# M.SC. MATHEMATICS

## MAL-514
## ORDINARY DIFFERENTIAL EQUATIONS – I

**DIRECTORATE OF DISTANCE EDUCATION**

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# CONTENTS

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## Chapter-1

# DIFFERENTIAL EQUATIONS AND EXISTENCE OF SOLUTIONS

### Objectives

The general purpose of this chapter is to provide an understanding of basic aspects of differential equations. A geometrical approach is used to prove the existence of solution of an initial value problem.

### Introduction

We live in a world of interrelated changing entities. The position of earth changes with time, the velocity of a falling body changes with distance, the path of a projectile changes with the velocity and angle at which it is fired.

In the language of mathematics, changing entities are called variables and the rate of change of one variable with respect to another a derivative. Equation which expresses a relationship among these variables and their derivatives are called differential equations. For example, from certain facts about the variable position of a particle and its rate of change with respect to time, we want to determine how the position of particle is related to the time, so that we can know where the particle was, is, or will be at any time t. Differential equations thus originate, whenever a universal law is expressed by means of variables and their derivatives.

### Relation of Differential Equations to various fields

Differential equations arise in many areas of science and technology: where a deterministic relationship involving some continuously changing quantities (modeled by functions) and their rates of change (expressed as derivatives) is known or postulated. Newton's Laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time.

The study of differential equations is a wide field in pure and applied mathematics, physics and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion to bridge design, to interactions between neurons. Many famous mathematicians have studied differential equations and contributed to the field, including Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, d' Alembert and Euler.

**Preliminaries**

In the real (x,y) plane, we denote x for independent variable and y for dependent variable. Then y will be a function of x and its value at $x \in [a, b]$ will be considered as y(x). Similarly in the real (t, x) or (t, y) plane, t will be treated as independent variable and x or y as dependent variables. We shall consider any such plane and the real or complex valued functions defined on any domain D in such plane, where by a domain we understand an open connected set.

**Differential Equation**

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. This relation between dependent variable (and its derivatives) and independent variables is non trivial i.e. some equations that appear to satisfy above definition are really identities. For example,

$$\left(\frac{dy}{dt} - y\right)^2 = \left(\frac{dy}{dt}\right)^2 - 2y\frac{dy}{dt} + y^2 \tag{1}$$

$$\sin^2\left(\frac{dy}{dt}\right) + \cos^2\left(\frac{dy}{dt}\right) = 1 \tag{2}$$

This equation (2) is satisfied by every differential function of one variable.

So, non trivial manner means we do not include these types of equations in the class of differential equations. We exclude such expressions as

$$\frac{d}{dx}(e^{ax}) = ae^{ax},$$

$$\frac{d}{dx}(uu) = u\frac{dv}{dx} + v\frac{du}{dx}$$

and so forth.

**Physical Meaning**

If y = f(x) is a given function then $\frac{dy}{dx}$ is interpreted as rate of change of y w.r.t. x. In any natural process the variables involved and their rates of change are connected with one another by means of the basic scientific principles (that govern the process). When this connection is expressed in mathematical symbols, the result is often a differential equation. For example, according to Newton's second law of motion, acceleration of a body of mass m is proportional to the total force F acting on it

i.e. $a \propto F$

or $a = \frac{1}{m} F$ with $\frac{1}{m}$ as constant of proportionality.

$\Rightarrow F = ma$ (1)

For instance, let a body of mass m falls freely under the influence of gravity alone. In this case only force acting on it is mg, where g is acceleration due to gravity. In most applications g is equal to 32 feet per second (or 980 centimeters per second per second).

If y is distance down to the body from some fixed height, than its velocity $v = \frac{dy}{dt}$ is rate of change of position and acceleration $a = \frac{d^2y}{dt^2}$ is rate of change of velocity. With this notation, (1) becomes

$$m\frac{d^2y}{dt^2} = mg$$

or
$$\frac{d^2y}{dt^2} = g \qquad (2)$$

If we alter the situation by assuming that air exerts a resisting force proportional to the velocity, then total force acting on the body is $mg - k\left(\frac{dy}{dt}\right)$ and (1) becomes

$$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}. \quad (3)$$

Equation (2) and (3) are differential equations that express the physical processes under consideration.

**General Form**

A differential equation having y as the dependent variable (unknown function) and t as the independent variable has the form

$F[t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^ny}{dt^n}] = 0$ for some positive integer n. if n = 0, the equation is algebraic or transcendental equation rather than a differential equation.

**Classification of Differential Equations**

**1. Ordinary Differential Equation**

A differential equation involving ordinary derivatives (total derivative $\frac{d}{dt}$) of one as more dependent variables w.r.t. a single independent variable is called an ordinary differential equation, e.g.

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \text{ and} \quad (1.1)$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \quad (1.2)$$

are ordinary differential equations. In (1.1) the variable x is the single independent variable and y is a dependent variable. In (1.2), the independent variable is t, whereas x is dependent variable.

Some general examples of ordinary differential equations

Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0 \quad (1.3)$$

Bessel's equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0 \quad (1.4)$$

are $2^{nd}$ order O.D. equations.

## 2. Partial Differential Equation

A differential equation involving partial derivatives $\left(\frac{\partial}{\partial t}\right)$ of one or more dependent variables w.r.t. more than one independent variables is called a Partial Differential Equation. e.g.

$$\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=u \tag{2.1}$$

In (2.1), s and t are independent variables and u is dependent variable.

Also, If u = f(x, y, z, t) is a function of time and three rectangular coordinates of a point in space, then following are partial differential equations of 2nd order.

Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \tag{2.2}$$

Heat equation

$$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t} \tag{2.3}$$

Wave equation

$$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial^2 u}{\partial t^2} \tag{2.4}$$

Generally partial differential equations arise in the physics of continuous media in problems involving electric fields, fluid dynamics, diffusion and wave motion.

### Order of a Differential Equation

The order of a differential equation is the order of the highest derivative present in the differential equation. Equations (1.1) and (2.2) are of 2nd order, (1.2) is of fourth order and (2.1) is of first order.

### Linear Ordinary Differential equation

A differential equation is said to be linear if it is linear in y and all its derivatives. i.e. a linear differential equation of order n, in the dependent variable y and the independent variable x is an equation that can be expressed in the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+.....+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)\,, \qquad \text{(A)}$$

where $a_0$ is not identically equal to zero.

Thus we observe here that

(1) The dependent variable y and its various derivatives occur to the first degree only.

(2) No products of y and/or any of its derivatives are present.

(3) No transcendental (i.e. vague) function of y and/or its derivatives occur.

Examples: (i) $\frac{d^2y}{dt^2}+7\frac{dy}{dt}+6y=0$ (equation with constant coefficients)

(ii) $t^7\frac{d^2y}{dt^2}+6\frac{dy}{dt}+y=\sin t$ (equation with variable coefficients)

(iii) $\frac{d^4y}{dx^4}+x^2\frac{d^3y}{dx^3}+x^3\frac{dy}{dx}=xe^x$ (equation with variable coefficients)

Here y and its derivatives occur to the first degree only and no product of y and/or any of its derivatives are present.

**Non Linear Differential Equation**

Equation F(x, y, $y^1$, ….. $y^{(n)}$,) (where F is a known function) is called a non linear differential equation of order n, if it cannot be written in the linear form as in (A)

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3+6y=0 \qquad (1)$$

$$\frac{d^2y}{dx^2}+5y\frac{dy}{dx}+6y=0 \qquad (2)$$

Equation (1) is non-linear due to presence of the term $5(dy/dx)^3$, which involves third power of first derivative.

Equation (2) is non-linear because of the term 5y(dy/dx), which involves the product of dependent variable and its first derivative.

**Application of Differential Equations**

Differential equations occur in numerous problems that are encountered in various branches of science and engineering. A few such problems are:

(1) The problem of determining the motion of a projectile, rocket, satellite or planet.

(2) The problem of determining the charge or current in an electric circuit.

(3) The problem of the conduction of heat in a rod or in a slab.

(4) The problem of determining the vibrations of a wire or a membrane.

(5) The study of the rate of growth of a population.

(6) The problem of the determination of curves that have certain geometrical properties.

**Solutions**

A functional relation between the dependent variable y and the independent variable x, that, in some interval J, satisfies the given differential equation i.e. reduces it to an identity in x is said to be a solution of the equation. The general solution of an nth order differential equation depends on x and on the n arbitrary real constants $c_1$, $c_1$, $c_2$, …… $c_n$. e.g. the function $y(x) = x^2 + ce^x$ is the general solution of the differential equation.

$$y' = y - x^2 + 2x \quad \text{in} \quad J = R$$

Also $y(x) = x^3 + \dfrac{c}{x^3}$ is a general solution of

$$xy' + 3y = 6x^3 \qquad \text{(B)}$$

and the function $y(x) = x^3$ is a particular solution of the equation (B), obtained by taking the particular value c = 0 in the general solution of (B).

**Singular Solution**

The solution which cannot be obtained by assigning particular values to the constants in general solution is called a singular solution.

For example in differential equation $y'^2 - xy' + y = 0$, the general solution is $y(x) = cx - x^2$, which represents a family of straight lines and $y(x) = \frac{1}{4}x^2$ is a singular solution which represents a parabola. Thus in 'General Solution' the word 'general' must not be taken in the sense of complete. A totality of all solutions of a differential equation is called a complete solution.

**Explicit Solution**

A differential equation of first order may be written as

$$F(x, y, y') = 0.$$

The function $y = \phi(x)$ is called an explicit solution of this differential equation provided $F(x, \phi(x), \phi'(x)) = 0$ in I.

A relation of the form $\psi(x,y) = 0$ is said to be an implicit solution of $F(x, y, y') = 0$ provided it determines one or more function $y = \phi(x)$ which satisfy $F(x, \phi(x), \phi'(x)) = 0$. It is frequently difficult (if not impossible) to solve $\psi(x, y) = 0$ for y. So, we can test the solution by obtaining y' by implicit differentiation $\psi_x = \psi_y y' = 0$ or $y' = -\psi_x/\psi_y$ and check if $F(x, y, -\psi_x/\psi_y) \cong 0$

Thus a relation $\phi(x,y) = 0$ is called an implicit solution of given differential equation, if this relation defines at least one real function of the variable x on an interval I such that this function is an explicit solution of given differential equation on this interval.

Thus $y = g(x)$ is a solution of differential equation $F[x, y, y', \ldots\ldots y^{(n)}] = 0$ on an interval I means that

$$F[x, g(x), g'(x), \ldots\ldots\ldots\ldots, g^{(n)}(x)] = 0,$$

for every choice of x in the interval I. In other words, a solution, when substituted into the differential equation makes the equation identically true for t in I.

**Some Definitions from Real Function Theory**

**Definition A**

A sequence $\{c_n\}$ of real numbers is said to converge to limit c if, given any $\varepsilon > 0$, there exists a positive number N such that

$$|c_n - c| < \varepsilon$$

for all n > N. We indicate this by writing $\lim_{n\to\infty} c_n = c$.

**Definition B**

Let $\{f_n\}$ be a sequence of real functions, each of which is defined on the real interval $a \le x \le b$. The sequence $\{f_n\}$ is said to converge pointwise to f an $a \le x \le b$, if, given any $\varepsilon > 0$, for each x such that $a \le x \le b$ there exists a positive number N (in general depending both on ε and on x) such that

$$|f_n(x) - f(x)| < \varepsilon$$

for every n > N.

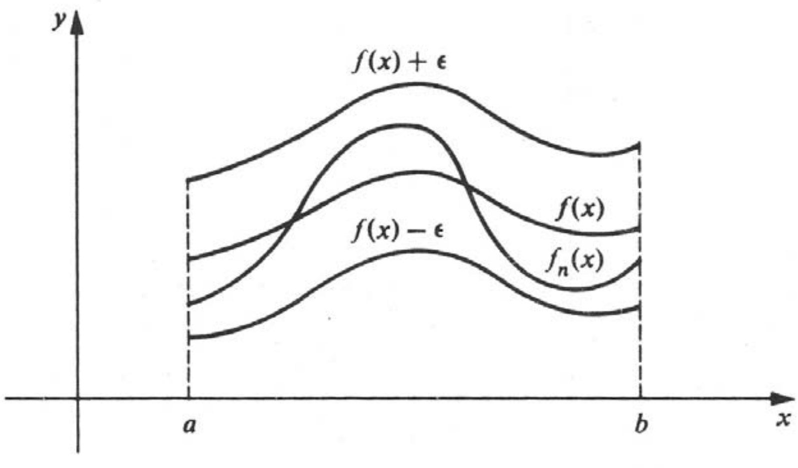
**Definition C**

Let $\{f_n\}$ be a sequence of real functions, each of which is defined on the real interval $a \le x \le b$. The sequence $\{f_n\}$ is said to converge uniformly to f on $a \le x \le b$, if, given any $\varepsilon > 0$, there exists a positive number N (which depends only upon ε) such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all n > N for every x such that $a \le x \le b$

Geometrically, this means that give $\varepsilon > 0$, the graphs of $y = f_n(x)$ for all n >N lie between the graphs of $y = f(x) + \varepsilon$ and $y = f(x) - \varepsilon$ for $a \le x \le b$ (see figure).




**Figure**

Statement of two important theorems to be used in existence and uniqueness theorem are as follows:

**Theorem - A**

**Hypothesis**

1. Let $\{f_n\}$ be a sequence of real functions which converges uniformly to f on the interval $a \leq x \leq b$.

2. Suppose each function $f_n (n=1, 2, 3, \ldots)$ is continuous on $a \leq x \leq b$.

**Conclusion** The limit function f is continuous on $a \leq x \leq b$.

**Theorem - B**

**Hypothesis**

1. Let $\{f_n\}$ be a sequence of real functions which converges uniformly to f on the interval $a \leq x \leq b$.

2. Suppose each function $f_n (n=1, 2, 3, \ldots)$ is continuous on $a \leq x \leq b$.

**Conclusion** Then for every $\alpha$ and $\beta$ such that $a \leq \alpha < \beta \leq b$.

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} f_n(x)\, dx = \int_{\alpha}^{\beta} \lim_{n\to\infty} f_n(x)\, dx \ .$$

**Definition D**

Consider the infinite series $\sum_{n=1}^{\infty} u_n$ of real functions $u_n$ (n = 1, 2, 3, …..) each of which is defined on a real interval $a \leq x \leq b$. Consider the sequence $\{f_n\}$ of partial sums of this series, defined as follows:

$f_1 = u_1,$

$f_2 = u_1 + u_2$

$f_3 = u_1 + u_2 + u_3,$

$f_n = u_1 + u_2 + u_3 + \ldots + u_n$

The infinite series $\sum_{n=1}^{\infty} u_n$ is said to converge uniformly to f on $a \leq x \leq b$ if its sequence of partial sums $\{f_n\}$ converges uniformly to f on $a \leq x \leq b$.

**Weierstrass M – Test**

**Hypothesis**

1. Let $\{M_n\}$ be a sequence of positive constants such that the series of constants $\sum_{n=1}^{\infty} M_n$ converges.

2. Let $\sum_{n=1}^{\infty} u_n$ be a series of real functions such that $|u_n(x)| \leq M_n$ for all x such that $a \leq x \leq b$ and for each n = 1, 2, 3, ......

**Conclusion**

The series $\sum_{n=1}^{\infty} u_n$ converges uniformly on $a \leq x \leq b$.

**Functions of two real variables**

**Definition E**

1. A set of points A of the xy plane will be called connected if any two points of A can be joined by a continuous curve which lies entirely in A.
2. A set of points A of the xy plane is called open if each point of A is the center of a circle whose interior lies entirely in A.
3. An open, connected set in the xy plane is called a domain.
4. A point P is called a boundary point of a domain D if every circle about P contains both points in D and points not in D.
5. A domain plus its boundary points will be called a closed domain.

**Definition F**

Let f be real function defined on a domain D of the xy plane, and let $(x_0, y_0)$ be an interior point of D. The function f is said to be continuous at $(x_0, y_0)$ if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon$$

for all $(x,y) \in D$ such that

$$|x - x_0| < \delta \text{ and } |y - y_0| < \delta.$$

**Definition G**

Let f be a real function defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to be bounded on D if there exists a positive number M such that $|f(x, y)| \leq M$ for all (x, y) in D.

**Theorem C**

**Hypothesis**

Let f be defined and continuous on a closed rectangle R: $a \leq x \leq b$, $c \leq y \leq d$.

**Conclusion** Then the function f is bounded on R.

**Definition H**

Let f be defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to satisfy a Lipschitz condition (with respect to y) in D if there exists a constant k> 0 such that

$$| f(x, y_1) - f(x, y_2) | \leq k | y_1 - y_2 |$$

for every $(x, y_1)$ and $(x, y_2)$ which belong to D. The constant k is called the Lipschitz constant.

**Initial Value Problem**

We now formulate the basic problem

Let D be a domain in the xy plane and let $(x_0, y_0)$ be an interior point of D. Consider the differential equation

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

where f is a continuous real function defined on D. We want to determine:

1. an interval $\alpha \leq x \leq \beta$ of the real x-axis such that $\alpha < x_0 < \beta$, and
2. a differentiable real function $\phi$ defined on this interval $[\alpha,\beta]$ and satisfying the following three requirements.

(i) $[x, \phi(x)] \in D$, and thus $f[x, \phi(x)]$ is defined, for all $x \in [\alpha,\beta]$.

(ii) $\frac{d\phi(x)}{dx} = f[x, \phi(x)]$, and thus $\phi$ satisfies the differential equation (1), for all $x \in [\alpha,\beta]$.

(iii) $\phi(x_0) = y_0$

This problem is called the initial-value problem associated with the differential equation (1) and the point $(x_0, y_0)$. We shall denote it by

$$\frac{dy}{dx} = f(x,y),$$

$$y(x_0) = y_0 \tag{2}$$

and a function $\phi$ satisfying the above requirements on an interval $[\alpha,\beta]$ is called a solution of the problem on the interval $[\alpha,\beta]$.

If $\phi$ is a solution of the problem on $[\alpha,\beta]$, then requirement (ii) shows that $\phi$ has a continuous first derivative $\phi'$ on $[\alpha,\beta]$

**Basic Lemma**

**Hypothesis**

1. Let f be a continuous function defined on a domain D of the xy plane.
2. Let $\phi$ be a continuous function defined on a real interval $\alpha \leq x \leq \beta$ and such that $[x, \phi(x)] \in D$ for all $x \in [\alpha, \beta]$.
3. Let $x_0$ be any real number such that $\alpha < x_0 < \beta$.

**Conclusion**

Then $\phi$ is a solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

on $[\alpha, \beta]$ and is such that $\phi(x_0) = y_0$ if and only if $\phi$ satisfies the integral equation

$$\phi(x) = y_0 + \int_{x_0}^{x} f[t, \phi(t)]\, dt \tag{2}$$

for all $x \in [\alpha,\beta]$.

**Proof** If $\phi$ satisfies the differential equation dy/dx = f(x, y) on [α, β], then

$$\frac{d\phi(x)}{dx} = f[x, \phi(x)] \tag{3}$$

for all $x \in [\alpha,\beta]$. So, $\phi(x)$ is a continuous function on [α, β], because it is differential on [α,β]. Since $\phi(x)$ is continuous on [α, β] and f is continuous on D, the function $F(x) = f[x, \phi(x)]$ is continuous on [α, β], so that it is integrable on [α, β]. Integrating (3) from $x_0$ to x.

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} f[t,\phi(t)]\, dt$$

Since $\phi(x_0) = y_0$, so $\phi$ satisfies the integral equation (2) for all $x \in [\alpha,\beta]$.

Conversely, let $\phi(x)$ satisfies the integral equation (2) for all $x \in [\alpha, \beta]$. Using the fundamental theorem of integral calculus, differentiation yields

$$\frac{d\phi(x)}{dx} = f[x,\phi(x)]$$

for all $x \in [\alpha, \beta]$ and so $\phi$ satisfies the differential equation $\frac{dy}{dx} = f(x,y)$ on [α, β]. Further from (2), we get $\phi(x_0) = y_0$.

Thus $\phi(x)$ is a solution of (1) if and only if, it satisfies the integral equation (2).

**Note** Equation (2) is called an integral equation because the unknown function $\phi(x)$ appears under the integral sign.

**Integral Equation**

An integral equation is an equation in which an unknown function appears under the integral sign. If the limits of integration are fixed, an integral equation is called a Fredholm Integral Equation. If one limit is variable, it is called a Volterra Integral equation. Volterra Integral equations are divided into two groups referred to as the $1^{st}$ and $2^{nd}$ kind.

A volterra integral equation of the first kind is

$$f(x)=\int_{a}^{x} K(x,t)\ \phi(t)dt \tag{1}$$

where $\phi(t)$ is the function to be solved for, f(x) is a specified function i.e. given known function and K(x, t) is the integral Kernel i.e. a known function of two variables. Thus if the unknown function is only under the integral sign, the equation is said to be of 1st kind.

If the unknown function is both inside and outside the integral, equation is called of 2nd kind. i.e.

$$\phi(x) = f(x) + \lambda\int_a^x K(x, t)\ \phi(t)dt$$

The parameter $\lambda$ is an unknown factor (may be $\lambda = 1$).

An example of integral equation is

$$f(x) = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)} + \frac{1}{2}\int_0^1 (x+1)e^{-xy}\ f(y)dy$$

which has solution $f(x) = e^{-x}$.

An integral equation is called homogeneous if f(x) = 0 i.e. known function f is identically zero. If f is non-zero, it is called inhomogeneous integral equation.

Of course, not all integral equations can be written in one of these forms.

**Reduction of Initial value problems to Volterra integral equations**

Now, we shall illustrate that how an initial value problem associated with a linear differential equation and auxiliary conditions reduce to a Volterra integral equation.

**Example 1** Convert the given IVP to an integral equation

$$y''(t) + y(t) = \cos(t), \quad y(0) = 0, y'(0) = 0 \qquad (1)$$

$y''(t) = -y(t) + \cos t$, integrate from 0 to t

$$\int_0^t y''(\xi)\ d\xi = \int_0^t [-y(\xi) + \cos(\xi)]\ d\xi$$

$$y'(t) - y'(0) = \int_0^t [-y(\xi) + \cos(\xi)]\ d\xi$$

$$y'(t) = -\int_0^t y(\xi)\, d\xi + [\sin(t) - \sin(0)] \quad [y'(0) = 0] \tag{2}$$

Integrating (2) again

$$\int_0^t y'(\xi)\, d\xi = -\int_0^t \int_0^s y(\xi)\, d\xi\, ds + \int_0^t \sin(\xi)\, d\xi$$

$$y(t) - y(0) = -\int_0^t y(\xi)(t - \xi) d\xi + [-\cos t + \cos 0]$$

The solution is

$$y(t) = -\int_0^t y(\xi)(t - \xi) d\xi + [1 - \cos t] \tag{3}$$

To check the solution, start differentiating (3)

$$y'(t) = -\frac{d}{dt}\int_0^t y(\xi)(t - \xi) d\xi + \sin t$$

$$= -\int_0^t \frac{\partial}{\partial t} y(\xi)(t - \xi) d\xi + \sin t$$

$$y'(t) = -\int_0^t y(\xi)\, d\xi + \sin t$$

Differentiating again

$$y''(t) = -\frac{d}{dt}\int_0^t y(\xi) d\xi + \cos t \qquad = -y(t) + \cos t$$

y''(t) + y(t) = cos (t), which is the given IVP.

**Example 2** Reduce IVP to integral equation

$y''(t) - 2ty'(t) - 3y(t) = 0, \quad y(0) = 1, y'(0) = 0$

$y''(t) = 2ty'(t) + 3y(t),$

Integrate w.r.t. t from 0 to t

$$y'(t) - y'(0) = \int_0^t 2\xi\, y'(\xi)\, d\xi + 3\int_0^t y(\xi)\, d\xi$$

$$y'(t) - 0 = 2\xi\, y(\xi)\Big|_0^t - 2\int_0^t y(\xi)d\xi + 3\int_0^t y(\xi)d\xi$$

$$y'(t) = 2[t\, y(t) - 0] + \int_0^t y(\xi)d\xi$$

Again integrating

$$y(t) - y(0) = 2\int_0^t \xi\, y(\xi)d\xi + \int_0^t \int_0^s y(\xi)\, d\xi\, ds$$

$$y(t) = 1 + 2\int_0^t \xi\, y(\xi)d\xi + \int_0^t y(\xi)\, (t - \xi)d\xi$$

$$y(t) = 1 + \int_0^t (t+\xi)\, y(\xi)d\xi$$, which is a Volterra integral equation of $2^{nd}$ kind with $K(t, \xi) = (t + \xi)$.

**Verification**

This Volterra integral equation satisfies the given initial conditions. We can obtain the original IVP by twice differentiating it.

Differentiating once

$$y'(t) = \int_0^t \frac{\partial}{\partial t}(t+\xi)\, y(\xi)d\xi + \left[(t+\xi)\, y(\xi)\right]_{\xi=t} \frac{dt}{dt} - \left[(t+\xi)\, y(\xi)\right]_{\xi=0} \frac{d0}{dt}$$

$$y'(t) = \int_0^t y(\xi)d\xi + 2t\, y(t)$$

Again differentiating

$$y''(t) = \int_0^t \frac{\partial}{\partial t}\, y(\xi)d\xi + \, y(\xi)\Big|_{\xi=t} \frac{dt}{dt} - \, y(\xi)\Big|_{\xi=0} \frac{d0}{dt} + 2[1.\, y(t) + t\, y'(t)]$$

$$= 0 + y(t) - 0 + 2\, y(t) + 2t\, y'(t)$$

$$= 3\, y(t) + 2t\, y'(t)$$

$$\Rightarrow \quad y''(t) - 2t\, y'(t) - 3\, y(t) = 0$$

which is the given IVP.

**Note Cauchy Method**

1. $$\int_a^x\int_a^x \phi(x)dx\, dx = \int_a^x (x-\xi)\, \phi(\xi)d\xi$$ or

$$\int_a^x\int_a^s \phi(y)dy\, ds = \int_a^x \phi(y)(x-y)\, dy$$

2. $$\int_0^x\int_0^{x_2}\int_0^{x_1} (\phi(x)dx_1)\, dx_2 dx = \frac{1}{2!}\int_0^x (x-\xi)^2\, \phi(\xi)\, d\xi$$

3. $$\int_0^x\int_0^{x_n}\int_0^{x_{n-1}} \ldots\ldots\int_0^{x_2} \phi(t)(dt)^n = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}\, \phi(\xi)\, d\xi$$

**Example 3**

$$\frac{d^2y}{dt^2} + A(t)\frac{dy}{dt} + B(t)y(t) = g(t) \tag{1}$$

$$y(a) = c_1, \qquad y'(a) = c_2. \tag{2}$$

We write

$$\frac{d^2y}{dt^2} = -A(t)\frac{dy}{dt} - B(t)y(t) + g(t)$$

We now integrate over the interval [a, t] to obtain

$$\frac{dy}{dt} - c_2 = -\int_a^t A(\xi)y'(\xi)d\xi - \int_a^t B(\xi)y(\xi)d\xi + \int_a^t g(\xi)d\xi$$

$$= [-A(\xi)y(\xi)]_a^t + \int_a^t A'(\xi)y(\xi)d\xi - \int_a^t B(\xi)y(\xi)d\xi + \int_a^t g(\xi)d\xi$$

$$= \int_a^t [A'(\xi)-B(\xi)]y(\xi)d\xi + \int_a^t g(\xi)d\xi - A(t)y(t) + c_1A(a) \tag{3}$$

Integrating (3) again, we obtain

$$y(t) - c_1 - c_2(t-a) = \int_a^t (t-s)[A'(s) - B(s)]y(s)ds - \int_a^t A(s)y(s)ds$$

$$+\int_a^t (t-s)g(s)ds + c_1A(a)(t\text{-}a)\,.$$

This implies

$$y(t) = \int_a^t [(t-s)\{A'(s)-B(s)\}\text{-}A(s)]y(s)ds + f(t) \tag{4}$$

where the non-homogeneous term f(t) is

$$f(t) = \int_a^t [(t-s)g(s)ds + (t\text{-}a)[c_1(A(a)) + c_2] + c_1 \tag{5}$$

Equation (4) is a Volterra integral equation of the second kind of the type

$$y(t) = f(t) + \int_a^t K(t,s)y(s)ds \tag{6}$$

in which the kernel K(t,s) is given by

$$K(t, s) = (t\text{-}s)\,[A'(s) - B(s)] - A(s). \tag{7}$$

Integral equation (4) is equivalent to the given initial value problem and it takes care of auxiliary condition in (2).

