{m} \frac{\partial F}{\partial \alpha_j} \delta \alpha_j$$

$\Rightarrow$ $v_i = \dfrac{\partial F}{\partial u_i}$ are satisfied provided

$$u_i = \frac{\partial G}{\partial v_i}$$

and $\dfrac{\partial G}{\partial \alpha_j} = \dfrac{-\partial F}{\partial \alpha_j}$

Lesson-4 Hamilton's Equations of Motion

4.1 Introduction

So far we have discussed about Lagrangian formulation and its application. In this lesson, we assume the formal development of mechanics turning our attention to an alternative statement of the structure of the theory known as Hamilton's formulation. In Lagrangian formulation, the independent variables are q_i and $\dot{q}_i$, whereas in Hamiltonian formulation, the independent variables are the generalized coordinates q_i and the generalized momenta p_i

4.2 Energy equation for conservative fields

Prove that for a dynamical system

T + V = constant

where T = K.E.

V = P.E. or ordinary potential

Proof : Here $V = V(q_1, q_2, \ldots, q_n)$

$T = T(q_1, q_2 \ldots Q_n, \dot{q}_1, \dot{q}_2 \ldots \dot{q}_n)$

$L = T - V = L(q_1, q_2 \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$

If Lagrangian function L of the system does not explicitly depend upon time t, then

$$\frac{\partial L}{\partial t} = 0$$

i.e. $L = L(q_j, \dot{q}_j)$ for j = 1, 2,…n

The total time derivative of L is

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \qquad \ldots(I)$$

We know that the Lagrange's equation is given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0 \qquad \ldots(II)$$

$$(I) \Rightarrow \frac{dL}{dt} = \sum_{j=1}^{n} \dot{q}_j\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \qquad \text{[using (ii)]}$$

$$= \sum_{j=1}^{n} \frac{d}{dt}\left[\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right]$$

$$\Rightarrow \qquad \frac{d}{dt}\left[\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = 0 \qquad \ldots(1)$$

$$\Rightarrow \qquad \frac{dH}{dt} = 0 \qquad \text{where } H = \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

is a function called Hamiltonian

$$H = \sum_{j=1}^{n} \dot{q}_j \, p_j - L \qquad \ldots(A)$$

$$\left[\because \frac{\partial L}{\partial \dot{q}_j} = p_j = \text{generalized component of momentum}\right]$$

Integrating (1),

$$\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{constant} \qquad \ldots(2)$$

Now $\displaystyle\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}$

$$= \frac{1}{2}\sum_{j=1}^{n} \dot{q}_j \left[\frac{\partial}{\partial \dot{q}_j}\left\{\sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i^{\,2}\right\}\right] \qquad \left[\because T = \frac{1}{2}\sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i^{\,2}\right]$$

$$= \sum_{j=1}^{n} \dot{q}_j \left[\sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i \left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right)\right]$$

$$= \sum_{j=1}^{n} \dot{q}_j \left[\sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \quad \left[\because \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}\right]$$

$$= \sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i \left[\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j\right] = \sum_{i=1}^{N} m_i \, \dot{\vec{r}}_i \, \dot{\vec{r}}_i$$

$$= 2T \qquad \ldots(3)$$

from (2) & (3),

$2T - L = \text{constant}.$

$\Rightarrow \quad 2T - (T-V) = \text{constant} \qquad [\because L = T-V]$

$\Rightarrow \quad T + V = \text{constant}.$

Also from (A), $H = T + V = \text{constant}$.

$\therefore$ Total energy $T + V = H$, when time t is explicitly absent.

4.3 Generalised potential

For conservative forces, Potential function $V = V(q_1, q_2 \ldots q_n)$, therefore

$\delta W = -\delta V$

$$= -\sum\left(\frac{\partial V}{\partial x_i}\delta x_i\right) = -\sum\left(\frac{\partial V}{\partial q_j}\right)\delta q_j$$

Also $\delta W = \Sigma Q_j \, \delta q_j$ where Q_j are generalized forces.

$$\Rightarrow \quad \sum Q_j \, \delta q_j = -\sum\left(\frac{\partial V}{\partial q_j}\right)\delta q_j$$

$$\Rightarrow \quad Q_j = -\frac{\partial V}{\partial q_j}$$

4.4 Cyclic or Ignorable co-ordinates

Lagrangian $\quad L = T - V$

If Lagrangian does not contain a co-ordinate explicitly, then that co-ordinate is called Ignorable or cyclic co-ordinate.

Let $\quad L = L(q_1, q_2 \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t)$

Let q_k is absent in L, then

$$\frac{\partial L}{\partial q_k} = 0$$

Lagrange's equation (equation of motion) corresponding to q_k becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - 0 = 0 \qquad \Rightarrow \frac{\partial L}{\partial \dot{q}_k} = \text{constant} = p_k$$

4.5 Hamiltonian and Hamiltonian variables:- In Lagrangian formulation, independent variable are generalized co-ordinates and time. Also generalised velocities appear explicitly in the formulation.

$\therefore$ $L(q_k, \dot{q}_k, t,)$

Like this Lagrangian $L(q_j, \dot{q}_j, t)$, a new function is Hamiltonian H which is function of generalized co-ordinates, generalized momenta and time ,i.e.,

$H(q_j, p_j, t)$, where $pj = \frac{\partial L}{\partial \dot{q}_j}$

Also we have shown that

$$H = \sum_j p_j \dot{q}_j - L$$

This quantity is also known as Hamiltonian. The independent variables $q_1, q_2, \ldots q_n$, $p_1, p_2 \ldots p_n$, t are known as Hamiltonian variables.

4.6 Hamilton's Canonical equations of motion

Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots. n$$

Now $H = H(q_j, p_j, t)$ ...(1)

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L(q_j, \dot{q}_j, t) \qquad \ldots(2)$$

The differential of H from (1),

$$dH = \sum \frac{\partial H}{\partial q_j} dq_j + \sum \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \qquad \ldots(3)$$

from (2) $\Rightarrow$

$$dH = \sum_{j=1}^{n} [p_j \, d\dot{q}_j + \dot{q}_j dp_j] - \sum \frac{\partial L}{\partial q_j} dq_j - \sum \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \frac{-\partial L}{\partial t} dt$$

$$\Rightarrow \quad dH = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j \, dp_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} d\, q_j \quad \left[\Theta \; p_j = \frac{\partial L}{\partial \dot{q}_j}\right] \qquad \ldots(4)$$

From Lagrange's equation, $\frac{d}{dt}(p_j) = \frac{\partial L}{\partial q_j}$

$$\Rightarrow \qquad \dot{p}_j = \frac{\partial L}{\partial q_j} \qquad \ldots(5)$$

Using (4) & (5), we get

$$dH = \sum_{j=1}^{n} \dot{q}_j \, dp_j - \Sigma \dot{p}_j \, dq_j - \frac{\partial L}{\partial t} dt \qquad \ldots(6)$$

Comparing equation (3) & (6), we get

$$\frac{\partial H}{\partial p_j} = \dot{q}_j, \; \dot{p}_j = \frac{-\partial H}{\partial q_j} \qquad \ldots(7)$$

and $\quad \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$ $\qquad$ where j = 1, 2,…n $\qquad$ …(8)

The equation (7) is called **Hamiltonian's canonical equations** of motion or Hamilton's equations.

Result:- To show that if a given co-ordinate is cyclic in Lagrangian L, then it will also be cyclic in Hamiltonian H.

If L is not containing q_k ,i.e., q_k is cyclic, then $\frac{\partial L}{\partial q_k} = 0$

then $\dot{p}_k = 0 \Rightarrow p_k$ = constant

From equation (1), $H(q_j, p_j, t)$

$\Rightarrow H(q_1, q_2, \ldots, q_{k-1}, q_{k+1}, \ldots, q_n, p_1, p_2, \ldots p_{k-1}, p_{k+1} \ldots p_n, t)$

If H is not containing t ,i.e.,

$$H = H(q_j, p_j)$$

then $\quad \frac{dH}{dt} = \sum \frac{\partial H}{\partial q_j} \dot{q}_j + \sum \frac{\partial H}{\partial p_j} \dot{p}_j$

Using equation (7) or Hamilton's equation

$$\frac{dH}{dt} = \sum \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p_j} - \sum \frac{\partial H}{\partial p_j} \left(\frac{\partial H}{\partial q_j} \right) = 0$$

$\Rightarrow \quad \frac{dH}{dt} = 0 \quad \Rightarrow \quad H = \text{constant}.$

If the equation of transformation are not depending explicitly on time & if P.E. is velocity independent, then H = E (total energy)

Which can also be seen from the expression as given under

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n)$$

P.E., $V = V(q_1, q_2, \ldots, q_n)$

K.E., $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^2$

Now $\dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$

$$\Rightarrow \quad T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \right)^2$$

$= (\text{quadratic function of } \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$

Therefore by Euler's theorem for Homogeneous function, we have

$$\sum \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$H = \sum p_j \dot{q}_j - L = \sum \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L = 2T - L$$

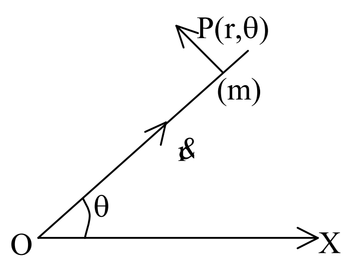
$\Rightarrow \quad H = 2T - (T - V) = T + V = E$

$\Rightarrow \quad H = E$

Example:- Write the Hamiltonian & Hamilton's equation of motion for a particle in central force field (planetary motion).

Solution : Let (r, θ) be the polar co-ordinates of a particle of mass 'm' at any instant of time t. Now $L = T - V(r)$ where $V(r)$ = P.E.

$$\Rightarrow \quad L = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] - V(r) \qquad \ldots(1)$$



As $q_j = r, \theta$

$\dot{q}_j = \dot{r}, \dot{\theta}$

$p_j = p_r, p_\theta$

Now $p_r = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ …(2)

$$H = \sum p_j \dot{q}_j - L = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2) - \frac{1}{2}mr^2\dot{\theta}^2 r + V(r)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \qquad \text{…(3)}$$

$\Rightarrow \quad H = T + V$

from (2), $\dot{r} = \dfrac{1}{m}p_r$

$\dot{\theta} = \dfrac{1}{mr^2}p_\theta$

Then $H = \dfrac{1}{2}m\left[\left(\dfrac{p_r}{m}\right)^2 + r^2\left(\dfrac{p_\theta}{mr^2}\right)^2\right] + V(r)$

$\Rightarrow \quad H = \dfrac{1}{2m}\left[p_r^2 + \dfrac{p_\theta^2}{r^2}\right] + V(r)$, which is required Hamiltonian

Hamilton's equations of motion are,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = \frac{-\partial H}{\partial q_j}$$

The two equations for $\dot{q}_j$ are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{q}_r$$

Similarly $\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2} = \dot{q}_\theta$

and two equations for $\dot{p}_j$ are,

$$\dot{p}_r = \frac{-\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V(r)}{\partial r}$$

and $\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = 0 \quad \Rightarrow \quad p_\theta = \text{constant}$

4.7 Routh's equations

Routh proposed for taking some of Lagrangian variables and some of Hamiltonian variables.

The Routh variables are the quantities

$t, q_j, q_\alpha, \dot{q}_j, p_\alpha$

$j = 1,2,\ldots.k$

$\alpha = k + 1, k + 2, \ldots.n$

k is arbitrary fixed number less than n. Routh's procedure involves cyclic and non-cyclic co-ordinates.

Suppose co-ordinates $q_1, q_2, \ldots, q_k$ ($k < n$) are cyclic (or Ignorable). Then we want to find a function R, called Routhian function such that it does not contain generalized velocities corresponding to cyclic co-ordinates.

$L = L(q_1, q_2 \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$

If $q_1, q_2 \ldots q_k$ are cyclic, then

$L(q_{k+1}, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots. \dot{q}_n, t)$

so that

$$dL = \sum_{j=k+1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad \left(dL - \sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) = \sum_{j=k+1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{k+1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \qquad \ldots(1)$$

Routhian function R, in which velocities $\dot{q}_1, \dot{q}_2 \ldots. \dot{q}_k$ corresponding to ignorable co-ordinate $q_1, q_2, \ldots, q_k$ are eliminated, can be written as

$R = R(q_{k+1}, q_{k+2}, \ldots, q_n, \dot{q}_{k+1}, \ldots. \dot{q}_n, t)$

so that

$$dR = \sum_{j=k+1}^{n} \frac{\partial R}{\partial q_j} dq_j + \sum_{j=k+1}^{n} \frac{\partial R}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial R}{\partial t} dt \qquad \ldots(2)$$

Further we can also define Routhian function as

$$R = L - \sum_{j=1}^{k} \dot{q}_j p_j$$

We want to remove $\sum_{j=1}^{k} q_j$ or $\sum_{j=1}^{k} \dot{q}_j$ from L to get R.

$$dR = dL - \sum_{j=1}^{k} \dot{q}_j \, dp_j - \sum_{j=1}^{k} p_j \, d\dot{q}_j$$

$$= dL - \sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \sum_{j=1}^{k} \dot{q}_j \, dp_j \qquad \text{[using (1)]}$$

$$\Rightarrow \qquad dR = \sum_{j=k+1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=k+1}^{n} \frac{\partial L}{\partial q_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt - \sum_{j=1}^{k} \dot{q}_j \, dp_j \qquad \ldots(3)$$

Comparing (2) & (3) by equating the coefficients of varied quantities as they are independent, we get

$$\frac{\partial L}{\partial q_j} = \frac{\partial R}{\partial q_j}, \qquad \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial R}{\partial \dot{q}_j} \qquad \ldots(4)$$

$$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t}, \qquad j = k + 1, k + 2, \ldots. n$$

Put (4) in Lagrangian's equations,

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0 \qquad j = 1, 2, \ldots. n$$

we get,

$$\sum_{j=k+1}^{n} \left[\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j} \right) - \frac{\partial R}{\partial q_j} \right] = 0$$

or $$\boxed{\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j} \right) - \frac{\partial R}{\partial q_j} = 0}, \; j = k + 1, \ldots. n$$

These are (n−k) 2nd order equations known as Routh's equations.

4.8 Generalised potential

When the system is not conservative. Let U is Generalised potential, say it depends on generalised velocities ($\dot{q}_j$) i.e. we consider the case when in place of ordinary potential V (q_j, t), there exits a generalised point U(q_j, t, $\dot{q}_j$) in terms of which the generalised forces Q_j are defined by

$$Q_j = \frac{d}{dt\left[\frac{\partial U}{\partial \dot{q}_j}\right] - \frac{\partial U}{\partial q_j}}, \qquad j = 1,2,\ldots n$$

[∵ L = T−V for conservative system, L = T−U for non-conservative system]

Here U is called generalised potential or velocity dependent potential.