**Exercise**

Obtain the Volterra integral equation corresponding to each of the following initial value problems.

(a) $y'' + \lambda y = 0;$ $y(0) = 1,\ y'(0) = 0$

(b) $y'' + y = \sin t;$ $y(0) = 1,\ y'(0) = 1$

(c) $y'' - y + t = 0;$ $y(0) = 1,\ y'(0) = 0$

(d) $y'' + \lambda y = f(t);$ $y(0) = 1,\ y'(0) = 0$

(e) $y'' + ty = 1;$ $y(0) = y'(0) = 0.$

**Existence Theorem**

Let I denote an open interval on the real line $-\infty < t < \infty$, that is, the set of all real t satisfying $a < t < b$ for some real constants a and b. The set of all complex-valued functions having k continuous derivatives on I is denoted by $C^k(I)$. If f is a member of this set, one writes $f \in C^k(I)$, or $f \in C^k$ on I. The symbol $f \in$ is to be read "is a member of" or "belongs to." It is convenient to extend the definition of $C^k$ to intervals I which may not be open. The real intervals $a < t < b$, $a \le t \le b$, $a \le t < b$, and $a < t \le b$ will be denoted by (a, b), [a, b], [a, b), and (a, b], respectively. If $f \in C^k$ on (a, b) and in addition the right hand kth derivative of f exists at a and is continuous from the right at a, then f is said to be of class $C^k$ on [a, b). Similarly, if the kth derivative is continuous from the left at b, then $f \in C^k$ on (a, b]. If both these conditions hold, one says $f \in C^k$ on [a, b].

If D is a domain in the real (t, y) plane, the set of all complex valued functions f defined on D such that all kth order partial derivatives $\partial^k f/\partial t^p \partial y^q$ $(p + q = k)$ exist and are continuous on D is denoted by $C^k(D)$, and one writes $f \in C^k$ on D or $f \in C^k(D)$.

The sets $C^0(I)$ and $C^0(D)$, the continuous functions on I and D, will be denoted by C(I) and C(D), respectively.

**Definition (ε - approximate solution)**

Let f be a real valued continuous function on a domain D in the (t, y) plane. An ε - approximate solution of an ODE of the first order

$$\frac{dy}{dt} = f(t, y) \qquad \text{(E)}$$

on a t-interval I is a function $\phi \in C(I)$ such that

i) $(t, \phi(t)) \in D$ for all $t \in I$

ii) $\phi \in C^1(I)$, except possibly for a finite set of points S on I where $\phi'(t)$ may have simple discontinuities (i.e., at such points of S, the right and left limits of $\phi'(t)$ exist but are not equal),

iii) $|\phi'(t) - f(t, \phi(t)| < \varepsilon$ for all $t \in I - S$

**Remark**

(1) In geometrical language, (E) prescribes a slope f(t, y) at each point of D. A solution $\phi$ on I is a function whose graph [the set of all points $(t, \phi(t))$, $t \in I$] has the slope $f(t, \phi(t))$ for each $t \in I$.

(2) When $\varepsilon = 0$, then it will be understood that the set S is empty, i.e. $S = \phi$. So (ii) holds for all $t \in I$.

(3) The statement (ii) implies that $\phi$ has a piecewise continuous derivative on I, and we shall denote it by

$$\phi \in C_p^1(I)$$

(4) Consider the rectangle

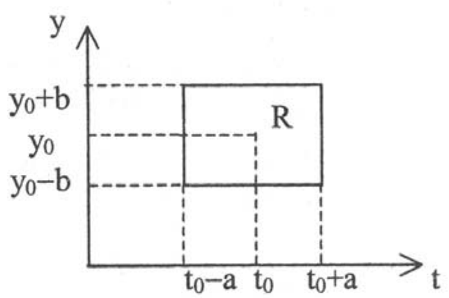
$$R = \{(t, y): |t-t_0| \leq a, |y-y_0| \leq b, a > 0, b > 0\} \quad (1)$$

about the point $(t_0, y_0)$.

Let $f \in C$ on the rectangle R. Since the rectangle R is a closed set, so the continuous function f on R is bounded. Let

$$M = \max |f(t, y)| \text{ on } R \quad (2)$$

and $\alpha = \min\left(a, \frac{b}{M}\right)$ (3)




**Theorem 1.1 (Cauchy – Euler construction of an approximate solution) or ε - Approximation Theorem**

Let $f \in C$ on the rectangle R. Given $\varepsilon > 0$, there exists an $\varepsilon$ - approximate solution $\phi$ of ODE of first order.

$$\frac{dy}{dt} = f(t, y) \quad (1)$$

on the interval $I = \{t: |t-t_0| \leq \alpha\}$ such that $\phi(t_0) = y_0$, $\alpha$ being some constant.

**Proof** Let ε > 0 be given. We shall construct an ε - approximate solution for the interval $[t_0, t_0 + \alpha]$. A similar construction will hold for $[t_0- \alpha, t_0]$.

This approximate solution will consist of a polygonal path starting at $(t_0, y_0)$, i.e. a finite number of straight- line segments joined end to end.

Since f is continuous on the closed rectangle

$$R = \{(t, y): |t-t_0| \leq a, |y-y_0| \leq b, a > 0, b > 0\} \tag{2}$$

So f is bounded and uniformly continuous on R. Let

$$M = \max_R |f(t,y)| \tag{3}$$

and $$\alpha = \min\left(a, \frac{b}{M}\right) \tag{4}$$

Then

(i) $\alpha = a \text{ if } M \leq \frac{b}{a}$ (fig 1(a))

(ii) $\alpha > \frac{b}{M} \text{ if } M \geq \frac{b}{a}$ (fig 1(b))

In the first case, we get a solution valid in the whole interval $|t-t_0| \leq a$, whereas in the second case, we get a solution valid only on I, a sub-interval of $|t-t_0| \leq a$.

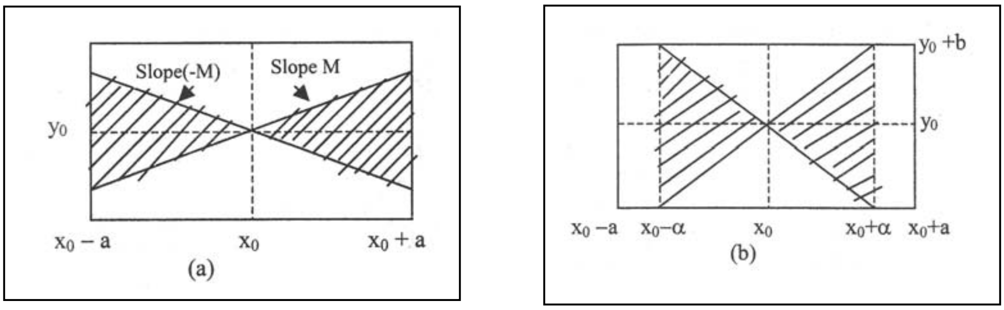



**Figure 1**

We consider the second case when $M \geq \frac{b}{a}$. Since f is uniformly continuous on R, therefore, for given ε > 0, there exists a real number $\delta_\varepsilon = \delta(\varepsilon) > 0$ such that

$$|f(t, y) - f(\bar{t}, \bar{y})| \leq \varepsilon \tag{5}$$

whenever

$$\left|t - \bar{t}\right| \ \leq \ \delta_{\varepsilon}, \ \left|y - \bar{y}\right| \ \leq \ \delta_{\varepsilon}, \text{ where } (t, y), \ (\bar{t}, \bar{y}) \ \in \ R \tag{6}$$

Now divide the interval $[t_0, t_0 + \alpha]$ into n subintervals such that

$$t_0 < t_1 < \ldots < t_n = t_0 + \alpha$$

and

$$\max\left|t_k - t_{k-1}\right| \leq \min\left(\delta_{\varepsilon}, \ \frac{\delta_{\varepsilon}}{M}\right). \tag{7}$$

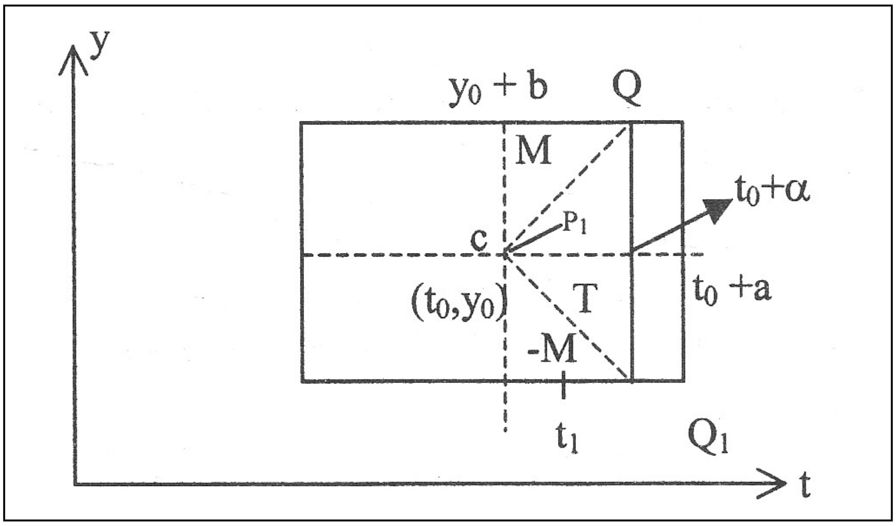



**Figure 2**

Draw two lines from $C(t_0, y_0)$ one having the slope M and the second line having the slope –M and then draw third line $t = t_0 + \alpha$. These three lines will enclose the region T.

Starting from the point $C(t_0, y_0)$ we draw a straight-line segment with slope $f(t_0, y_0)$ proceeding to the right of $t_0$ until it intersects the line $t = t_1$ at some point $P_1$ $(t_1, y_1)$. Here, slope of line $CP_1$ is $f(t_0, y_0)$. This line segment, $CP_1$, must lie inside the triangular region T, as shown in the figure 2 above because, we have, in this case,

$$\alpha = \frac{b}{M} \tag{8}$$

Now, at the point $P_1$ $(t_1, y_1)$, we construct to the right of $t_1$ a straight-line segment with slope $f(t_1, y_1)$ upto the intersection with line $t = t_2$, say at the point $P_2(t_2, y_2)$.

Continuing in this way, after a finite number of steps, the resultant path $\phi(t)$ will meet the line $t = t_0 + \alpha$ at the point $P_n$ $(t_n, y_n)$. Further, this polygon path (fig. 3) will lie entirely within the region T. This $\phi$ is the required $\varepsilon$-approximate solution.

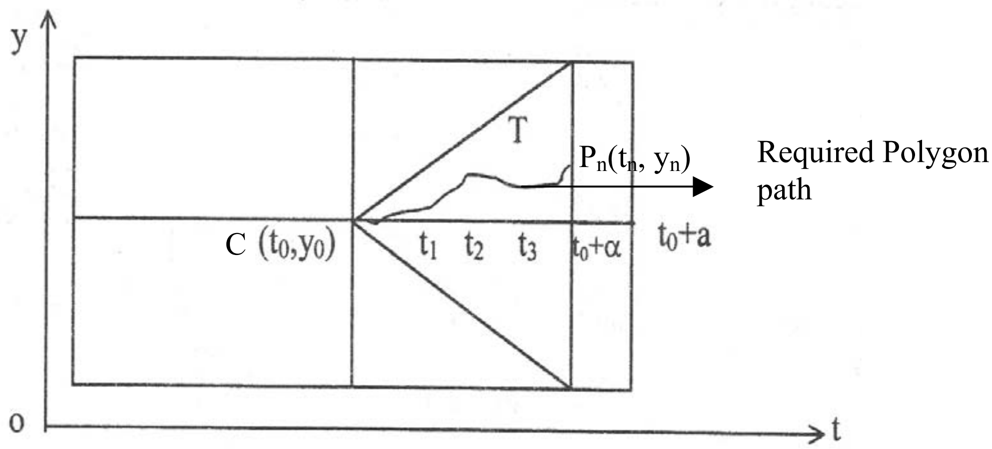



**Figure 3**

The analytical expression for the solution function $\phi(t)$ is

$$\phi(t) = \phi(t_{k-1}) + f\,(t_{k-1}, \phi(t_{k-1}))\,(t - t_{k-1}) \tag{9}$$

for $t \in [t_{k-1}, t_k]$ and $k = 1, 2, \ldots, n$ and $\phi(t_0) = y_0$. From the construction of the function $\phi$, it is clear than $\phi \in C_p^1$ on $[t_0, t_0 + \alpha]$, and that

$$|\phi(t) - \phi(\overline{t})| \leq M|t - \overline{t}| \tag{10}$$

where $t$, $\overline{t}$ are in $[t_0, t_0 + \alpha]$.

If t is such that $t_{k-1} \leq t \leq t_k$, then equations (7) and (10) together imply that

$$|\phi(t) - \phi(t_{k-1})| \leq M|t - t_{k-1}| \tag{11}$$

$$\leq M|t_k - t_{k-1}|$$

$$\leq M\frac{\delta_\varepsilon}{M} = \delta_\varepsilon$$

From equations (4), (5), (7) and (9), we obtain

$$|\phi'(t) - f(t, \phi(t))| = |f(t_{k-1}, \phi(t_{k-1})) - f(t, \phi(t)| \leq \varepsilon, \tag{12}$$

where $t_{k-1} \le t \le t_k$.

This shows that $\phi$ is an $\varepsilon$- approximate solution as desired. By similar arguments, the theorem can be proved on the interval $[t_0-\alpha, t_0]$. Hence $\phi$ is an $\varepsilon$-approximate solution on the interval $[t_0-\alpha, t_0+\alpha]$.

**Remark**

After finding an "$\varepsilon$- approximate solution" of an IVP, one may prove that there exists a sequence of these approximate solutions which tend to a solution. For this, the notion of an equicontinuous set of functions is required.

**Equicontinuous Set of Functions**

A set of functions F = {f} defined on a real interval I is said to be equicontinuous on I if, for given any $\varepsilon > 0$, there exists a real number $\delta_\varepsilon = \delta(\varepsilon) > 0$, independent of $f \in F$, such that

$$|f(t) - f(\overline{t})| < \varepsilon$$

whenever $|t - \overline{t}| < \delta_\varepsilon$ for $t, \overline{t} \in I$

**Theorem 1.2 (Due to Ascoli)**

On a bounded interval I, let F = {f} be an infinite, uniformly bounded, equicontinuous set of functions. Prove that F contains a sequence which is uniformly convergent on I.

**Proof** Let $\{r_k\}$, k = 1,2........, be all the rational numbers present in the bounded interval I listed in some order. The set of numbers $\{f(r_1): f \in F\}$is bounded, hence there exists a sequence of distinct functions $\{f_{n1}\}$, $f_{n1} \in F$, such that sequence $\{f_{n1}(r_1)\}$ is convergent (By Bolazano Weierstrass Theorem, every infinite bounded sequence has a subsequence, which is convergent).

Similarly, the set of number $\{f_{n1}(r_2)\}$ has a convergent subsequence $\{f_{n2}(r_2)\}$.

Continuing in this way, we obtain, an infinite set of functions $f_{nk} \in F$, n, k = 1, 2, ..., such that $\{f_{nk}\}$ converges at $r_1, r_2, ...., r_k$.

Define

$$g_n = f_{nn}. \quad (1)$$

Now, it will be shown that $\{g_n)$ is the required sequence which is uniformly convergent on I. Clearly, $\{g_n\}$ converges at each of the rationals $r_1, r_2, \ldots r_k$ on I. Thus, given any $\varepsilon > 0$, and each rational number $r_k \in I$, there exists an integer $N_\varepsilon(r_k)$ such that

$$|g_n(r_k) - g_m(r_k)| < \varepsilon \text{ for } n, m > N_\varepsilon(r_k). \tag{2}$$

Since the set F is equicontinuous, there exists a real number $= \delta_\varepsilon = \delta(\varepsilon) > 0$, which is independent of $f \in F$, such that

$$\left|f(t) - f(\overline{t})\right| < \varepsilon, \tag{3}$$

whenever

$$\left|t - \overline{t}\right| < \delta_\varepsilon \text{ and } t, \overline{t} \in I.$$

We divide the interval I into a finite number of subintervals, $I_1, I_2, \ldots I_p$ such that the length of the largest subinterval is less than $\delta_\varepsilon$, i.e.,

$$\max\{l(I_k) : k = 1, 2, \ldots, p\} < \delta_\varepsilon. \tag{4}$$

For each such subinterval $I_k$, choose a rational number $\overline{r}_k \in I_k$. If $t \in I$, then $t \in I_k$ for some suitable k. Hence, by (2) and (3), it follows that

$$\left|g_n(t)\text{-}g_m(t)\right| \le \left|g_n(t)\text{-}g_n(\overline{r}_k)\right| + \left|g_n(\overline{r}_k)\text{-}g_m(\overline{r}_k)\right| + \left|g_m(\overline{r}_k) - g_m(t)\right| \tag{5}$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

provided that

$$m, n > \max\{N_\varepsilon(\overline{r}_1), N_\varepsilon(\overline{r}_2), \ldots, N_\varepsilon(\overline{r}_p)\}$$

This proves the uniform convergence of the sequence $\{g_n\}$ on I, where $g_n \in F$ for each $n \in N$. This completes the proof.

**Remark**

The existence of a solution to the initial value problem, without any further restriction on the function f(t, y) is guaranteed by the following Cauchy-Peano theorem.

**Theorem 1.3 Cauchy-Peano Existence Theorem**

If f $\in$ C on the rectangle R, then there exists a solution $\phi \in C^1$ of the differential equation.

$$\frac{dy}{dt} = f(t,y)$$

on the interval $|t - t_0| \leq \alpha$ for which $\phi(t_0) = y_0$, where

$$R = \{(t, y): |t - t_0| \leq a, |y - y_0| \leq b, a > 0, b > 0\}, \ \alpha = \min\left(a, \frac{b}{M}\right),$$

$$M = \max |f(t,y)| \text{ on } R .$$

**Proof.** Let $\{\varepsilon_n\}$, n = 1,2,.........., be a monotonically decreasing sequence of positive real numbers which tends to 0 as $n \to \infty$. By Cauchy-Euler Construction Theorem, for each such $\varepsilon_n$ there exists an $\varepsilon_n$- approximate solution, say $\phi_n$ of ODE

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

on the interval

$$|t - t_0| \leq \alpha \text{ with } \phi_n(t_0) = y_0 \tag{2}$$

It is being given that

$$\alpha = \min\left(a, \frac{b}{M}\right) \tag{3}$$

$$M = \max |f(t, y)| \text{ for } (t, y) \in R \tag{4}$$

$$R = \{(t, y): |t - t_0| \leq a, |y - y_0| \leq b, a > 0, b > 0\} \tag{5}$$

Further, from Cauchy Euler Construction Theorem

$$|\phi_n(t) - \phi_n(\overline{t})| \leq M|t - \overline{t}| \tag{6}$$

for t, $\overline{t}$ in $[t_0, t_0 + \alpha]$.

Applying (6) to $\overline{t} = t_0$ and since,

$$|t - t_0| \leq \alpha \leq \frac{b}{M}, \tag{7}$$

it follows that

$|\phi_n(t) - y_0| < b$ for all t in $|t - t_0| < \alpha$. (8)

This implies that the sequence $\{\phi_n\}$ is uniformly bounded by $|y_0| + b$.

Further, (6) implies that the sequence $\{\phi_n\}$ is an equicontinuous set. Therefore, by Ascoli Lemma there exists a subsequence $\{\phi_{nk}\}$, k= 1, 2, ...., of $\{\phi_n\}$, converging uniformly on the interval $[t_0- \alpha, t_0+\alpha]$ to a limit function $\phi$, which must be continuous, since each $\phi_n$ is continuous.

Now, we shall show that this limit function $\phi$ is a solution of (1) which satisfies the required specification. For this, we write the relation defining $\phi_n$ as an $\varepsilon_n$-approximate solution in an integral form, as follows:

$$\phi_n(t) = y_0 + \int_{t_0}^{t} [f(s, \phi_n(s)) + \Delta_n(s)]ds \tag{9}$$

where

$$\Delta_n(s) = \phi'_n(s) - f(s,\ \phi_n(s)) \tag{10}$$

at those points where $\phi'_n$ exists and $\Delta_n(s) = 0$ otherwise.

Because $\phi_n$ is an $\varepsilon_n$ - approximate solution, so

$$|\Delta_n(s)| \leq \varepsilon_n. \tag{11}$$

Since f is uniformly continuous on R, and $\phi_{nk} \to \phi$ uniformly on $[t_0 - \alpha, to + \alpha]$ as $k \to \infty$, it follows that

$$f(t, \phi_{nk}(t)) \to f(t, \phi(t))$$

uniformly on $[t_0 - \alpha, to + \alpha]$ as $k \to \infty$.

Replacing n by $n_k$ in (9) and letting $k \to \infty$,

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s,\phi(s))ds \tag{12}$$

From (12), we get

$$\phi(t_0) = y_0 \tag{13}$$

and upon differentiation, as f is continuous,

$$\frac{d\phi}{dt} = f(t, \phi(t)). \tag{14}$$

It is clear from (13) and (14) that $\phi$ is a solution of ODE (1) through the point $(t_0, y_0)$ on the interval $|t - t_0| \le \alpha$ of class $C^1$. This completes the proof of the theorem.

**Remarks**

(1) If uniqueness of solution is assured, the choice of a subsequence in ε-approximation theorem is unnecessary.

(2) It can happen that the choice of a subsequence is unnecessary even though uniqueness is not satisfied. Consider the example.

$$\frac{dy}{dt} = y^{1/3} \tag{1}$$

There are an infinite number of solutions starting at the point C(0,0) which exist on I = [0,1].

For any c, $0 \le c \le 1$, the function $\phi_c$ defined by

$$\phi_c(t) = \begin{cases} 0 & 0 \le t \le c \\ \left[\dfrac{2(t-c)}{3}\right]^{3/2} & c < t \le 1 \end{cases} \tag{2}$$

is a solution of (1) on I. If the construction of ε-approximation theorem is applied to equation (1), one finds that the only polygonal path starting at the point C(0,0) is $\phi_1$. This shows that this method cannot, in general, gives all solution of (1)

**Theorem 1.4** Let $f \in C$ on a domain D in the (t, y) plane, and suppose $(t_0, y_0)$ is any point in D. Then there exists a solution $\phi$ of

$$\frac{dy}{dt} = f(t, y) \quad \text{for } (t, y) \in D, \; y(t_0) = y_0 \tag{1}$$

on some t-interval containing $t_0$ in this interior.

**Proof** Since domain D is open, there exists an $r > 0$ such that all points whose distance from $C(t_0, y_0)$ is less than r, are contained in D. Let R be any closed rectangle containing $C(t_0, y_0)$ and let R is contained in this open circle of radius r. Then required result is obtained by applying Cauchy-Peano Existence Theorem on (1).

**Summary**

In this chapter, first of all the reader is made familiar with certain basic concepts of differential equations and real function theory. Existence theorems namely Cauchy-Euler construction theorem and Cauchy-Peano Existence theorem are proved to show the existence of solutions of initial value problems. A relation between initial value problem and Volterra Integral equation is established with suitable illustrations.

**Keywords**

Differential equations, Initial value problem, $\varepsilon$ - approximate solution, Existence of solutions.

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## Chapter-2

# EXISTENCE AND UNIQUENESS OF SOLUTIONS

### Objectives

This chapter is concerned with a very important property of ordinary differential equations i.e. existence and uniqueness of solutions. Picard's method of successive approximations, which apart from being a mere numerical technique to approximate solutions has far reaching theoretical implications as well, is applied, to obtain approximate solutions of initial value problems.

### Introduction

In the previous chapter, we did not device a general method which can assert theoretically the existence and uniqueness of solutions of a wider class of first order ordinary differential equations. This chapter discusses in detail the approximation method of Picard to the solution of the initial value problem of the general first order non-linear differential equations of the type.

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0, \qquad (1)$$

where f(x,y) is some arbitrary function defined and continuous in some neighbourhood of $(x_0, y_0)$. This method is also useful even when exact solutions are available, especially, if the exact solution is quite involved and we are only interested in the numerical value of the solution function at different points. The Picard's theorem gives the unique solution of the above initial value problem (1) by the method of successive approximations, using the integral equation equivalent to the given non-linear differential equation.

**Theorem 2.1 The Existence and Uniqueness Theorem (Picard-Lindelof Theorem)**

**Hypothesis**

1. Let D be a domain of the xy plane, and let f be a real function satisfying the following two requirements.

(i) f is continuous in D;

(ii) f satisfies a Lipschitz condition (with respect to y) in D; that is, there exists a constant $k > 0$ such that

$$|f(x,y_1) - f(x, y_2)| \leq k| y_1 - y_2| \tag{1}$$

for all $(x,y_1), (x, y_2) \in D$.

2. Let $(x_0, y_0)$ be an interior point of D; let a and b be such that the rectangle R: $|x - x_0| \leq a$, $| y - y_0| \leq b$, lies in D; let $M = \max |f(x,y)|$ for $(x, y) \in R$, and let $h = \min (a, b/M)$.

**Conclusion**

There exists a unique solution $\phi$ of the initial value problem

$$\frac{dy}{dx} = f(x,y),$$

$$y(x_0) = y_0, \tag{2}$$

on the interval $|x - x_0| \leq h$

**Proof** We shall prove this theorem by the method of successive approximations. Let x be such that $| x - x_0| \leq h$. We define a sequence of functions.

$$\phi_1, \phi_2, \phi_3, \ldots, \phi_n, \ldots$$

called the successive approximations (Picard iterants) as follows:

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0]\, dt,$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f[t, \phi_1(t)]\, dt,$$

$$\phi_3(x) = y_0 + \int_{x_0}^{x} f[t, \phi_2(t)]\, dt, \tag{3}$$

.
.
.

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \phi_{n-1}(t)]\, dt.$$

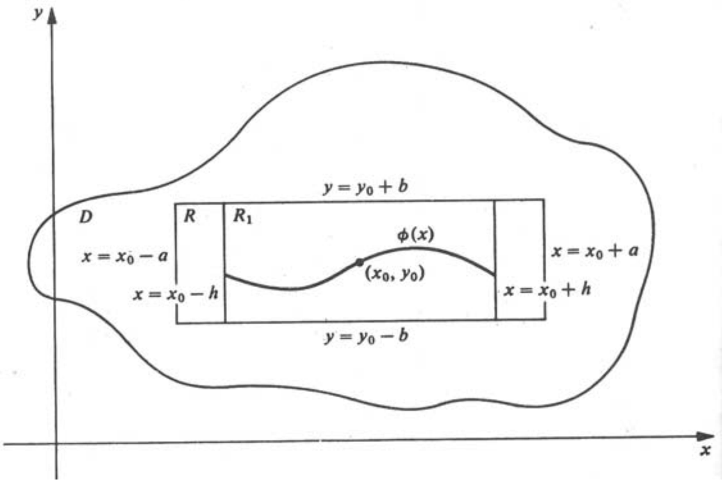


**Figure** The case in which $\frac{b}{M} < a$ so that h = min $\left(a, \frac{b}{M}\right) = \frac{b}{M}$ smaller interval $|x - x_0| \leq h < a$ associated with the smaller rectangle $R_1$ defined by $|x - x_0| \leq h < a$, $|y - y_0| \leq b$.

We shall divide the proof into five main steps.

1. The functions $\{\phi_n\}$ defined by (3) actually exist, have continuous derivatives, and satisfy the inequality $|\phi_n(x) - y_0| \leq b$ on $|x - x_0| \leq h$; and thus $f[x, \phi_n(x)]$ is defined on this interval.

2. The functions $\{\phi_n\}$ satisfy the inequality

$$|\phi_n(x) - \phi_{n-1}(x)| \leq \frac{M}{k} . \frac{(kh)^n}{n!} \text{ on } |x - x_0| \leq h$$

3. As $n \to \propto$, the sequence of functions $\{\phi_n\}$ converges uniformly to a continuous function $\phi$ on $|x - x_0| \leq h$.

4. The limit function $\phi$ satisfies the differential equation $dy/dx = f(x,y)$ on $|x - x_0| \leq h$ and is such that $\phi(x_0) = y_0$.

5. This function $\phi$ is the only differentiable function on $|x - x_0| \le h$ which satisfies the differential equation dy/dx=f(x,y) and is such that $\phi(x_0) = y_0$.

Throughout the entire proof we shall consider the interval $[x_0, x_0 + h]$; similar arguments hold for the interval $[x_0 - h, x_0]$.

1. We shall prove the first step by using mathematical induction. Assume that $\phi_{n-1}$ exists, has a continuous derivative, and is such that $|\phi_{n-1}(x) - y_0| \le b$ for all x such that $x_0 \le x \le x_0 + h$. Thus $[x, \phi_{n-1}(x)]$ lies in the rectangle R and so $f[x, \phi_{n-1}(x)]$ is defined and continuous and satisfies

$$|f[x, \phi_{n-1}(x)]| \le M \text{ on } [x_0, x_0 + h].$$

Since $$\phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \phi_{n-1}(t)]dt,$$

we see that $\phi_n$ also exists and has a continuous derivative on $[x_0, x_0+h]$. Also,

$$|\phi_n(x) - y_0| = |\int_{x_0}^{x} f[t, \phi_{n-1}(t)]dt|$$

$$\le \int_{x_0}^{x} |f[t, \phi_{n-1}(t)]| dt \le \int_{x_0}^{x} M\ dt$$

$$= M(x - x_0) \le Mh \le b.$$

Thus, $[x, \phi_n(x)]$ lies in R and hence $f[x, \phi_n(x)]$ is defined and continuous on $[x_0, x_0 + h]$. Clearly $\phi_1$ defined by

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0]dt$$

exists and has a continuous derivative on this interval. Also,

$$|\phi_1(x) - y_0| \le \int_{x_0}^{x} |f[t, y_0]| dt \le M(x - x_0) \le b$$

and so $f[x, \phi_1(x)]$ is defined and continuous on the interval under consideration. Thus, by mathematical induction, each function $\phi_n$ of the sequence (3) possesses these desired properties on $[x_0, x_0+ h]$.

2. In this step we again employ mathematical induction. We assume that

$$|\phi_{n-1}(x) - \phi_{n-2}(x)| \le \frac{Mk^{n-2}}{(n-1)!}(x - x_0)^{n-1} \text{ on } [x_0, x_0 + h]. \quad (4)$$

Then

$$|\phi_n(x) - \phi_{n-1}(x)| = \left|\int_{x_0}^{x} \{f[t, \phi_{n-1}(t)] - f[t, \phi_{n-2}(t)]\} dt\right|$$

$$\leq \int_{x_0}^{x} \left| f[t, \phi_{n-1}(t)] - f[t, \phi_{n-2}(t)] \right| dt$$

Since by step 1, $|\phi_n(x) - y_0| \leq b$ for all n on $[x_0, x_0 + h]$, using the Lipschitz condition (1) we have

$$| f [t, \phi_{n-1}(t)] - f [t, \phi_{n-2}(t)]| \leq k |\phi_{n-1}(t) - \phi_{n-2}(t)|$$

Thus

$$|\phi_n(x) - \phi_{n-1}(x)| \leq \int_{x_0}^{x} k \left|\phi_{n-1}(t) - \phi_{n-2}(t)\right| dt$$

Now using the assumption (4), we have

$$|\phi_n(x) - \phi_{n-1}(x)| \leq k \int_{x_0}^{x} \frac{Mk^{n-2}}{(n-1)!} (t - x_0)^{n-1} dt$$

$$= \frac{Mk^{n-1}}{(n-1)!} \int_{x_0}^{x} (t - x_0)^{n-1} dt$$

$$= \frac{Mk^{n-1}}{(n-1)!} \left[\frac{(t - x_0)^n}{n}\right]\Bigg|_{x_0}^{x} = \frac{Mk^{n-1}}{n!} (x - x_0)^n$$

Thus, we have

$$|\phi_n(x) - \phi_{n-1}(x)| \leq \frac{Mk^{n-1}}{n!} (x - x_0)^n, \qquad (5)$$

which is precisely inequality (4) with (n-1) replaced by n. When n =1, we have as in Step 1:

$$|\phi_1(x) - y_0| \leq M(x - x_0).$$

This is inequality (5) when n = 1. Thus by induction the inequality (5) is satisfied on $[x_0, x_0 + h]$ for all n.

Since

$$\frac{Mk^{n-1}}{n!} (x - x_0)^n \leq \frac{Mk^{n-1}}{n!} h^n = \frac{M}{k} \frac{(kh)^n}{n!},$$

we have

$$|\phi_n(x) - \phi_{n-1}(x)| \le \frac{M}{k}\frac{(kh)^n}{n!} \qquad (6)$$

for n = 1, 2, 3, ………… on $[x_0, x_0 + h]$.

3. Now the series of positive constants

$$\frac{M}{k}\sum_{n=1}^{\infty}\frac{(kh)^n}{n!} = \frac{M}{k}\frac{kh}{1!} + \frac{M}{k}\frac{(kh)^2}{2!} + \frac{M}{k}\frac{(kh)^3}{3!} + \ldots$$

converges to $\frac{M}{k}[e^{kh} - 1]$.

Also the series

$$\sum_{i=1}^{n}[\phi_i(x) - \phi_{i-1}(x)]$$

is such that (6) is satisfied for all x on the interval $x_0 \le x \le x_0 + h$, for each n = 1, 2, 3… Thus by the Weierstrass M –test, the series

$$y_0 + \sum_{i=1}^{n}[\phi_i(x) - \phi_{i-1}(x)]$$

converges uniformly on $[x_0, x_0 + h]$. Therefore its sequence of partial sums $\{S_n\}$ converges uniformly to a limit function $\phi$ on $[x_0, x_0 + h]$. But

$$S_n(x) = y_0 + \sum_{i=1}^{n}[\phi_i(x) - \phi_{i-1}(x)] = \phi_n(x).$$

In other words, the sequence $\phi_n$ converges uniformly to $\phi$ on $[x_0, x_0 + h]$. Thus, each $\phi_n$ is continuous for $[x_0, x_0 + h]$. Theorem A shows that the limit function $\phi$ is also continuous on $[x_0, x_0 + h]$.

4. Since each $\phi_n$ satisfies $|\phi_n(x) - y_0| \le b$ on $[x_0, x_0 + h]$, we also have $|\phi(x) - y_0| \le b$ on $[x_0, x_0 + h]$. Thus $f[x, \phi(x)]$ is defined on this interval and we can further apply the Lipschitz condition (1) and obtain

$$|f[x, \phi(x)] - f[x, \phi_n(x)]| \le k|\phi(x) - \phi_n(x)| \qquad (7)$$

for $x \in [x_0, x_0 + h]$. By step 3, given $\varepsilon > 0$, there exists $N > 0$ such that $|\phi(x) - \phi_n(x)| < \varepsilon/k$ for all $n > N$ and all x on $[x_0, x_0 + h]$. Thus

$$k|\phi(x) - \phi_n(x)| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon \qquad (8)$$

for all n > N and all x on the interval under consideration. Thus from (7) and (8) we see that the sequence of functions defined by f $[x, \phi_n(x)]$ for n = 1, 2, 3, ….. converges uniformly to the function defined by f $[x, \phi(x)]$ on $[x_0, x_0 + h]$. Also, each function defined by f $[x, \phi_n(x)]$ for n = 1, 2, 3, …… is continuous on this interval. Thus theorem B applies and

$$\phi(x) = \lim_{n\to\alpha} \phi_{n+1}(x) = y_0 + \lim_{n\to\alpha} \int_{x_0}^{x} f[t,\ \phi_n(t)]\, dt$$

$$= y_0 + \int_{x_0}^{x} \lim_{n\to\alpha} f[t,\ \phi_n(t)]\, dt$$

$$= y_0 + \int_{x_0}^{x} f[t,\ \phi(t)]\, dt.$$

Thus the limit function $\phi$ satisfies the integral equation

$$\phi(x) = y_0 + \int_{x_0}^{x} f[t,\ \phi(t)]\, dt$$

on $[x_0, x_0 + h]$. Thus by the basic lemma, the limit function $\phi$ satisfies the differential equation dy/dx = f(x, y) on $[x_0, x_0 + h]$ and is such that $\phi(x_0) = y_0$. We have thus proved the existence of solution of the basic initial value problem (2) on the interval $[x_0, x_0 + h]$.

5. We now prove that the solution $\phi$ is unique. Assume that $\psi$ is another differentiable function defined on $[x_0, x_0 + h]$ such that

$$\frac{d\psi}{dx} = f[x, \psi(x)]$$

and $\psi(x_0) = y_0$. Then certainly

$$|\psi(x) - y_0| < b \qquad (9)$$

on some interval $[x_0, x_0 + \delta]$. Let $x_1$ be such that $|\psi(x) - y_0| < b$ for $x_0 \le x < x_1$ and $|\psi(x_1) - y_0| = b$. Suppose $x_1 < x_0 + h$. Then

$$M_1 = \left|\frac{\psi(x_1) - y_0}{x_1 - x_0}\right| = \frac{b}{x_1 - x_0} > \frac{b}{h} \ge M$$

But by the mean – value theorem there exists $\xi$, where $x_0 < \xi < x_1$, such that

$$M_1 = |\psi'(\xi)| = |f[\xi, \psi(\xi)]| \leq M,$$

a contradiction. Thus $x_1 \geq x_0 + h$ and the inequality (9) holds for $x_0 \leq x < x_0 + h$, and so

$$|\psi(x) - y_0| \leq b \tag{10}$$

on the interval $x_0 \leq x \leq x_0 + h$.

Since $\psi$ is a solution of $dy/dx = f(x,y)$ on $[x_0, x_0+h]$ such that $\psi(x_0) = y_0$, from the basic lemma we see the $\psi$ satisfies the integral equation.

$$\psi(x) = y_0 + \int_{x_0}^{x} f[t, \psi(t)]dt \tag{11}$$

on $[x_0, x_0 + h]$. We shall now prove by mathematical induction that

$$|\psi(x) - \phi_n(x)| \leq \frac{k^n b(x - x_0)^n}{n!} \tag{12}$$

on $[x_0, x_0 + h]$. We assume that

$$|\psi(x) - \phi_{n-1}(x)| \leq \frac{k^{n-1} b(x - x_0)^{n-1}}{(n-1)!} \tag{13}$$

on $[x_0, x_0 + h]$. Then from (3) and (11) we have

$$|\psi(x) - \phi_n(x)| = \left| \int_{x_0}^{x} \{ f[t, \psi(t)] - f[t, \phi_{n-1}(t)] \} dt \right|$$

$$\leq \int_{x_0}^{x} \left| f[t, \ \psi(t)] - f[t, \ \phi_{n-1}(t)] \right| dt$$

Using Lipschitz condition, we have

$$|\psi(x) - \phi_n(x)| \leq \int_{x_0}^{x} k \left| \psi(t) - \phi_{n-1}(t) ] \right| dt$$

Using assumption (13), we have

$$|\psi(x) - \phi_n(x)| \leq k \int_{x_0}^{x} \frac{k^{n-1} b(t - x_0)^{n-1}}{(n-1)!} dt$$

$$= \frac{k^n b}{(n-1)!} \left[ \frac{(t - x_0)^n}{n} \right] \Bigg|_{x_0}^{x} = \frac{k^n b(x - x_0)^n}{n!}$$

which is (13) with (n-1) replaced by n. When n = 1, we have

$$|\psi(x) - \phi_1(x)| \le \int_{x_0}^{x} \left| f[t,\psi(t)] - f[t,\ y_0] \right| dt$$

$$\le k\int_{x_0}^{x} \left|\psi(t) - y_0\right| dt \le kb(x - x_0),$$

which is (12) for n = 1. Thus by induction the inequality (12) holds for all n on $[x_0, x_0 + h]$. Hence we have.

$$|\psi(x) - \phi_n(x)| \le b\frac{(kh)^n}{n!} \tag{14}$$

for n = 1, 2, 3…. on $[x_0, x_0 + h]$.

Now the series $\sum_{n=0}^{\infty} b\frac{(kh)^n}{n!}$ converges, and so $\lim_{n\to\infty} b\frac{(kh)^n}{n!} = 0$. Thus from (14) $\psi(x) = \lim_{n\to\infty} \phi_n(x)$ on $[x_0, x_0 + h]$. But $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ on this interval Thus,

$$\psi(x) = \phi(x)$$

on $[x_0, x_0 + h]$. Thus the solution $\phi$ of the basic initial value problem is unique on $[x_0, x_0 + h]$.

We have thus proved that the basic initial – value problem has a unique solution on $[x_0, x_0 + h]$, we can carry through similar arguments on the interval $[x_0 - h, x_0]$. We thus conclude that the differential equation dy/dx = f(x,y) has a unique solution $\phi$ such that $\phi(x_0) = y_0$ on $|x - x_0| \le h$.

**The Method of Successive Approximation or Picard Iteration Method**

This is very useful method to deduce the existence of solution. We know that $\phi$ is a solution of IVP

$$x' = f(t, x) \tag{E}$$

on $|t - \tau| \le \alpha$ s.t. $\phi(\tau) = \xi$

iff $\phi$ satisfies the integral equation

$$\phi(t) = \xi + \int_{\tau}^{t} f(s, \phi(s))\, ds \quad , \quad |t - \tau| \le \alpha.$$

**Definition**

The successive approximations for (E) are defined to be the functions i.e. a sequence of functions $\phi_0(t)$, $\phi_1(t)$, ……… (called successive approximations) as follows

$$\phi_0(t) = \xi$$

and $$\phi_{k+1}(t) = \xi + \int_\tau^t f(s, \phi_k(s))\, ds \;, \quad k = 0, 1, 2, \ldots\ldots; \qquad |t - \tau| \le \alpha$$

i.e. $$\phi_0(t) = \xi$$

$$\phi_1(t) = \xi + \int_\tau^t f(s, \phi_0(s))\, ds$$

$$= \xi + \int_\tau^t f(s, \xi)\, ds$$

$$\phi_2(t) = \xi + \int_\tau^t f(s, \phi_1(s))\, ds$$

…………………………

…………………………

$$\phi_n(t) = \xi + \int_\tau^t f(s, \phi_{n-1}(s))\, ds\, .$$

**Explanation**

If $\phi(\tau) = \xi$, let us define the constant function $\phi_0(t) = \xi$. Though this constant function satisfies the initial condition, it does not in general satisfies the integral equation. But if we find

$\phi_1(t) = \xi + \int_\tau^t f(s, \phi_0(s))\, ds$, then $\phi_1(t)$ may be a little more closer to $\phi(t)$. In a similar manner, we can find $\phi_2(t)$, $\phi_3(t)$, ……… Continuing this process successively, we can obtain

$$\phi_n(t) = \xi + \int_{\tau}^{t} f(s, \phi_{n-1}(s))\, ds$$

Now the crux of the Picard Theorem is that $\phi_n(t) \to \phi(t)$ as $n \to \infty$ giving the unique solution of the IVP.

**Note:** If one solves the given linear equation, the Picard Successive Approximations converge to the exact solution of the initial value problem.

### Geometrical Interpretation

In geometrical language, we are to devise a method for constructing a function $x = x(t)$ whose graph passes through the point $(\tau, \xi)$ and that satisfies the differential equation $x' = f(t, x)$ in some neighborhood of $\tau$.

### Process of Iteration

We begin with a crude approximation to a solution and prove it step by step by applying a repeatable operation, which will bring us as close as we please to an exact solution. In the integral equation

$$x(t) = \xi + \int_{\tau}^{t} f(s, x(s))\, ds, \tag{1}$$

the dummy variable s is used to avoid confusion with the variable upper limit t on the integral I. A rough approximation to a solution is given by the constant $\phi_0(t) = \xi$, which is simply a horizontal straight line through the point $(\tau, \xi)$. We insert this approximation in the right side of equation (1) in order to obtain a new and perhaps better approximation $x_1(t)$ as follows.

$$x_1(t) = \xi + \int_{\tau}^{t} f(s, x_0(s))\, ds$$

The next step is to use $x_1(t)$ to generate another and perhaps even better approximation $x_2(t)$ in the same way

$$x_2(t) = \xi + \int_{\tau}^{t} f(s, x_1(s))\, ds$$

This procedure is called Picard's method of successive approximation.

**Solution of Initial Value Problems by Picard method**

Solve the following IVPs by Picard method of successive approximation.

**1.** Consider the IVP $\frac{dx}{dt} = x$ , x(0) = 1.

Integrating over the interval [0, t], we obtain

$$x(t) = 1 + \int_0^t x(s)\,ds$$

which is (Volterra) integral equation of $2^{nd}$ kind.

Let $\phi_0(t) = 1$, then by Picard's method

$$\phi_1(t) = 1 + \int_0^t \phi_0(s)\,ds = 1 + \int_0^t 1\,ds = 1 + t$$

$$\phi_2(t) = 1 + \int_0^t \phi_1(s)\,ds = 1 + \int_0^t (1+s)\,ds$$

$$= 1 + t + \frac{t^2}{2!}$$

$$\phi_3(t) = 1 + \int_0^t \phi_2(s)\,ds = 1 + \int_0^t (1+s+\frac{s^2}{2!})\,ds$$

$$= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

Continuing like this, we obtain

$$\phi_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots\ldots + \frac{t^n}{n!}$$

Take limit as $n \to \infty$

$$\lim_{n\to\infty} \phi_n(t) = e^t$$

$\Rightarrow$ $\phi(t) = e^t$ is the unique solution of the given IVP, by the Picard's method of successive approximations.

**2.** $\frac{dx}{dt} = t^2 x$ , $x(0) = 1$

**Sol** The corresponding integral equation is

$$x(t) = 1 + \int_0^t s^2\ x(s)\ ds$$

Picard iterates are

$$\phi_0(t) = 1, \quad \phi_1(t) = 1 + \int_0^t s^2 .\ 1\ ds\ = 1 + \frac{t^3}{3}$$

$$\phi_2(t) = 1 + \int_0^t s^2\ (1+ \frac{s^3}{3})\ ds\ = 1 + \frac{t^3}{3} + \frac{1}{2!}\left(\frac{t^3}{3}\right)^2$$

$$\phi_3(t) = 1 + \int_0^t s^2 \left(1+ \frac{s^3}{3} + \frac{1}{2!}\left(\frac{s^3}{3}\right)^2\right) ds$$

$$= 1 + \frac{t^3}{3} + \frac{1}{2!}\left(\frac{t^3}{3}\right)^2 + \frac{1}{3!}\left(\frac{t^3}{3}\right)^3$$

……………………………………………….

……………………………………………….

$$\phi_n(t) = 1 + \int_0^t s^2\ \phi_{n-1}(s)\ ds$$

$$= 1 + \frac{t^3}{3} + \frac{1}{2!}\left(\frac{t^3}{3}\right)^2 + \ldots\ldots\ldots. + \frac{1}{n!}\left(\frac{t^3}{3}\right)^n$$

The exact solution is obtained by taking limit $n \to \infty$, i.e. $\lim_{n\to\infty} \phi_n(t) = \phi(t) = e^{t^3/3}$, to which the above approximate solution converges.

**3.** $\frac{dx}{dt} = t(x - t^2 + 2)$ , $x(0) = 1$

**Sol** The integral equation, equivalent to the above IVP is

$$x(t) = 1 + \int_0^t s\ [x(s)\ -\ s^2\ + 2]\ \ ds$$

The approximate solutions are

$$\phi_0(t) = 1, \quad \phi_1(t) = 1 + \int_0^t s\,(1 - s^2 + 2)\, ds$$

$$= 1 + \int_0^t s\,(3 - s^2)\, ds = 1 + \frac{3t^2}{2} - \frac{t^4}{4}$$

$$\phi_2(t) = 1 + \int_0^t s\left[\left(1 + \frac{3s^2}{2} - \frac{s^4}{4}\right) - s^2 + 2\right] ds$$

$$= 1 + \int_0^t s\left(3 + \frac{s^2}{2} - \frac{s^4}{4}\right) ds$$

$$= 1 + \frac{3t^2}{2} + \frac{t^4}{8} - \frac{t^6}{24}$$

$$\phi_3(t) = 1 + \int_0^t s\left(3 + \frac{s^2}{2} + \frac{s^4}{8} - \frac{s^6}{24}\right) ds$$

$$= 1 + \frac{3t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} - \frac{t^8}{192},$$

and so on. Thus

$$\phi(t) = t^2 + (1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} + \frac{t^8}{384} + \ldots \infty)$$

i.e. $\phi(t) = t^2 + e^{t^2/2}$ is exact solution.

**4.** $\frac{dx}{dt} = t\,x\,, \quad x(0) = 1$

**Sol** We write f (t, x) = t x and the integral equation corresponding to the initial value problem is

$$x(t) = 1 + \int_0^t t\,x(t)\, dt \;\text{ or }\; = 1 + \int_0^t s\,x(s)\, ds$$

The successive approximations are given by

$$\phi_0(t) = 1$$

$$\phi_n(t) = 1 + \int_0^t t\, \phi_{n-1}(t)\, dt \qquad \text{for } n = 1, 2, 3, \ldots\ldots\ldots$$

Thus $\phi_0(t) = 1$

$$\phi_1(t) = 1 + \int_0^t 1.\, s\, ds = 1 + \frac{t^2}{2}$$

$$\phi_2(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2}\right) ds = 1 + \int_0^t \left(s + \frac{s^3}{2}\right) ds$$

$$= 1 + \frac{t^2}{2} + \frac{t^4}{2\,.\,4}$$

We shall prove by induction, that

$$\phi_n(t) = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \ldots\ldots\ldots\ldots + \frac{1}{n!}\left(\frac{t^2}{2}\right)^n \quad \forall\, n \qquad \text{(A)}$$

For n = 0, 1, 2, we have already checked the relation A. Suppose that

$$\phi_{n-1}(t) = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \ldots\ldots\ldots\ldots + \frac{1}{(n-1)!}\left(\frac{t^2}{2}\right)^{n-1}$$

Then

$$\phi_n(t) = 1 + \int_0^t s\left[1 + \left(\frac{s^2}{2}\right) + \frac{1}{2!}\left(\frac{s^2}{2}\right)^2 + \ldots\ldots + \frac{1}{(n-1)!}\left(\frac{s^2}{2}\right)^{n-1}\right] ds$$

$$= 1 + \int_0^t \left[s + \frac{s^3}{2} + \frac{1}{2!}\frac{s^5}{2^2} + \ldots\ldots + \frac{1}{(n-1)!}\frac{s^{2n-1}}{2^{n-1}}\right] ds$$

$$= 1 + \left[\frac{t^2}{2} + \frac{t^4}{2.4} + \frac{1}{2!}\frac{t^6}{2^2.6} \ldots\ldots + \frac{1}{(n-1)!}\frac{t^{2n}}{2^{n-1}(2n)}\right]$$

$$\Rightarrow \qquad \phi_n(t) = 1 + \left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \ldots\ldots\ldots + \frac{1}{n!}\left(\frac{t^2}{2}\right)^n$$

Therefore, by the principle of mathematical induction the equality (A) is true $\forall$ n = 1, 2, 3, …… Moreover, we observe that $\phi_n(t)$ is the partial sum of the first (n + 1) terms of the infinite series expansion of the function $\phi(t) = e^{t^2/2}$.

**5.** $\frac{dx}{dt} = x\,,\ x(1) = 1$

**Sol** The given equation is equivalent to the integral equation

$$x(t) = 1 + \int_1^t x(s)\ ds$$

The successive approximations are given by

$$\phi_0(t) = 1$$

$$\phi_{n+1}(t) = 1 + \int_1^t \phi_n(s)\ ds \qquad \text{for } n = 1, 2, 3, \ldots\ldots\ldots$$

We find $\phi_0(t) = 1$

$$\phi_1(t) = 1 + \int_1^t ds = t$$

$$\phi_2(t) = 1 + \int_1^t s\ ds = 1 + \int_1^t \left[(s - 1) + 1\right]\ ds$$

Here, it is convenient to have integrand occurring in the successive approximations in powers of (s – 1) rather than in powers of s (since $t_0 = 1$ and not zero).

$$\Rightarrow \qquad \phi_2(t) = 1 + \left\{ s + \frac{(s-1)^2}{2} \right\}_1^t$$

$$= 1 + (t - 1) + \frac{(t-1)^2}{2!}$$

$$\phi_3(t) = 1 + \int_1^t \left[ 1 + (s - 1) + \frac{(s-1)^2}{2} \right] ds$$

$$= 1 + (t - 1) + \frac{(t-1)^2}{2!} + \frac{(t-1)^3}{3!}$$

By induction, we shall obtain

$$\phi_n(t) = 1 + (t-1) + \frac{(t-1)^2}{2!} + \frac{(t-1)^3}{3!} + \ldots\ldots + \frac{(t-1)^n}{n!} \qquad (A)$$

We note that $\phi_n(t)$ is the partial sum of the first (n+1) terms of the infinite series expansion of the function

$$\phi(t) = e^{t-1}.$$

Moreover, this series converges for all real t.

$$\therefore \quad \phi_n(t) \to \phi(t) = e^{t-1} \quad \forall\, t.$$

Hence, the function $\phi(t)$, is the required solution of the given problem.

**6.** Find the first four approximations of the initial value problem $x'(t) = 1 + tx$, $x(0) = 1$.

Let us find the first approximation as

$$x_0(t) = x(0) = 1.$$

The second approximation is

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds = x_0 + \int_0^t [1 + s\, x_0(s)]\, ds.$$

Using the approximation $x_0(t) = 1$, we get

$$x_1(t) = x_0 + \int_0^t (1 + s)\, ds = 1 + t + \frac{t^2}{2}. \qquad (1)$$

Now $$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\, ds = x_0 + \int_0^t [1 + s\, x_1(s)]\, ds \qquad (2)$$

Using the approximation $x_1(t)$ in (2), we get

$$x_2(t) = 1 + \int_0^t 1 + s\left[1 + s + \frac{s^2}{2}\right] ds$$

$$= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8}$$

Now $$x_3(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\, ds = x_0 + \int_0^t [1 + s\, x_2(s)]\, ds \qquad (3)$$

Using the approximation $x_2(t)$ in (3), we get

$$x_3(t) = 1 + \int_0^t \left[1 + s\left(1 + s + \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{8}\right)\right] ds$$

$$= 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} + \frac{t^5}{15} + \frac{t^6}{48}$$

**7.** Find the first four Picard successive approximations of the initial value problem

$$x'(t) = x + t, \quad x(0) = 1$$

and find the n-th approximation $x_n(t)$. Find the limit of the sequence $x_n(t)$.