Here Lagrange equations are $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$

$$\Rightarrow \qquad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(T - U)\right] - \frac{\partial}{\partial q_j}(T - U) = 0$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad [\because L = T - U \text{ for non-conservative system}]$$

4.9 Poisson's Bracket

Let A and B are two arbitrary function of a set of canonical variables (or conjugate variables) $q_1, q_2, \ldots, q_n, p_1, p_2 \ldots p_n$, then Poisson's Bracket of A & B is defined as

$$[A, B]_{q,p} = \sum_j \left(\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j}\right)$$

If F is a dynamical variable, i.e.,

F = F(q_j, p_j, t), then

$$\frac{dF}{dt} = \frac{dF}{dt}(q_j, p_j, t) = \sum_j \left(\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial p_j}\dot{p}_j \right) + \frac{\partial F}{\partial t} \qquad \ldots(1)$$

Using Hamilton's canonical equations,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = \frac{-\partial H}{\partial q_j}$$

$\therefore$ from (1), $\displaystyle\frac{dF}{dt} = \sum \left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \right) + \frac{\partial F}{\partial t}$

$\Rightarrow \qquad \displaystyle\frac{dF}{dt} = [F, H]_{q,p} + \frac{\partial F}{\partial t}$

If F is not depending explicitly on t, then

$$\frac{\partial F}{\partial t} = 0,$$

so $\displaystyle\frac{dF}{dt} = \sum \left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \right)$

$$= [F, H]_{q,p}$$

4.9 Properties

I. $[X, Y]_{q,p} = -[Y, X]_{q,p}$

II. $[X, X] = 0$

III. $[X, Y+Z] = [X, Y] + [X, Z]$

IV. $[X, YZ] = Y[X, Z] + Z[X, Y]$

Solution :- I. By definition $[X, Y]_{q,p} = \displaystyle\sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j} \right)$

$$[Y, X]_{q,p} = \sum_j \left(\frac{\partial Y}{\partial q_j}\frac{\partial X}{\partial p_j} - \frac{\partial Y}{\partial p_j}\frac{\partial X}{\partial q_j} \right)$$

$$= -\sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j} \right)$$

$\Rightarrow \qquad [Y, X]_{q,p} = -[X, Y]$

II. $$[X, X]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial X}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial X}{\partial q_j} \right) = 0$$

Also $$[X, C]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial C}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial C}{\partial q_j} \right) = 0$$

III. $$[X, Y + Z]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial (Y + Z)}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial (Y + Z)}{\partial q_j} \right)$$

$$= \sum_j \left[\frac{\partial X}{\partial q_j} \left(\frac{\partial Y}{\partial p_j} + \frac{\partial Z}{\partial p_j} \right) - \frac{\partial X}{\partial p_j} \left(\frac{\partial Y}{\partial q_j} + \frac{\partial Z}{\partial q_j} \right) \right]$$

$\Rightarrow$ $$[X, Y + Z]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right) + \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Z}{\partial q_j} \right)$$

$$= [X, Y] + [X, Z]$$

IV. $$[X, YZ]_{q,p} = \sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial (YZ)}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial (YZ)}{\partial q_j} \right)$$

$$= \left[\frac{\partial X}{\partial q_j} \left(Y \frac{\partial Z}{\partial p_j} + Z \frac{\partial Y}{\partial p_j} \right) - \frac{\partial X}{\partial p_j} \left(Z \frac{\partial Y}{\partial q_j} + Y \frac{\partial Z}{\partial q_j} \right) \right]$$

$$= 4 \left[\sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Z}{\partial q_j} \right) \right] + Z \left[\sum_j \left(\frac{\partial X}{\partial q_j} \frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j} \frac{\partial Y}{\partial q_j} \right) \right]$$

$$= Y[X, Z] + Z[X, Y]$$

Also

(i) $[q_i, q_j]_{q,p} = 0$

(ii) $[p_i, p_j]_{q,p} = 0$

(iii) $$[q_i, p_j]_{q,p} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Solution:-

(i) $$[q_i, q_j]_{q,p} = \sum_k \left[\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right] \qquad \ldots(1)$$

Because q_i or q_j is not function of p_k

$$\Rightarrow \qquad \frac{\partial q_i}{\partial p_k} = 0, \qquad \frac{\partial q_j}{\partial p_k} = 0$$

(1) $\Rightarrow$ $[q_i, q_j]_{q,p} = 0.$

(ii) $\quad [p_i, p_j]_{q,p} = \sum_k \left[\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right]$

As p_i, p_j is not a function of q_k

$$\therefore \qquad \frac{\partial p_i}{\partial q_k} = 0, \qquad \frac{\partial p_j}{\partial q_k} = 0$$

$\Rightarrow \qquad [p_i, p_j]_{q,p} = 0$

(iii) Now $\quad [q_i, p_j]_{q,p} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right)$

$$= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - 0\right) = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k}$$

$$= \sum_k \delta_{ik}\ \delta_{jk} = \sum_k \delta_{ij} = \delta_{ij}$$

$$\Rightarrow \qquad [q_i, p_j]_{q,p} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Some other properties:-

If $[\phi, \psi]$ be the Poisson Bracket of ϕ & ψ, then

(1) $\quad \frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \varphi}{\partial t}, \psi\right] + \left[\varphi, \frac{\partial \psi}{\partial t}\right]$

(2) $\quad \frac{d}{dt}[\phi, \psi] = \left[\frac{d\varphi}{dt}, \psi\right] + \left[\varphi, \frac{d\psi}{dt}\right]$

Solution:- (1) $\quad \frac{\partial}{\partial t}[\phi, \psi] = \frac{\partial}{\partial t}\left[\sum_i \left(\frac{\partial \varphi}{\partial q_i}\frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial \psi}{\partial q_i}\right)\right]$

$$= \sum_i \frac{\partial}{\partial t}\left[\frac{\partial \varphi}{\partial q_i}\frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial \psi}{\partial q_i}\right]$$

$$= \sum_i \left[\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial q_i}\right)\left(\frac{\partial \psi}{\partial p_i}\right) + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p_i}\right)\right]$$

$$- \left[\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial p_i}\right)\frac{\partial \psi}{\partial q_i} + \frac{\partial \varphi}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial q_i}\right)\right]$$

$$\sum_i \left[\frac{\partial}{\partial q_i}\frac{\partial \varphi}{\partial t}\left(\frac{\partial \psi}{\partial p_i}\right) - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right] + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}$$

$$= \sum_i \left[\frac{\partial}{\partial q_i}\left(\frac{\partial \varphi}{\partial t}\right)\frac{\partial \psi}{\partial p_i} + \frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial q_i}\left(\frac{\partial \psi}{\partial t}\right)\left(\frac{\partial \varphi}{\partial p_i}\right) - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right]$$

$$= \sum_i \left[\frac{\partial}{\partial q_i}\left(\frac{\partial \varphi}{\partial t}\right)\frac{\partial \psi}{\partial p_i} - \frac{\partial \psi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \varphi}{\partial t}\right)\right]$$

$$+ \sum_i \left[\frac{\partial \varphi}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial \varphi}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial \psi}{\partial t}\right)\right]$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \varphi}{\partial t}, \psi\right] + \left[\varphi, \frac{\partial \psi}{\partial t}\right].$$

(2) Similarly, we can prove

$$\frac{d}{dt}[\phi, \psi] = \left[\frac{d\varphi}{dt}, \psi\right] + \left[\varphi, \frac{d\psi}{dt}\right]$$

4.9.1 Hamilton's equations of motion in Poisson's Bracket:-

If $\quad H \rightarrow$ Hamiltonian

then $\quad [q, H]_{q,p} = \dfrac{\partial H}{\partial p} = \dot{q}$

$$[p, H]_{q,p} = \frac{-\partial H}{\partial q} = \dot{p}$$

From Hamilton's equations,

$$\frac{-\partial H}{\partial q} = \dot{p}, \qquad \frac{\partial H}{\partial p} = \dot{q}$$

$$[q_j, H] = \sum_i \left[\frac{\partial q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right]$$

$$[q_j, H] = \sum_i \delta_{ji} \frac{\partial H}{\partial p_i} \qquad \left[\Theta \ \frac{\partial q_j}{\partial p_i} = 0\right]$$

$$= \frac{\partial H}{\partial p_j}$$

Also $[p_j, H] = \dot{p}_j$

But $[p_j, H] = 0$

$\Rightarrow \quad \dot{p}_j = 0 \quad \Rightarrow \quad p_j = \text{constant}$

4.10 Jacobi's Identity on Poisson Brackets (Poisson's Identity):-

If X, Y, Z are function of q & p only, then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof : $[X, [Y, Z]] + [Y, [Z, X]] = [X, [Y, Z]] - [Y, [X, Z]]$

$$= \left[X, \sum_j \left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

$$\left[Y, \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right] \qquad \ldots(1)$$

Let $\sum_j \frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} = E, \quad \sum_j \frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j} = F$

$\sum_j \frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j} = G, \qquad \sum_j \frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j} = H$

$\therefore$ (1) $\Rightarrow [X, [Y, Z]] - [Y, [X, Z]]$

$$= [X, E-F] - [Y, G-H]$$

$$= [X, E] - [X, F] - [Y, G] + [Y, H] \qquad \ldots(2)$$

Let $E = \sum_j \frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j} = \left(\sum_j \frac{\partial Y}{\partial q_j}\right)\left(\sum_j \frac{\partial Z}{\partial p_j}\right)$

$\therefore \quad E = E_1 E_2$

Similarly $F = F_1 F_2$, $G = G_1 G_2$, $H = H_1 H_2$

$\therefore$ RHS of (2) becomes

$$[X, E] - [X, F] - [Y, G] + [Y, H] = [X, E_1 E_2] + [Y, H_1 H_2] - [X, F_1 F_2] - [Y, G_1 G_2]$$

$$= [X, E_1] E_2 + [X, E_2] E_1 - [X, F_1] F_2 - [X, F_2] F_1 - [Y, G_1] G_2$$

$$-[Y, G_2] G_1 + [Y, H_1] H_2 + [Y, H_2] H_1$$

$\therefore$ RHS of (2) is $= \left[X, \sum_j \left(\frac{\partial Y}{\partial q_j}\frac{\partial Z}{\partial p_j}\right)\right] - \left[X, \sum_j \left(\frac{\partial Y}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$

$$- \left[Y, \sum_j \left(\frac{\partial X}{\partial q_j}\frac{\partial Z}{\partial p_j}\right)\right] + \left[Y, \sum_j \left(\frac{\partial X}{\partial p_j}\frac{\partial Z}{\partial q_j}\right)\right]$$

Using properties $[X, E_1 E_2] = [X, E_1] E_2 + [X, E_2] E$,

$$= \left[X, \sum \frac{\partial Y}{\partial q_j}\right] \sum \frac{\partial Z}{\partial p_j} + \left[X, \sum \frac{\partial Z}{\partial p_j}\right] \sum \frac{\partial Y}{\partial q_j}$$

$$- \left[X, \sum \frac{\partial Y}{\partial p_j}\right] \sum \frac{\partial Z}{\partial q_j} - \left[X, \sum \frac{\partial Z}{\partial q_j}\right] \sum \frac{\partial Y}{\partial p_j}$$

$$- \left[Y, \sum \frac{\partial X}{\partial q_j}\right] \sum \frac{\partial Z}{\partial q_j} - \left[Y, \sum \frac{\partial Z}{\partial p_j}\right] \sum \frac{\partial X}{\partial q_j}$$

$$+ \left[Y, \sum \frac{\partial X}{\partial p_j}\right] \sum \frac{\partial Z}{\partial q_j} + \left[Y, \sum \frac{\partial Z}{\partial q_j}\right] \sum \frac{\partial X}{\partial p_j}$$

$$= \sum_j \left\{ \frac{-\partial Z}{\partial q_j}\left(\left[\frac{\partial X}{\partial p_j}, Y\right] + \left[X, \frac{\partial Y}{\partial p_j}\right]\right) + \frac{\partial Z}{\partial p_j}\left(\left[\frac{\partial X}{\partial q_j}, Y\right]\left[X, \frac{\partial Y}{\partial q_j}\right]\right)\right\}$$

$$+ \sum_j \left\{ \frac{\partial Y}{\partial q_j}\left[X, \frac{\partial Z}{\partial p_j}\right] - \frac{\partial Y}{\partial p_j}\left[X, \frac{\partial Z}{\partial q_j}\right] - \frac{\partial X}{\partial q_j}\left[Y, \frac{\partial Z}{\partial p_j}\right] + \frac{\partial X}{\partial p_j}\left[Y, \frac{\partial Z}{\partial q_j}\right]\right\} \quad \ldots(3)$$

Using the identity,

$$\frac{\partial}{\partial t}[X, Y] = \left[\frac{\partial X}{\partial t}, Y\right] + \left[X, \frac{\partial Y}{\partial t}\right]$$

Then, we find that R.H.S. of equation (3) reduces to

$$= \sum_j \left\{ -\frac{\partial Z}{\partial q_j} \frac{\partial}{\partial p_j}[X, Y] + \frac{\partial Z}{\partial p_j} \frac{\partial [X, Y]}{\partial q_j} \right\}$$

+ 0 (All other terms are cancelled)

$$= -\sum_j \left\{ \frac{\partial Z}{\partial q_j} \frac{\partial [X, Y]}{\partial p_j} - \frac{\partial Z}{\partial p_j} \frac{\partial [X, Y]}{\partial q_j} \right\}$$

$$= -[Z, [X, Y]]$$

or $\quad [X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]]$

$\Rightarrow \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Particular Case

Let Z = H, then

$[X, [Y, H]] + [Y, [H, X]] + [Y, [X, Y]] = 0$

Suppose X & Y both are constants of motion, then

$[X, H] = 0, \ [Y, H] = 0$

Then Jacobi's identity gives

$[H, [X, Y]] = 0$

$\Rightarrow$ [X, Y] is also a constant of Motion. Hence poisson's Bracket of two constants of Motion is itself a constant of Motion.

4.11 Poisson's Theorem

The total time rate of evolution of any dynamical variable F(p, q, t) is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]$$

Solution : $\quad \frac{dF}{dt}(p, q, t) = \frac{\partial F}{\partial t} + \sum_j \left[\frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial p_j} \dot{p}_j \right]$

$$= \frac{\partial F}{\partial t} + \sum_j \left[\frac{\partial F}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial q_j} \right]$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]$$

If F is constant of motion, then $\frac{dF}{dt} = 0$.

Then by Poisson's theorem,

$$\frac{\partial F}{\partial t} + [F, H] = 0$$

Further if F does not contain time explicitly, then $\frac{\partial F}{\partial t} = 0$

$\Rightarrow \quad [F, H] = 0$

This is the requirement condition for a dynamical variable to be a constant of motion.

4.12 Jacobi-Poisson Theorem :- (or Poisson's Second theorem)

If u and v are any two constants of motion of any given Holonomic dynamical system, then their Poisson bracket [u, v] is also a constant of motion.

Proof:- We consider $\frac{d}{dt}[u, v] = \frac{\partial}{\partial t}[u, v] + [[u, v], H]$ …(1)

using the following results,

$$\frac{\partial}{\partial t}[u, v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] \quad \ldots(2)$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \quad \ldots(3)$$

$$\therefore \quad (1) \Rightarrow \frac{d}{dt}[u, v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [[u, v], H] \quad \ldots(4)$$

Put w = H in (3), we get

$$[H, [u, v]] = -[u, [v, H]] - [v, [H, u]]$$

$$\Rightarrow \quad -[[v, H], u] - [[H, u], v] = [[u, v], H] \quad \ldots(5)$$

from (4) & (5), we get

$$\frac{d}{dt}[u, v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] - [[v, H], u] - [[H, u], v]$$

$$= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [u, (v, H]] + [[v, H], v]$$

$$= \left[\frac{\partial u}{\partial t} + [u, H], v\right] + \left[u, \frac{\partial v}{\partial t} + [v, H]\right]$$

$\Rightarrow$ $$\frac{d}{dt}[u, v] = \left[\frac{du}{dt}, v\right] + \left[u, \frac{dv}{dt}\right] \qquad \ldots(6)$$

Because $\frac{du}{dt}$ and $\frac{dv}{dt}$ both are zero as u & v were constants of motion.

$\therefore$ (6) $\Rightarrow$ $\frac{d}{dt}[u, v] = 0$

$\Rightarrow$ The Poisson bracket [u, v] is also a constant of motion.

4.13 Derivation of Hamilton's Principle from Lagrange's equation:- We know that Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \ldots(1)$$

Now $$\delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L(q_j\, \dot{q}_j) dt$$

$\Rightarrow$ $$\delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \left[\sum_j \left(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right)\right] dt$$

$$= \int_{t_1}^{t_2} \left[\sum \frac{\partial L}{\partial q_j}\delta q_j\right] dt + \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\, dt$$

$$= \int_{t_1}^{t_2} \sum \frac{\partial K}{\partial q_j}\delta q_j\, dt + \sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j\, dt \qquad \ldots(2)$$

Since, there is no coordinate variation at the end points.