The given equation is equivalent to the integral equation

$$x(t) = 1 + \int_0^t (s + x(s))\, ds$$

Let us first define the first approximation $x_0(t) = x(0) = 1$. The successive approximations are given by

$$x_1(t) = 1 + \int_0^t (s + 1)\, ds = 1 + t + \frac{t^2}{2!}$$

$$x_2(t) = 1 + \int_0^t \left(s + 1 + s + \frac{s^2}{2!}\right) ds$$

$$= 1 + t + t^2 + \frac{t^3}{3!}$$

$$x_3(t) = 1 + \int_0^t \left[s + \left(1 + s + s^2 + \frac{s^3}{3!}\right)\right] ds$$

$$= 1 + t + \left(t^2 + \frac{t^3}{3}\right) + \frac{t^4}{4!}$$

$$x_4(t) = 1 + \int_0^t \left[ s + \left( 1 + s + s^2 + \frac{s^3}{3} + \frac{s^4}{4!} \right) \right] ds$$

$$= 1 + t + \left( t^2 + \frac{t^3}{3} + \frac{t^3}{3.4} \right) + \frac{t^5}{5!}$$

Proceeding in this manner, we get

$$x_n(t) = 1 + t + 2\left( \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots.\frac{t^n}{n!} \right) + \frac{t^{n+1}}{(n+1)!}$$

Taking the limit as $n \to \infty$, we get

$$x_n(t) = 1 + t + 2\,(e^t - t - 1), \quad \text{since } \frac{t^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

since $0 < t < 1$

Hence $x(t) = 2e^t - t - 1$ which can be easily seen to be the exact solution of the given differential equation.

**Note 1.** If one solves the given linear equation, the Picard successive approximations converge to the exact solution of the initial value problem.

**Note 2.** In the calculation of the Picard successive approximations, we can take the initial approximation other than a constant function also. In general we cannot say that such a sequence of approximation will converge to a solution of the initial value problem.

**Exercise**

Use the method of successive approximations to find the first three members $\phi_1$, $\phi_2$, $\phi_3$ of a sequence of functions that approaches the exact solution of the problem.

1. $\frac{dy}{dx} = xy$, $y(0) = 1$
2. $\frac{dy}{dx} = x + y$, $y(0) = 1$
3. $\frac{dy}{dx} = x + y^2$, $y(0) = 0$
4. $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 0$
5. $\frac{dy}{dx} = e^x + y^2$, $y(0) = 0$
6. $\frac{dy}{dx} = \sin x + y^2$, $y(0) = 0$

**Summary**

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition, provided, the function in given differential equation satisfies the Lipschitz condition. Also, we present a method due to E. Picard, which gives approximate solution curves of a differential equation passing through a given point.

**Keywords**

Existence and Uniqueness, solution, Lipschitz condition, Successive approximations.

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## Chapter-3

# APPROXIMATE METHODS OF SOLVING FIRST ORDER EQUATIONS

### Objectives

In many cases, the existence of solution is guaranteed by the theorems, but no method for obtaining those solutions in explicit and closed form is known. The main objective of this chapter is to gain ability to solve first order ordinary differential equations by using some approximate methods.

### Introduction

The graphical methods of the preceding section for solving initial value problems are very general but they suffer from several serious disadvantages. Not only are they tedious and subject to possible errors of construction, but they merely provide us with the approximate graphs of the solutions and do not furnish any analytical expression for these solutions. Though we have studied an approximate method in the form of Picard's method in Chapter 2, it involves evaluation of integrals at each step which may not be easy in certain cases. The advantage of the methods described in this chapter over the earlier methods, will lie in the fact that in getting the approximate numerical values of the unknown functions we shall only require the numerical values of the functions appearing as coefficients in the differential equation, apart from mere elementary arithmetical computations. We present some approximate methods of solving first order equations.

### Power Series Methods

In order to explain the power series methods we shall assume that power series solutions actually do exist.

We consider the initial value problem consisting of the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

and the initial condition

$$y(x_0) = y_0 \tag{2}$$

and assume that the differential equation (1) possesses a solution that is representable as a power series in powers of $(x - x_0)$. That is, we assume that the differential equation (1) has a solution of the form

$$y = c_0 + c_1(x - x_0) + c_2 (x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{3}$$

that is valid in some interval about the point $x_0$. We now consider methods of determining the coefficients $c_0$, $c_1$, $c_2$,……. in (3) so that the series (3) actually does satisfy the differential equation (1)

**A. The Taylor Series Method**

We thus assume that the initial value problem consisting of the differential equation (1) and the initial condition (2) has a solution of the form (3) that is valid in some interval about $x_0$. Then by Taylor's theorem, for each x in this interval the value y(x) of this solution is given by

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n \tag{4}$$

From the initial condition (2) we have

$$y(x_0) = y_0$$

and from the differential equation (1)

$$y'(x_0) = f(x_0, y_0)$$

Substituting these values of $y(x_0)$ and $y'(x_0)$ into the series in (4) we obtain the first two coefficients of the desired series solution (3). Now differentiating the differential equation (1), we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[f(x, y)] = f_x(x, y) + f_y(x, y)\frac{dy}{dx}$$

$$= f_x(x, y) + f_y(x, y)\, f(x, y). \tag{5}$$

where we use subscripts to denote partial differentiations. From this we get

$$y''(x_0) = f_x(x_0, y_0) + f_y(x_0, y_0)\, f(x_0, y_0)$$

Substituting this value of $y''(x_0)$ into (4) we obtain the third coefficient in the series solution (3). Proceeding in like manner, we differentiate (5) successively to obtain

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n} \ldots\ldots$$

From these we obtain the values

$$y'''(x_0),\ y^{(iv)}(x_0) \ldots,\ y^{(n)}(x_0), \ldots$$

Substituting these values into (4) we obtain the fourth and following coefficients in the series solution (3).

**Example**

Use the Taylor series method to obtain a power series solution of the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \tag{1}$$

$$y(0) = 1 \tag{2}$$

in powers of x.

**Solution**

Since we seek a solution in powers of x, we set $x_0 = 0$ and thus assume a solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{n=0}^{\infty} c_n x^n$$

By Taylor's theorem, we know that for each x in the interval where this solution is valid

$$y(x) = y(0) + y'(0)\, x + \frac{y''(0)}{2!} x^2 + \ldots = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \tag{3}$$

Using the initial condition (2) in the differential equation (1) we see that

$$y'(0) = 0^2 + 1^2 = 1 \qquad (4)$$

Differentiating (1) successively, we obtain

$$\frac{d^2y}{dx^2} = 2x + 2y\frac{dy}{dx} \qquad (5)$$

$$\frac{d^3y}{dx^3} = 2 + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 \qquad (6)$$

$$\frac{d^4y}{dx^4} = 2y\frac{d^3y}{dx^3} + 6\frac{dy}{dx}\frac{d^2y}{dx^2} \qquad (7)$$

Substituting $x = 0, y = 1, \frac{dy}{dx} = 1$ into (5), we obtain

$$y''(0) = 2(0) + 2(1)(1) = 2. \qquad (8)$$

Substituting $y = 1, \frac{dy}{dx} = 1, \frac{d^2y}{dx^2} = 2$ into (6), we obtain

$$y'''(0) = 2+2(1)(2) + 2(1)^2 = 8. \qquad (9)$$

Finally, substituting $y = 1, \frac{dy}{dx} = 1, \frac{d^2y}{dx^2} = 2, \frac{d^3y}{dx^3} = 8$ into (7), we find that

$$y^{(iv)}(0) = (2)(1)(8) + (6)(1)(2) = 28. \qquad (10)$$

By successive differentiation of (7), we could proceed to determine

$$\frac{d^5y}{dx^5}, \frac{d^6y}{dx^6}, \ldots\ldots,$$

and hence obtain

$$y^{(v)}(0), y^{(vi)}(0), \ldots..$$

Substituting the values given by (2), (4), (8), (9) and (10) into (3), we obtain the first five coefficients of the desired series solution. We thus find the solution

$$y = 1 + x + \frac{2}{2!}x^2 + \frac{8}{3!}x^3 + \frac{28}{4!}x^4 + \ldots$$

$$= 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$$

## B. The Method of Undetermined Coefficients

We consider an alternative method for obtaining the coefficients $c_0$, $c_1$, $c_2$,... in the assumed series solution

$$y = c_0 + c_1(x - x_0) + c_2 (x - x_0)^2 + .... = \sum_{n=0}^{\infty} c^n (x - x_0)^n \qquad (1)$$

of the problem consisting of differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (2)$$

and the initial condition

$$y(x_0) = y_0 \, . \qquad (3)$$

We shall refer to this alternative method as the method of undetermined coefficients. In order to apply it we assume that f(x,y) in the differential equation (2) can be represented in the form

$$f(x,y) = a_{00} + a_{10}(x - x_0) + a_{01}(y-y_0) + a_{20}( x - x_0)^2$$
$$+ a_{11}(x - x_0) (y - y_0) + a_{02} (y - y_0)^2 + ... \qquad (4)$$

The coefficients $a_{ij}$ in (4) may be found by Taylor's formula for functions of two variables. Using the representation (4) for f(x,y) the differential equation (2) takes the form

$$\frac{dy}{dx} = a_{00} + a_{10}(x - x_0) + a_{01}(y - y_0) + a_{20}(x - x_0)^2$$
$$+ a_{11}(x - x_0) (y - y_0) + a_{02} (y - y_0)^2 + ... \qquad (5)$$

Now assuming that the series (1) converges in some interval $|x - x_0| < r$ ($r > 0$) about $x_0$, we may differentiate (1) term by term and the resulting series will also converge on $|x - x_0| < r$ and represent y'(x) there. Doing this we thus obtain

$$\frac{dy}{dx} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + ... \qquad (6)$$

We note that in order for the series (1) to satisfy the initial condition (2) that $y = y_0$ at $x = x_0$, we must have $c_0 = y_0$ and hence

$$y - y_0 = c_1(x - x_0) + c_2\,(x - x_0)^2 + \ldots \quad (7)$$

Now substituting (1) and (6) into the differential equation (5), and making use of (7), we obtain

$$\begin{aligned} & c_1 + 2c_2\,(x - x_0) + 3c_3\,(x - x_0)^2 + \\ & \quad = a_{00} + a_{10}\,(x - x_0) + a_{01}\,[c_1\,(x - x_0) + c_2\,(x - x_0)^2 + \ldots] \\ & \qquad + a_{20}\,(x - x_0)^2 + a_{11}\,(x - x_0)\,[c_1\,(x - x_0) + c_2\,(x - x_0)^2 + \ldots] \\ & \qquad + a_{02}\,[c_1\,(x - x_0) + c_2\,(x - x_0)^2 + \ldots]^2 + \ldots \end{aligned} \quad (8)$$

Performing the multiplications and then combining like powers of $(x - x_0)$, we see that (8) takes the form

$$\begin{aligned} & c_1 + 2c_2\,(x - x_0) + 3c_3\,(x - x_0)^2 + \ldots\ldots. \\ & \quad = a_{00} + (a_{10} + a_{01}c_1)\,(x - x_0) \\ & \qquad + (a_{01}c_2 + a_{20} + a_{11}c_1 + a_{02}\,c_1^2\,(x - x_0)^2 + \ldots\,. \end{aligned} \quad (9)$$

In order that (9) be satisfied for all values of x in the interval $|x - x_0| < r$, the coefficients of like powers of $(x - x_0)$ on both side of (9) must be equal. Equating these coefficients, we obtain

$$\begin{aligned} & c_1 = a_{00}, \\ & 2c_2 = a_{10} + a_{01}c_1, \\ & 3c_3 = a_{01}c_2 + a_{20} + a_{11}c_1 + a_{02}\,c_1^2\ , \end{aligned} \quad (10)$$

From the conditions (10) we determine successively the coefficients $c_1$, $c_2$, $c_3$... of the series solution (1). From the first condition, we obtain $c_1$ as the known coefficient $a_{00}$. Then from the second condition, we obtain $c_2$ in terms of the known coefficients $a_{10}$ and $a_{01}$ and the coefficient $c_1$ just determined. Thus we obtain $c_2 = \frac{1}{2}(a_{10} + a_{01}a_{00})$. Similarly, we proceed to determine $c_3$, $c_4$.... We observe that in general each coefficient $c_n$ is thus given in terms of the known coefficients $a_{ij}$ in the expansion (4) and the previously determined coefficients $c_1$, $c_2$, .....$c_{n-1}$ .

Finally, we substitute the coefficients $c_0$, $c_1$, $c_2$,..... so determined into the series (1) and thereby obtain the desired solution.

**Example**

Use the method of undetermined coefficients to obtain a power series solution of the initial value problem.

$$\frac{dy}{dx} = x^2 + y^2 \quad (1)$$

$$y(0) = 1, \ (2)$$

in powers of x .

**Solution**

Since $x_0 = 0$, the assumed solution is of the form

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... \quad (3)$$

In order to satisfy the initial condition (2) we must have $c_0 = 1$ and hence the series (3) takes the form

$$y = 1 + c_1 x + c_2 x^2 + c_3 x^3 + .... \quad (4)$$

Differentiating (4) we obtain

$$\frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + ... \quad (5)$$

For the differential equation (1) we have $f(x, y) = x^2 + y^2$. Since $x_0 = 0$ and $y_0 = 1$, we must expand $x^2 + y^2$ in the form

$$\sum_{i,j=0}^{\infty} a_{ij} x^i (y-1)^j .$$

Since $\qquad y^2 = (y - 1)^2 + 2(y - 1) + 1,$

the desired expansion is given by

$$x^2 + y^2 = 1 + 2(y - 1) + x^2 + (y - 1)^2.$$

Thus the differential equation (1) takes the form

$$\frac{dy}{dx} = 1 + 2(y-1) + x^2 + (y-1)^2 \,. \tag{6}$$

Now substituting (4) and (5) into the differential equation (6), we obtain

$$c_1 + 2c_2\,x + 3c_3x^2 + 4c_4x^3 + ....$$

$$= 1 + 2\,(c_1x + c_2x^2 + c_3x^3 + ...) + x^2 + (c_1x + c_2x^2 + ...)^2. \tag{7}$$

Collecting like powers of x in the right hand side of (7), it takes the form

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3$$

$$= 1 + 2c_1x + (2c_2 + 1 + c_1^2)\,x^2 + (2c_3 + 2c_1c_2)x^3 + ... \tag{8}$$

Equating the coefficients of the like powers of x in (8), we obtain the conditions

$$c_1 = 1$$

$$2c_2 = 2c_1$$

$$3c_3 = 2c_2 + 1 + c_1^2,$$

$$4c_4 = 2c_3 + 2c_1c_2, \tag{9}$$

From the conditions (9) we obtain successively

$$c_1 = 1$$

$$c_2 = c_1 = 1,$$

$$c_3 = \frac{1}{3}(2c_2 + 1 + c_1^2) = \frac{4}{3},$$

$$c_4 = \frac{1}{4}(2c_3 + 2c_1c_2) = \frac{1}{4}\left(\frac{14}{3}\right) = \frac{7}{6} \tag{10}$$

Substituting these coefficients into the series (4), we obtain the first five terms of the desired series solution. We thus find

$$y = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + ...$$

We note that this is of course the same series previously obtained by the Taylor series method.

**Exercise**

Obtain a power series solution in power of x of each of the initial value problems in questions 1-5 by (a) the Taylor series method and (b) the method of undetermined coefficients.

1. $\frac{dy}{dx} = x + y,$ y(0) = 1

2. $\frac{dy}{dx} = x^2 + 2y^2,$ y(0) = 4

3. $\frac{dy}{dx} = 1 + xy^2,$ y(0) = 2

4. $\frac{dy}{dx} = x^3 + y^3,$ y(0) = 3

5. $\frac{dy}{dx} = x + \sin y,$ y(0) = 0

Obtain a power series solution in powers of x – 1 of each of the initial value problems in questions 6-9 by (a) the Taylor series method and (b) method of undetermined coefficients.

6. $\frac{dy}{dx} = x^2 + y^2,$ y(1) = 4

7. $\frac{dy}{dx} = x^3 + y^2,$ y(1) = 1

8. $\frac{dy}{dx} = x + y + y^2,$ y(1) = 1

9. $\frac{dy}{dx} = x + \cos y,$ y(1) = π

**Numerical Methods**

**Introduction**

In this section we introduce certain basic numerical methods for approximating the solution of the initial value problem consisting of the differential equation.

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

and the initial condition

$$y(x_0) = y_0 . \tag{2}$$

Numerical methods employ the differential equation (1) and the condition (2) to obtain approximations to the values of the solution corresponding to various, selected values of x. To be more specific, let y denote the solution of the problem and let h denote a positive increment in x. The initial condition tells us that $y = y_0$ at $x = x_0$. A numerical method will employ the differential equation (1) and the condition (2) to approximate successively the values of y at $x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$, ....

## A. The Euler Method

The Euler method is very simple but not very practical. Let y denote the exact solution of the initial value problem that consists of the differential equation.

$$\frac{dy}{dx} = f(x,y) \tag{1}$$

and the initial condition $y(x_0) = y_0.$ (2)

Let h denotes a positive increment in x and let $x_1 = x_0 + h$. Then

$$\int_{x_0}^{x_1} f(x,y)dx = \int_{x_0}^{x_1} \frac{dy}{dx}dx = y(x_1) - y(x_0) \ .$$

Since $y_0$ denotes the value $y(x_0)$ of the exact solution y at $x_0$, we have

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x,y)dx. \tag{3}$$

If we assume that f(x, y) varies slowly on the interval $x_0 \le x \le x_1$, then we can approximate f(x, y) in (3) by its value $f(x_0, y_0)$ at the left endpoint $x_0$. Then

$$y(x_1) \approx y_0 + \int_{x_0}^{x_1} f(x_0,y_0)dx$$

But $$\int_{x_0}^{x_1} f(x_0,y_0)dx = f(x_0,y_0)(x_1 - x_0) = hf(x_0,y_0).$$

Thus $$y(x_1) \approx y_0 + hf(x_0,y_0).$$

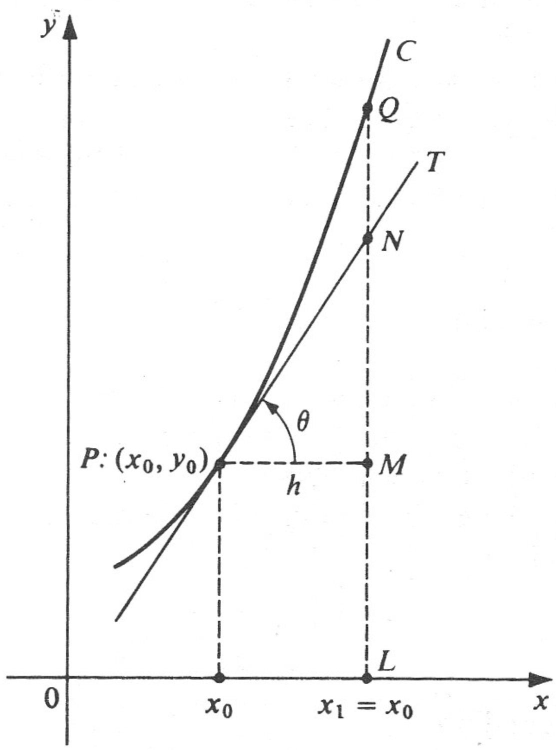
Thus we obtain the approximate value $y_1$ of y at $x_1 = x_0 + h$ by the formula

$$y_1 = y_0 + hf(x_0, y_0). \tag{4}$$

Having obtained $y_1$, we proceed in like manner to obtain $y_2$ by the formula $y_2 = y_1 + hf(x_1, y_1)$, $y_3$ by the formula $y_3 = y_2 + hf(x_2, y_2)$, and so forth. In general we find $y_{n+1}$ in terms of $y_n$ by the formula

$$y_{n+1} = y_n + hf(x_n, y_n). \tag{5}$$

The graph of the exact solution y is a curve C in the xy plane (see figure) Let P denotes the initial point $(x_0, y_0)$ and let T be the tangent to C at P. Let Q be the point at which the line $x = x_1$ intersects C and let N be the point at which this line intersects T. Then the exact value of y at $x_1$ is represented by LQ. The approximate value $y_1$ is represented by LN, since LN = LM + MN = $y_0$ + PM tanθ = $y_0$ + $hf(x_0, y_0)$. The error in approximating the exact value of y at $x_1$ by $y_1$ is thus represented by NQ. The figure suggests that if h is sufficiently small, then this error NQ will also be small and hence the approximation will be good.




**Figure**

But if the increment h is very small, then the computations will be more lengthy and so the method will involve tedious and time consuming labour. Thus, this method is not very practical.

**Example**

Apply the Euler method to the initial value problem

$$\frac{dy}{dx} = 2x + y \qquad (1)$$

$$y(0) = 1 \qquad (2)$$

Employ the method to approximate the value of the solution y at x = 0.2, 0.4, 0.6, 0.8, and 1.0 using (1) h = 0.2, and (2) h = 0.1. Obtain results to three figures after the decimal point. Compare with the exact value obtained.

**Solution 1** We have f (x,y) = 2x + y and h = 0.2. From initial condition (2), we have $x_0 = 0$, $y_0 = 1$. We now proceed with the calculations.

(a) $x_1 = x_0 + h = 0.2$, $f(x_0, y_0) = f(0, 1) = 1.000$,

$y_1 = y_0 + hf(x_0, y_0) = 1.000 + 0.2\ (1.000) = 1.200$

(b) $x_2 = x_1 + h = 0.4$, $f(x_1, y_1) = f(0.2, 1.200) = 1.600$,

$y_2 = y_1 + hf(x_1, y_1) = 1.200 + 0.2\ (1.600) = 1.520.$

(c) $x_3 = x_2 + h = 0.6$, $f(x_2, y_2) = f(0.4, 1.520) = 2.320$,

$y_3 = y_2 + hf(x_2, y_2) = 1.520 + 0.2\ (2.320) = 1.984.$

(d) $x_4 = x_3 + h = 0.8$, $f(x_3, y_3) = f(0.6, 1.984) = 3.184$,

$y_4 = y_3 + hf(x_3, y_3) = 1.984 + 0.2\ (3.184) = 2.621.$

(e) $x_5 = x_4 + h = 1.0$, $f(x_4, y_4) = f(0.8, 2.621) = 4.221$,

$y_5 = y_4 + hf(x_4, y_4) = 2.621 + 0.2\ (4.221) = 3.465.$

These results, corresponding to the various values of $x_n$, are collected in the second column of Table 1.

**Table 1**

| $x_n$ | $y_n$ using h = 0.2 | $y_n$ using h = 0.1 | y |
|---|---|---|---|
| 0.0 | 1.000 | 1.000 | 1.000 |
| 0.1 | - | 1.100 | 1.116 |
| 0.2 | 1.200 | 1.230 | 1.264 |
| 0.3 | - | 1.393 | 1.450 |
| 0.4 | 1.520 | 1.592 | 1.675 |
| 0.5 | - | 1.831 | 1.946 |
| 0.6 | 1.984 | 2.114 | 2.266 |
| 0.7 | - | 2.445 | 2.641 |
| 0.8 | 2.621 | 2.830 | 3.076 |
| 0.9 | - | 3.273 | 3.579 |
| 1.0 | 3.465 | 3.780 | 4.155 |

2. In this case we have $f(x, y) = 2x + y$ and $h = 0.1$. Again we have $x_0 = 0$, $y_0 = 1$. The calculations are as follows:

(a) $x_1 = x_0 + h = 0.1$, $f(x_0, y_0) = f(0, 1) = 1.000$,

$y_1 = y_0 + hf(x_0, y_0) = 1.000 + 0.1\ (1.000) = 1.100$

(b) $x_2 = x_1 + h = 0.2$, $f(x_1, y_1) = f(0.1, 1.100) = 1.300$,

$y_2 = y_1 + hf(x_1, y_1) = 1.100 + 0.1\ (1.300) = 1.230$.

(c) $x_3 = x_2 + h = 0.3$, $f(x_2, y_2) = f(0.2, 1.230) = 1.630$,

$y_3 = y_2 + hf(x_2, y_2) = 1.230 + 0.1\ (1.630) = 1.393$.

(d) $x_4 = x_3 + h = 0.4$, $f(x_3, y_3) = f(0.3, 1.393) = 1.993$,

$y_4 = y_3 + hf(x_3, y_3) = 1.393 + 0.1\ (1.993) = 1.592$.

(e) $x_5 = x_4 + h = 0.5$, $f(x_4, y_4) = f(0.4, 1.592) = 2.392$,

$y_5 = y_4 + hf(x_4, y_4) = 1.592 + 0.1\ (2.392) = 1.831$.

(f) $x_6 = x_5 + h = 0.6$, $f(x_5, y_5) = f(0.5, 1.831) = 2.831$,

$y_6 = y_5 + hf(x_5, y_5) = 1.831 + 0.1\ (2.831) = 2.114$.

(g) $x_7 = x_6 + h = 0.7$, $f(x_6, y_6) = f(0.6, 2.114) = 3.314$,

$y_7 = y_6 + hf(x_6, y_6) = 2.114 + 0.1\ (3.314) = 2.445$.

(h) $x_8 = x_7 + h = 0.8$, $f(x_7, y_7) = f(0.7, 2.445) = 3.845$,

$y_8 = y_7 + hf(x_7, y_7) = 2.445 + 0.1\ (3.845) = 2.830$.

(i) $x_9 = x_8 + h = 0.9$, $f(x_8, y_8) = f(0.8, 2.830) = 4.430$,

$y_9 = y_8 + hf(x_8, y_8) = 2.830 + 0.1\ (4.430) = 3.273$.

(j) $x_{10} = x_9 + h = 1.0$, $f(x_9, y_9) = f(0.9, 3.273) = 5.073$,

$y_{10} = y_9 + hf(x_9, y_9) = 3.273 + 0.1\ (5.073) = 3.780$.

These results are collected in the third column of Table 1. The values of the exact solution y, computed to three figures after the decimal point, are listed in the fourth column of Table 1. From this table we compute the errors involved in both approximations at x = 0.2, 0.4, 0.6, 0.8, and 1.0. These errors are tabulated in Table 2.

A study of these tables illustrates two important facts concerning the Euler method. First, for a fixed value of h, the error becomes greater and greater as we proceed over a larger and larger range away from the initial point. Second, for a fixed value of $x_n$, the error is smaller if the value of h is smaller.

**Table 2**

| $x_n$ | Error using h = 0.2 | Error using h = 0.1 |
|---|---|---|
| 0.2 | 0.064 | 0.034 |
| 0.4 | 0.155 | 0.083 |
| 0.6 | 0.282 | 0.152 |
| 0.8 | 0.455 | 0.246 |
| 1.0 | 0.690 | 0.375 |

**Exercise**

1. Consider the initial value problem

$$\frac{dy}{dx} = x - 2y, \quad y(0) = 1.$$

(a) Apply the Euler method to approximate the values of the solution y at x = 0.1, 0.2, 0.3, and 0.4, using h = 0.1. Obtain results to three figures after the decimal point.

(b) Proceed as in part(a) using h = 0.05

(c) Find the exact solution of the problem and determine its values at x = 0.1. 0.2, 0.3 and 0.4 (to three figures after the decimal point).