$\Rightarrow$ $$\delta q_j\Big|_{t_1} = \delta q_j\Big|_{t_2} = 0$$

So (2) ⇒ $\delta\int_{t_1}^{t_2} L\ dt = \int_{t_1}^{t_2}\left[\sum_j \frac{\partial L}{\partial q_j} \frac{-d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j\ dt$

⇒ $\delta\int_{t_1}^{t_2} L\ dt = \int_{t_1}^{t_2} 0\ \delta q_j\ dt$ [Using (1)]

$= 0$

4.14 Derivation of Lagrange's equations from Hamilton's principle

We are given $\delta\int_{t_1}^{t_2} L\,dt = 0$

As δq_j are arbitrary & independent of each other, So its coefficients should be zero separately. So we have

$$\sum_j\left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] = 0$$

⇒ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$ for $j = 1, 2, \ldots. n$

4.15 Principle of Least action

Action of a dynamical system over an interval $t_1 < t < t_2$ is

$$A = \int_{t_1}^{t_2} 2T\ dt$$

where T = K.E.

This principle states that the variation of action along the actual path between given time interval is least, i.e.,

$$\delta\int_{t_1}^{t_2} 2T\ dt = 0 \qquad \ldots(1)$$

Now we know that $T + V = E$ (constant)

V = P.E. and $L = T - V$

By Hamilton's principle, we have

$$\int_{t_1}^{t_2} \delta L \; dt = 0 \;\; \text{or} \;\; \int_{t_1}^{t_2} \delta(T - V) dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta\,(T - E + T)\; dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} [\delta(2T) - \delta E] \; dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta(2T)\, dt = 0 \qquad [\text{using } E = \text{constant} \quad \therefore \;\; \delta E = 0]$$

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} 2T \; dt = 0$$

4.16 Distinction between Hamilton's Principle and Principle of least action:- Hamilton's principle $\delta S = 0$ is applicable when the time interval $(t_2 - t_1)$ in passing from one configuration to the other is prescribed whereas the principle of least action i.e. $\delta A = 0$ is applicable when the total energy of system in passing from one configuration to other is prescribed and the time interval is in no way restricted. This is the essential distinction between two principles.

4.17 Poincare–Cartan Integral Invariant :- We derive formula for δW in the general case when the initial and terminal instant of time, just like initial & terminal co-ordinates are not fixed but are functions of a parameter α.

$$W(\alpha) = \int_{t_1}^{t_2} L\,[t_1\, q_j(t, \alpha),\; \dot{q}_j(t, \alpha)]\, dt$$

let $\quad t_1 = t_1(\alpha), \;\; t_2 = t_2(\alpha)$

$$\dot{q}_j = \dot{q}_j(\alpha) \;\; \text{at } t = t_1$$

$$q_j^2 = q_j^2(\alpha) \;\; \text{at } t = t_2$$

Now $\delta W = \delta \int_{t_1}^{t_2} L \; dt = L_2\, \delta t_2 - L_1\, \delta t_1 + \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right] dt$

Integrating by parts

Then $\delta W = L_2\, \delta t_2 + \sum_j p_j^2 [\delta q_j]_{t=t_2} - L_1\, \delta t_1 - \sum_j p'_j [\delta q_j]_{t=t_1}$

$$+ \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j \, dt \qquad \ldots(1)$$

Now $q_j = q_j(t, \alpha)$

$$\left.\begin{aligned} \therefore \quad & [\delta q_j]_{t=t_1} = \left[\frac{\partial q_j(t,\alpha)}{\partial \alpha} \right]_{t=t_1} \delta\alpha \\ \& \quad & [\delta q_j]_{t=t_2} = \left[\frac{\partial}{\partial \alpha} q_j(t,\alpha) \right]_{t=t_2} \delta\alpha \end{aligned}\right\} \qquad \ldots(2)$$

On the other hand, for the variation of terminal co-ordinates

$$q_j^2 = q_j^2\, q[t(\alpha), \alpha]$$

$$\therefore \quad \delta q_j^2 = \dot{q}_j^2\, \delta t_2 + \left[\frac{\partial q_j}{\partial \alpha}(t,\alpha) \right]_{t=t_2} \delta\alpha$$

$$\Rightarrow \quad \delta q_j^2 = [\delta q_j]_{t=t_2} + \dot{q}_j^2\, \delta t_2 \qquad \text{[using (2)]}$$

$$\Rightarrow \quad [\delta q_j]_{t=t_2} = \delta q_j^2 - \dot{q}_j^2\, \delta t_2 \qquad \ldots(3)$$

Similarly $[\delta q_j]_{t=t_1} = \delta q'_j - \dot{q}'_j\, \delta t_1$ $\qquad \ldots(4)$

Put (3) & (4) in (1), we get

$$\delta W = L_2\, \delta t_2 + \sum_j p_j^2 (\delta q_j^2 - \dot{q}_j^2\, \delta t_2) - L_1\, \delta t_1$$

$$- \sum_j p'_j (\delta q'_j - \dot{q}'_j \delta t_1) + \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j dt$$

$$\Rightarrow \quad \delta W = \left[\sum_{j=1}^{n} p_j\, \delta q_j - H\, \delta t \right]_1^2 + \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j \, dt \qquad \ldots(5)$$

where $\left[\sum_{j=1}^{n} p_j\,\delta q_j - H\,\delta t\right]_1^2 = \sum_j p_j^2\,\delta q_j^2 - H_2\,\delta t_2 \;-\; \sum_j p'_j\,\delta q'_j \;+ H_1\,\delta t_1$

Now we know that $-H = L - \sum_j p_j\,\dot{q}_j$

$\therefore \qquad -H_1 = L_1 - \sum_j p'_j\,\dot{q}'_j$

and $\qquad -H_2 = L_2 - \sum_j p_j^2\,\dot{q}_j^2$

In the special case for any α, the path is extremum, the integral on R.H.S. of equation (5) is equal to zero and formula for variation of W takes the form

$$\delta W = \left[\sum_{j=1}^{n} p_j\;\delta q_j - H\,\delta t\right]_1^2 \qquad \ldots(6)$$

Integrating, we get

$$W = \int \left[\sum_{j=1}^{n} p_j\,\delta q_j - H\delta t\right] dt$$

which is known as Poincare Cartan Integral Invariant.

4.18 Whittaker's Equations :- We consider a generalised conservative system i.e. an arbitrary system for which the function H is not explicitly dependent on time. For it, we have

$$H(q_j, p_j) = E_0 \text{ (constant)} \qquad \ldots(1)$$

where $j = 1, 2, \ldots. n$

(2n – dimensional phase space in which q_j, p_j are coordinates)

Then basic integral invariant I will becomes

$$I = \int \left(\sum p_j\;\delta q_j - H\,\delta t\right)$$

$$\Rightarrow \qquad I = \int \sum_{j=1}^{n} p_j\;\delta q_j \qquad [\Theta \text{ for a conservative system, } \delta t = 0] \qquad \ldots(2)$$

solving (1) for one of the momenta, for example p_1.

$$p_1 = -\,k_1(q_1, q_2, \ldots., q_n, p_2 \ldots p_n, E_0) \qquad \ldots(3)$$

Put the expression obtained in (2) in place of p_1, we get

$$I = \int\left[\sum_{j=2}^{n} p_j\ \delta q_j + p_1\,\delta q_1\right]$$

$$= \int\left[\sum_{j=2}^{n} p_j\ \delta q_j - k_1\ \delta q_1\right] \qquad \ldots(4)$$

But this integral invariant (4) again has the form of Poincare – Cartan integral if it is assumed that the basic co-ordinate and momenta are quantities q_j & p_j (j = 2, 3...n) & variable q_1 plays the role of time variable (and in place of H, we have function k_1). Therefore the motion of a generalised conservative system should satisfy the following Hamilton's system of differential equations (2n – 2).

$$\dot{q}_j = \frac{dq_j}{dt} = \frac{\partial k_1}{\partial p_j};\ \frac{-\partial k_1}{\partial q_j} = \frac{dp_j}{dq_1} \qquad j = 2, 3\ldots. \qquad \ldots(5)$$

The equations (5) were obtaioned by Whittaker and are known as Whittaker's equations.

4.19 Jacobi's equations :-

Integrating the system of Whittaker's equations, we find q_j & p_j (j = 2, 3,….n) as functions of variables q_1 and (2n –2) arbitrary constant $C_1, C_2\ldots. C_{2n-2}$. Moreover, the integrals of Whittaker's equations will contain an arbitrary constant E_0 ,i.e., they will be of the form

$$\left.\begin{aligned} q_j &= \phi_j\,(q_1, E_0, C_1, C_2\ldots. C_{2n-2}) \\ p_j &= \psi_j(q_1, E_0, C_1, C_2\ldots. C_{2n-2}) \end{aligned}\right\} \quad (j = 2, 3,\ldots.n) \qquad \ldots(6)$$

Putting expression (6) in (3), we find

$$p_1 = \psi_1\,(q_1, E_0, C_1, C_2\ldots.. C_{2n-2}) \qquad \ldots(7)$$

From equation

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1} \qquad \Rightarrow \qquad dt = \frac{dq_1}{\left(\dfrac{\partial H}{\partial p_1}\right)}$$

$$\Rightarrow \qquad t = \int \frac{dq_1}{\left(\frac{\partial H}{\partial p_1}\right)} + C_{2n-1} \qquad \ldots(8)$$

where all the variables in partial derivative $\left(\frac{\partial H}{\partial p_1}\right)$ are expressed in terms of q_1 with the help of (6) & (7). The Hamiltonian system of Whittaker's equations (5) may be replaced by an equivalent system of equations of the Lagrangian type:

$$\frac{d}{dq_1}\left(\frac{\partial P}{\partial q'_j}\right) - \frac{\partial P}{\partial q_j} = 0, \qquad j = 2, 3, \ldots. n \qquad \ldots(9)$$

These are (n −1) second order equations where

$$q'_j = \frac{dq_j}{dq_1}$$

and the function P(analogous to Lagrangian function) is connected with the function K_1 (analogous of Hamiltonian function) by the equation

$$p = p(q_1, q_2 \ldots. q_n, q'_2, \ldots q'_n)$$

$$P = \sum_{j=2}^{n} p_j\, q'_j - K_1 \qquad \ldots(10)$$

The momenta p_j must be replaced by their expression in temrs of $q'_2 = \frac{dq_2}{dq_1}, \ldots\ldots$

$$q'_n = \frac{dq_n}{dq_1}$$

which may be obtained from first (n−1) equation (5).

From (3) & (10),

$$P = \sum_{j=2}^{n} p_j\, q'_j + p_1 = \frac{1}{\dot{q}_1} \sum_{i=1}^{n} p_i\, \dot{q}_i$$

$$\Rightarrow \qquad P = \frac{1}{\dot{q}_1}\,(L + H) \qquad \ldots(11)$$

For conservative system,

$$L = T - V, \;\; H = T + V$$

$\Rightarrow \qquad L + H = 2T$

Then $\quad P = \dfrac{2T}{\dot{q}_1}$ …(12)

& K.E., $T = \dfrac{1}{2}\sum_{i,k=1}^{n} a_{ik}\,\dot{q}_i\dot{q}_k$

$= \dot{q}_1^2\, G(q_1, q_2 .....q_n, q'_2 ...q'_n)$ …(13)

where

$G(q_1, q_2 ….q_n,\ q'_2 ...q'_n ...q'_n) = \dfrac{1}{2}\sum_{i,k=1}^{n} a_{ik}\, q'_i q'_k$ …(14)

From (1) & (13), we find

$H = E$

$T = G\dot{q}_1^2$

$\dot{q}_1 = \sqrt{\dfrac{T}{G}} = \sqrt{\dfrac{H - V}{G}}$

$\therefore \qquad \dot{q}_1 = \sqrt{\dfrac{E - V}{G}}$ …(15)

and $\quad P = \dfrac{2T}{\dot{q}_1} = \dfrac{2\dot{q}_1^2 G}{\dot{q}_1} = 2G\dot{q}_1$

$\Rightarrow \qquad P = 2G\sqrt{\dfrac{E - V}{G}}$ [from (15)]

$= 2\sqrt{G(E - V)}$ …(16)

Differential equation (9) in which function P is of the form (16) and which belong to ordinary conservative system are called Jacobi's equations.

4.20 Theorem of Lee-Hwa-Chung

If $\quad I' = \int \sum_{i=1}^{n} [A_i\,(q_k, p_k, t)\,\delta q_i + B_i(t, q_k, p_k)\delta p_i]$

is a universal relative integral invariant, then $I' = c\,I$, where c is a constant and I_1 is Poincare integral.

For $n = 1$, $I' = \int (A\delta q + B\delta p)$

$\Rightarrow$ $I' = c\int p\delta q = cI_1$

$$\left[I_1 = \int \sum_{i=1}^{n} [p_i \,\delta q_i] \right]$$

and $I_1 = \int p\delta q - H\delta t$ from Poincare Cartan integral.

Lesson-5 Canonical Transformations

5.1 Introduction:-The Hamiltonian formulation if applied in a straightforward way, usually does not decrease the difficulty of solving any given problem in mechanics; we get almost the same differential equations as are provided by the Lagrangian procedure. The advantages of the Hamiltanian formation lie not in its use as a calculation tool, but rather in the deeper insight it affords into the formal structure of mechanics.

We first derive a specific procedure for tranforming one set of variables into some other set which may be more suitable.

In dealing with a given dynamical system defined physically, we are free to choose whatever generalised coordinates we like. The general dynamical theory is invariant under transformations $q_i \rightarrow Q_i$, by which we understand a set of n variable expressing one set of n generalised co-ordinates q_i in term of another set Q_i. Invariant means that any general statement in dynamical theory is true no matter which system of coordinates is used.

In Hamiltonian formulation, we can make a transformation of the independent coordinates q_i , p_i to a new set Q_i, P_i with equations of transformation

$Q_i = Q_i\ (q,p,t)$, $P_i = P_i\ (q,p,t)$

Here we will be taking transformations which in the new coordinates Q, P are canonical.

5.2 Point transformation

$$Q_j = Q_j(q_j, t)$$

Transformation of configuration space is known as point transformation.

5.3 Canonical transformation

old variables → new set of variable

$q_j, p_j \rightarrow Q_j, P_j$

Here $Q_j = Q_j\ (q_j, p_j, t)$

$$P_j = P_j(q_j, p_j, t) \qquad \ldots(1)$$

If the transformation are such that the Hamilton's canonical equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j},\ p_j = \frac{-\partial H}{\partial q_j}$$

Preserve their form in the new variables i.e.

$$\dot{Q}_j = \frac{\partial K}{\partial P_j}, \qquad \dot{P}_j = \frac{-\partial K}{\partial Q_j}$$

K being Hamiltonian in the new variable, then transformations are said to be **canonical transformation**.

Also if

$H = \sum p_j \dot{q}_j - L$ in old variable, then in new variable,

$K = \sum P_j \dot{Q}_j - L'$

where L′ = new Lagrangian

Now $\quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}_j}\right) - \frac{\partial L'}{\partial Q_j} = 0$

i.e. Lagrange's equations are covariant w.r.t. point transformation & if we define P_j as

$$P_j = \frac{\partial L'(Q_j, \dot{Q}_j)}{\partial \dot{Q}_j}$$

$$\dot{Q}_j = \frac{\partial K(Q_j P_j, t)}{\partial P_j}$$

and $\quad \dot{P}_j = \frac{-\partial K(Q_j, P_j, t)}{\partial Q_j}$

Hamilton's principle in old variable, $\delta \int_{t_1}^{t_2} L\, dt = 0$

$$\delta \int_{t_1}^{t_2} \left[\sum p_j \dot{q}_j - H(q,p,t)\right] dt = 0 \qquad \ldots(2)$$

and in new variable,

$$\delta\int_{t_1}^{t_2}\left[\sum P_j \dot{Q}_j - k(Q,P,t)\right]dt = 0 \qquad \ldots(3)$$

$$\delta\int_{t_1}^{t_2}\left\{\left(\sum p_j \dot{q}_j - H\right) - \left(\sum P_j \dot{Q}_j - k\right)\right\}dt = 0 \qquad \ldots(4)$$

Let $\quad F = F(q, p, t)$

$$\therefore \qquad \delta\int_{t_1}^{t_2}\frac{d}{dt}F(q,p,t) = \delta[F(q,p,t)]_{t_1}^{t_2}$$

$$= \delta F$$

$$= \frac{\partial F}{\partial q_j}\delta q_j + \frac{\partial F}{\partial p_j}\delta p_j \Big|_{t_1}^{t_2}$$

$$= \frac{\partial F}{\partial q_j}\delta q_j \Big|_{t_1}^{t_1} + \frac{\partial F}{\partial p_j}\delta p_j \Big|_{t_1}^{t_2} = 0$$

[Since the variation in q_j and p_j at end point vanish]

$$\therefore (4) \Rightarrow \qquad \delta\int_{t_1}^{t_2}\left\{\left(\sum p_j \dot{q}_j - H\right) - \left(\sum P_j \dot{Q}_j - K\right)\frac{dF}{dt}\right\}dt = 0$$

$$\Rightarrow \qquad \left(\sum p_j \dot{q}_j - H\right) - \left(\sum P_j \dot{Q}_j - K\right) = \frac{dF}{dt} \qquad \ldots(5)$$

In (5), F is considered to be function of (4n + 1) variables i.e. q_j, p_j, Q_j, P_j, t.

But two sets of variables are connected by 2n transformation equation (1) & thus out of 4n variables besides t, only 2n are independent.

Thus F can be fuinction of $F_1(q, Q, t)$, $F_2(q, P, t)$, $F_3(p, Q, t)$, or $F_4(p, P, t)$

So transformation relation can be derived by the knowledge of function F. Therefore it is known as **Generating Function**.

Let $\quad F_1 = F_1(q, Q, t)$

$$\sum p_j \dot{q}_j - H = \sum P_j \dot{Q}_j - k + \frac{dF_1}{dt}(q,Q,t) \qquad \ldots(6)$$

and $$\frac{dF_1}{dt}(p,Q,t) = \sum\left(\frac{\partial F_1}{\partial q_j}\dot{q}_j + \frac{\partial F_1}{\partial Q_j}\dot{Q}_j\right) + \frac{\partial F_1}{\partial t} \qquad \ldots(7)$$

$\therefore$ from (6) & (7), we get

$$\sum p_j \dot{q}_j - H = \sum_j P_j \dot{Q}_j - k + \sum_j \frac{\partial F_1}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial F_1}{\partial Q_j}\dot{Q}_j + \frac{\partial F_1}{\partial t}$$

or $$\sum_j \left[\frac{\partial F_1}{\partial q_j} - p_j\right]\dot{q}_j + \sum_j \left[P_j + \frac{\partial F_1}{\partial Q_j}\right]\dot{Q}_j + H - K + \frac{\partial F_1}{\partial t} = 0 \qquad \ldots(8)$$

Since q_j and Q_j are to be considered as independent variables, equation (8) holds if the coefficients of q_j and Q_j separately vanish i.e.

$$\frac{\partial F_1}{\partial q_j}(q, Q, t) = p_j \qquad \ldots(9)$$

and $$P_j = \frac{-\partial F_1(q,Q,t)}{\partial Q_j} \qquad \ldots(10)$$

and $$K = H + \frac{\partial F_1(q,Q,t)}{\partial t} \qquad \ldots(11)$$

equation (11) gives relation between old and new Hamiltonian. Solving (9), we an find $Q_j = Q_j(q_j, p_j, t)$ which when put in (10) gives

$$P_j = P_j(q_j, p_j, t) \qquad \ldots(12)$$

equation (12) are desired canonical transformation.

5.4 Hamilton–Jacobi Equation :-If the new Hamiltonian vanish, i.e., $K = 0$, then

$$Q_j = \alpha_j \text{ (constant)}$$

$$P_j = \beta_j \text{ (constant)}$$

Also $$H + \frac{\partial F_1}{\partial t} = K = 0$$

$$\Rightarrow \quad H(q_j, p_j, t) + \frac{\partial F_1}{\partial t} = 0$$

Using (11), $$\boxed{H\left(q_j, \frac{\partial F_1}{\partial q_j}, t\right) + \frac{\partial F_1}{\partial t} = 0}$$

This partial differential equation together with equation (9) is known as Hamilton-Jacobi Equation.

Generating function is also called characteristic function

Let F_1 is replaced by S, then

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, ..., q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, ..., \frac{\partial S}{\partial q_n}, t\right) = 0, \qquad ...(9)$$

The solution S to above equation is called Hamilton principle function or characteristic function. Equation (9) is first order non-linear partial differential equation in (n + 1) independent variables (t, q_1, q_2,…,q_n) and one dependent variables S. So Therefore, there will be (n + 1) arbitrary constants out of which one would be simply an additive constant and remaining n arbitrary constants may appear as arguments of S so that complete solution has the form

$$S = S(q, t, \alpha) + A \qquad ...(10)$$

where $\alpha_i = \alpha_1, \alpha_2, \ldots, \alpha_n$ are n constants and A is additive constant.

Jacobi proved a theorem known as Jacobi'c theorem that the system would volve in such a way that the derivatives of S w.r.t. α's remain constant in time and we write

$$\frac{\partial S}{\partial \alpha_i} = \beta_i \text{ (constant) } (i = 1, 2, …., n)$$

α_i = Ist integrals of motion

β_i = IInd integrals of motion

5.5 Statement of Jacobi's Theorem

If $S(t, q_i, \alpha_i)$ is some complete integral fo Hamilton Jacobi equation (9) then the final equations of motion of a holonomic system with the given function. H may be written in the form

$$\frac{\partial S}{\partial q_i} = p_i \quad \text{and} \quad \frac{\partial S}{\partial \alpha_i} = \beta_i$$

where β_i, α_i are arbitrary constants

Proof: Given the complete integral for S given by (10), we wish to prove

$$\frac{\partial S}{\partial \alpha_i} = \beta_i$$

Consider

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \alpha_i}\right) = \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial \alpha_i}\right) + \sum \frac{\partial}{\partial q_j}\left(\frac{\partial S}{\partial \alpha_i}\right)\dot{q}_j$$

$$= \frac{\partial}{\partial \alpha_i}\left(\frac{\partial S}{\partial t}\right) + \frac{\partial}{\partial \alpha_i}\left(\frac{\partial S}{\partial q_j}\right)\dot{q}_j$$

$$= \frac{\partial}{\partial \alpha_i}\left[-H\left(t, q_j, \frac{\partial S}{\partial q_j} = p_j\right)\right] + \frac{\partial^2 S}{\partial \alpha_i \partial q_j}\dot{q}_j \quad \text{[using (9)]}$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \alpha_i}\right) = \frac{-\partial H}{\partial q_j}\frac{\partial q_j}{\partial \alpha_i} - \frac{\partial H}{\partial\left(\frac{\partial S}{\partial q_j}\right)}\frac{\partial^2 S}{\partial \alpha_i \partial q_j} + \frac{\partial^2 S}{\partial \alpha_i \partial q_j}\dot{q}_j$$

Since q_j's and α_i's are independent, we get $\frac{\partial q_j}{\partial \alpha_i} = 0$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \alpha_i}\right) = \left[-\frac{\partial H}{\partial p_j} + \dot{q}_j\right]\frac{\partial^2 S}{\partial \alpha_i \partial q_j} = 0, \text{ Using Hamilton's equation of motion}$$

$$\left[\dot{q}_j = \frac{\partial H}{\partial p_j}\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial S}{\partial \alpha_i}\right) = 0$$

$$\Rightarrow \frac{\partial S}{\partial \alpha_i} = \text{constant} = \beta_i \qquad (i = 1, 2, \ldots, n)$$

Remark : Consider total time derivative of $S(q_j, \alpha_j, t)$

$$\frac{dS}{dt} = \sum \frac{\partial S}{\partial q_j}\dot{q}_j + \sum \frac{\partial S}{\partial \alpha_j}\dot{\alpha}_j + \frac{\partial S}{\partial t}$$

But we know that $\alpha_j = 0$ since α_j are constant. Also $\frac{\partial S}{\partial q_j} = p_j$ & $H + \frac{\partial S}{\partial t} = 0$ gives

$\frac{\partial S}{\partial t} = -H$, we have

$$\frac{dS}{dt} = \sum p_j \dot{q}_j - H = L$$

$$\Rightarrow \quad S = \int L \, dt + \text{constant}$$

The expression differs from Hamilton's principle in a constant show that this time integral is ofindefinite form. Thus the same integral when indefinite form shapes the Hamilton's principle.

5.6 Method of sepration of variables:-

For a generalised conservative system $\frac{\partial H}{\partial t} = 0$. Then Hamilton's Jacobi equation has the form

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0$$

If Hamiltonian does not explicitly contain time, one can linearly decouple time from rest of variables in S and we write

$S(q_1, q_2, \ldots\ldots, q_n, t) = S_1(t) + V_1(q_1, q_2, \ldots\ldots q_n)$

and its complete solution is of the form

$$S = -Et + V(q_1, q_2, \ldots, q_n, \alpha_1, \ldots, \alpha_{n-1}, E)$$

[S = function of t + function of q

$$S = -Et + V(q_1, q_2, \ldots, q_n, \alpha_1, \ldots, \alpha_{n-1}, E)$$

Example:- Write Hamiltonian for one-dimensional harmonic oscillator of mass m & solve Hamilton-Jacobi equation for the same.

Solution :- Let q be the position co-ordinates of harmonic oscillator then $\dot{q}$ is its velocity and

$$\text{K.E.,} \quad T = \frac{1}{2} m \dot{q}^2$$

P.E., $V = \frac{1}{2}\ kq^2$ [k is some constant]

Lagrangian

$$L = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k\, q^2$$

The momentum is

$$p = \frac{\partial L}{\partial \dot{q}_j} = m\dot{q}$$

$$\Rightarrow \dot{q} = \frac{p}{m}$$

Then, the Hamiltonian is

$$H = \sum p_i \dot{q}_i - L$$

$$= p\ \dot{q} - \left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2} kq^2\right)$$

$$= p\ \frac{p}{m} - \frac{1}{\alpha} m\ \frac{p^2}{m^2} + \frac{1}{2} k\ q^2$$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2} kq^2$$

Also for a conservative system,

$$\text{Total Energy} = \text{P.E.} + \text{K.E.} = \frac{1}{2} m \frac{p^2}{m^2} + \frac{1}{2} kq^2 = \frac{1}{2}\left[\frac{p^2}{m} + kq^2\right] = \text{Hamiltonian}$$

$$\Rightarrow \quad H\,(q, p) = \frac{p^2}{2m} + \frac{kq^2}{2} \qquad \ldots(1)$$

Replacing p by $\frac{\partial S}{\partial q}$ in H,

$H\left(q, \frac{\partial S}{\partial q}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2}$. Then from $\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0$, we get

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{kq^2}{2} = 0 \qquad \ldots(2)$$

We separate variables

$$S(q, t) = V(q) + S_1(t) \Rightarrow \frac{\partial S}{\partial t} = \frac{dS_1}{dt}, \ \& \ \frac{\partial S}{\partial q} = \frac{dV}{dq}$$

Putting in (2), we get

$$(2) \Rightarrow \frac{1}{2m}\left[\frac{dV}{dq}\right]^2 + \frac{kq^2}{2} + \frac{dS_1}{dt} = 0$$

$$\Rightarrow \frac{dS_1}{dt} = \frac{-1}{2m}\left(\frac{dV}{dq}\right)^2 - \frac{kq^2}{2}$$

L.H.S. is function of t only and R.H.S. is function of q only,

But it is possible only when each side is equal to a constant (–E) (say)

$$\text{Let} \quad \frac{dS_1}{dt} = -E \qquad \ldots(i)$$

$$\& \quad \frac{1}{2m}\left(\frac{dV}{dq}\right)^2 + \frac{1}{2}kq^2 = E \qquad \ldots(ii)$$

(i) $\Rightarrow S_1 = -Et + \text{constant}$

(ii) $\Rightarrow \left(\frac{dV}{dq}\right)^2 = 2m\left(E - \frac{1}{2}kq^2\right)$

$$\frac{dV}{dq} = \sqrt{2m\left(E - \frac{1}{2}kq^2\right)}$$

$$V(q) = \int \sqrt{2m}\left(E - \frac{1}{2}kq^2\right)^{\frac{1}{2}} dq + \text{constant}$$

Therefore, complete integral is

$$S(q, t) = S_1(t) + V(q)$$

$$S(q, t) = -Et + \int \sqrt{2m}\left(E - \frac{1}{2}Kq^2\right)^{\frac{1}{2}} dq + \text{constant}$$

Further by Jacobi's theorem

Here $\alpha_1 = E$

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 \quad \Rightarrow \frac{\partial S}{\partial E} = \beta_1$$

$$\beta_1 = \frac{\partial S}{\partial E} \quad \Rightarrow \beta_1 = -t + \frac{\sqrt{2m}}{2}\int \frac{dq}{\left(E - \frac{1}{2}Kq^2\right)^{\frac{1}{2}}}$$

$$\beta_1 = -t + \sqrt{\frac{m}{2}}, \frac{\sqrt{2}}{\sqrt{K}}\int \frac{dq}{\left(\frac{2E}{K} - q^2\right)^{\frac{1}{2}}}$$

$$= -t + \frac{\sqrt{m}}{\sqrt{K}}\int \frac{dq}{\sqrt{\left(\sqrt{\frac{E}{K}}\right)^2 - (q)^2}} \qquad \left[\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$

$$= -t + \sqrt{\frac{m}{K}}\sin^{-1}\left(q\sqrt{\frac{K}{2E}}\right)$$

$$\sqrt{\frac{K}{m}}(t + \beta_1) = \sin^{-1}\left(\sqrt{\frac{K}{2E}}q\right)$$

$$\sqrt{\frac{k}{2E}}q \qquad = \sin\left[\sqrt{\frac{K}{m}}(t + \beta_1)\right]$$

$$q(E, t) \qquad = \sqrt{\frac{2E}{K}}\sin\left[\sqrt{\frac{K}{m}}(t + \beta_1)\right]$$

The constants β_1, E can be found from initial conditions
The momentum is given by

$$p = \frac{\partial S}{\partial q} = \frac{\partial(S_1(t) + V(q))}{\partial q}$$

$$= \frac{\partial V}{\partial q}$$

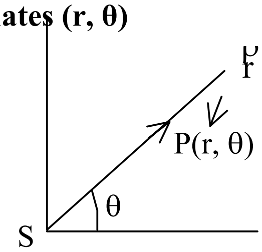
$$= \sqrt{2m}\left(E - \frac{1}{2}Kq^2\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{2m}}{\sqrt{2}}\left(2E - Kq^2\right)^{\frac{1}{2}}$$

$$= \sqrt{m\left(2E - Kq^2\right)},$$

which can be expressed as a function of t if we put q = q(t).