(d) Compare the results obtained in parts (a), (b), and (c). Tabulate errors as in table 2.

2. Proceed as in question 1 for the initial value problem

$$\frac{dy}{dx} = x + y$$

$$y(0) = 2.$$

## B. The Modified Euler Method

In Section A we observed that the value $y(x_1)$ of the exact solution y of the initial value problem

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

$$y(x_0) = y_0, \tag{2}$$

at $x_1 = x_0 + h$ is given by

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y)dx. \tag{3}$$

In the Euler method we approximated f(x, y) in (3) by its value $f(x_0, y_0)$ at the left endpoint of the interval $x_0 \leq x \leq x_1$ and obtained the approximation

$$y_1 = y_0 + hf(x_0, y_0) \tag{4}$$

for y at $x_1$. It seems that a more accurate value would be obtained if we were to approximate f(x, y) by the average of its values at the left and right endpoints of $x_0 \leq x \leq x_1$, instead of simply by its value at the left endpoint $x_0$. This is essentially what is done in the modified Euler method. In order to approximate f(x, y) by the average of its values at $x_0$ and $x_1$, we need to know its value $f[x_1, y(x_1)]$ of y at $x_1$. We must find a first approximation $y_1^{(1)}$ for $y(x_1)$, and to find this, we take

$$y_1^{(1)} = y_0 + hf(x_0, y_0) \tag{5}$$

as the first approximation to the value of y at $x_1$. Then we approximate $f[x_1, y(x_1)]$ by $f(x_1, y_1^{(1)})$, using the value $y_1^{(1)}$ found by (5). From this we obtain.

$$\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2} \tag{6}$$

which is approximately the average of the values of f(x, y) at the endpoints $x_0$ and $x_1$. We now replace f(x, y) in (3) by (6) and thereby obtain

$$y_1^{(2)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}h \tag{7}$$

as the second approximation to the value of y at $x_1$.

We now use the second approximation $y_1^{(2)}$ to obtain a second approximation $f(x_1, y_1^{(2)})$ for the value of f (x, y) at $x_1$. From this we obtain

$$y_1^{(3)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(2)})}{2} h \tag{8}$$

as the third approximation of the value of y at $x_1$. Proceeding in this way we obtain a sequence of approximations

$$y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, .....$$

to the value of the exact solution y at $x_1$. We proceed to compute the successive terms of the sequence until we obtain two consecutive members that have the same value to the number of decimal places required. Let that value of the solution y at $x_1$ be denoted by $y_1$. We now proceed to approximate y at $x_2 = x_1 + h$, in exactly the same way as we did in finding $y_1$. We find successively

$$y_2^{(1)} = y_1 + hf(x_1, y_1),$$

$$y_2^{(2)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(1)})}{2} h,$$

$$y_2^{(3)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(2)})}{2} h, \tag{9}$$

….. ….. ….. ….. …..

until two consecutive members of this sequence agree, thereby obtaining an approximation $y_2$ to the value of y at $x_2$.

Proceeding further in like manner one obtains an approximation $y_3$ to the value of y at $x_3$, and so forth.

**Example**

Apply the modified Euler method to the initial value problem

$$\frac{dy}{dx} = 2x + y, \tag{1}$$

$$y(0) = 1. \tag{2}$$

Employ the method to approximate the value of the solution y at x = 0.2 and x = 0.4 using h = 0.2. Obtain results to three figures after the decimal point. Compare with the results obtained using the basic Euler method with h = 0.1 and with the exact values.

**Solution**

Here f(x, y) = 2x + y, $x_0$ = 0, and $y_0$ = 1, and we are to use h = 0.2. We begin by approximating the value of y at $x_1 = x_0 + h = 0.2$. A first approximation $y_1^{(1)}$ is found using Euler Method. Since $f(x_0, y_0) = f(0, 1) = 1.000$, we have

$$y_1^{(1)} = y_0 + hf(x_0, y_0) = 1.000 + 0.2(1.000) = 1.200.$$

We now use modified Euler method to find a second approximation $y_1^{(2)}$ to the desired value. Since $f(x_1, y_1^{(1)}) = f(0.2, 1.200) = 1.600$, we have

$$y_1^{(2)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2} h = 1.000 + \frac{1.000 + 1.600}{2}(0.2) = 1.260$$

We next find a third approximation $y_1^{(3)}$. Since

$$f(x_1, y_1^{(2)}) = f(0.2, 1.260) = 1.660, \text{ we find}$$

$$y_1^{(3)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(2)})}{2} h = 1.000 + \frac{1.000 + 1.660}{2}(0.2) = 1.266.$$

Similarly, we find

$$y_1^{(4)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(3)})}{2} h = 1.000 + \frac{1.000 + 1.666}{2}(0.2) = 1.267$$

and

$$y_1^{(5)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(4)})}{2} h = 1.000 + \frac{1.000 + 1.667}{2}(0.2) = 1.267.$$

Since the approximations $y_1^{(4)}$ and $y_1^{(5)}$ are the same to the number of decimal places required, we take their common value as the approximation $y_1$ to the value of the solution y at $x_1$ = 0.2. That is, we take

$$y_1 = 1.267 \qquad (3)$$

We now proceed to approximate the value of y at $x_2 = x_1 + h = 0.4$. We find successively

$$y_2^{(1)} = y_1 + hf(x_1, y_1) = 1.267 + 0.2(1.667) = 1.600$$

$$y_2^{(2)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(1)})}{2} h = 1.267 + \frac{1.667 + 2.400}{2}(0.2) = 1.674.$$

$$y_2^{(3)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(2)})}{2} h = 1.267 + \frac{1.667 + 2.474}{2}(0.2) = 1.681.$$

$$y_2^{(4)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(3)})}{2} h = 1.267 + \frac{1.667 + 2.481}{2}(0.2) = 1.682.$$

$$y_2^{(5)} = y_1 + \frac{f(x_1, y_1) + f(x_2, y_2^{(4)})}{2} h = 1.267 + \frac{1.667 + 2.482}{2}(0.2) = 1.682.$$

Since the approximations $y_2^{(4)}$ and $y_2^{(5)}$ are both the same to the required number of decimal places, we take their common values as the approximation $y_2$ to the value of the solution y at $x_2 = 0.4$. That is, we take

$$y_2 = 1.682 \tag{4}$$

We compare the results (3) and (4) with those obtained using the basic Euler method with h = 0.1 and with the exact values. For this purpose the various results and the corresponding errors are listed in Table 3.

The principal advantage of the modified Euler method over the basic Euler method is immediately apparent from a study of Table 3.

**Table 3**

| $x_n$ | Exact value of y (to three decimal places) | Using basic Euler method with h = 0.1 | | Using modified Euler with h = 0.2 | |
|---|---|---|---|---|---|
| | | Approximation | | Approximation | |
| | | $y_n$ | Error | $y_n$ | Error |
| 0.2 | 1.264 | 1.230 | 0.034 | 1.267 | 0.003 |
| 0.4 | 1.675 | 1.592 | 0.083 | 1.682 | 0.007 |

**Exercise**

1. Consider the initial-value problem

$$\frac{dy}{dx} = 3x + 2y, \qquad y(0) = 1.$$

(a) Apply the modified Euler method to approximate the values of the solution y at x = 0.1, 0.2 and 0.3 using h = 0.1. Obtain results to three figures after the decimal point.

(b) Proceed as in part (a) using h = 0.05

(c) Find the exact solution of the problem and determine its value at x = 0.1, 0.2 and 0.3 (to three figures after the decimal point).

(d) Compare the results obtained in parts (a), (b) and (c) and the tabulate errors.

2. Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \qquad y(0) = 1$$

(a) Apply the modified Euler method to approximate the values of the solution y at x = 0.1, 0.2 and 0.3 using h = 0.1. Obtain results to three figures after the decimal point.

(b) Apply the Euler method to approximate the values of the solution y at x = 0.1, 0.2 and 0.3 using h = 0.1. Obtain results to three figures after the decimal point.

(c) Compare the results obtained in parts (a) and (b) and the tabulate errors.

## C. The Runge-Kutta Method

We now consider the Runge-Kutta method for approximating the values of the solution of the initial value problem.

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

$$y(x_0) = y_0 \tag{2}$$

at $x_1 = x_0 + h$, $x_2 = x_1 + h$, and so forth. This method gives accurate results without the need of using extremely small values of the interval h.

To approximate the value of the solution of the initial value problem under consideration at $x_1 = x_0 + h$ by the Runge-Kutta method, we calculate successively the coefficients $k_1$, $k_2$, $k_3$, $k_4$, and $K_0$ defined by the formulas

$$k_1 = hf(x_0, y_0),$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), \qquad (3)$$

$$k_4 = hf(x_0 + h, y_0 + k_3),$$

and

$$K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Then we set

$$y_1 = y_0 + K_0 \qquad (4)$$

and take this as the approximate value of the exact solution at $x_1 = x_0 + h$.

We proceed to approximate the value of the solution at $x_2 = x_1 + h$ in an exactly similar manner. Using $x_1 = x_0 + h$ and $y_1$ as determined by (4), we calculate successively the coefficients $k_1$, $k_2$, $k_3$, $k_4$ and $K_1$ defined by

$$k_1 = hf(x_1, y_1),$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right), \qquad (5)$$

$$k_4 = hf(x_1 + h, y_1 + k_3),$$

and

$$K_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Then we set

$$y_2 = y_1 + K_1 \qquad (6)$$

and take this as the approximate value of the exact solution at $x_2 = x_1 + h$.

Similarly, we proceed to approximate the value of the solution at $x_3 = x_2 + h$, $x_4 = x_3 + h$, and so forth. Let $y_n$ denotes the approximate value obtained for the solution at $x_n = x_0 + nh$, we calculate successively $k_1$, $k_2$, $k_3$, $k_4$, and $K_n$ defined by

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$

$$k_4 = hf(x_n + h, y_n + k_3),$$

and $$K_n = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Then we set $$y_{n+1} = y_n + K_n$$

and take this as the approximate value of the exact solution at $x_{n+1} = x_n + h$.

**Example**

Apply the Runge-Kutta method to the initial value problem

$$\frac{dy}{dx} = 2x + y \quad (1)$$

$$y(0) = 1 \quad (2)$$

Employ the method to approximate the value of the solution y at x = 0.2 and x = 0.4 using h = 0.2. Carry the intermediate calculations in each step to five figures after the decimal point, and round off the final results of each step to four such places. Compare with the exact value.

**Solution**

Here f(x, y) = 2x + y, $x_0 = 0$, $y_0 = 1$, and we are to use h = 0.2. Using these quantities we calculate successively $k_1$, $k_2$, $k_3$, $k_4$, and $K_0$. We first find

$$k_1 = hf(x_0, y_0) = 0.2\ f(0, 1) = 0.2(1) = 0.20000$$

Then since

$$x_0 + \frac{h}{2} = 0 + \frac{1}{2}(0.2) = 0.1$$

and $$y_0 + \frac{k_1}{2} = 1.00000 + \frac{1}{2}(0.20000) = 1.10000,$$

we find $$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\,f(0.1,\ 1.10000)$$

$$= 0.2\ (1.30000) = 0.26000.$$

Next, since

$$y_0 + \frac{k_2}{2} = 1.00000 + \frac{1}{2}(0.26000) = 1.13000,$$

we find

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\,f(0.1,\ 1.13000)$$

$$= 0.2(1.33000) = 0.26600.$$

Since $x_0 + h = 0.2$ and $y_0 + k_3 = 1.00000 + 0.26600 = 1.26600$, we obtain

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.2f\ (0.2,\ 1.26600)$$

$$= 0.2\ (1.66600) = 0.33320.$$

Finally, we find

$$K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.20000 + 0.52000 + 0.53200 + 0.33320)$$

$$= 0.26420.$$

Then the approximate value of the solution at $x_1 = 0.2$ is

$$y_1 = 1 + 0.2642 = 1.2642 \qquad (3)$$

Now using $y_1$ as given by (3), we calculate successively $k_1$, $k_2$, $k_3$, $k_4$, and $K_1$. We first find

$$k_1 = hf(x_1,\ y_1) = 0.2f(0.2,\ 1.2642) = 0.2\ (1.6642) = 0.33284$$

Then since

$$x_1 + \frac{h}{2} = 0.2 + \frac{1}{2}(0.2) = 0.3$$

and

$$y_1 + \frac{k_1}{2} = 1.26420 + \frac{1}{2}(0.33284) = 1.43062$$

we find

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2\,f\,(0.3,\ 1.43062)$$

$$= 0.2(2.03062) = 0.40612$$

Next, since

$$y_1 + \frac{k_2}{2} = 1.26420 + \frac{1}{2}(0.40612) = 1.46726$$

we find

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.2\,f\,(0.3,\ 1.46726)$$

$$= 0.2(2.06726) = 0.41345$$

Since $x_1 + h = 0.4$ and $y_1 + k_3 = 1.26420 + 0.41345 = 1.67765$, we obtain

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.2f\,(0.4,\ 1.67765)$$

$$= 0.2\ (2.47765) = 0.49553$$

Finally, we find

$$K_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.33284 + 0.81224 + 0.82690 + 0.49553) = 0.41125\,.$$

Then, the approximate value of the solution at $x_2 = 0.4$ is

$$y_2 = 1.2642 + 0.4112 = 1.6754 \qquad (4)$$

Rounded off to four places after the decimal point, the exact values at x = 0.2 and x = 0.4 are 1.2642 and 1.6754, respectively. The approximate value at x = 0.2 as given by (3) is therefore correct to four places after the decimal point and the approximate value at x = 0.4 as given by (4) is likewise correct to four place.

The remarkable accuracy of the Runge-Kutta method in this problem is certainly apparent. In fact, we employ the method to approximate the solution at x = 0.4 using h = 0.4 (that is, in only one step), we obtain the value 1.6752, which differs from the exact value 1.6754 by merely 0.0002.

**Exercise**

1. Consider the initial value problem

$$\frac{dy}{dx} = 3x + 2y,$$

$$y(0) = 1$$

(a) Apply the Runge-Kutta method to approximate the values of the solution y at x = 0.1, 0.2 and 0.3 using h = 0.1. Carry the intermediate calculations in each step to five figures after the decimal point and round off the final results of each step to four such places.

(c) Find the exact solution of the problem and compare the results obtained in part(a) with the exact values.

**Summary**

In this chapter several more approximate methods, namely, power series method, Taylor Series method, method of undetermined coefficients, Euler method and Runge-Kutta method, for the solution of arbitrary first order ordinary differential equations are considered. In the study of each method in this chapter, the primary concern is to obtain familiarity with the procedure itself and to develop skill in applying it.

**Keywords**

Approximate methods, Taylor Series, Undetermined coefficients, Euler method, Runge-Kutta method.

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## Chapter- 4

# CONTINUATION OF SOLUTIONS AND MATRIX METHOD FOR SYSTEM OF EQUATIONS

### Objectives

The main objective of this chapter is to find out the maximal interval of existence of solution of initial value problems. Equal emphasis is on the study of dependence of solution on initial conditions and functions of IVP.

### Introduction

In the Picard's theorem, we obtained the solution of the initial value problem near the initial point $(x_0, y_0)$ in a closed rectangle R contained in the domain D and the interval of convergence was determined by $h = \min\left(a, \frac{b}{M}\right)$, where $|f(x, y)| \leq M$ on R. We obtained the best possible h determined by the closed rectangle R as a part of the domain and we obtained the solution on $(x_0 - h, x_0)$ on the left side and $(x_0, x_0 + h)$ on the right side unmindful of points outside the rectangle. Hence the question arises whether the solution exists outside the best possible interval, the answer is yes and is explained in the first three theorems of this chapter. The fact that we are using initial conditions to construct Picard successive approximations shows that the solutions are functions of the initial conditions. Thus, it is evident that we get different solutions of same equations for different initial conditions. Also, we shall obtain the relation between two initial value problems of two different functions and this will explain how the solutions change when the functions are slightly changed. This chapter also extends the theory to a system of equations, which give rise to the study of matrix differential equation. The existence and uniqueness of solutions of the initial value problem of such a vector differential equation will be studied.

**Theorem 4.1**

The largest open interval over which the solution y(x) with $y(x_0) = y_0$ is defined is any one of the following two types.

(i) (a, b) where both a and b are finite or either a is finite or b is finite.

(ii) The entire x-axis in the sense $-\infty < x < \infty$.

**Proof**

Suppose that f satisfies the hypothesis of Picard theorem in D and that $(x_0, y_0) \in D$. Let $R: |x - x_0)| \leq a$, $|y - y_0| \leq b$ be a rectangle lying in D which gives rise to the "best possible" h of the conclusion of Picard theorem. The Picard theorem asserts that the initial value problem.

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0 \tag{1}$$

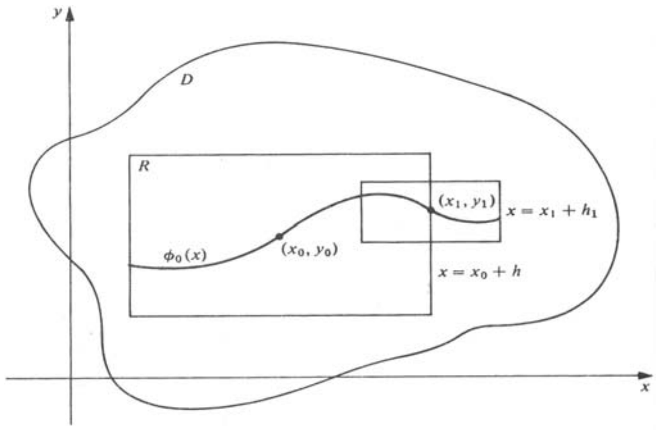
possesses a unique solution $\phi_0$ on $|x - x_0| \leq h$, but nothing is implied about $\phi_0$ outside this interval. Now let us consider the extreme right hand point for which $\phi_0$ is defined. This is the point $(x_1, y_1)$, where $x_1 = x_0 + h$, $y_1 = \phi_0(x_1)$. Since this point is a point of R, it is certainly a point of the domain D in which the hypotheses of Picard Theorem are satisfied. Thus we can reapply Picard Theorem at the point $(x_1, y_1)$ and can prove that the differential equation $dy/dx = f(x, y)$ possesses a unique solution $\phi_1$ such that $\phi_1(x_1) = y_1$, which is defined on some interval $x_1 \leq x \leq x_1 + h_1$, where $h_1 > 0$.

Now let us define $\phi$ as follows:

$$\phi(x) = \begin{cases} \phi_0(x), & x_0 - h \leq x \leq x_0 + h = x_1, \\ \phi_1(x), & x_1 \leq x \leq x_1 + h_1 \end{cases}$$

We now assert that $\phi$ is a solution of problem (1) on the extended interval $x_0 - h \leq x \leq x_1 + h_1$ (see Figure). The function $\phi$ is continuous on this interval and is such that $\phi(x_0) = y_0$. For $x_0 - h \leq x \leq x_0 + h$ we have

$$\phi_0(x) = y_0 + \int_{x_0}^{x} f[t, \phi_0(t)]dt$$




**Figure**

and hence

$$\phi(x) = y_0 + \int_{x_0}^{x} f[t, \phi(t)]dt \tag{2}$$

on this interval. On the interval $x_0 + h < x \le x_1 + h_1$, we have

$$\phi_1(x) = y_1 + \int_{x_1}^{x} f[t, \phi_1(t)]dt,$$

or
$$\phi(x) = y_1 + \int_{x_1}^{x} f[t, \phi(t)]dt.$$

Since $y_1 = \phi_0(x_1) = y_0 + \int_{x_0}^{x_1} f[t, \phi(t)]dt$, we thus have

$$\phi(x) = y_0 + \int_{x_0}^{x} f[t, \phi(t)]dt \tag{3}$$

on the interval $x_0 + h < x \le x_1 + h_1$. Thus, combining the results of (2) and (3) we see that $\phi$ satisfies the integral equation (3) on the extended interval $x_0 - h \le x \le x_1 + h_1$. Since $\phi$ is continuous on this interval, so is $f[x, \phi(x)]$. Thus,

$$\frac{d\phi(x)}{dx} = f[x, \phi(x)]$$

on $[x_0 - h, \ x_1 + h_1]$. Therefore $\phi$ is a solution of problem (1) on this larger interval.

The function $\phi$ so defined is called a continuation of the solution $\phi_0$ to the interval $[x_0 - h , x_1 + h_1]$. If we now apply Picard Theorem again at the point $[x_1+h_1, \phi(x_1 + h_1)]$, we may thus obtain the continuation over the still larger interval

$x_0 - h \le x \le x_2 + h_2$ where $x_2 = x_1 + h_1$ and $h_2$ is positive. Repeating this process further, we may continue the solution over successively larger intervals $x_0 - h \le x \le x_n + h_n$ extending farther and farther to the right of $x_0 + h$. Also, in like manner, it may be continued over successively larger intervals extending farther and farther to the left of $x_0 - h$.

Thus repeating the process indefinitely on both the left and the right, we extend the solution to successively larger intervals $[a_n, b_n]$, where

$$[x_0 - h, x_0 + h] = [a_0, b_0] \subset [a_1, b_1] \subset [a_2, b_2] \subset \cdots \subset [a_n, b_n] \subset \cdots$$

Let $a = \lim_{n\to\infty} a_n$ *and* $b = \lim_{n\to\infty} b_n$, where the limit exists finitely or infinitely.

We thus obtain a largest open interval $a < x < b$ over which the solution $\phi$ such that $\phi(x_0) = y_0$ may be defined. It is clear that two cases are possible.

1. $a = -\infty$ and $b = +\infty$, in which case $\phi$ is defined for all x, $-\infty < x < +\infty$.

2. Either a is finite or b is finite or both.

This completes the proof of theorem.

We can be more definite concerning the largest open interval over which the solution of this initial-value problem is defined. In this connection we state the following theorem.

**Theorem 4.2**

**Hypothesis**

1. Let f be continuous in the unbounded domain D: $a < x < b$, $-\infty < y < +\infty$.

2. Let f satisfies a Lipschitz condition (with respect to y) in this unbounded domain. That is, assume there exists $k > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le k\,|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

**Conclusion** A solution $\phi$ of $dy/dx = f(x, y)$ such that $\phi(x_0) = y_0$, where $(x_0, y_0)$ is any point of D, is defined on the entire open interval $a < x < b$. In particular, if $a = -\infty$ and $b = +\infty$, then $\phi$ is defined for all x, $-\infty < x < +\infty$.

**Example** Consider the IVP

$$\frac{dy}{dx} = y^2,$$

$$y(-1) = 1.$$

It has a solution $\phi(x) = \frac{-1}{x}$ through the point (-1, 1) and this solution exists on the interval [-1, 0] but cannot be continued further to the right. Because, in that case $\phi$ does not stay within the region D, where $f(x, y) = y^2$ is bounded.

**Maximal Interval of Existence**

Let f(x, y) be a continuous function on a (x, y)- set E. Let $\phi = \phi(x)$ be a solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

on an interval I.

The interval I is called a right maximal interval of existence for $\phi$ if there does not exist an extension of $\phi(x)$ over an interval, say $I_1$, so that $\phi = \phi(x)$ remains a solution of (1), where I is a proper subset of $I_1$ and I, $I_1$ have different right end points. Similarly, a left maximal interval of existence for $\phi$ can be defined.

**Definition** A maximal interval of existence of a solution of ODE (1) is an interval which is both left and right maximal interval.

**Theorem 4.3 Extension Theorem**

Let f (x, y) be continuous on an open (x, y) - set E and let y(x) be a solution of differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

on some interval. Then y(x) can be extended as a solution over a maximal interval of existence $(\omega_-, \omega_+)$. Also if $(\omega_-, \omega_+)$ is a maximal interval of existence, then y(x) tends to the boundary $\partial E$ of E as $x \to \omega_-$ or $x \to \omega_+$.

**Proof** Let $E_1, E_2, \ldots\ldots. E_n$ be open subsets of the given set E such that $E = \bigcup_n E_n$ and let $\overline{E}_1, \overline{E}_2, \ldots\ldots. \overline{E}_n$ be the closures of these open sets. Then $\overline{E}_1, \overline{E}_2, \ldots\ldots. \overline{E}_n$ are compact subsets of E and let us choose $\overline{E}_n$ s.t. $\overline{E}_n \subset E_{n+1}$. By Cauchy Peano Existence theorem, if $(x_0, y_0)$ is any point of $\overline{E}_n$, there exists an $\varepsilon_n > 0$ such that all solutions of given differential equation through $(x_0, y_0)$ exist on the interval $|x - x_0| \le \varepsilon_n$.

Now consider a given solution y = y(x) of differential equation (1) on an open interval J. If J is the right maximal interval of existence, then nothing to prove. If J is not the right maximal interval of existence, then by the theorem on continuation of solution, this solution y(x) can be extended to an interval such that interval contains the right end point of interval J. Thus in proving the existence of a right maximal interval of existence, it can be supposed that y(x) is defined on a closed interval $[a, b_0]$.

Let us denote n(1) be the integer so large that $(b_0, y(b_0)) \in \overline{E}_{n(1)}$. Then y(x) can be extended over an interval $[b_0, b_0 + \varepsilon_{n(1)}]$. Now if $(b_0 + \varepsilon_{n(1)}, y(b_0 + \varepsilon_{n(1)})) \in \overline{E}_{n(1)}$, then y(x) can be extended over an interval (another) $[b_0 + \varepsilon_{n(1)}, b_0 + 2\varepsilon_{n(1)}]$ of length $\varepsilon_{n(1)}$. Continuing this way, we can say that there exist an integer $j(1) \ge 1$ such that y(x) can be extended over $a \le x \le b_1$, where $b_1 = b_0 + j(1)\varepsilon_{n(1)}$ and this $(b_1, y(b_1)) \notin \overline{E}_{n(1)}$.

Let n(2) be so large that $(b_1, y(b_1)) \in \overline{E}_{n(2)}$. Applying the same procedure we can say that there exist an integer $j(2) \ge 1$ such that y(x) can be extended over $a \le x \le b_2$, where $b_2 = b_1 + j(2)\varepsilon_{n(2)}$ such that $(b_2, y(b_2)) \notin \overline{E}_{n(2)}$. Continuing in this way, we get a sequence of integers n(1), n(2), ……… such that n(1) < n(2) < n(3), …… and numbers $b_0 < b_1 < b_2$ ……. such that y(x) can be extended over $[a, \omega_+]$ where $b_k \to \omega_+$ as $k \to \infty$ and $(b_k, y(b_k)) \notin \overline{E}_{n(k)}$. This sequence of points $(b_1, y(b_1))$, $(b_2, y(b_2))$, ……. $(b_k, y(b_k))$, ………. is either unbounded or bounded. If bounded, then by Bolzano Weierstrass Theorem this sequence has a limit point (say $y_0$). [If unbounded, we can apply theorem 2]. We claim that this limit point lies on the boundary of set E and cannot be an interior point of E.

If this is possible, then there exist a neighbourhood $N_\varepsilon$ of the limit point $(\omega_+, y_0)$ such that $N_\varepsilon$ is contained in E i.e. $N_\varepsilon(\omega_+, y_0) \subset E$. Since $(\omega_+, y_0)$ is the limit point of the sequence, therefore $N_\varepsilon(\omega_+, y_0)$ contains infinitely many terms of the sequence $(b_1, y(b_1))$, $(b_2, y(b_2))$, ……. If $(\omega_+, y_0)$ is not on $\partial E$, then $N_\varepsilon(\omega_+, y_0)$ contains some term $(b_k, y(b_k))$ corresponding to $\delta_1$, on the right of $\omega_+$.

This is a contradiction as $[a, \omega_+]$ is right maximal interval of existence.