Example:- Central force problem in Polar co-ordinates (r, θ)



Here K.E,

$$T = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

P.E., V(r)

Then L = T − V

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} \qquad [q_1 = r, \dot{q}_1 = \dot{r}, q_2 = \theta, \dot{q}_2 = \dot{\theta}, p_j = p_r, p_\theta]$$

$$p_r = m\dot{r} \qquad \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

Therefore, Hamiltonian is

$$H = \sum p_i \dot{q}_i - L$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$= p_{\dot{r}}\frac{p_r}{m} + p_{\dot{\theta}}\frac{p_Q}{mr^2} - \frac{mp_r^2}{2mr^2} - \frac{mr^2p^2_{\dot{\theta}}}{2m^2r^2} + V(r)$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} + V(r)$$

$$= \frac{1}{2}\left[\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2}\right] + V(r)$$

$$= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{mr^2}\right] + V(r)$$

H − J equation is

$$\frac{\partial S}{\partial r} + H = 0$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right]$$
$$+ V(r) = 0$$

Using the Method of separation of variable, we have

$$S = S_1(t) + R(r) + \Phi(\theta)$$

$$\Rightarrow \qquad \frac{dS_1}{dt} = \frac{-1}{2m}\left[\left(\frac{dR}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{d\Phi}{d\theta}\right)^2\right] - V(r)$$

L.H.S. is function of t and R.H.S. is function of r & θ but not of t, therefore it is not possible only where each is equal to constant = −E (say)

$$\Rightarrow \qquad \frac{dS_1}{dt} = -E \quad \Rightarrow S_1(t) = -Et + \text{constant}$$

$$\text{and } \frac{-1}{2m}\left[\left(\frac{dR}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{d\Phi}{d\theta}\right)^2\right] - V(r) = -E$$

$$\Rightarrow \qquad \frac{-r^2}{2m}\left(\frac{dR}{dr}\right)^2 - r^2V(r) + r^2E = \frac{1}{2m}\left(\frac{d\Phi}{d\theta}\right)^2$$

L.H.S is function of r only and R.H.S is function of θ only

So each = constant = $\frac{h^2}{2m}(\text{say})$

$$\frac{d\Phi}{d\theta} = h \qquad \Rightarrow \Phi(\theta) = h\theta + \text{constant}$$

Then r equation gives

$$\frac{-r^2}{2m}\left(\frac{dR}{dr}\right)^2 - r^2V(r) + Er^2 = \frac{h^2}{2m}$$

$$\left(\frac{dR}{dr}\right)^2 = \left[2m(E - V) - \frac{h^2}{r^2}\right]$$

$$\frac{dR}{dr} = \left[2m(E - V) - h^2r^{-2}\right]^{\frac{1}{2}}$$

$$R = \int \sqrt{2m(E-V) - h^2 r^{-2}}\, dr + \text{constant}$$

Therefore, complete solution is

$$S = S_1 + R + \Phi$$

$$S = -Et + h\theta + \int \sqrt{2m(E-V) - h^2 r^{-2}}\, dr + \text{constant}$$, is required solution

Now, $\dfrac{\partial S}{\partial E} = \text{constant}$

$$\Rightarrow \quad -t + \int \frac{m\, dr}{\sqrt{2m(E-V) - \dfrac{h^2}{r^2}}} = \beta_1 (\text{say})$$

The other equation is

$$\frac{\partial S}{\partial h} = \text{constant}$$

$$\Rightarrow \Phi + \frac{1}{2} \int \frac{(-2hr^{-2})dr}{\left[2m(E-V) - h^2 r^{-2}\right]^{\frac{1}{2}}} = \beta_2 (\text{say})$$

Example:-When a particle of mass m moves in a force field of potential V. Write the Hamiltonian.

Solution:- Here K.E. is

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$\Rightarrow \dot{x} = \frac{p_x}{m}, \dot{y} = \frac{p_y}{m}, \dot{z} = \frac{p_z}{m}$$ and P.E. is V (x,y,z)

$\Rightarrow \quad H = T + V$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z)$$

Example :- A particle of mass m moves in a force whose of potential in spherical coordinates V is $-\mu\cos\theta/r^2$. Write Hamiltonian in spherical coordinate (r, θ, ϕ). Also find solution of H.J. equation.

Solution:- $L = \dfrac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2\right] - V(r, \theta, \varphi)$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad , p_{\dot{\phi}} = mr^2 \sin^2\theta\,\dot{\phi}$$

Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{\mu\cos\theta}{r^2}$$

Writing $p_r = \frac{\partial S}{\partial r}$, $p_\theta = \frac{\partial S}{\partial \theta}$, $p_\varphi = \frac{\partial S}{\partial \varphi}$

Required Hamilton Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{\mu\cos\theta}{r^2} = 0 \qquad \ldots(1)$$

Let S (t, r, θ, ϕ) = $S_1(t) + S_2(r) + S_3(\theta) + S_4(\phi)$ in (1)

$$\frac{\partial S_1}{\partial t} = -\frac{1}{2m}\left[\left(\frac{dS_2}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_3}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{ds_4}{d\phi}\right)^2 + \frac{\mu\cos\theta}{r^2}\right]$$

L.H.S. is function of t only, R.H.S. is function of r, θ, ϕ and not of t, it is possible only when each is constant (= – E) (say)

$$\Rightarrow \frac{dS_1}{dt} = -E \quad \Rightarrow \qquad S_1 = -Et + \text{constant}$$

$$\& \quad \frac{1}{2m}\left\{\left(\frac{dS_2}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_3}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{dS_4}{d\phi}\right)\right\} - \frac{\mu\cos\theta}{r^2} = E$$

Multiplying $2mr^2$ and rearranging terms, we get

$$r^2\left(\frac{dS_2}{dr}\right)^2 - 2mr^2E = -\left(\frac{dS_3}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_4}{d\phi}\right)^2 + 2m\mu\cos\theta$$

L.H.S. is function of r only and R.H.S. is function of θ and ϕ. It is possible only if each is equal to constant.

$$\Rightarrow \qquad r^2\left(\frac{dS_2}{dr}\right)^2 - 2mEr^2 = \beta_1 \qquad \ldots(2)$$

$$\& \quad -\left(\frac{dS_3}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_4}{d\phi}\right)^2 + 2m\mu\cos\theta = \beta_1 \qquad \ldots(3)$$

$$\Rightarrow \quad S_2 = \int\sqrt{2mE + \frac{\beta_1}{r^2}}\ dr + \text{constant}$$

$$\Rightarrow \left(\frac{dS_4}{d\phi}\right)^2 = 2m\mu\cos\theta\sin^2\theta - \beta_1\sin^2\theta - \sin^2\theta\left(\frac{dS_3}{d\theta}\right)^2$$

L.H.S. is function of ϕ whereas R.H.S. is function of θ and it is possible when each is equal to constant.

$$\Rightarrow \left(\frac{dS_4}{d\varphi}\right)^2 = \beta_2 \quad \Rightarrow \frac{dS_4}{d\varphi} = \sqrt{\beta_2} \qquad \ldots(4)$$

and $\quad p_\phi = \dfrac{dS_4}{d\varphi}$ $\qquad \ldots(5)$

$\Rightarrow S_4 = p_\phi\ \phi$ + constant

$p_\varphi^2 = \beta_2$ [from (4) & (5)]

and $S_3 = \int\sqrt{2m\mu\cos\theta - p_\phi^2\,\text{cosec}^2\phi - \beta_1}\,d\theta$ + constant [using (5)]

The complete solution is

$$S = -\,Et + \int\sqrt{2mE + \frac{\beta_1}{r^2}}dr + \int\sqrt{2m\mu\cos\theta - \beta_1 - -p_\phi^2\,\text{cosec}^2\theta}\,d\theta$$

$$+\ \phi\ p_\phi + \text{constant}$$

5.7 Lagrange's Brackets

Lagrange's bracket of (u, v) w.r.t. the basis (q_j, p_j) is defined as

$$\{u, v\}_{q,p} \text{ or } (u, v)_{q,p} = \sum_j\left[\frac{\partial q_j}{\partial u}.\frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v}\right]$$

Properties: (1) (u, v) = −(v, u)

(2) $(q_i, q_j) = 0$

(3) $(p_i, p_j) = 0$

(4) $(q_i, p_j) = \delta_{ij}$

(u, v) $= \sum_j \left(\frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v} \right) = -\sum_j \left(\frac{\partial p_j}{\partial u}\frac{\partial q_j}{\partial v} - \frac{\partial q_j}{\partial u}\frac{\partial p_j}{\partial v} \right)$ = -(v, u)

(2) $(q_i, q_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i} . \frac{\partial p_{k'}}{\partial q_j} - \frac{\partial q_{k'}}{\partial q_j}\frac{\partial p_k}{\partial q_i} \right)$ [Since q's and p's are independent

= 0. $\Rightarrow \frac{\partial p_{k'}}{\partial q_j} = o$ and $\frac{\partial p_k}{\partial q_i}\, 0$]

(3) Similarly we can prove that

$\{p_i, p_j\} = 0$

(4) $\{q_i, p_j\} = \sum_k \left(\frac{\partial q_k}{\partial q_i} . \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j}\frac{\partial p_k}{\partial q_i} \right) = \sum_k \frac{\partial q_k}{\partial q_i} . \frac{\partial p_k}{\partial p_j} = \sum_k \delta_{ki}\delta_{kj} = \delta_{ij}$

5.8 Invariance of Poisson's Bracket under Canonical transformation:-

Poisson's bracket is

$$(u, v)_{q,\,p} = \sum_j \left(\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} \right)$$

The transformation of co- ordinates in a 2n – dimensional phase space is called canonical if the transformation carries any Hamiltonian into a new hamiltonian system

To show :- $[F,G]_{q\,,\,p} = [F, G]_{Q,P}$

Poisson's brackets is

$$[F, G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right)$$

If q, p are functions of Q & P then q = q (Q, P) & p = p (Q, P) and F & G will also function of (q, p), we have, $G = G(Q_k, P_k)$, we have

$$[F, G]_{q,p} = \sum_i \left\{ \begin{array}{l} \dfrac{\partial F}{\partial q_i}\left[\dfrac{\partial G}{\partial Q_k}\cdot\dfrac{\partial Q_k}{\partial P_i} + \dfrac{\partial G}{\partial P_k}\cdot\dfrac{\partial p_k}{\partial P_i}\right] \\ -\dfrac{\partial F}{\partial p_i}\left[\dfrac{\partial G}{\partial Q_k}\cdot\dfrac{\partial Q_k}{\partial q_i} + \dfrac{\partial G}{\partial P_k}\cdot\dfrac{\partial P_k}{\partial q_i}\right] \end{array} \right\}$$

$$= \sum_{i,k} \left\{ \begin{array}{l} \dfrac{\partial G}{\partial Q_k}\left[\dfrac{\partial F}{\partial q_i}\cdot\dfrac{\partial Q_k}{\partial p_i} - \dfrac{\partial F}{\partial p_i}\cdot\dfrac{\partial Q_k}{\partial q_i}\right] \\ +\dfrac{\partial G}{\partial P_k}\left[\dfrac{\partial F}{\partial q_i}\cdot\dfrac{\partial P_k}{\partial p_i} - \dfrac{\partial F}{\partial p_i}\cdot\dfrac{\partial P_k}{\partial q_i}\right] \end{array} \right\}$$

$$= \sum_{i,k} \left\{ \frac{\partial G}{\partial Q_k}\left[F, Q_q\right]_{q,p} + \frac{\partial G}{\partial P_k}\left[F, P_k\right]_{q.p} \right\} \qquad \ldots(1)$$

To find $[F, Q_k]_{q,p}$ & $[F, P_k]_{q,p}$

Replacing F by Q_i in (1)

$$[Q_i, G]_{q,p} = \sum_{i,k} \frac{\partial G}{\partial Q_k}\left[Q_i, Q_k\right]_{q,p} + \frac{\partial G}{\partial P_R}\left[Q_i, P_k\right]_{q,p}$$

$$= 0 + \sum_k \frac{\partial G}{\partial P_k}\delta_{ik}$$

$$[Q_i, G]_{q,p} = \frac{\partial G}{\partial P_i}$$

$$\Rightarrow \quad [G, Q_i]_{q,p} = -\frac{\partial G}{\partial P_i}$$

and $\quad [F, Q_k]_{q,p} = -\dfrac{\partial F}{\partial P_R} \qquad \ldots(2)$

Replacing F by P_i in (1)

$$[P_i, G] = -\frac{\partial G}{\partial \varphi_i} \qquad \Rightarrow \qquad [G, P_i] = \frac{\partial G}{\partial q_i}$$

and $\quad [F, P_k] = \dfrac{\partial F}{\partial q_k} \qquad \ldots(3)$

Put these values from (2) and (3) in (1), we get:-

$$[F, G]_{q,p} = \sum_{i,k}\left(-\frac{\partial G}{\partial \phi_K}\frac{\partial F}{\partial P_K} + \frac{\partial G}{\partial P_K}\cdot\frac{\partial F}{\partial \phi_K}\right).$$

$= [F, G]_{Q,P}$

5.9 Poincare integral Invariant:-

Under Canonical transformation, the integral

$$J = \iint_S \sum dq_i dp_i \qquad \ldots(1)$$

Where S is any 2 – D (surface) phase space remains Invariant

Proof :- The position of a point on any 2– D surface is specified completely by two parameters, e.g. u, v

Then
$$\left.\begin{aligned} q_i &= q_i(u, v) \\ p_i &= p_i(u, v) \end{aligned}\right\} \qquad \ldots(2)$$

In order to transform integral (1) into new variables (u, v), we take the relation

$$dq_i\, dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} du\, dv \qquad \ldots(3)$$

where $\dfrac{\partial(q_i, p_i)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial q_i}{\partial u} & \dfrac{\partial p_i}{\partial u} \\ \dfrac{\partial q_i}{\partial v} & \dfrac{\partial p_i}{\partial v} \end{vmatrix}$ as the Jacobian.

Let Canonical transformation be

$$Q_k = Q_k(q, p, t),\ P_k = P_k(q, p, t) \qquad \ldots(4)$$

then $$dQ_k\, dP_k = \frac{\partial(\varphi_K P_K)}{\partial(u, v)} du\, dv \qquad \ldots(5)$$

if J is invariant under canonical transformation (4), then we can write

$$\iint_S \sum_i dq_i dp_i = \iint_S \sum_K dQ_K dP_K$$

or $$\iint_S \sum_i \frac{\partial(q_i p_i)}{\partial(u, v)} du\, dv = \iint_S \sum_K \frac{\partial(Q_K, P_K)}{\partial(u, v)} dudv$$

Because the surface S is arbitrary the expressions are equal only if the integrands are identicals,

i.e.,

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_K \frac{\partial(Q_K, P_K)}{\partial(u, v)}$$

…(6)

In order to prove it, we would transform(q,p) basis to (Q, P) basesthrough the generating function $F_2(q, p, t)$, With this form of generating function, we have

$$p_i = \frac{\partial F_2}{\partial q_i} \;\&\; Q_K = \frac{\partial F_2}{\partial P_k}$$

$$\frac{\partial p_i}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_i}\right) = \sum_K \left(\frac{\partial^2 F_2}{\partial q_i . \partial q_k} \cdot \frac{\partial q_k}{\partial u} + \frac{\partial^2 F_2}{\partial q_i . \partial p_k}\frac{\partial P_k}{\partial u}\right)$$

and $\frac{\partial p_i}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial q_i}\right) = \sum_K \left(\frac{\partial^2 F_2}{\partial q_i . \partial q_k} \cdot \frac{\partial q_k}{\partial v} + \frac{\partial^2 F_2}{\partial q_i . \partial p_k}\frac{\partial P_k}{\partial v}\right)$

Now, $\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix}$

$$= \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial u} + \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial v} + \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial v} \end{vmatrix}$$

$$= \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F_2}{\partial q_i \partial q_k}\frac{\partial q_k}{\partial v} \end{vmatrix} + \sum_{i,k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial^2 F_2}{\partial q_i \partial P_k}\frac{\partial P_k}{\partial v} \end{vmatrix}$$

L. H. S of (6) = $\sum_{i,k} \frac{\partial^2 F}{\partial q_i \partial q_i} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix}$

$$+ \sum_{i,k} \frac{\partial^2 F}{\partial q_i \partial P_k} \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} \qquad \ldots(7)$$

We see that first term on R.H.S. is antisymmetric expression under interchange of i and k, its value will be zero, i.e.,

$$\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial q_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix} = \sum_{k,i}\frac{\partial^2 F}{\partial q_k \partial q_i}\begin{vmatrix}\frac{\partial q_k}{\partial u} & \frac{\partial q_i}{\partial u}\\ \frac{\partial q_k}{\partial v} & \frac{\partial q_i}{\partial v}\end{vmatrix}$$

$$= -\sum_{k,i}\frac{\partial^2 F}{\partial q_k \partial q_i}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix}$$

or $$\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial q_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial q_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial q_k}{\partial v}\end{vmatrix} = 0 \qquad \ldots(8)$$

Similarly replacing q by P we have from (8)

$$\sum_{i,k}\frac{\partial^2 F}{\partial P_i \partial P_k}\begin{vmatrix}\frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix} = 0$$

Now equation (7) can be written as

$$\sum_i \frac{\partial(q_i p_i)}{\partial(u,v)} = \sum_{i,k}\frac{\partial^2 F}{\partial P_i \partial P_k}\begin{vmatrix}\frac{\partial P_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial P_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$+\sum_{i,k}\frac{\partial^2 F}{\partial q_i \partial P_k}\begin{vmatrix}\frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$= \sum_{i,k}\begin{vmatrix}\frac{\partial^2 F_2}{\partial P_k \partial P_i}\frac{\partial P_i}{\partial u} + \frac{\partial^2 F_2}{\partial P_k \partial q_i}\frac{\partial q_i}{\partial u} & \frac{\partial P_k}{\partial u}\\ \frac{\partial^2 F}{\partial P_k \partial P_i}\frac{\partial P_i}{\partial v} + \frac{\partial^2 F_2}{\partial P_k \partial q_i}\frac{\partial q_i}{\partial v} & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

$$= \sum_k \begin{vmatrix}\frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial u}\\ \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_k}\right) & \frac{\partial P_k}{\partial v}\end{vmatrix}$$

Put $\frac{\partial F_2}{\partial P_k} = Q_k$

$$= \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix} = \sum_k \frac{\partial (Q_k, P_k)}{\partial (u, v)} = \text{R.H.S of (6).}$$

Which proves that integral is invariant under canonical transformation.

5.10 Lagrange's bracket is invariant under Canonical transformation:-

The Lagrange's bracket of u & v is defined as

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} \right)$$

$$= \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_i}{\partial v} \\ \frac{\partial p_i}{\partial u} & \frac{\partial p_i}{\partial v} \end{vmatrix}$$

Since $\sum_i \frac{\partial (q_i p_i)}{\partial (u, v)}$ is invariant under Canonical transformation.

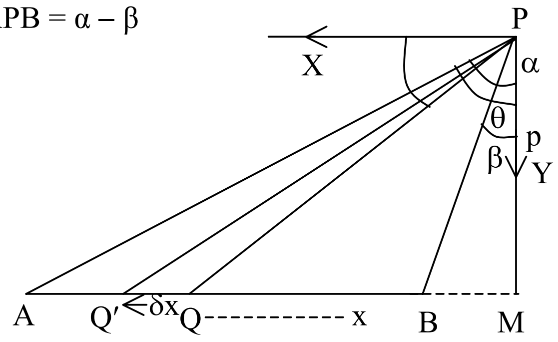
So Lagrange's bracket is also invariant under canonical transformation

Lesson-6 Attraction and Potential

6.1. Attraction of a uniform straight rod at an external point:-

AB be a rod

$\angle APB = \alpha - \beta$



Let m be the mass per unit length of a uniform rod AB.

It is required to find the components of attraction of the rod AB at an external point P.

$MP = p$

Consider an element QQ′ of the rod where

$MQ = x$

$QQ' = dx$

$\angle MPQ = \theta$,

In Δ MPQ ,

$$\tan\theta = \frac{MQ}{MP} = \frac{x}{p} \quad \Rightarrow \quad x = p\tan\theta \qquad \ldots(*)$$

$$\cos\theta = \frac{MP}{PQ} = \frac{p}{pQ}$$

$$\Rightarrow \quad PQ = \frac{p}{\cos\theta} \quad \Rightarrow \quad PQ = p\sec\theta \qquad \ldots(**)$$

Mass of element QQ′ of rod = m dx

$$= mp\sec^2\theta\, d\theta \qquad \ldots(\text{using}*)$$

The attraction at P of the element QQ′ is $= \dfrac{\text{mass}}{(\text{distance})^2} = \dfrac{mp\sec^2 d\theta}{(PQ)^2}$ along PQ

Therefore, Force of attraction at P of the element QQ′ is

$$= \frac{mp\sec^2\theta\, d\theta}{p^2\sec^2\theta} \qquad \text{…[using(**)]}$$

$$= \frac{m}{p}d\theta \qquad \text{along PQ}$$

…(1)

Let $\angle MPA = \alpha$ and $\angle MPB = \beta$

$$f = \int_{\beta}^{\alpha} \frac{m}{p} d\theta$$

let X and Y be the components of attraction of the rod parallel & $\perp_r$ to rod, then

$$X = \int_{\beta}^{\alpha} \frac{m}{p}\sin\theta\, d\theta$$

& $$Y = \int_{\beta}^{\alpha} \frac{m}{p}\cos\theta\, d\theta$$

$$X = \frac{m}{p}\left[-\cos\theta\right]_{\beta}^{\alpha} = \frac{m}{p}\left[\cos\beta - \cos\alpha\right]$$

$$= \frac{m}{p}\left[2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\beta}{2}\right] \qquad \text{…(2)}$$

and $$Y = \frac{m}{p}\left[\sin\theta\right]_{\beta}^{\alpha} = \frac{m}{p}\left[\sin\alpha - \sin\beta\right]$$

$$= \frac{m}{p}\left[2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right] \qquad \text{…(3)}$$

Resultant force of Attraction R is given by

$$R = \sqrt{X^2 + Y^2}$$

$$R = \frac{2m}{p}\sin\frac{\alpha-\beta}{2} \qquad \text{[…using 2\&3]}$$

$$= \frac{2m}{p}\sin\angle\frac{APB}{2}$$

Resultant R makes angle $\tan^{-1}\dfrac{X}{Y}$

or $\frac{1}{2}(\alpha+\beta)$ with PM $\left[\Theta \tan^{-1}\left(\frac{X}{Y}\right) = \left[\tan^{-1}\left(\tan\frac{\alpha+\beta}{2}\right)\right]\right]$

i.e. it acts along bisector of angle ∠APB.

Also $X = \frac{m}{PB} - \frac{m}{PA}$ $\left[\Theta \cos\beta = \frac{p}{PB}, \cos\alpha = \frac{p}{PA} \ \& \ \text{using}(2)\right]$

Cor :- If the rod is infinitely long, the angle APB is two right angles & Resultant attraction = $\frac{2m}{p}$ $\perp_r$ to the rod.

6.2 Potential of uniform rod :-

By definition, the potential at P is given by

$$V = \int \frac{m}{PQ} dx$$

$$V = \int_{\beta}^{\alpha} \frac{mp\sec^2\theta}{p\sec\theta} d\theta$$

$$= \int_{\beta}^{\alpha} m\sec\theta \, d\theta$$

$$= m\left[\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]_{\beta}^{\alpha}$$

$$= m\left[\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right]$$

$$= m\log\left[\frac{\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\beta}{2}+\frac{\pi}{4}\right)}\right]$$

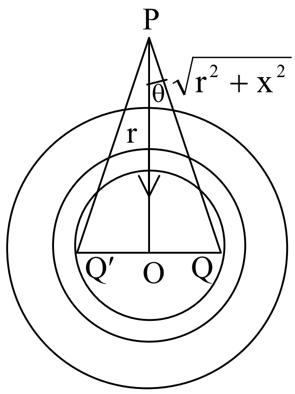
6.3 Potential at a point P on the axis of a Uniform circular disc or plate:-

We consider a uniform circular disc of radius 'a' & P is a pt. on the axis of disc & the pt. P is at a distance r from the centre 0, i.e., OP = r, OQ = x, PQ = $\sqrt{r^2+x^2}$ let us divide the disc into a number of concentric rings & let one such ring has radius 'x' & width dx

Then, Mass of ring is = $2\Pi x \, dx\rho$

where ρ density of material of disc ρ = mass/Area

Therefore, Potential at P due to this ring is given by ,dv = $\frac{2\pi x\, dx\rho}{\sqrt{r^2+x^2}}$



OP = r

Hence, the potential at P due to the whole disc is given by

$$V = 2\Pi\, \rho \int_0^a \frac{x\, dx}{\sqrt{x^2+r^2}}$$

$$V = \frac{2\pi\rho}{2}\int_0^a 2x\left(x^2+r^2\right)^{-\frac{1}{2}} dx$$

$$V = 2\Pi\, \rho \left[\sqrt{a^2+r^2} - r\right]$$

Let Mass of disc = M

$= \Pi a^2\, \rho$

$\Rightarrow \quad \Pi\, \rho = \frac{M}{a^2}$

Then V = $\frac{2m}{a^2}\left[\sqrt{a^2+r^2} - r\right]$ is requiredpotential at any pt P which lies on the axis of disc.

6.4 Attraction at any point on the axis of Uniform circular disc :-

Here radius of disc = a

OP = r, PQ = $\sqrt{x^2+r^2}$

OQ = x

We consider two element of masses d_m at the two opposite position Q and Q′as shown. Now element d_m at Q causes attraction on unit mass at P in the direction PQ. Similarly other mass d_m at Q′ causes attraction on same unit mass at P in the direction P Q′ and the force of attraction is same in magnitude.

These two attraction forces when resolved into two direction one along the axes PO and other at right angle PO. Components along PO are additive and component along perpendicular to PO canceling each other

Mass of ring = 2Πx dx ρ

Attraction at P due to ring along PO is given by

$$d\overset{\rho}{f} = \frac{\left(\sum dm\right)\cos\theta}{(PQ)^2}$$

$$d\overset{\rho}{f} = \frac{\cos Q.2\pi x\, dx\rho}{(PQ)^2} = \frac{r.2\pi x dx\,\rho}{(PQ)^3} \quad \text{along PO} \qquad [\Theta \cos\theta = \frac{r}{PQ} \text{in} \Delta OPQ\,]$$

$$= \frac{2\pi\rho .rx\, dx}{\left(r^2 + x^2\right)^{\frac{3}{2}}}$$

Therefore, the resultant attraction at P due to the whole disc along PO is given by

$$\overset{\rho}{f} = \pi\rho r \int_0^a (2x)\left(r^2 + x^2\right)^{-\frac{3}{2}} dx$$

$$= \pi\rho r \left[-2\left(x^2 + r^2\right)^{-\frac{1}{2}}\right]_0^a$$

$$= 2\pi\rho r \left[\frac{1}{r} - \frac{1}{\sqrt{a^2 + r^2}}\right] \text{along PO}$$

Let M = mass of disc of radius a

$=\Pi a^2$

$$\Pi\,\rho = \frac{M}{a^2}\rho$$

$$\overset{\rho}{f} = \frac{2M}{a^2}\left[1 - \frac{r}{\sqrt{a^2 + r^2}}\right]$$

$$= \frac{2M}{a^2}[1 - \cos\alpha]$$

Where α is the angle which any radius of disc subtends at P

Particular cases :-

1. If radius of disc becomes infinite, then $\alpha = \frac{\pi}{2}$

and

$$\vec{f} = \frac{2M}{a^2}\left[1 - \cos\frac{\pi}{2}\right]$$

$$= \frac{2M}{a^2} = \text{constant [here, it is independent of position of P]}$$

2. When P is at a very large distance from the disc, then $\alpha \to 0$

Therefore, $\vec{f} = \frac{2M}{a^2}(1 - \cos 0)$

$= 0$

6.5 Potential of a thin spherical shell :-

We consider a thin spherical shell of radius 'a' & surface density 'ρ' let P be a point at a distance 'r' from the center O of the shell. We consider a slice BB′C′C in the form of ring with two planes close to each other and perpendicular to OP.

Area of ring (slice) BB′C′C

$= 2\Pi BD \times BB'$

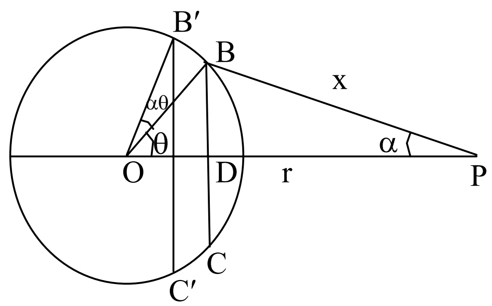
where Radius of ring, $BD = a\sin\theta$

width of ring, $BB' = a\, d\theta$

Therefor, Mass of slice (ring) is

$= 2\Pi\, a\sin\theta\, a d\theta\, \rho$

$= 2\Pi a^2 \rho \sin\theta\, d\theta$



Hence, Potential at P due to slice (ring) is

$$dV = \frac{2\pi a^2 \rho \sin\theta d\theta}{x} \qquad \ldots(1)$$

Now, from ΔBOP,

$$BP^2 = OP^2 + OB^2 - 2OP.OB\cos\theta$$

$$x^2 = r^2 + a^2 - 2ar\cos\theta$$

differentiating

$$2x\,dx = 2ar\sin\theta\,d\theta$$

$$\frac{x}{ar}dx = \sin\theta d\theta$$

Putting in (1), we get $dV = \dfrac{2\pi a^2 \rho x\,dx}{x.ar}$

$$= \frac{2\pi a\rho dx}{r} \qquad (2)$$

Therefore, Potential for the whole spherical shell is obtained by integrating equation (2), we have

$$V = \int \frac{2\pi a\rho}{r}dx$$

$$= \frac{2\pi a\rho}{r}\int dx$$

Now, we consider the following cases :-

Case(i) The point P is outside the shell. In this case, the limit of integration extends from $x = (r - a)$ to $x = (r + a)$

Hence $\qquad V = \dfrac{2\pi a\rho}{r}\displaystyle\int_{r=a}^{r+a} dx$

$$V = \frac{4\pi a^2 \rho}{r}$$

Here, Mass of spherical shall $= 4\Pi a^2\,\rho$

Then $V = \dfrac{M}{r}$

Case(ii) When P is on the spherical shell, then limits are from $x = 0$ to $x = 2a$ (here $r = a$)

Then $V = \frac{2\pi a\rho}{a}\int_0^{2a} dx$

$$= \frac{4\pi a^2\rho}{a} = \frac{M}{a}$$

Case(iii) When P is inside the spherical shell, limit are from $x = (a - r)$ to $(a + r)$

$$V = 4\Pi a\,\rho = \frac{M}{a}$$

6.6 Attraction of a spherical shell

Let us consider a slice BB′C′C at point P , the attraction due to this slice is

$$d\vec{f} = \frac{2\pi a^2\rho\sin\theta\, d\theta}{x^2} \text{ along PB}$$

The resultant attraction directed along PO is given by

$$d\vec{f} = \frac{2\pi a^2\rho\sin\theta\, d\theta}{x^2}\cos\alpha$$

We know that $\sin\theta\ d\theta = \frac{xdx}{ar}$

In ΔBDP, $\cos\alpha = \frac{PD}{PB} = \frac{r - a\cos\theta}{x}$.

$$d\vec{f} = \frac{2\pi a^2\rho x dx}{ar.x^2}\left(\frac{r - a\cos\theta}{x}\right)$$

We know that

$$x^2 = a^2 + r^2 - 2ar\cos\theta$$

$$x^2 - a^2 + r^2 = 2r^2 - 2ar\cos\theta$$

$$\frac{x^2 - a^2 + r^2}{2r} = r - a\cos\theta$$

Then, $d\vec{f} = \frac{2\pi a^2\rho x dx}{ar.x^2}\frac{\left(x^2 - a^2 + r^2\right)}{2r, x}$

$$= \frac{\pi a\rho}{r^2}\left(\frac{x^2 - a^2 + r^2}{x^2}\right)dx$$

$$= \frac{a\pi\rho}{r^2}\left(1 + \frac{r^2 + a^2}{x^2}\right)dx$$

Hence the attraction for the whole spherical shell is obtained by integration

Therefore, $$\vec{f} = \frac{\pi a\rho}{r^2}\int\left[1 + \frac{r^2 - a^2}{x^2}\right]dx$$