$\therefore$ $(\omega_+, y_0)$ lies on $\partial E$ and $y(x) \to \partial E$ as $x \to \omega_+$.

Similarly, it can be proved that y(x) can be extended over a left maximal interval of existence say $(\omega_-, a)$.

Therefore, y(x) can be extended over a maximal interval of existence $(\omega_-, \omega_+)$ and $y(x) \to \partial E$ as $x \to \omega_-$ or $x \to \omega_+$.

This completes the proof of theorem.

**Dependence of solutions on Initial conditions and on the function f**

**A. Dependence on Initial Conditions**

We now consider how the solution of the differential equation $dy/dx = f(x, y)$ depends upon a slight change in the initial conditions or upon a slight change in the function f. We will show that under suitable restrictions such slight changes would cause only slight changes in the solution.

We first consider the result of a slight change in the initial condition $y(x_0) = y_0$. Let f be continuous and satisfies a Lipschitz condition with respect to y in a domain D and let $(x_0, y_0)$ be a fixed point of D. Then by existence and uniqueness theorem, the initial value problem

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

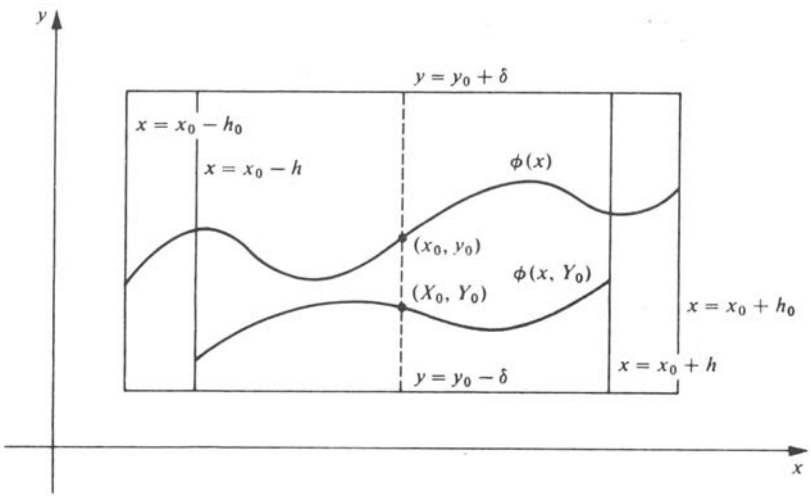
has a unique solution $\phi$ defined on some sufficiently small interval $|x - x_0| \le h_0$. Now suppose the initial y value is changed from $y_0$ to $Y_0$. If $Y_0$ is such that $|Y_0 - y_0|$ is sufficiently small, then we can be certain that the new initial value problem

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = Y_0 \qquad (1)$$

also has a unique solution on some sufficiently small interval $|x - x_0| \le h_1$.

In fact, let the rectangle, R: $|x - x_0| \le a$,$| y - y_0| \le b$, lies in D and let $Y_0$ be such that $|Y_0 - y_0| \le b/2$. Then by existence and uniqueness theorem, this problem (1) has a unique solution $\psi$ which is defined and contained in R for $|x - x_0| \le h_1$, where $h_1 = \min(a, b/2M)$ and $M = \max |f(x, y)|$ for $(x, y) \in R$. Thus we may assume that there exists $\delta > 0$ and $h > 0$ such that for each $Y_0$ satisfying $|Y_0 - y_0| \le \delta$, problem (1) possesses a unique solution $\phi(x, Y_0)$ on $|x - x_0| \le h$ (see Figure).




**Figure**

We are now in a position to state the basic theorem concerning the dependence of solutions on initial conditions.

**Theorem 4.4**

**Hypothesis**

1. Let f be continuous and satisfies Lipschitz condition with respect to y, with Lipschitz constant k, in a domain D of the xy plane and let $(x_0, y_0)$ be a fixed point of D.

2. Assume there exists $\delta > 0$ and $h > 0$ such that for each $Y_0$ satisfying $|Y_0 - y_0| \le \delta$, the initial value problem.

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = Y_0 \qquad (1)$$

possesses a unique solution $\phi(x, Y_0)$ defined and contained in D on $|x - x_0| \le h$

**Conclusion**

If $\phi$ denotes the unique solution of (1) when $Y_0 = y_0$, and $\bar{\phi}$ denotes the unique solution of (1) when $Y_0 = \bar{y}_0$, where $|\bar{y}_0 - y_0| = \delta_1 \le \delta$, then

$$|\bar{\phi}(x) - \phi(x)| \le \delta_1 e^{kh} \qquad \text{on } |x - x_0| \le h.$$

**Proof** From Existence and Uniqueness theorem, we know that

$$\phi = \lim_{n \to \infty} \phi_n$$

where

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f[t, \phi_{n-1}(t)]\, dt \quad (n = 1, 2, 3, ...)$$

and $\phi_0(x) = y_0$; $|x - x_0| \le h$.

Similarly,

$$\bar{\phi} = \lim_{n \to \infty} \bar{\phi}_n$$

where $$\bar{\phi}_n(x) = \bar{y}_0 + \int_{x_0}^{x} f[t, \bar{\phi}_{n-1}(t)]\, dt \quad (n = 1, 2, 3, ...)$$

and $\bar{\phi}_0(x) = \bar{y}_0$; $|x - x_0| \le h$.

We shall show by induction that

$$|\bar{\phi}_n(x) - \phi_n(x)| \le \delta_1 \sum_{j=0}^{n} \frac{k^j (x - x_0)^j}{j!} \qquad (2)$$

on $[x_0, x_0 + h]$, where k is the Lipschitz constant. We assume that on $[x_0, x_0 + h]$,

$$\left|\bar{\phi}_{n-1}(x) - \phi_{n-1}(x)\right| \le \delta_1 \sum_{j=0}^{n-1} \frac{k^j (x - x_0)^j}{j!} \tag{3}$$

Then

$$\left|\bar{\phi}_n(x) - \phi_n(x)\right| = \left|\bar{y}_0 + \int_{x_0}^{x} f[t, \bar{\phi}_{n-1}(t)]\, dt - y_0 - \int_{x_0}^{x} f[t, \phi_{n-1}(t)]\, dt\right|$$

$$\le \left|\bar{y}_0 - y_0\right| + \int_{x_0}^{x} \left|f[t, \bar{\phi}_{n-1}(t)] - f[t, \phi_{n-1}(t)]\right| dt.$$

Applying Lipschitz condition, we have

$$\left|f[x, \bar{\phi}_{n-1}(x)] - f[x, \phi_{n-1}(x)]\right| \le k \left|\bar{\phi}_{n-1}(x) - \phi_{n-1}(x)\right|$$

and since $\left|\bar{y}_0 - y_0\right| = \delta_1$, so

$$\left|\bar{\phi}_n(x) - \phi_n(x)\right| \le \delta_1 + k \int_{x_0}^{x} \left|\bar{\phi}_{n-1}(t) - \phi_{n-1}(t)\right| dt .$$

Using the assumption (3), we have

$$\left|\bar{\phi}_n(x) - \phi_n(x)\right| \le \delta_1 + k \int_{x_0}^{x} \delta_1 \sum_{j=0}^{n-1} \frac{k^j (t - x_0)^j}{j!}\, dt$$

$$= \delta_1 + k\,\delta_1 \sum_{j=0}^{n-1} \frac{k^j}{j!} \int_{x_0}^{x} (t - x_0)^j\, dt$$

$$= \delta_1 \left[1 + \sum_{j=0}^{n-1} \frac{k^{j+1} (x - x_0)^{j+1}}{(j+1)!}\right].$$

Since

$$\delta_1 \left[1 + \sum_{j=0}^{n-1} \frac{k^{j+1} (x - x_0)^{j+1}}{(j+1)!}\right] = \delta_1 \sum_{j=0}^{n} \frac{k^j (x - x_0)^j}{j!},$$

we have

$$\left|\bar{\phi}_n(x) - \phi_n(x)\right| \le \delta_1 \sum_{j=0}^{n} \frac{k^j (x - x_0)^j}{j!} ,$$

which is (3) with (n – 1) replaced by n.

Also, on $[x_0, x_0 + h]$, we have

$$|\bar{\phi}_1(x) - \phi_1(x)| = \left| \bar{y}_0 + \int_{x_0}^{x} f[t, \bar{y}_0]\, dt - y_0 - \int_{x_0}^{x} f[t, y_0]\, dt \right|$$

$$\leq |\bar{y}_0 - y_0| + \int_{x_0}^{x} |f[t, \bar{y}_0] - f[t, y_0]|\, dt$$

$$\leq \delta_1 + \int_{x_0}^{x} k|\bar{y}_0 - y_0|\, dt = \delta_1 + k\,\delta_1 (x - x_0).$$

Thus (2) holds for n = 1. Hence the induction is complete and (2) holds on $[x_0, x_0 + h]$. Using similar arguments on $[x_0 - h, x_0]$, we have

$$|\bar{\phi}_n(x) - \phi_n(x)| \leq \delta_1 \sum_{j=0}^{n} \frac{k^j\, |x - x_0|^j}{j!} \leq \delta_1 \sum_{j=0}^{n} \frac{(kh)^j}{j!}$$

for all x on $|x - x_0| \leq h$, n = 1, 2, 3……. Letting $n \to \infty$, we have

$$|\bar{\phi}(x) - \phi(x)| \leq \delta_1 \sum_{j=0}^{\infty} \frac{(kh)^j}{j!}.$$

But $\sum_{j=0}^{\infty} \frac{(kh)^j}{j!} = e^{kh}$; and so we have the desired inequality

$$|\bar{\phi}(x) - \phi(x)| \leq \delta_1 e^{kh} \text{ on } |x - x_0| \leq h.$$

This completes the proof of the theorem.

**Remark** Thus under the conditions stated, if the initial values of the two solutions $\phi$ and $\bar{\phi}$ differ by a sufficiently small amount, then their values will differ by an arbitrary small amount at every point of $|x - x_0| \leq h$. Geometrically, this means that if the corresponding integral curves are sufficiently close to each other initially, then they will be arbitrarily close to each other for all x such that $|x - x_0| \leq h$.

## B. Dependence on the Function f

We now consider how the solution of $dy/dx = f(x, y)$ will change if the function f is slightly changed. In this connection we have the following theorem.

**Theorem 4.5**

**Hypothesis**

1. In a domain D of the xy plane, assume that

   (i) f is continuous and satisfies Lipschitz condition with respect to y, with Lipschitz constant k.

   (ii) F is continuous.

   (iii) |F(x, y)- f(x, y)| ≤ ∈ for (x, y) ∈ D.

2. Let $(x_0, y_0)$ be a point of D and let

   (i) ϕ be the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

   (ii) ψ be a solution of the initial value problem

$$\frac{dy}{dx} = F(x, y),$$

$$y(x_0) = y_0$$

   (iii) [x, ϕ(x)] and [x, ψ(x)] ∈ D for $|x - x_0| \le h$.

**Conclusion** Then

$$|\phi(x) - \psi(x)| \le \frac{\varepsilon}{k}(e^{kh} - 1) \ on \ |x - x_0| \le h.$$

**Proof** Let $\bar{\phi}_0(x) = \psi(x)$ and define a sequence of function $\{\bar{\phi}_n\}$ by

$$\bar{\phi}_n(x) = y_0 + \int_{x_0}^{x} f[t, \ \bar{\phi}_{n-1}(t)]dt, \quad |x - x_0| \le h \quad (n = 1, 2, 3, \ldots\ldots)$$

Then $\bar{\phi} = \lim_{n\to\infty} \bar{\phi}_n$ is a solution of dy/dx = f(x, y) such that $\bar{\phi}(x_0) = y_0$ on $|x - x_0| \le h$.

By Hypothesis 1(i), the initial-value problem dy/dx = f(x, y), $y(x_0) = y_0$, has a unique solution on $|x - x_0| \le h$. Thus from Hypothesis 2(i) $\bar{\phi}(x) = \phi(x)$ on $|x - x_0| \le h$, and so $\lim_{n\to\infty} \bar{\phi}_n = \phi$.

From Hypothesis 2(ii) we have

$$\psi(x) = y_0 + \int_{x_0}^{x} F\ [t,\ \psi(t)]dt, \qquad |x - x_0| \le h.$$

We shall show by induction that

$$\left|\bar{\phi}_n(x) - \psi(x)\right| \le \varepsilon \sum_{j=1}^{n} \frac{k^{j-1}\ (x - x_0)^j}{j!} \tag{1}$$

on $[x_0, x_0 + h]$. We assume that on this interval

$$\left|\bar{\phi}_{n-1}(x) - \psi(x)\right| \le \varepsilon \sum_{j=1}^{n-1} \frac{k^{j-1}\ (x - x_0)^j}{j!} \tag{2}$$

Then

$$\left|\bar{\phi}_n(x) - \psi(x)\right| = \left| y_0 + \int_{x_0}^{x} f[t,\ \bar{\phi}_{n-1}(t)]dt - y_0 - \int_{x_0}^{x} F[t,\ \psi(t)]\ dt \right|$$

$$\le \int_{x_0}^{x} \left| f[t,\ \bar{\phi}_{n-1}(t)] - F[t,\ \psi(t)] \right| dt\ .$$

We write $F(x, y) = f(x, y) + \delta(x, y)$. Then

$$\left|\bar{\phi}_n(x) - \psi(x)\right| \le \int_{x_0}^{x} \left| f[t,\ \bar{\phi}_{n-1}(t)] - f\ [t,\ \psi(t)] - \delta\ [t,\ \psi(t)] \right| dt.$$

Applying the inequality $|A - B| \le |A| + |B|$ and then the Lipschitz condition satisfied by f, we have

$$\left|\bar{\phi}_n(x) - \psi(x)\right| \le \int_{x_0}^{x} \left| f[t,\ \bar{\phi}_{n-1}(t)] - f\ [t,\ \psi(t)] \right| dt + \int_{x_0}^{x} \left| \delta\ [t,\ \psi(t)] \right| dt$$

$$\le k \int_{x_0}^{x} \left| \bar{\phi}_{n-1}(t) - \psi(t) \right| dt + \int_{x_0}^{x} \left| \delta\ [t,\ \psi(t)] \right| dt.$$

Now using the assumption (2) and the fact that

$$|\delta\ (x, y)| = |F(x, y) - f\ (x, y)| \le \varepsilon,$$

we obtain

$$\left|\bar{\phi}_n(x) - \psi(x)\right| \le k\varepsilon \sum_{j=1}^{n-1} \frac{k^{j-1}}{j!} \int_{x_0}^{x} (t - x_0)^j\ dt + \int_{x_0}^{x} \varepsilon\ dt$$

$$= \varepsilon \sum_{j=1}^{n-1} \frac{k^j\ (x - x_0)^{j+1}}{(j+1)!} + \varepsilon\ (x - x_0)$$

$$= \quad \varepsilon \sum_{j=1}^{n} \frac{k^{j-1} (x - x_0)^j}{j!} \quad .$$

Thus (2) holds with (n – 1) replaced by n. Also Hypothesis 1 (iii) shows that

$$\left|\bar{\phi}_1(x) - \psi(x)\right| \;\le\; \int_{x_0}^{x} \left| f[t,\ \psi(t)] - F[t,\ \psi(t)] \right| dt \;\le\; \int_{x_0}^{x} \varepsilon\, dt = \varepsilon\, (x - x_0)$$

on $[x_0, x_0 + h]$. Thus (1) holds for n = 1. Thus the induction is complete and so (1) holds on $[x_0, x_0 + h]$ for n = 1, 2, 3, …..

Using similar arguments on $[x_0 – h, x_0]$, we thus have

$$\left|\bar{\phi}_n(x) - \psi(x)\right| \;\le\; \varepsilon \sum_{j=1}^{n} \frac{k^{j-1}\ |x - x_0|^j}{j!} \;\le\; \frac{\varepsilon}{k} \sum_{j=1}^{n} \frac{(kh)^j}{j!}$$

for all x on $|x – x_0| \le h$, n = 1, 2, 3, …..

Letting $n \to \infty$, we obtain

$$\left|\phi(x) - \psi(x)\right| \;\le\; \frac{\varepsilon}{k} \sum_{j=1}^{\infty} \frac{(kh)^j}{j!}$$

But $\sum_{j=1}^{\infty} \frac{(kh)^j}{j!} = e^{kh} – 1$. Thus we obtain the desired inequality

$$\left|\phi(x) - \psi(x)\right| \;\le\; \frac{\varepsilon}{k}\ (e^{kh} - 1) \text{ on } |x – x_0| \le h.$$

Thus, under the hypotheses stated, if ε is sufficiently small, the difference between the solutions ϕ and ψ will be arbitrarily small on $|x – x_0| \le h$. This completes the proof of the theorem.

**Systems of Linear Differential Equations**

**Introduction**

Upto now, we obtained solutions of single differential equation of different types and obtained the existence and uniqueness of solution of the initial value problem of first order equation which are not necessarily linear. However, we come across practical situations where we have to deal with more than one differential equation with many variables or depending upon a single variable. For example, if we consider the motion of a particle in three dimensions, we get one such situation.

**Systems of First Order Equations**

In analogy with the system of the single equation $x'(t) = f(t, x)$, $t \in I$, we shall study the system of equations by considering the following n-equations:

$$x_1' = f_1(t, x_1, x_2, \ldots\ldots. x_n)$$

$$x_2' = f_2(t, x_1, x_2, \ldots\ldots. x_n)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_n' = f_n(t, x_1, x_2, \ldots\ldots. x_n) \qquad (1)$$

where $f_1, f_2, \ldots f_n$ are the given n-real valued functions defined on the domain D in $R^{n+1}$ and $x_1, x_2, \ldots\ldots., x_n$ are functions of t to be determined from the equations. Our problem is to find an interval I and n-differentiable functions $\phi_1, \phi_2, \ldots \phi_n$ on I such that

(i) $(t, \phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ is in D for $t \in I$ and

(ii) $\phi_i'(t) = f_i(t, \phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ for all $t \in I$ and $i = 1, 2, 3, \ldots. n$.

When n such differentiable functions $(\phi_1, \phi_2, \ldots \phi_n)$ exist, $(\phi_1, \phi_2, \ldots \phi_n)$ is called a solution of (1) on I.

Using the vector notation $x = (x_1, x_2, \ldots\ldots., x_n)$ and $F = (f_1(t), f_2(t), \ldots. f_n(t))$, the system can be written in the form $x' = F(t, x)$ and the solution is denoted by $\phi = (\phi_1, \phi_2, \ldots \phi_n)$.

Let us represent a linear system in the following form:

$$x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + \ldots\ldots + a_{1n}(t)\, x_n + b_1(t)$$

$$x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + \ldots\ldots + a_{2n}(t)\, x_n + b_2(t)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots. \quad \ldots.. \quad \ldots.. \quad \ldots.$$

$$x_n' = a_{n1}(t)\, x_1 + a_{n2}(t)\, x_2 + \ldots\ldots + a_{nn}(t)\, x_n + b_n(t), \qquad (2)$$

which is a linear system of n first order equations in n unknown functions $x_1, x_2, \ldots x_n$ and $a_{ij}(t)$, $b_j(t)$ $i, j = 1, 2, \ldots n$ are all given functions on I. The above system (2) is a particular case of (1), since $x_i'(t)$ can be taken as $f_i(t, x_1, x_2, \ldots\ldots. x_n)$ where each $f_i$ is linear on I. By using matrices, we can represent (2) as

$$x'(t) = A(t)\, x(t) + B(t), \quad t \in I \qquad (3)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \ A(t) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } \quad B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \dots \\ b_n(t) \end{bmatrix}.$$

(3) is a linear equation in x and it is a matrix representation of a linear non-homogeneous system. If B(t) = 0, then (3) reduces to a homogeneous system $x'(t) = A(t)\, x(t)$.

**Matrix Preliminaries**

Let $A(t) = [a_{ij}(t)]$, i,j = 1, 2, 3, ….. n be an n x n matrix of functions defined for $t \in I = [a, b]$.

**Definition 1**

The matrix $A(t) = [a_{ij}(t)]$, $t \in I$ is said to be continuous or differentiable, if every element $a_{ij}(t)$, i, j = 1, 2, ... n of the matrix A(t) is continuous or differentiable on I. The derivative A'(t) is obtained by differentiating every element of A(t), that is, $A'(t) = [a'_{ij}(t)]$, i, j = 1, 2, 3, ... n.

**Definition 2**

If $x = (x_1, x_2, \ldots\ldots x_n) \in R^n$, then the norm of x denoted by |x| is defined as

$$|x| = |x_1| + |x_2| + \ldots\ldots + |x_n| = \sum_{i=1}^{n} |x_i| \ .$$

**Definition 3**

Let $A = [a_{ij}]$ denotes an n x n matrix then the norm of the matrix A denoted by |A| is defined as $|A| = \sum_{i,j=1}^{n} |a_{ij}|$ .

If $A(t) = [a_{ij}(t)]$ is a continuous matrix on I, then |A(t)| is also continuous on I. Also,

(i) $|A + B| \le |A| + |B|$

(ii) $|AB| \le |A|\,|B|$

(iii) $|\alpha A| = |\alpha|\,|A|$ for every scalar $\alpha$

(iv) $|Ax| \le |A|\,|x|$ for any vector x.

**Definition 4**

A sequence $\{A_n\}$ of matrices is said to be convergent, if given any $\varepsilon > 0$, there exists a positive integer $n_0$ such that $|A_m - A_n| < \varepsilon$ for all m, $n \ge n_0$.

**Definition 5**

A sequence $\{A_n\}$ of matrices is said to tend to a limit A, if given any $\varepsilon > 0$, there exists a positive integer $n_0$ such that $|A_n - A| < \varepsilon$ for all $n \ge n_0$.

**Definition 6**

The infinite series $\sum_{n=1}^{\infty} A_n$ of matrices is said to be convergent, if the sequence of partial sums of the series is convergent and the sum of the series is the limit matrix of partial sums.

The exponential of a matrix A is defined as $e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$, where the convergence of the series is in the sense of the Definition 6 and $A^n$ represents the n-th power of A and I is the identity matrix.

From the above definition, the following important properties are observed:

(i) $|e^A| \le (n - 1) + e^{|A|}$ where | | denotes the norm.

**Proof** $e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$, where I is the identity matrix of order n.

Hence $|e^A| \le |I| + \sum_{n=1}^{\infty} \frac{|A|^n}{n!}$

Now for any matrix $A = [a_{ij}]$, $|A| = \sum_{i,j} |a_{ij}|$ so that we have $|I| = n$. Hence we have

$$|e^A| \le n - 1 + 1 + \sum_{n=1}^{\infty} \frac{|A|^n}{n!} = (n-1) + \sum_{n=0}^{\infty} \frac{|A|^n}{n!} = (n-1) + e^{|A|}$$

which gives $|e^A| \le (n-1) + e^{|A|}$.

(ii) For matrices A and B, it is not true in general that $e^{A+B} = e^A \cdot e^B$ but this relation is valid when A and B commute.

**Definition 7**

The sum of the diagonal elements of a matrix A is called the trace of A. If A is the given matrix, then the trace of A is denoted by trA.

**Representation of n-th Order Equation as a System**

Any differential equation of order n can be written as a system of n first – order differential equations.

**Theorem 4.6**

The general n-th order initial value problem

$$x^{(n)} = f(t, x, x', \ldots\ldots x^{(n-1)}),\ t \in I \tag{1}$$

$$x(t_0) = a_0,\ x'(t_0) = a_1,\ \ldots\ x^{(n-1)}(t_0) = a_{n-1},\ t_0 \in I$$

where $a_0, a_1, a_2, \ldots\ldots a_{n-1}$, are given constants, is equivalent to a system of n-linear equations.

**Proof** We define a new family of unknown functions $x_n = x^{(n-1)}$.

i.e. $x_1 = x,\ x_2 = x',\ x_3 = x'',\ \ldots\ldots\ x_{n-1} = x^{(n-2)},\ x_n = x^{(n-1)}$.

We can then rewrite the original differential equation as a system of differential equations with order 1 and dimension n. Thus we have the system of equations

$$x_1' = x' = x_2$$

$$x_2' = x'' = x_3$$

$$x_3' = x''' = x_4$$

$$\ldots \quad \ldots. \quad \ldots.$$

$$x_{n-1}' = x^{(n-1)} = x_n$$

and $$x_n' = x^{(n)}. \tag{2}$$

Using the above change of variables, we get

$$x_n'(t) = f(t, x_1, x_2, \ldots, x_n). \tag{3}$$

Let $\phi = (\phi_1, \phi_2, \ldots \phi_n)$ be a solution of (2). Then

$$\phi_2 = \phi_1', \phi_3 = \phi_2' = \phi_1'', \ldots, \phi_n = \phi_1^{(n-1)} \tag{4}$$

Hence $\qquad f(t, \phi_1, \phi_2, \ldots \phi_n) = f[t, \phi_1, \phi_1', \ldots \phi_1^{(n-1)}(t)] = \phi_1^{(n)}(t)$

which shows that the component $\phi_1$ is a solution of (1). Conversely, if $\phi_1(t)$ is a solution of (1), then the vector $\phi = (\phi_1, \phi_2, \ldots \phi_n)$ is a solution of (2). Thus, system (2) is equivalent to (1).

Further we obtain the initial conditions of the system as a vector

$$\phi_1(t_0) = a_0, \ \phi_1'(t_0) = a_1, \ \ldots\ldots \ \phi_1^{(n-1)}(t_0) = a_{n-1}$$

so that the vector $\phi(t_0) = (a_0, a_1, a_2, \ldots a_{n-1})$ gives the initial condition.

**Now, we shall transform the linear equations of order n into a system of equations.**

The general n-th order equation is

$$a_0(t)\, x^{(n)} + a_1(t)\, x^{(n-1)} + \ldots + a_n(t)\, x = b(t), \ t \in I \tag{1}$$

where $a_0(t) \neq 0$ for any $t \in I$.

Now make the following substitutions:

$$x(t) = x_1(t)$$

$$x_1'(t) = x'(t) = x_2$$

$$x_2'(t) = x''(t) = x_3$$

$$x_3'(t) = x'''(t) = x_4$$

$$\ldots \ \ldots \ \ldots \ \ldots$$

$$x_{n-1}'(t) = x^{(n-1)}(t) = x_n$$

$$x_n'(t) = x^{(n)}(t). \tag{2}$$

Rewriting equation (1), we get

$$x^{(n)} = \frac{-a_n(t)}{a_0(t)}\, x - \frac{a_{n-1}(t)}{a_0(t)}\, x' - \ldots - \frac{a_1(t)}{a_0(t)}\, x^{(n-1)} + \frac{b(t)}{a_0(t)}$$

Using (2) in the above equation

$$x^{(n)} = \frac{-a_n(t)}{a_0(t)}\, x_1 - \frac{a_{n-1}(t)}{a_0(t)}\, x_2 - \ldots - \frac{a_1(t)}{a_0(t)}\, x_n + \frac{b(t)}{a_0(t)}$$

Using the matrix notation, the above system can be rewritten as

$$x_1'(t) = 0\, x_1 + 1\, x_2 + \ldots\ldots + 0$$

$$x_2'(t) = 0\, x_1 + 0\, x_2 + 1\, x_3 + \ldots\ldots + 0$$

$$\ldots \;\ldots.\; \ldots.\; \ldots.. \;\; \ldots.\; \ldots. \;\; \ldots\ldots \;\; \ldots\ldots.$$

$$x_n'(t) = \frac{-a_n(t)}{a_0(t)}\, x_1 - \frac{a_{n-1}(t)}{a_0(t)}\, x_2 - \ldots - \frac{a_1(t)}{a_0(t)}\, x_n + \frac{b(t)}{a_0(t)} \qquad (3)$$

Thus, the above system can be written in the matrix form as

$$\begin{bmatrix} x_1' \\ x_2' \\ .. \\ .. \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \frac{-a_{n-2}}{a_0} & \ldots & \frac{-a_1}{a_0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ .. \\ .. \\ \frac{b(t)}{a_0} \end{bmatrix}$$

Using the vector notation, we get

$$x' = A(t)x + B(t) \qquad (4)$$

**Note** The n-th order linear equation (1) is equivalent to the system (2) leading to the matrix differential equation (4).

**Example 1** The matrix form of the system of equations

$$x_1' = x_1 + 2\, x_2\,, \quad x_2' = 4\, x_1 + 3\, x_2$$

is
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

**Example 2** Find the matrix form of the linear equation

$$x''' - 4x'' + 10x' - 6x = 9t \qquad (1)$$

Let us make the following substitution

$$x_1 = x, \quad x_2 = x', \quad x_3 = x'' \tag{2}$$

and express x, x', x'' and x''' in terms of $x_1$, $x_2$, $x_3$ and their derivatives.

From (2) $\quad x' = x_1' = x_2$

$$x'' = x_2' = x_3 \text{ and } x''' = x_3'. \tag{3}$$

Using (3) in (1), we get

$$x_3' = 4x_3 - 10x_2 + 6x_1 + 9t$$

and

$$x_1' = 0 + 1x_2 + 0 \tag{4}$$

$$x_2' = 0 + 0 + 1x_3 \tag{5}$$

$$x_3' = 6x_1 - 10x_2 + 4x_3 + 9t \tag{6}$$

With the help of (4), (5) and (6), we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -10 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 9t \end{bmatrix}$$

**Existence and Uniqueness of Solutions of System of Equations**

In this section we will solve the initial value problem of the vector differential equation

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t \in I$$

where A(t) is an n x n matrix defined over I and x(t) is a vector function on I. For a fixed $t_0$, $x(t_0)$ is a fixed vector. We shall state and prove the following theorem for a system of equations given in the vector form.

**Theorem 4.7** Let A(t) be a continuous n x n matrix defined on a closed and bounded interval I. Then the initial value problem

$$x'(t) = A(t)\, x(t), \quad x(t_0) = x_0, \quad t, t_0 \in I \tag{1}$$

has a unique solution on I.

**Proof** We first note that the initial value problem (1) is equivalent to the solution of the vector integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds \tag{2}$$

which gives $x(t_0) = x_0$. On differentiating (2), we get

$$x'(t) = A(t)\, x(t),\ t \in I$$

If x(t) is a vector solution of (1), then integrating (1), we get

$$x(t) = x(t_0) + \int_{t_0}^{t} A(s)\, x(s)\, ds.$$

With the help of the successive approximations, we shall show that $\{x_n(t)\}$ is a Cauchy sequence in $R^n$ for every $t \in 1$. Since $R^n$ is complete, $\{x_n(t)\}$ converges uniformly to a limit x(t) and then we prove its uniqueness. Let us define the approximations by

$$x(t_0) = x_0\ , \qquad x_{n+1}(t) = x_0 + \int_{t_0}^{t} A(s)\, x_n(s)\, ds\ ,\ \ t \geq t_0,\ t \in I$$

for n = 0, 1, 2, 3, ……..

Since $x_0$ is a given vector, the sequence $\{x_n(t)\}$ is well-defined. First we shall show that $\{x_n(t)\}$ converges uniformly on I to a function x(t) which is the solution of the initial value problem (1) and for this it is enough to show that $\{x_n(t)\}$ is a Cauchy sequence in $R^n$. Since $R^n$ is complete $\{x_n(t)\}$ converges to x(t). To show that $x_n(t)$ is a Cauchy sequence, let us consider

$$x_0(t) + \sum_{n=1}^{\infty} \left(x_{n+1}(t) - x_n(t)\right)$$

where sequence of partial sums is $x_n(t)$.

Hence, consider

$$x_{n+1}(t) - x_n(t) = \int_{t_0}^{t} A(s)\ \left(x_n(s) - x_{n-1}(s)\right)\ ds$$

We shall construct an upper bound for $|\,x_{n+1}(t) - x_n(t)\,|$ by induction method where | | is the norm.

Since A(t) is a continuous matrix defined on I, there exists a constant M such that

$$|A(t)| \leq M,\ t \in I \tag{4}$$

Now $$x_1(t) - x_0(t) = \int_{t_0}^{t} A(s)\, x_0(s)\, ds$$

so that $$|x_1(t) - x_0(t)| \le \int_{t_0}^{t} |A(s)|\, |x_0|\, ds$$

Using (4), we have $$|x_1(t) - x_0(t)| \le M\, |x_0|\, (t - t_0) \tag{5}$$

Similarly, we find

$$x_2(t) - x_1(t) = \int_{t_0}^{t} A(s)\, (x_1 - x_0)\, ds$$

$$|x_2(t) - x_1(t)| \le \int_{t_0}^{t} |A(s)|\, |x_1 - x_0|\, ds$$

$$|x_2(t) - x_1(t)| \le M \int_{t_0}^{t} M\, |x_0|\, |s - t_0|\, ds$$

$$= M^2 \frac{|x_0|\, (t - t_0)^2}{2!}$$

Proceeding in same manner, we have by induction

$$|x_{n+1}(t) - x_n(t)| \le \frac{M^{n+1} |x_0|\, (t - t_0)^{n+1}}{(n+1)!} \tag{6}$$

which is the (n + 2)-th term of a convergent series $\sum_{n=1}^{\infty} \frac{M^{n-1} |x_0|\, (t - t_0)^{n-1}}{(n-1)!}$ of positive constants whose sum is $|x_0|\, e^{M(t - t_0)}$. Since the n-th term of a convergent series tends to zero as $n \to \infty$, the right hand side of (6) tends to zero as $n \to \infty$. Hence $\{x_n(t)\}$ is a Cauchy sequence in $R^n$ for each $t \in I$. Since $R^n$ is complete, $x_n(t) \to x(t)$ in the norm $|\ |$. Since the norm convergence is uniform, $x_n \to x$ uniformly on I.

Now consider

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} A(s)\, x_n(s)\, ds \tag{7}$$

Taking the limit as $n \to \infty$ on both sides of (7), we have

$$x(t) = x_0 + \lim_{n\to\infty} \int_{t_0}^{t} A(s)\, x_n(s)\, ds \tag{8}$$

Since $x_n(t) \to x(t)$ uniformly on I, we can take the limit under the integral sign of (8) to get

$$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds$$

which proves that x(t) is the solution of the integral equation. Since the solution of the above integral equation and the initial value problem (1) are equivalent, x(t) gives the solution of the initial value problem.

Now, we shall prove the uniqueness of the solution. If the solution x(t) is not unique, let y(t) be another solution of (1).

Then we get $$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds$$

$$y(t) = x_0 + \int_{t_0}^{t} A(s)\, y(s)\, ds$$

Or $$x(t) - y(t) = \int_{t_0}^{t} A(s)\, \big(x(s) - y(s)\big)\, ds$$

so that $$|x(t) - y(t)| \le M \int_{t_0}^{t} |x(s) - y(s)|\, ds$$

Thus, for any given $\varepsilon > 0$, we get from the above inequality

$$|x(t) - y(t)| < \varepsilon + M \int_{t_0}^{t} |x(s) - y(s)|\, ds$$

Let us take $z(t) = |x(t) - y(t)|$. Then we have

$$z(t) < \varepsilon + M \int_{t_0}^{t} z(s)\, ds \;,\; t \in I$$

If $r(t) = \varepsilon + M \int_{t_0}^{t} z(s)\, ds$, then $r(t_0) = \varepsilon$ and $z(t) < r(t)$. From the definition $r'(t) = M\, z(t) < M\, r(t)$ so that we have $r'(t) - M\, r(t) < 0$.

Integrating the left hand side of the above inequality, we get

$$r(t)\, e^{-Mt} - r(t_0)\, e^{-Mt_0} < 0$$

from which we have $$r(t) < r(t_0)\, e^{M(t - t_0)}$$

Hence $z(t) < r(t) < \varepsilon e^{M(t - t_0)}$, since $r(t_0) = \varepsilon$.

Since the above inequality is true for each $\varepsilon > 0$, we get $z(t) < 0$ which implies $|x(t) - y(t)| = 0$ which gives $x(t) = y(t)$ on I. This proves that the solution is unique. Hence the proof of the theorem is complete.

**Note** The zero vector function on I is always a solution of (1). If the solution of (1) is zero for any to $t_0 \in I$, and since the solution is unique, then it must be zero throughout I.

**Corollary** $x(t) = 0$ is the only solution of the initial value problem $x' = A(t) x$, $x(t_0) = 0$ where $t, t_0 \in I$ and $A(t)$ is a continuous matrix on I.

**Proof** Let us find the successive approximations

$$x_1(t) = x_0 + \int_{t_0}^{t} A(s)\, x_0(s)\, ds = 0 + \int_{t_0}^{t} A(s)\, 0\, ds = 0 \qquad (1)$$

$$x_2(t) = x_0 + \int_{t_0}^{t} A(s)\, x_1(s)\, ds = 0 + \int_{t_0}^{t} A(s)\, 0\, ds = 0 \qquad (2)$$

In a similar manner, we can show that $x_n(t) = 0$ for all n.

Hence $x_n(t) \to 0$ as $n \to \infty$ for all n implies $x(t) = 0$. Thus $x(t) = 0$ is the only solution.

**Summary**

This chapter is devoted to the study of finding the largest open interval over which the solution of IVP is defined and then establishing the conditions for the continuation of solution in a general domain. Further, dependence of solutions of initial value problems on initial conditions and functions are also proved. In last, the study of solution of the initial value problem of first order equation is extended to analyse systems of first order differential equations. The criterion for existence and uniqueness of solutions for linear system of equations is presented.

**Keywords**

Continuation of solutions, Maximal interval, Existence, Uniqueness, System of equations.

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## Chapter-5

# TOTAL DIFFERENTIAL EQUATIONS AND DIFFERENTIAL INEQUATIONS

### Objectives

The purpose of this chapter is to introduce the reader to Pfaffian differential equations, their condition of integrability and some methods to solve such equations. Some theorems for differential inequations are also presented.

### Introduction

The purpose of this chapter is to study Pfaffian (Total) differential equations and differential inequations. Certain methods to solve Pfaffian differential equations in three variables are explained. The most important techniques in the theory of differential equations involve the integration of differential inequalities Gronwall inequality is proved in this context. Comparison theorems are also presented to compare the unknown solutions of one differential equation with the known solutions of another.

### Pfaffian Differential Forms and Equations

The expression

$$\sum_{i=1}^{n} F_i(x_1, x_2, ........, x_n)\, dx_i \qquad (1)$$

where $F_i$ (i = 1, 2, ….., n) are continuous functions of some or all of the n independent variables $x_1$, $x_2$, …..$x_n$, is called a Pfaffian differential form of n variables. The relation

$$\sum_{i=1}^{n} F_i\ dx_i = 0 \qquad (2)$$

is called a Pfaffian differential equation or total differential equation in n variables.

We shall consider Pfaffian differential equations in two variables and those in a higher number of variables, separately.

In the case of two variables we may write equation (2) in the form

$$P(x, y)\, dx + Q(x, y)\, dy = 0 \tag{3}$$

which is equivalent to

$$\frac{dy}{dx} = f(x, y), \tag{4}$$

if we write f(x, y) = - P/Q. Now the functions P(x, y) and Q(x, y) are known functions of x and y, so that f(x, y) is defined uniquely at each point of the xy plane at which the functions P(x, y) and Q(x, y) are defined.

The fundamental existence theorem in the theory of ordinary differential equations is of the form:

**Theorem 5.1**

A Pfaffian differential equation in two variables always possesses an integrating factor.

**Proof**

A Pfaffian differential equation in two variables is

$$P(x, y)\, dx + Q(x, y)\, dy = 0 \tag{1}$$

where the functions P and Q are continuously differentiable.

If $\quad Q(x, y) \neq 0, \quad$ then $\dfrac{dy}{dx} = \dfrac{-P(x, y)}{Q(x, y)}$

Then, by existence theorem, this equation has a solution.

Let ψ(x, y) = c be a solution of this equation.

$$\Rightarrow \quad d\psi(x, y) = 0$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = 0 \tag{2}$$

Comparing (1) and (2)

$$P(x, y) = \frac{\partial \psi}{\partial x}, \qquad Q(x, y) = \frac{\partial \psi}{\partial y},$$

$\Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, which is the condition for given equation to be exact.

In case, given equation (1) is not exact, then multiplying (1) by μ i.e.

$$\mu(x, y)\, P(x, y)\, dx + \mu(x, y)\, Q(x, y)\, dy = 0$$

Now, using condition of exactness, we get

$$\frac{\partial}{\partial y}[\mu(x, y)\, P(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)\, Q(x, y)] \tag{3}$$

Find value of μ(x, y) from (3), which when multiplied to given equation, makes it exact. Then the equation is said to be integrable and will possess an integrating factor.

**Pfaffian Differential Equation in Three Variables**

When there are three variables, the Pfaffian differential equation is of the form

$$P\, dx + Q\, dy + R\, dz = 0, \tag{1}$$

where P, Q and R are functions of x, y and z. If we introduce the vector **X** = (P, Q, R) and **dr** = (dx, dy, dz), we may write this equation in the vector notation as

$$\mathbf{X} . \mathbf{dr} = 0 \tag{2}$$

Before discussing this equation, we first consider two lemmas:

**Lemma 1**

A necessary and sufficient condition that there exists between two functions u(x, y) and v(x, y) a relation F(u, v) = 0, not involving x or y explicitly is that

$$\frac{\partial\,(u,v)}{\partial\,(x,y)} = 0.$$

**Proof**

First, the condition is necessary. We have a relation of the form

$$F(u, v) = 0 \tag{1}$$

Differentiating this identity with respect to x, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{2}$$

and differentiation with respect to y yields

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{3}$$

Eliminating $\frac{\partial F}{\partial v}$ from these two equations, we get

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right\} = 0$$

But the relation (1) involves both u and v, it follows that $\frac{\partial F}{\partial u}$ is not identically zero, so that

$$\frac{\partial (u,v)}{\partial (x,y)} = 0 \tag{4}$$

Second, the condition is sufficient. We may easily eliminate y from the equations

$$u = u(x, y), \quad v = v(x, y).$$

Let the resultant eliminant be

$$F(u, v, x) = 0. \tag{5}$$

Differentiating this relation with respect to x, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0$$

and differentiating with respect to y, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0$$

Eliminating $\frac{\partial F}{\partial v}$ from these two equations, we find that

$$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial(u,v)}{\partial(x, y)}\frac{\partial F}{\partial u} = 0$$

If the condition (4) is satisfied, we see that

$$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} = 0\,.$$

The function v is a function of both x and y, so that $\partial v/\partial y$ can not be identically zero. Hence

$$\frac{\partial F}{\partial x} = 0\,,$$

which shows that the function F does not contain the variable x explicitly. Thus from (5), we get

$$F(u, v) = 0.$$

**Lemma 2**

If **X** is a vector such that **X.** curl **X** = 0 and μ is an arbitrary function of x, y, z then (μ**X**). curl (μ**X**) = 0.

**Proof**

From the definition of curl, we have

$$\mu\mathbf{X}.\ \text{curl}\ \mu\mathbf{X} = \sum_{x,y,z} (\mu P) \left\{\frac{\partial(\mu R)}{\partial y} - \frac{\partial(\mu Q)}{\partial z}\right\}$$

where **X** has components (P, Q, R). The right-hand side of equation may be written in the form

$$\mu^2 \sum_{x,y,z} P \left\{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right\} - \mu \sum_{x,y,z}\left\{PQ\,\frac{\partial \mu}{\partial z} - PR\,\frac{\partial \mu}{\partial y}\right\}$$

and the second of these sums is identically zero. Hence

$$\mu\mathbf{X}.\ \text{curl}\ \mu\mathbf{X} = \{\mathbf{X}.\ \text{curl}\ \mathbf{X}\}\ \mu^2 = 0 \quad \text{as } \mathbf{X}.\ \text{curl}\ \mathbf{X} = 0$$

The converse of this theorem is also true, as can be seen by applying the factor 1/μ to the vector μ**X.**

**Note** Equations of the form

$$P\ dx + Q\ dy + R\ dz = 0,$$

do not always possess integrals. To find out the criterion for determining, whether or not an equation of this type is integrable, the following theorem is proved.

**Theorem 5.2**

A necessary and sufficient condition that the Pfaffian differential equation **X**. **dr** = 0 should be integrable is that **X**. curl **X** = 0.

**Proof**

The condition is necessary, for if the equation

$$P\ dx + Q\ dy + R\ dz = 0 \tag{1}$$

is integrable, there exists among the variables x, y, z a relation of the type

$$F\ (x, y, z) = C \tag{2}$$

where C is a constant. Writing this in the differential form

$$\frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy + \frac{\partial F}{\partial z}\ dz = 0 \tag{3}$$

Now, equations (1) and (3) must be identical, i.e. there must exist a function μ(x, y, z) such that

$$\frac{\frac{\partial F}{\partial x}}{P} = \frac{\frac{\partial F}{\partial y}}{Q} = \frac{\frac{\partial F}{\partial z}}{R} = \mu(x, y, z)\ (\text{say})$$

i.e. $\mu P = \frac{\partial F}{\partial x}, \quad \mu Q = \frac{\partial F}{\partial y}, \quad \mu R = \frac{\partial F}{\partial z}$

i.e., such that $\mu\mathbf{X} = \text{grad}\ F$

Now, since $\text{curl grad}\ F = 0$

we have $\operatorname{curl}(\mu \mathbf{X}) = 0$

so that $\mu \mathbf{X}. \operatorname{curl}(\mu \mathbf{X}) = 0$

From Lemma 2 it follows that

$$\mathbf{X}. \operatorname{curl} \mathbf{X} = 0$$

Sufficient condition. If z is treated as a constant, the differential equation (1) becomes

$$P(x,y,z)\, dx + Q(x,y,z)\, dy = 0, \tag{4}$$

which is a Pfaffian differential equation in two variables and by Lemma 1, it possesses a solution of the form

$$U(x, y, z) = c_1 \tag{5}$$

where the "constant" $c_1$ may involve z. In differential form (5) can be written as

$$d\,U(x,y,z) = 0 \text{ i.e. } \quad \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = 0 \tag{6}$$

Now (4) and (6) must be identical. Thus there must exist a function μ such that

$$\frac{\partial U}{\partial x} = \mu P, \qquad \frac{\partial U}{\partial y} = \mu Q\,. \tag{7}$$

Substituting from the equations (7) into equation (1), we see that the latter equation may be written in the form

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz + \left(\mu R - \frac{\partial U}{\partial z}\right) dz = 0$$

which is equivalent to the equation

$$dU + K\, dz = 0, \tag{8}$$

if we write

$$K = \mu R - \frac{\partial U}{\partial z} \tag{9}$$

Now we are given that $\mathbf{X} . \operatorname{curl} \mathbf{X}=0$, and it follows from Lemma 2 that

$$\mu\mathbf{X}.\ \text{curl}\ \mu\mathbf{X} = 0$$

Now

$$\mu\mathbf{X} = (\mu P, \mu Q, \mu R) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right)$$

$$= \text{grad } U + (0, 0, K)$$

Hence

$$\mu\mathbf{X}.\ \text{curl}\ (\mu\mathbf{X}) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right).\left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}, 0\right)$$

$$= \frac{\partial U}{\partial x}\frac{\partial K}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial K}{\partial x}.$$

Thus the condition **X**. curl **X** = 0 is equivalent to the relation

$$\frac{\partial(U,K)}{\partial(x,y)} = 0$$

From Lemma 1, it follows that there exists between U and K a relation independent of x and y but not necessarily of z. In other words, K can be expressed as a function K (U, z) of U and z alone, and equation (8) takes the form

$$\frac{dU}{dz} + K(U,z) = 0$$

which, by Theorem 1, has a solution of the form

$$\phi\ (U, z) = c$$

where c is an arbitrary constant. Now, replacing U by its expression in terms of x, y, and z, we obtain the solution in the form

$$F(x, y, z) = c$$

which shows that the equation (1) is integrable.

Once it has been established that the equation is integrable, it only remains to determine an appropriate integrating factor μ (x, y, z). We shall discuss the solution of Pfaffian differential equations in three variables.

**Solution of Pfaffian Differential Equations in Three Variables**

We shall now consider some methods by which the solutions of Pfaffian differential equations in three variables x, y, z may be derived.

**(a) By Inspection** Once the condition of integrability is satisfied, one may find by rearranging the terms or dividing or multiplying by some suitable function to reduce to a form containing some or several parts, which are exact differentials. It is often possible to derive the primitive of the equation by inspection. In particular, we may also have curl **X** = 0, then **X** must be of the form grad u, and hence the equation **X . dr** = 0 reduces to the form

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = 0, \text{ so that}$$

$$u(x, y, z) = c \quad \text{is a solution.}$$

**Example 1** Solve the equation

$$(x^2z - y^3)\,dx + 3xy^2\,dy + x^3\,dz = 0$$

first showing that it is integrable.

**Solution** To test for integrability, we note that **X** = $(x^2z - y^3, 3xy^2, x^3)$, so that curl **X** = $(0, -2x^2, 6y^2)$, and hence **X.** curl **X** = 0.

We may write the equation in the form

$$x^2(z\,dx + x\,dz) - y^3\,dx + 3xy^2\,dy = 0$$

i.e.

$$z\,dx + x\,dz - \frac{y^3}{x^2}\,dx + \frac{3y^2}{x}\,dy = 0$$

i.e.

$$d(xz) + d\left(\frac{y^3}{x}\right) = 0$$

so that the primitive of the equation is

$$x^2z + y^3 = cx$$

where c is a constant.

**Example 2** Solve the equation

$$(y + x)\,dz + dx + dy = 0,$$

by showing that it is integrable.

**Solution** Here P = 1, Q = 1, R = y + x

$$\mathbf{X}.\ \text{curl}\ \mathbf{X} = P\,[Q_z - R_y] + Q\,[R_x - P_z] + R\,[P_y - Q_x] = 0$$

Hence, equation is integrable.

Dividing the given equation by (y + x) throughout, we obtain

$$dz + \frac{dx + dy}{x + y} = 0$$

Hence $x + y = ce^{-z}$ is required solution.

**(b) Variables Separable Method** In this case it is possible to write the Pfaffian differential equation in the form

$$P(x)\,dx + Q(y)\,dy + R(z)\,dz = 0$$

which obviously gives on integration

$$\int P(x)\,dx + \int Q(y)\,dy + \int R(z)\,dz = c$$

where c is a constant.

**Example 3** Solve the equation

$$a^2y^2z^2\,dx + b^2z^2x^2\,dy + c^2x^2y^2\,dz = 0$$

**Solution** If we divide both sides of this equation by $x^2y^2z^2$, we have

$$\frac{a^2}{x^2}\,dx + \frac{b^2}{y^2}\,dy + \frac{c^2}{z^2}\,dz = 0$$

On integration, we obtain the integral surfaces as

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = k$$

where k is a constant.

**Example 4** Solve the equation

$$a^2yz\,dx + b^2zx\,dy + c^2xy\,dz = 0$$

**Solution** It is easy to verify the condition of integrability. Dividing the given equation by xyz, we get

$$\frac{a^2}{x}\,dx + \frac{b^2}{y}\,dy + \frac{c^2}{z}\,dz = 0,$$

which on integration gives

$$x^{a^2}\, y^{b^2}\, z^{c^2} = k$$

as the required solution.

**(c) One Variable Separable** It may happen that the equation is of the form

$$P(x, y)\,dx + Q(x, y)\,dy + R(z)\,dz = 0 \qquad (1)$$

For this equation

$$\mathbf{X} = \{P(x, y), Q(x, y), R(z)\}$$

and a simple calculation shows that

$$\text{curl } \mathbf{X} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

so that the condition for integrability, **X**. curl **X** = 0, implies that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In other words, P dx + Q dy is an exact differential, du say, and equation (1) becomes as

$$du + R(z)\,dz = 0$$

with solution

$$u(x, y) + \int R(z)\,dz = c$$

**Example 5** Verify that the equation

$$x(y^2 - a^2)\,dx + y(x^2 - z^2)\,dy - z(y^2 - a^2)\,dz = 0$$

is integrable and solve it.

**Solution** If we divide throughout by $(y^2 - a^2)(x^2 - z^2)$, we see that the equation assumes the form

$$\frac{x\,dx - z\,dz}{x^2 - z^2} + \frac{y\,dy}{y^2 - a^2} = 0$$

showing that it is separable in y. By the above argument it is therefore integrable if

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

which is readily shown to be true. To determine the solution of the equation we note that it is

$$\frac{1}{2} d \log (x^2 - z^2) + \frac{1}{2} d \log (y^2 - a^2) = 0$$

so that the solution is $(x^2 - z^2)(y^2 - a^2) = c$

where c is a constant.

**(d) Homogeneous Equations** The equation

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0 \qquad (2)$$

is said to be homogeneous if the functions P, Q, R are homogeneous in x, y, z of the same degree n. To solve such an equation we make the substitutions

$$y = ux, \quad z = vx. \qquad (3)$$

Substituting (3) into (2), we see that equation (2) takes the form

$P(1,u,v)\,dx + Q(1,u,v)\,(u\,dx + x\,du) + R(1,u,v)\,(x\,dv + v\,dx) = 0$, a factor $x^n$ cancelling out. If we now write

$$A(u,v) = \frac{Q(1,u,v)}{P(1,u,v) + uQ(1,u,v) + vR(1,u,v)}$$

$$B(u,v) = \frac{R(1,u,v)}{P(1,u,v) + uQ(1,u,v) + vR(1,u,v)}$$

we find that this equation is of the form

$$\frac{dx}{x} + A(u,v)du + B(u,v)dv = 0$$

and can be solved by method (c).

**Example 6** Verify that the equation

$$yz\,(y + z)\,dx + xz\,(x + z)\,dy + xy\,(x + y)\,dz = 0$$

is integrable and find its solution.

**Solution**

It is easy to show that the condition of integrability is satisfied. Making the substitution y = ux, z = vx, we find that the equation satisfied by x, u, v is

$$uv\,(u + v)\,dx + v\,(v + 1)\,(u\,dx + x\,du) + u\,(u + 1)\,(v\,dx + x\,dv) = 0$$

which reduces to

$$\frac{dx}{x} + \frac{v(v+1)\,du + u(u+1)dv}{2uv\,(1+u+v)} = 0.$$

Splitting the factors of du and dv into partial fractions,

$$2\frac{dx}{x} + \left\{\frac{1}{u} - \frac{1}{1+u+v}\right\}du + \left\{\frac{1}{v} - \frac{1}{1+u+v}\right\}dv = 0$$

or,

$$2\frac{dx}{x} + \frac{du}{u} + \frac{dv}{v} - \frac{d\,(1 + u + v)}{1 + u + v} = 0.$$

The solution of this equation is obviously

$$x^2uv = c(\,1+ u + v),$$

where c is a constant. In terms of original variables, we see that the solution of the given equation is

$$xyz = c(x + y + z).$$

**Example 7** Solve the equation

$$(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - yx)dz = 0$$

**Solution** Here

$$P = y^2 + yz, \qquad Q = z^2 + zx, \qquad R = y^2 - yx$$

and $\quad \mathbf{X} \cdot \text{curl}\, \mathbf{X} = 0$ i.e. condition of integrability is satisfied.