Now we consider the following cases depending upon the position of P

Case(i) When point P is inside the shell, the limits of integration are x = (r − a) to (r + a)

$$\vec{f} = \frac{\pi a\rho}{r^2}\int_{r-a}^{r+a}\left(1 + \frac{r^2 - a^2}{x^2}\right)dx$$

$$\vec{f} = \frac{\pi a\rho}{r^2}\left[x + \left(r^2 - a^2\right)\left(\frac{-1}{x}\right)\right]_{r-a}^{r+a}$$

$$= \frac{4\pi a^2\rho}{r^2} = \frac{M}{r^2}$$

Case(ii) When pt. P is on the shell, the limit of integration are x = 0 to 2a

$$\vec{f} = \frac{\pi a\rho}{r^2}\int_0^{2a}\left(1 + \frac{r^2 - a^2}{x^2}\right)dx$$

Here integration is not possible (due to second term is becoming indeterminant), because when P is on the shell, then

$r = a;\ x = 0$

Hence to evaluate the integral, we consider that pt. P is situated not on the surface but very near to the surface

Let $r = a + \delta$, where δ is very small

Then attraction is $$\vec{f} = \frac{a\pi\rho}{r^2}\left[\int_{\delta}^{2a+\delta}dx + \int_{\delta}^{2a+\delta}\left\{\frac{(a+\delta)^2 - a^2}{x^2}\right\}dx\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a + \int_{\delta}^{2a+\delta}\frac{2a\delta}{x^2}dx\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a + 2a\delta\left(\frac{-1}{x}\right)_{\delta}^{2a+\delta}\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a - \frac{2a\delta}{2a+\delta} + \frac{2a\delta}{\delta}\right]$$

$$= \frac{2\pi a^2\delta}{r^2}\left[2 - \frac{\delta}{2a+\delta}\right] \text{ as } \delta \to o, \text{ then } r = a$$

$$= \frac{4\pi a^2\rho}{a^2} = \frac{M}{a^2}$$

Case (iii) Poin. P is inside the shell, limits are x = a – r to a + r

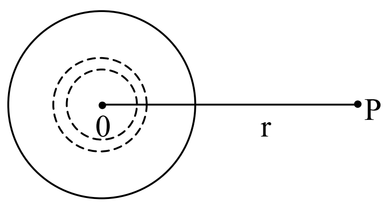
$$\vec{f} = \frac{\pi a\rho}{r^2}\int_{a-r}^{a+r}\left[1 + \frac{r^2 + a^2}{x^2}\right]dx$$

$$= \frac{\pi a\rho}{r^2}\left[x - \left(r^2 - a^2\right)\left(\frac{1}{x}\right)\right]_{a-r}^{a+r} = 0$$

So, there is no resultant attraction inside the shell.

6.7 Potential of a Uniform solid sphere:- A uniform solid sphere may be supposed to be made up of a number of thin uniform concentric spherical shells. The masses of spherical shells may be supposed to be concentric at centre O.

Case I :- At an external point



Therefore the potential due to all such shells at an external point P is given by

$$V = \frac{m_1}{r} + \frac{m_2}{r} + \ldots$$

where m_1, m_2 … etc are the masses of shells.

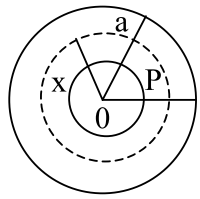
$$V = \frac{1}{r}(m_1 + m_2 + \ldots) = \frac{M}{r}$$

where M is the mass of solid sphere.

Case II:- The point P is on the sphere.

In case I, put r = a

$$V = \frac{M}{a}, \text{ where } a = \text{radius of sphere}$$



Case III:- At an internal point. Here point P is considered to be external to solid sphere of radius r & internal to the shell of internal radius r, external radius = a.

Let V_1 = potential due to solid sphere of radius r

V_2 = potential due to thick shell ofinternal radius r and external radius a

Then $V_1 = \dfrac{\text{mass of sphere of radius r}}{r}$

$$= \frac{4}{3}\frac{\pi r^3 \rho}{r} = \frac{4}{3}\pi r^2 \rho$$

To calculate V_2

We consider a thin concentration shell of radius 'x' & thicknes dx. The potential at P due to thin spherical shell under consideration is given by

$$\frac{4\pi x^2 dx\rho}{x} = 4\pi x \, dx \, \rho$$

Hence for the thick shell of radius r & a, the potential is given by

$$V_2 = 4\pi\rho \int_r^a x \, dx$$

$$= 4\pi\rho\left(\frac{a^2 - r^2}{2}\right) = 2\pi\rho(a^2 - r^2)$$

Therefore, the potential at P due to given solid sphere.

$$V = V_1 + V_2 = \frac{2}{3}\pi\rho(3a^2 - r^2)$$

where M = Mass of given solid sphere = $\dfrac{4}{3}\pi a^3 \rho$

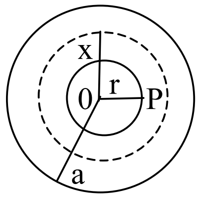
$$\Rightarrow \qquad \pi\rho = \frac{3M}{4\pi a^3}$$

Hence $V = \frac{2}{3}.\frac{3M}{4\pi a^3}(3a^2 - r^2) = \frac{M}{2a^3}(3a^2 - r^2)$

6.8 Attraction for a uniform solid sphere

Case I : At an external point

$$\vec{F} = \frac{m_1}{r^2} + \frac{m_2}{r^2} + \ldots.$$



$$\vec{F} = \frac{M}{r^2}, \quad M = m_1 + m_2 + \ldots.$$

M = Mass of sphere and m_1 , m_2…. Are masses of concentric spherical shells

Case II: At a point on the sphere,

Here we put r = a in above result

We get $\vec{F} = \frac{M}{a^2}$

Case III: At a point inside the sphere.

The point P is external to the solid sphere of radius r and it is internal to thick spherical shell of radii r and a.

And we know that attraction (forces of attraction) at an internal point in case of spherical shell is zero. Hence the resultant attraction at P is only due to solid sphere of radius r and is given by

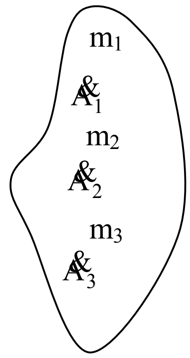
$$\vec{F} = \frac{\text{mass of sphere of radius r}}{r^2}$$

$$= \frac{4}{3}\frac{\pi r^3 \rho}{r^2} = \frac{4}{3}\pi r \rho$$

If $M = \frac{4}{3}\pi a^3 \rho \ \Rightarrow \ \pi\rho = \frac{3M}{4a^3}$

Then $\vec{F} = \frac{Mr}{a^3}$

6.9 Self attracting systems:- To find the work done by the mutual attractive forces of the particles of a self-attracting system while the particles are brought from an infinite distance to the positions, they occupy in the given system. System consists of particles of masses m_1, m…. at A_1, A_2….. etc. in the given system A.



We first being m_1 from infinity to the position A_1. Then the work done in this process is zero. Since there is no particle in the system to exact attraction on it next m_2 is brought from infinity to its position A_2. Then the work done on it by m_1 is = potential of m_1 at $A_2 \times m_2$

$$= \frac{m_1}{r_{12}} m_2 = \frac{m_1 m_2}{r_{12}}$$

where r_{12} = distances between m_1 & m_2 ($r_{12} = r_{21}$)

Then these two particles m_1 and m_2 attracts the third particle.

Work done on m_3 by m_1 & m_2 is

$$= \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$$

when m_4 is brought from infinity to its position $A_4 = \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}$

Hence the total work done in collecting all the particles from rest at infinity to their positions in the sysem A.

$$= \frac{m_1 m_2}{r_{12}} + \left(\frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \right) \ldots\ldots..$$

$$= \sum \frac{m_s m_t}{r_{st}}$$, where summation extends to every pair of particles.

Let $V_1 = \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} + \ldots$

= potential at A_1 of m_2, m_3 ….. s one

V_2 = potential at A_2 of m_1, m_3, m_4….

$$= \frac{m_1}{r_{21}} + \frac{m_3}{r_{23}} + \ldots$$

$$V_3 = \frac{m_1}{m_{13}} + \frac{m_2}{r_{23}} + \frac{m_4}{r_{43}}$$

$$\sum \frac{m_s m_t}{r_{st}} = \frac{1}{2}[V_1\, m_1 + V_2\, m_2 + \ldots + V_3 m_3]$$

Total work done = $\frac{1}{2}\Sigma mv$

This represents the work done by mutual attraction of the system of particles. If the system from a cont. body, then work done will be

$$= \frac{1}{2}\int v\, dm$$

Conversely (if particle is scattered) the work done by the mutual attraction forces of the system as its particles are scattered at infinite distance from confinguration A, then work done = $\frac{-1}{2}\sum mv = \frac{-1}{2}\int v\, dm$

We can find the work done as the body changes from one configuration A to another configuration B. The work done in changing of its from A to state at infinity + work done in collecting particles in a state at infinity to configuration B

$$= \frac{-1}{2}\int_A V\, dm + \frac{1}{2}\int_B V'\, dm'$$

$$A \to \infty \to B$$

Example. A self attracting sphere of uniform density ρ & radius 'a' changes to one of uniform density & radius 'b'. Show that the work done by its mutual attractive forces is given by

$$\frac{3}{5}M^2\left(\frac{1}{b} - \frac{1}{a}\right)$$

where M is mass of sphere.

Solution: Here the work done by mutual attractive forces of the system. As the particle which constitute the sphere of radius 'a' are scattered to infinite distance

$$w_1 = \frac{-1}{2}\int v\,dm$$

We consider a point within the system at a distance x. The potential at this point within the spheres

$$V = \frac{2}{3}\pi\rho(3a^2 - x^2)$$

Let us now consider at this point, a spherical shell of radius x & thickness dx, then

$$dm = 4\pi\, x^2\, dx\, \rho$$

$$V\,dm = \frac{2}{3}\pi\rho\,(3a^2 - x^2)\,4\pi x^2\rho\,dx$$

$$= \frac{8}{3}\pi^2\rho^2x^2\,(3a^2 - x^2)\,dx$$

$$\int v\,dm = \frac{8}{3}\pi^2\rho^2\int_0^a x^2\,(3a^2 - x^2)\,dx$$

$$= \frac{8}{3}\pi^2\rho^2\left[3a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{8}{3}\pi^2\rho^2\left[a^5 - \frac{a^5}{5}\right] = \frac{8}{3}\pi^2\rho^2\frac{4a^5}{5}$$

$$= \frac{32}{15}\pi^2\rho^2a^5$$

M = Mass of sphere of radius a

$$= \frac{4}{3}\pi a^3\ \rho \Rightarrow \rho = \frac{3m}{4\pi a^3}$$

$$\int v\,dm = \frac{6}{5}\frac{m^2}{a}$$

$$\Rightarrow \qquad W_1 = \frac{-1}{2}\int v\,dm = \frac{-3}{5}\frac{m^2}{a}$$

Similarly if W_2 is work done in bringing the particle ∞ to the second configuration (a sphere of radius b)

Then $W_2 = \frac{1}{2}\int v'\, dm' = \frac{3}{5}\frac{m^2}{b}$

Total work done is given by

$$W = W_1 + W_2 = \frac{3}{5}m^2\left(\frac{1}{b} - \frac{1}{a}\right)$$

6.10 Laplace's equation for potential

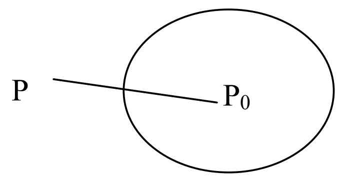
Let V be the potential of the system of attracting particles at a point P(x, y, z) not in contact with the particles so that

$$V = \sum \frac{m}{r} \qquad \ldots(1)$$

where m is the mass of particle at $P_0(a,b,c)$

r = distance of P from the P_0,

and $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2 \qquad \ldots(2)$



$$(1) \Rightarrow \frac{\partial V}{\partial x} = -\sum \frac{m}{r^2}\frac{\partial r}{\partial x} = -\sum \frac{m}{r^2}\frac{(r-a)}{r^2}$$

$$\left[\Theta\ (2) \Rightarrow 2r\frac{\partial r}{\partial x} = 2(x-a) \Rightarrow \frac{\partial r}{\partial x} = \frac{x-a}{r}\right]$$

and $\quad \frac{\partial V}{\partial y} = -\sum \frac{m(y-b)}{r^3}$

$$\frac{\partial V}{\partial z} = -\sum \frac{m(z-c)}{r^3}$$

$\therefore \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}[\, -\Sigma m\,(x-a)\, r^{-3}]$

$$= -\Sigma m(x-a)\,(-3\, r^{-4})\, \frac{\partial r}{\partial x}$$

$$- \Sigma m\, r^{-3}\, (1)$$

$$= \Sigma m\, 3\frac{(x-a)^2}{r^5} \; - - \sum \frac{m}{r^3}$$

and $\quad \frac{\partial^2 V}{\partial y^2} = \sum \frac{m(y-b)^2}{r^5} - \sum \frac{m}{r^3}$

$$\frac{\partial^2 V}{\partial z^2} = 3\sum \frac{m(z-c)^2}{r^5} - \sum \frac{m}{r^3} \text{ ss}$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which is Laplace equation.

$V \to$ potential

dv = small volume elemnt

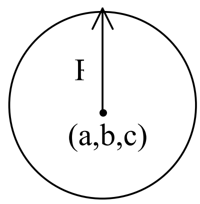
$\therefore$ dm = ρ dv

So $\quad V = \int \frac{\rho dv}{r}$

$$\frac{\partial V}{\partial x} = \int \left(\frac{-1}{r^2}\right) \frac{\partial r}{\partial x} \ \rho \ dv$$

6.11 Poisson's equation for potential

Let the point P(x, y, z) be in contact (inside) the attracting matter. We describe a sphere of small radius R & centre (a, b, c) contains the point P.



ρ = density of material (sphere)

Since the sphere we describe is very small, therefore we cosndier the matter inside this sphere is of uniform density ρ.

So potential at P may be due to

(i) the matter inside the sphere

(ii) the matter outside the sphere.

V_1 = contribution towards potential at P by the matter outside the sphere

V_2 = contribution towards potential at P by the matter inside the sphere.

Since the point P is not in contact with the matter outside the sphere. Therefore by Laplace equation $\nabla^2 V_1 = 0$.

Here V_2 = potential at P(x, y, z) inside the sphere of radius R.

$$V_2 = \frac{2}{3}\pi\rho\,(3R^2 - r^2)$$

where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

$$\frac{\partial V_2}{\partial x} = \frac{2}{3}\pi\rho\left(-2r\frac{\partial r}{\partial x}\right) = \frac{2}{3}\pi\rho(-2)\frac{r(x-a)}{r}$$

$$= \frac{-4}{3}\pi\rho(x-a)$$

$$\therefore \quad \frac{\partial^2 V_2}{\partial x^2} = \frac{-4}{3}\pi\,\rho$$

Similarly $\dfrac{\partial^2 V_2}{\partial y^2} = \dfrac{-4}{3}\pi\rho, \qquad \dfrac{\partial^2 V}{\partial z^2} = \dfrac{-4}{3}\pi\rho$

$$\therefore \quad \frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_2}{\partial z^2} = -4\pi\rho$$

$$\Rightarrow \quad \nabla^2 V_2 = -4\pi\rho$$

Since total potential $V = V_1 + V_2$

$$\therefore \quad \nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \boxed{\nabla^2 V = -4\pi\rho}$$

This equationis known as Poisson's equation

6.12 Equipotential Surfaces

The potential V of a given attracting system is a function of coordinates x,y,z. The equation

$$V(x, y, z) = \text{constant}$$

represents a surface over which the potential is constant. Such surfaces are known equipotential surfaces. Condition that a family of given surfaces is a possible family of equipotential surfaces in a free space.

To find the condition that the equation

$$f(x, y, z) = \text{constant}$$

may represent the family of equipotential surface.

If the potential V is constant whenever f(x, y, z) is constant, then there must be a functional relation between V and f(x, y, z) say

$V = \phi\{f(x, y, z)\}$

$V = \phi(f)$

$$\frac{\partial V}{\partial x} = \phi'(f)\ \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2} = \phi''(f)\left(\frac{\partial f}{\partial x}\right)^2 + \phi'(f)\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \phi''(f)\left(\frac{\partial f}{\partial y}\right)^2 + \phi'(f)\ \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 V}{\partial z^2} = \phi''(f)\left(\frac{\partial f}{\partial z}\right)^2 + \phi'(f)\ \frac{\partial^2 f}{\partial z^2}$$

Adding

$$\nabla^2 V = \phi'(f)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + \phi''(f)\left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}$$

But in free space, $\nabla^2 V = 0$

$$\Rightarrow \qquad \frac{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2} = \frac{-\varphi''(f)}{\varphi'(f)} = \text{a function of f}$$

$$= \psi(f) \text{ (say)} \qquad \ldots(1)$$

This is the necessary condition and when it is satisfied, the potential V can be expressed in terms of f(x, y, z).

Then $V = \phi(f)$, where $\dfrac{\varphi''(f)}{\varphi'(f)} + \psi(f) = 0$

Integrating, $\log \phi'(f) = \log A - \int \psi(f) df$

$$\Rightarrow \qquad \log\left(\frac{\varphi'(f)}{A}\right) = -\int \psi(f)\, df$$

$$\Rightarrow \qquad \phi'(f) = Ae^{-\int \psi(f) df}$$

Again integrating,

$$V = \phi(f) = A\int e^{\int \psi(f)df}\, df + B \qquad \ldots(2)$$

which is required expression in terms of f(x, y, z) for V.

Example :- Show that a family of right circular cones with a common axis & vertex is a possible family of equipotential surfaces & find the potential function.

Solution : Taking axis of z for common axis. The equation of family of cones is

$$f(x, y, z) = \frac{x^2 + y^2}{z^2} = \text{constant} \qquad \ldots(3)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{z^2}, \qquad \frac{\partial f}{\partial x^2} = \frac{2}{z^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{z^2}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2}{z^2}$$

$$\frac{\partial f}{\partial z} = (x^2 + y^2)\,(-2)\,(z^{-3})$$

$$\frac{\partial^2 f}{\partial z^2} = 6(x^2 + y^2)\, z^{-4}$$

Therefore condition (1) becomes

$$\frac{-\varphi''(f)}{\varphi'(f)} = \frac{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}$$

$$= \frac{\dfrac{2}{z^2} + \dfrac{2}{z^2} + \dfrac{6(x^2 + y^2)}{z^4}}{\dfrac{4x^2}{z^4} + \dfrac{4y^2}{z^4} + \dfrac{4(x^2 + y^2)^2}{z^6}}$$

$$= \frac{2z^2 + 2z^2 + 6(x^2 + y^2)}{z^4} \frac{z^6}{4z^2(x^2 + y^2) + 4(x^2 y^2)^2}$$

$$= \frac{4z^2 + 6(x^2 + y^2)}{[4z^2(x^2 + y^2) + (x^2 + y^2)^2]}$$

$$= \frac{2z\left[2+\frac{3(x^2+y^2)}{z^2}\right]}{4z^4\left[\frac{x^2+y^2}{z^2}+\left(\frac{x^2+y^2}{z^2}\right)^2\right]}$$

$$= \frac{2+3f}{2(f+f^2)} = \frac{2+3f}{2f(f+1)} = \text{function of f}$$

$\Rightarrow$ $$\frac{-\varphi''(f)}{\varphi'(f)} = \frac{2+3f}{2f(1+f)}$$ [function of $f \to 0$, condition (1) is satisfied]

$\Rightarrow$ $$\frac{\varphi''(f)}{\varphi'(f)} + \frac{2+3f}{2f(1+f)} = 0$$

$\Rightarrow$ $$\frac{\varphi''(f)}{\varphi'(f)} + \frac{1}{f} + \frac{1}{2(1+f)} = 0$$

Integrating,

$$\log \phi'(f) + \log f + \frac{1}{2} \log (1+f) = \log C$$

$\Rightarrow$ $$\log \phi'(f) = \log \frac{C}{f\sqrt{1+f}}$$

$\Rightarrow$ $$\phi'(f) = \frac{C}{f\sqrt{1+f}}$$

$\Rightarrow$ $$\frac{d\varphi}{df} = \frac{C}{f\sqrt{1+f}}$$

$\Rightarrow$ $$\int d\,\phi = \int \frac{C}{f\sqrt{1+f}}\,df + C'$$

Put $f = \tan^2\theta$

$\Rightarrow$ $df = 2\tan\theta \sec^2\theta\, d\theta$

$\therefore$ $$\phi = C\int \frac{2\tan\theta \sec^2\theta d\theta}{\tan^2\theta.\sqrt{1+\tan^2\theta}} + C'$$

$$= C\int \frac{2}{\tan\theta}.\frac{\sec^2\theta}{\sec\theta} d\theta + C'$$

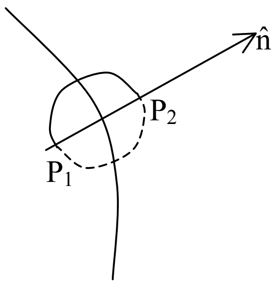
$$= 2C \int \frac{\sec\theta}{\tan\theta} d\theta + C'$$

$$= 2C \int \operatorname{cosec}\theta \; d\theta + C'$$

$\therefore \qquad V = \phi(f) = 2C \log (\operatorname{cosec} \theta - \cot \theta) + C'$

or $\qquad V = \phi(f) = 2C \log \left(\tan\frac{\theta}{2}\right) + C'$ is the required potential function. So V is constant when θ is constant.

6.13 Variation in attraction in crossing a surface on which there exist a thin layer of attracting matter.



Let P_1 and P_2 be two points on the opposite side of surface. $\sigma \to$ surface density of small circular disc of the surface between P_1 and P_2. For potential at surface, $V_1 = V_2$

$$\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = -4\pi\sigma$$

To find the attraction of matter, when the potential is given at all points of space, then Poisson's equation

$$\nabla^2 V = -4\pi\rho$$

gives the volume density of matter

$$\therefore \qquad \rho = \frac{-1}{4\pi} \nabla^2 V$$

If potential is given by different functions V_1, V_2 on opposite side of a surface S, then surface density σ is given by

$$\sigma = \frac{-1}{4\pi}\left[\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n}\right]$$

Example. The potential outside a certain cylindrical boundary is zero, inside it is

$V = x^3 - 3xy^2 - 9x^2 + 3ay^2$. Find the distribution of matter.

Solution : Since V_2 = outside potential and V_1 = Inside potential

Here $V_2 = 0$

We find the boundary.

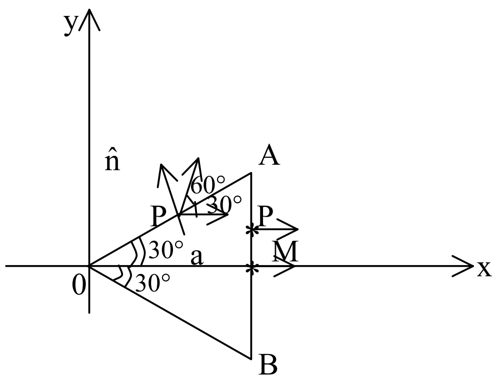
Since the potential is continuous across the boundary & zero outside the boundary.

The boundary may be given by

$$x^3 - 3xy^2 - ax^2 + 3ay^2 = 0$$

or $(x - a)(x^2 - 3y^2) = 0$

$$\Rightarrow \quad (x - a)(x + \sqrt{3}\,y)(x - \sqrt{3}\,y) = 0$$



AB is equation of line $x = a$

OB is equation of line $x + \sqrt{3}\,y = 0$

OA is equation of line $x - \sqrt{3}\,y = 0$

The section is an equilateral ΔOAB of height ‘a’.

$$\frac{\partial V_1}{\partial x} = 3x^2 - 3y^2 - 2ax$$

$$\frac{\partial V_1}{\partial y} = -6xy + 6ay$$

$$\frac{\partial^2 V_1}{\partial x^2} = 6x - 2a$$

$$\frac{\partial^2 V_1}{\partial y^2} = -6x + 6a$$

$$\frac{\partial^2 V_1}{\partial z^2} = 0$$

so that inside the region,

$$\rho = \frac{-1}{4\pi}\nabla^2 V_1$$

$$= \frac{-1}{4\pi}[4a] \quad \Rightarrow \quad \boxed{\rho = \frac{-a}{\pi}}$$

and outside, $\rho = 0$ since $V_2 = 0$

At P on AB $(x = a)$,

$$\sigma = \frac{-1}{4\pi}\left[\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x}\right]_{x=a}$$

$$= \frac{-1}{4\pi}[\,0 - 3x^2 + 3y^2 + 2ax]$$

$$= \frac{-1}{4\pi}[3y^2 - a^2] = \frac{3}{4\pi}\left[\frac{a^2}{3} - y^2\right] \qquad \ldots(1)$$

In ΔOAM, $OA^2 = OM^2 + AM^2$

$\Rightarrow \quad OA^2 = a^2 + \frac{1}{4}(OA)^2$

$\Rightarrow \quad \frac{3}{4}(OA)^2 = a^2 \quad \Rightarrow \quad (OA)^2 = \frac{4}{3}a^2$

$\Rightarrow \quad (2\,MA)^2 = \frac{4}{3}a^2 \quad \Rightarrow \quad (MA)^2 = \frac{1}{3}a^2 \qquad \ldots(2)$

$\therefore$ from (1) & (2), we have

$$\sigma = \frac{3}{4\pi}[MA^2 - MP^2]$$

$$= \frac{3}{4\pi}(MA + MP)\,(MA - MP)$$

$$= \frac{3}{4\pi}(PB)\,(AP)$$

At P on OA ($x = \sqrt{3}\,y$)

$$\sigma = \frac{1}{4\pi}\left[\frac{\partial V_1}{\partial n}\right]$$

$$= \frac{1}{4\pi}\left[-\sin 30°\frac{\partial V_1}{\partial x} + \cos 30°\frac{\partial V_1}{\partial y}\right]_{x=\sqrt{3}\,y}$$

$$= \frac{1}{4\pi}\left[\frac{-1}{2}\frac{\partial V_1}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial V_1}{\partial y}\right]_{x=\sqrt{3}\,y}$$

$$= \frac{1}{4\pi}\left[\frac{-3}{2}x^2\,\frac{3}{2}y^2 + ax - 3\sqrt{3}xy + 3\sqrt{3}ay\right]_{x=\sqrt{3}y}$$

$$= \frac{1}{4\pi}\left[\frac{-3}{2}x^2 + \frac{3}{2}\frac{x^2}{3} + ax - 3x^2 + 3\sqrt{3}\,a.\frac{x}{\sqrt{3}}\right]$$

$\Rightarrow \qquad \sigma = \frac{1}{\pi}x(a - x).$

6.14 Harmonic functions

Any solution of Laplace equation ∇^2 V = 0 in x, y, z is called Harmonic function or spherical harmonic.

i.e. $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If V is a Harmonic function fo degree n, then

$\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}\frac{\partial^t}{\partial z^t}$ is a harmonic function of degree n – p – q – t.

Now $\nabla^2 V = 0$ [Laplace equation]

p times w.r.t. x

q times w.r.t. y

t times w.r.t. z

So $$\nabla^2\left[\frac{\partial^q}{\partial x^p}\frac{\partial^q}{\partial y^q}\frac{\partial^t}{\partial z^t}\right] = 0$$

6.15 Surface and solid Harmonics :- In spherical polar coordinates (r, θ, φ) Laplace equation is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0 \qquad \ldots(1)$$

Let V = r^n Sn where S_n is independent of r or $S_n(\theta, \phi)$.

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = \frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}(r^n\ S_n)\right]$$

$$= \frac{\partial}{\partial r}[r^2\ S_n\ n\ r^{n-1}]$$

$$= \frac{\partial}{\partial r}[S_n\ n\ r^{n+1}] = nS_n\frac{\partial}{\partial r}(r^{n+1})$$

$$= n\,(n + 1)\, r^n\, S_n$$

$$(1) \Rightarrow n\,(n + 1)\, r^n\, S_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta . r^n \frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta} r^n \frac{\partial^2 S_n}{\partial\theta^2} = 0$$

$$\Rightarrow \quad n(n + 1)\, r^n\, S_n + \frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S_n}{\partial\theta}\right) + \frac{r^n}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2} = 0$$

$$\Rightarrow \quad n(n +1)\, S_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2} = 0$$

$$\Rightarrow \quad n(n +1)\, S_n + \cot\theta\, \frac{\partial S_n}{\partial\theta} + \frac{\partial^2 S_n}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2} = 0 \qquad \ldots(2)$$

If $\cos\theta = \mu$

$$N(n +1)\, S_n + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial S_n}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial\varphi^2} = 0 \qquad \ldots(3)$$

A solution S_n of equation (2) is called a Laplace function or a **surface harmonic** of order n. Since n(n+1) remains unchanged when we write −(n +1) for n. So there are two solutios of (1) of which S_n is a factor namely $r^n S_n$ and $r^{-n-1} S_n$.

These are known as solid Harmonic of degree n & −(n +1) respectively.

Remark:

1. If U is a Harmonic function of degree n, then $\frac{U}{r^{2n+1}}$ is also Harmonic function.

$$U = r^n S_n$$

so that $\frac{U}{r^{2n+1}} = \frac{r^n S_n}{r^{2n+1}} = \frac{S_n}{r^{n+1}} = S_n\, r^{-(n+1)}$

which is Harmonic.

Let xyz → 3rd degree is a solutionof Laplace equation, then $\frac{xyz}{r^7}$ is also Harmonic.

2. If U is a Harmonic function of degree −(n +1), then Ur^{2n+1} is also a Harmonic function, are may write

$$U = r^{-n-1} S_n$$

so that $r^{2n+1} U = r^{2n+1} r^{-n-1} S_n = r^n S_n$
which is Harmonic.

6.16 Surface density in terms of surface Harmonics

The potential at any point P due to a number of particles situated on the surface of sphere of radius 'a' can be ut in the form

$$V_1 = \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} U_n, \quad \text{when } r < a \qquad \ldots(1)$$

$$V_2 = \sum \frac{a^n}{r^{n+1}} U_n \quad \text{when } r > a \qquad \ldots(2)$$

where U_n denotes the sum of a number of surface Harmonics (for each particle) & therefore itself a surface harmonic. We assume (1) & (2) to represent potential of a certain distribution of mass & want to find it (density) on the surface.

Here U_1 is Harmonic

$\Rightarrow \quad \nabla^2 V_1 = 0, \ \nabla^2 V_2 = 0$

Here on the surface of sphere, density is given by

$$-4\pi\sigma = \left[\frac{\partial V_2}{\partial r} - \frac{\partial V_1}{\partial r}\right]_{r=a}$$

$$\Rightarrow \quad \sigma = \frac{1}{4\pi}\left[\frac{\partial V_1}{\partial r} - \frac{\partial V_2}{\partial r}\right]_{r=a}$$

$$= \frac{1}{4\pi}\left[\sum \frac{U_n n\, r^{n-1}}{a^{n+1}} + \sum \frac{U_n a^n (n+1)}{r^{n+2}}\right]_{r=a}$$

$$= \frac{1}{4\pi}\left[\sum \frac{U_n n\, a^{n-1}}{a^{n+1}} + \sum \frac{U_n a^n (n+1)}{a^{n+2}}\right]$$

$$= \frac{1}{4\pi}\left[\sum U_n \frac{n}{a^2} + \sum U_n \frac{(n+1)}{a^2}\right]$$

$$\Rightarrow \quad \sigma = \sum \frac{(2n+1)U_n}{4\pi a^2} \qquad \ldots(3)$$

If potential is given by (1) & (2), then surface density is given by (3).