Substituting $y = ux,\ z = vx$, we obtain

$$\frac{dx}{x} + \frac{(v^2 + v)\,du + (u^2 - u)\,dv}{(1 + v)(u + v)\,u} = 0.$$

Splitting the factors of du and dv into partial fractions, we see that this is equivalent to

$$\frac{dx}{x} + \frac{du}{u} + \frac{dv}{v + 1} - \frac{du + dv}{u + v} = 0,$$

which on integration, gives the solution as

$$\frac{x\,u(v + 1)}{u + v} = C \quad \text{i.e.}$$

$$\frac{y(z + x)}{z + y} = C.$$

**(e) Natani's Method** In this method, we assume one of the variables as constant. Let us treat the variable z as constant, so that the resulting differential equation

$Pdx + Qdy = 0$, is easily integrable.

Let its integral be given by

$\phi(x, y, z) = c_1$, $c_1$ may involve z. (1)

The solution of equation $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ is then of the form

$$\Phi(\phi, z) = c_2 \tag{2}$$

where $c_2$ is a constant, and we can express this solution in the form

$$\phi(x, y, z) = \psi(z)$$

where $\psi$ is a function of z alone. To determine the function $\psi(z)$ we observe that, if we give the variable x a fixed value, $\alpha$ say, then

$$\phi(\alpha, y, z) = \psi(z) \tag{3}$$

is a solution of the differential equation

$$Q(\alpha, y, z)\, dy + R(\alpha, y, z)\, dz = 0 \qquad (4)$$

This equation will always have a solution of the form

$$K(y, z) = c \qquad (5)$$

by using the methods of the theory of first order differential equations.

Since equations (3) and (5) represent general solutions of the same differential equation (4), they must be equivalent. Therefore, if we eliminate the variable y between (3) and (5), we obtain an expression for the function $\psi(z)$. Substituting this expression in equation (3), we obtain the solution of the Pfaffian differential equation.

The method is often simplified by choosing a value for $\alpha$, such as 0 or 1, which makes the solving of differential equation (4) easy. It is necessary to verify in advance that the equation is integrable before using Natani's method.

**Example 8** Verify that the equation

$$z(z + y^2)\, dx + z\,(z + x^2)\, dy - xy\,(x + y)\, dz = 0$$

is integrable and find its primitive.

**Solution** For this equation

$$\mathbf{X} = \{z(z + y^2),\ z(z + x^2),\ - xy\,(x + y)\}$$

$$\text{Curl } \mathbf{X} = 2\,(-x^2 - xy - z,\quad y^2 + xy + z,\quad zx - zy)$$

and **X.** curl **X** = 0, showing that the equation is integrable. It is probably simplest to take dy = 0 in Natani's method. The equation then becomes

$$\left\{\frac{1}{x} - \frac{1}{x+y}\right\} dx + \left\{\frac{1}{z+y^2} - \frac{1}{z}\right\} dz = 0$$

showing that it has the solution

$$\frac{x(y^2 + z)}{z(x+y)} = f(y) \qquad (1)$$

Now, let z = 1 in the original equation, it reduces to the simple form

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0 \qquad (2)$$

This can be easily integrated to

$$\tan^{-1}x \; + \; \tan^{-1}y = \text{constant} = \tan^{-1}\frac{1}{c} \text{ (say)}$$

Now

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

we see that the solution of equation (2) is

$$\frac{1-xy}{x+y} = c \qquad (3)$$

This solution must be the form assumed by (1) in the case z = 1; or (3) must be equivalent to the relation

$$\frac{x(y^2+1)}{x+y} = f(y) \qquad (4)$$

Eliminating x between equations (3) and (4), we find that

$$f(y) = 1 - cy.$$

Substituting this expression in equation (1), we find that the solution of the equation is

$$x(y^2 + z) = z(x + y)(1 - cy).$$

**Example 9** Solve by checking the integrability

$$y(1 + z^2)\, dx - x\,(1 + z^2)\, dy + (x^2 + y^2)\, dz = 0$$

**Solution** Divide throughout by $(1 + z^2)$ and treating z as constant, so that we obtain

$$y\, dx - x dy = 0$$

This implies $$\frac{x}{y} = f(z) \qquad (1)$$

Putting y = 1, we obtain

$$x = f(z) \qquad (2)$$

and $$(1 + z^2)\, dx + (1 + x^2)\, dz = 0$$

which on integration gives

$$(x + z)/ (1 - xz) = c \tag{3}$$

Elimination of x from (2) and (3) gives

$$f(z) = \frac{Cz}{1 + Cz}$$

Thus from (1), the primitive is

$$\frac{x}{y} = \frac{Cz}{1 + Cz}.$$

**Problems**

Verify that the following equations are integrable and find their primitives:

1. $2y(a - x)\,dx + [z - y^2 + (a - x)^2]\,dy - y\,dz = 0$
2. $zy^2dx + zx^2dy - x^2y^2dz = 0$
3. $(y^2 + yz + z^2)\,dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$
4. $yz\,dx + xz\,dy + xy\,dz = 0$
5. $(1 + yz)\,dx + x\,(z - x)\,dy - (1 + xy)\,dz = 0$
6. $y\,(x + 4)\,(y + z)\,dx - x\,(y + 3z)\,dy + 2xy\,dz = 0$
7. $yz\,dx + (x^2y - zx)\,dy + (x^2z - xy)\,dz = 0$
8. $2yz\,dx - 2xz\,dy - (x^2 - y^2)\,(z - 1)\,dz = 0$
9. $y(x + y)\,(z + y)\,dx - x\,(y + z^2)\,dy + 2xy\,dz = 0$
10. $(2xyz + z^2)\,dx + x^2zdy + (xz + 1)\,dz = 0$

**Differential Inequations**

The most important techniques in the theory of differential equations involve the ‘integration’ of differential inequalities. The following integral inequality known as ‘Gronwall’s inequality’ is fundamental in the study of ordinary differential equations. It is one of the simplest and most useful result involving an integral inequation.

In the following r, u, v, U, V are scalars while y, z, f, g are n–dimensional vectors.

**Theorem 5.3 (Gronwall's Inequality)**

**Statement** Let u(t) and v(t) be two non-negative continuous functions defined on closed interval [a,b]. Let c be any non-negative constant. Then the inequality $v(t) \le c + \int_a^t v(s)u(s)ds, \quad \text{for a } \le t \le b$

implies the inequality

$$v(t) \le c \exp\left[\int_a^t u(s)ds\right] \qquad for\ a \le t \le b$$

and, in particular, if c = 0, then v(t) ≅ 0.

**Proof** Case I : When c > 0.

$$\text{Let} \quad V(t) = c + \int_a^t v(s)u(s)ds. \tag{1}$$

Then $$V(a) = c \tag{2}$$

and by hypothesis

$$v(t) \le V(t), \tag{3}$$

and

$$V(t) \ge c > 0 \quad \text{on } [a, b], \tag{4}$$

as u and v are non-negative functions. Also from (1), we have, on [a, b],

$$\begin{aligned} V'(t) &= v(t)\,u(t) \\ &\le V(t)\,u(t), \end{aligned}$$

using (3). This implies, using (4)

$$\frac{V'(t)}{V(t)} \le u(t). \tag{5}$$

Integrating (5) over [a, t], we get

$$\left[\log V(t)\right]_a^t \leq \int_a^t u(s)ds$$

which gives

$$\log V(t) - \log V(a) \leq \int_a^t u(s)ds$$

Using V(a) = c, we have

$$\log V(t) - \log c \leq \int_a^t u(s)ds .$$

We can rewrite the above inequality as

$$\log V(t) \leq \log c + \log\left[\exp \int_a^t u(s)ds\right].$$

Taking exponential on both sides of the above, we get

$$V(t) \leq c \exp\left[\int_a^t u(s)ds\right]. \qquad (6)$$

Replacing the left hand side of (6) by lesser term, as in (3), we get

$$v(t) \leq V(t) \leq c \exp\left[\int_a^t u(s)ds\right]$$

or

$$v(t) \leq c \exp\left[\int_a^t u(s)ds\right], \qquad (7)$$

which is the Gronwall's inequality.

**Case II** When c = 0. Letting $c \to 0^+$ in (7) we get the desired result.

**Restatement** Another form of Gronwall's inequality is given below.

Let r(t) be continuous for $|t - t_0| \leq \delta$ and satisfies the inequalities

$$0 \leq r(t) \leq \varepsilon + \delta\left|\int_{t_0}^t r(s)ds\right|$$

for some non-negative constants ε and δ. Then

$0 \leq r(t) \leq \varepsilon \exp\{\delta| t- t_0|\}$.

**Proof** On taking c = ε, $t_0$ = a and u(t) = δ in above theorem, the result follows immediately.

**Cor. 1** Let f(t, y) satisfies a Lipschitz condition with constant k for y ∈ D and $|t - t_0| \leqslant \delta$. Let y(t) and z(t) be solutions of problem

$$\frac{dy}{dt} = f(t, y)$$

for $|t - t_0| \leqslant \delta$ such that

$$y(t_0) = y_0, \qquad z(t_0) = z_0, \tag{1}$$

where $y_0, z_0 \in D$. Then

$$|z(t) - y(t)| \leq |z_0 - y_0| \exp\{k|t - t_0|\}.$$

**Proof** Given that y(t) and z(t) are solutions of (1), the corresponding integral equations are:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds, \qquad z(t) = z_0 + \int_{t_0}^{t} f(s, z(s))ds, \tag{2}$$

Subtracting, we see that

$$z(t) - y(t) = z_0 - y_0 + \int_{t_0}^{t} [f(s, z(s)) - f(s, y(s))]ds. \tag{3}$$

Taking the norm of both sides and applying the Lipschitz condition, it follows that

$$|z(t) - y(t)| \leq |z_0 - y_0| + \left| \int_{t_0}^{t} [f(s, z(s)) - f(s, y(s))]ds \right|$$

$$\leq |z_0 - y_0| + \int_{t_0}^{t} |f(s, z(s)) - f(s, y(s))| ds$$

$$\Rightarrow \qquad 0 \leq |z(t) - y(t)| \leq |z_0 - y_0| + \int_{t_0}^{t} k|z(s) - y(s)| ds$$

From Gronwall's inequality, with $r(t) = |z(t) - y(t)|$, $\varepsilon = |z_0 - y_0|$ and $\delta = k$, the result follows immediately.

**Cor. 2** Let f(t,y) satisfies a Lipschitz condition for $y \in D$ and $|t - t_0| \le \delta$. Then the initial value problem has a unique solution, that is, there is at most one continuous function y(t) which satisfies

$$\frac{dy}{dt} = f(t, y),$$

$$y(t_0) = y_0$$

**Proof** Putting $z_0 = y_0$ in Cor. 1, we see that $z(t) = y(t)$ for all $|t - t_0| \leqslant \delta$, which shows the uniqueness of the solution of initial value problem, whenever f(t, y) satisfies a Lipzchitz condition. Hence the result.

**The Comparison Theorems**

Since most differential equations can not be solved in terms of elementary functions, it is important to compare the unknown solutions of one differential equation with the known solutions of another. i.e. compare functions satisfying the differential inequality

$$f'(x) \le F(x, f(x)),$$

with exact solution of the differential equation

$$y' = F(x, y),$$

which is a normal first order differential equation. The following theorems give such comparisons.

**Theorem 5.4** Let f(t, y) satisfies a Lipschitz condition for $t \ge a$. If the function $u = u(t)$ satisfies the differential inequality.

$$\frac{dy}{dt} \le f(t, y) \qquad \text{for } t \ge a \tag{1}$$

and $v = v(t)$ is a solution of differential equation

$$\frac{dy}{dt} = f(t, y) \tag{2}$$

satisfying the initial conditions

$$u(a) = v(a) = c_0 \tag{3}$$

then $u(t) \leqslant v(t)$ for $t \geq a$.

**Proof** If possible, suppose that

$$u(t_1) > v(t_1) \tag{4}$$

for some $t_1$ in the given interval. Let $t_0$ be the largest t in the interval $[a, t_1]$ such that

$$u(t) \leqslant v(t).$$

Then

$$u(t_0) = v(t_0).$$

Let $\sigma(t) = u(t) - v(t)$ . (5)

Then $\sigma(t_0) = 0, \quad \sigma(t_1) > 0$ (6)

and $\sigma(t) \geq 0$ for $[t_0, t_1]$. (7)

Also for $t_0 \leqslant t \leqslant t_1$,

$$\begin{aligned} \sigma'(t) &= u'(t) - v'(t) \\ &\leqslant f(t,u(t)) - f(t, v(t)), \quad \text{using (1) \& (2).} \\ &\leqslant K\,|u(t) - v(t)|, \qquad (8) \\ &= K\,\sigma(t), \end{aligned}$$

where K is the Lipschitz constant for the function f.

Multiplying both sides of (8) by $e^{-Kt,}$ we write

$$\begin{aligned} 0 &\geq e^{-Kt}\{\sigma'(t) - K\sigma(t)\} \\ &= \frac{d}{dt}\left\{\sigma(t)\, e^{-Kt}\right\}. \end{aligned}$$

This implies

$$\frac{d}{dt}\left\{\sigma(t)\, e^{-Kt}\right\} \leq 0 \quad \text{in } [t_0, t_1] \tag{9}$$

So, $\sigma(t).\ e^{-Kt}$ is a decreasing function for $[t_0, t_1]$.

Therefore $\quad \sigma(t)\ e^{-Kt} \leqslant \sigma(t_0)\ e^{-Kt_0}$ for all t in $[t_0, t_1]$

$$\Rightarrow \sigma(t) \le \sigma(t_0)\ e^{K(t - t_0)}$$

$$\Rightarrow \sigma(t) \le 0 \qquad \text{for all t in } [t_0, t_1]\text{, using (6).}$$

$$\Rightarrow \sigma(t) \text{ vanishes identically in } [t_0, t_1].$$

This contradicts the assumption that $\sigma(t_1) > 0$. Hence, we conclude that

$$u(t) \leqslant v(t)$$

for all t in the given interval.

This completes the proof

**Theorem 5.5 (Comparison Theorem)**

Let u = u(t) and v = v(t) be solutions of differential equations

$$\frac{dy}{dt} = U(t, y), \qquad \frac{dz}{dt} = V(t, z) \tag{1}$$

respectively, where

$$U(t, y) \leqslant V(t, y) \tag{2}$$

in the strip $a \leqslant t \leqslant b$ and U or V satisfies a Lipschtiz condition, and

$$u(a) = v(a). \tag{3}$$

Then $\quad u(t) \leqslant v(t) \qquad$ for all $t \in [a, b]$.

(4)

**Proof** (i) Let V satisfies a Lipschitz condition. Since

$$\frac{dy}{dt} = U(t, y) \le V(t, y),$$

the functions u(t) and v(t) satisfy the conditions of theorem 5.4 with V in place of f. Therefore, the inequality (4) follows immediately.

(ii) If U satisfies a Lipschitz condition, the functions

$$f(t) = -u(t), \qquad g(t) = -v(t) \tag{5}$$

satisfy the differential equations

$$\frac{du}{dt} = -U(t, -u),$$

and

$$\frac{dv}{dt} = -V(t, -v),$$

$$\leqslant -U(t, -v), \text{ using (2)} \tag{6}$$

As $g(t)$ is a solution of

$$\frac{dv}{dt} \leq -U(t, -v)$$

and $f(t)$ is a solution of

$$\frac{du}{dt} = -U(t, -u).$$

Therefore, by Theorem 5.4, we obtain the inequality

$$g(t) \leqslant f(t) \qquad \text{for } t \geq a$$

$$\Rightarrow -g(t) \geq -f(t) \qquad \text{for } t \geq a$$

$$\Rightarrow v(t) \geq u(t) \qquad \text{for } t \geq a$$

$$\Rightarrow u(t) \leqslant v(t) \qquad \text{for } t \geq a$$

This completes the proof.

**Remark** The inequality $u(t) \leqslant v(t)$ in this comparison theorem 5.5 can often be replaced by a strict inequality.

**Corollary 1** In theorem 5.5, for any $t_1 > a$, either

$$u(t_1) < v(t_1) \quad \text{or} \quad u(t) \equiv v(t) \qquad \text{for } a \leqslant t \leqslant t_1$$

**Proof** By theorem 5.5, $u(t) \leqslant v(t)$ for all $t > a$. (1)

Let $t_1 > a$ be any value of t.

If $u(t_1)$ is not less than $v(t_1)$, then $u(t_1) = v(t_1)$. (2)

Then, either u and v are identically equal for $a \leqslant t \leqslant t_1$, or else

$$u(t_0) < v(t_0) \quad (3)$$

for some $t_0$ in the interval $(a, t_1)$.

Let $\sigma_1(t) = v(t) - u(t)$, for $t \in [a, t_1]$. (4)

Then $\sigma_1(t_0) > 0$, (5)

and by theorem 5.5,

$$u(t) \leqslant v(t) \qquad \text{for } t \in [a, t_1]$$

$$\Rightarrow \sigma_1(t) \geq 0. \qquad \text{for } t \in [a, t_1]. \quad (6)$$

Further, for $t \in [a, t_1]$

$$\begin{aligned} \sigma_1'(t) &= v'(t) - u'(t) \\ &= V(t, v(t)) - U(t, u(t)) \\ &\geq V(t, v(t)) - V(t, u(t)) \ (\because \ U \leq V \text{ given}) \\ &\geq -K\{v(t) - u(t)\} \end{aligned}$$

$$\Rightarrow \quad \sigma_1'(t) \geq -K\sigma_1 \quad (7)$$

$$\Rightarrow \quad (\sigma_1' + K\sigma_1) \geq 0.$$

Hence $\left\{e^{Kt} \cdot \sigma_1(t)\right\}' = e^{Kt}\left\{\sigma_1'(t) + K\sigma_1(t)\right\} \geq 0,$

using (7) for $t \in [a, t_1]$. This shows that the function $\phi(t) = e^{kt}\sigma_1(t)$ is an increasing function on the interval $[a, t_1]$. So

$$\phi(t) \geq \phi(t_0) \qquad \text{for } t \in [t_0, t_1]$$

$$\Rightarrow e^{kt}\sigma_1(t) \geq e^{Kt_0}\sigma_1(t_0)$$

$$\Rightarrow \sigma_1(t) \geq \sigma_1(t_0)\, e^{-K(t-t_0)} > 0$$

$$\Rightarrow \sigma_1(t) > 0 \text{ in } [t_0, t_1]$$

$\Rightarrow v(t) - u(t) > 0$ in $[t_0, t_1]$

$\Rightarrow v(t) > u(t)$ in $[t_0, t_1] \subseteq [a, b]$,

which is a contradiction. Hence, u and v are identical for $a \leqslant t \leqslant t_1$. This completes the proof.

**Cor 2** In theorem 5.5, assume that U as well as V, satisfies a Lipschitz conditions and, instead of u(a) = v(a), that u(a) < v(a).

Then

$$u(t) < v(t) \qquad \text{for } t > a.$$

**Proof** The proof will be by contradiction.

If we had $u(t) \geq v(t)$ for some $t > a$, there would be a first $t = t_1 > a$, where

$$u(t) \geq v(t). \tag{1}$$

We define two functions

$$y = \phi(t) = u(-t),$$

$$z = \psi(t) = v(-t). \tag{2}$$

Then $\phi$ and $\psi$ satisfy the differential equations

$$\frac{dy}{dt} = -U(-t, y), \qquad \frac{dz}{dt} = -V(-t, z) \tag{3}$$

and the respective initial conditions,

$$\phi(-t_1) = \psi(-t_1). \tag{4}$$

Since

$$-U(-t, y) \geq -V(-t, y),$$

we can apply theorem 5.5 in the interval $[-t_1, -a]$, knowing that the function $-U(-t, y)$ satisfies a Lipschitz condition. So, by theorem 5.5, we conclude that

$$\phi(t) \geq \psi(t) \text{ in } [-t_1, -a]$$

$\Rightarrow \phi(-a) \geq \psi(-a)$

$\Rightarrow u(a) \geq v(a),$ (5)

which is a contradiction (to the given assumption that u(a) < v(a)). Therefore, the assumption that u(t) ≥ v(t) is wrong. Thus,

$u(t) < v(t) \qquad \text{for } t > a.$

Hence the result.

**Summary**

In this chapter, after proving some preliminary results about Pfaffian differential equations, the criterion for determining the condition of integrability of such equations is explained. Some methods of solutions of Pfaffian differential equations in three variables are also discussed. Some theorems for differential inequations along with Gronwall inequality are presented at the end of chapter.

**Keywords**

Total differential equations, Integrability, Comparison theorems, Gronwall inequality.

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## Chapter-6

# STURM THEORY AND RICCATI EQUATIONS

### Objectives

In this chapter, the reader is made familiar with Sturm theory, Sturm separation and comparison theorems describing the location of roots of homogeneous second order linear differential equations. Riccati Equations and Pruffer transformation are also important parts of this chapter.

### Introduction

We know that a second order differential equation can be expressed in the form

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right]+Q(t)x=0, \qquad (1)$$

(called self adjoint form) where P(t) has continuous derivate, Q(t) is continuous and P(t) > 0 on a ⩽ t ⩽ b. We shall need a well-known theorem on point sets known as the Bolzano-Weierstrass theorem. Suppose E is a set of points on the t axis. A point $t_0$ is called a limit point of E if there exists a sequence of distinct points $t_1$, $t_2$, $t_3$… of E such that $\lim_{n\to\infty} t_n = t_0$. The Bolzano-Weierstrass theorem states that every bounded infinite set E has at least one limit point.

### Theorem A

If the function ϕ is a solution of the homogeneous equation

$$a_0(t)\frac{d^nx}{dt^n}+a_1(t)\frac{d^{n-1}x}{dt^{n-1}}+......+a_n(t)x=0$$

Such that

$$\phi(t_0) = 0, \quad \phi'(t_0) = 0, \ \ldots\ldots, \ \phi^{(n-1)}(t_0) = 0,$$

where $t_0$ is a point of an interval $a \leqslant t \leqslant b$ on which the co-efficients $a_0, a_1, \ldots, a_n$ are all continuous and $a_0(t) \neq 0$. Then $\phi(t) = 0$ for all t such that $a \leqslant t \leqslant b$.

**Theorem B**

Let $f_1, f_2, \ldots, f_n$ be n solutions of the homogeneous linear differential equation $a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots\ldots + a_n(t)x = 0$ on [a, b]. Then these n solutions are linearly dependent on [a, b] iff $W(f_1, f_2, .., f_n)(t) = 0,\ \forall\ t \in [a,\ b]$, where $W(f_1, f_2, .., f_n)(t)$ denotes the Wronskian of solutions $f_1, f_2, \ldots, f_n$ at t.

**Sturm Theory**

**Theorem 6.1**

1. Let f be a solution of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q(t)x = 0,$$

having first derivative f ' on $a \leqslant t \leqslant b$, (1)

2. and let f has an infinite number of zeros on $a \le t \le b$.

**Conclusion** Then $f(t) = 0$ for all t on $a \le t \le b$.

**Proof** Since f has an infinite number of zeros on [a, b], by the Bolzano-Weierstrass theorem the set of zeros has a limit point $t_0 \in [a, b]$. Thus there exists a sequence $\{t_n\}$ of zeros which converges to $t_0$ (where $t_n \neq t_0$). Since f is continuous $\lim_{t \to t_0} f(t) = f(t_0)$, where $t \to t_0$ through any sequence of points on [a, b]. Let $t \to t_0$ through the sequence of zeros $\{t_n\}$. Then

$$\lim_{t \to t_0} f(t) = 0 = f(t_0).$$

Now since $f'(t_0)$ exists,

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0},$$

where $t \to t_0$ through the sequence $\{t_n\}$. For such points

$$\frac{f(t)-f(t_0)}{t-t_0}=0 \text{ and thus } f'(t_0\} = 0$$

Thus f is a solution of Equation (1) such that $f(t_0) = f'(t_0) = 0$. Hence, using theorem A

$$f(t) = 0 \quad \text{for all t on} \quad a \leqslant t \leqslant b.$$

**Theorem 6.2 Abel's Formula**

**Hypothesis** Let f and g be any two solutions of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right]+Q(t)x=0, \tag{1}$$

on the interval $a \leqslant t \leqslant b$.

**Conclusion** Then for all t on $a \leqslant t \leqslant b$,

$$P(t)\,[f(t)g'(t) - f'(t)g(t)] = k, \tag{2}$$

where k is a constant.

**Proof** Since f and g are solutions of (1) on $a \leqslant t \leqslant b$, we have

$$\frac{d}{dt}\left[P(t)f'(t)\right]+Q(t)f(t)=0 \tag{3}$$

and

$$\frac{d}{dt}\left[P(t)g'(t)\right]+Q(t)g(t)=0 \tag{4}$$

for all $t \in [a, b]$. Multiply (3) by $-g(t)$ and (4) by $f(t)$ and adding

$$f(t)\frac{d}{dt}\left[P(t)g'(t)\right]-g(t)\frac{d}{dt}\left[P(t)f'(t)\right]=0 \tag{5}$$

Integrating (5) from a to t, we obtain

$$f(s)P(s)g'(s)\Big|_a^t-\int_a^t P(s)g'(s)f'(s)ds-g(s)f'(s)P(s)\Big|_a^t+\int_a^t P(s)f'(s)g'(s)ds=0,$$

or

$P(t)[f(t)g'(t) - f'(t)g(t)] = P(a)[f(a)g'(a) - f'(a)g(a)] = k$ (constant) and thus we have Abel's formula (2).

**Theorem 6.3**

**A. Hypothesis** Let f and g be two solutions of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right]+Q(t)x=0 \qquad (1)$$

such that f and g have a common zero on $a \leqslant t \leqslant b$.

**Conclusion** Then f and g are linearly dependent on $a \leqslant t \leqslant b$.

**B. Hypothesis** Let f and g be nontrivial linearly dependent solutions of Equation (1) on $a \leqslant t \leqslant b$, and suppose $f(t_0) = 0$, where $t_0$ is such that $a \leqslant t_0 \leqslant b$.

**Conclusion** Then $g(t_0) = 0$

**Proof**

**A.** Abel's formula is

$$P(t)[f(t)g'(t) - f'(t)g(t)] = k \qquad (2)$$

Let $t_0 \in [a, b]$ be the common zero of f and g, then $f(t_0)= g(t_0) = 0$. Letting $t = t_0$ in the Abel's formula, we obtain $k = 0$. Thus

$$P(t)[f(t)g'(t) - f'(t)g(t)] = 0 \text{ for } t \in [a, b].$$

Since we have assumed throughout that $P(t) > 0$ on $a \leqslant t \leqslant b$, the quantity in brackets above must be zero for all t on $a \leqslant t \leqslant b$. But this quantity is $W(f, g)(t)$. Thus by using the theorem B, the solution f and g are linearly dependent on $a \leqslant t \leqslant b$.

**B.** Since f and g are linearly dependent on $a \leqslant t \leqslant b$, there exist constants $c_1$ and $c_2$ not both zero, such that

$$c_1f(t) + c_2g(t) = 0, \qquad (3)$$

for all t on $a \leqslant t \leqslant b$. Now given that neither f nor g is zero for all t on $a \leqslant t \leqslant b$. If $c_1 =0$, then $c_2\, g(t) = 0$ for all t on $a \leqslant t \leqslant b$. Since g is not zero for all t on [a, b], we must have $c_2 = 0$, which is a contradiction. Thus $c_1 \neq 0$, and likewise $c_2 \neq 0$. Thus neither $c_1$ nor $c_2$ in (3) is zero. Since $f(t_0) = 0$, letting $t = t_0$ in (3) we have $c_2\, g(t_0) = 0$. Thus $g(t_0) = 0$.

**Example 1**

The equation

$$\frac{d^2x}{dt^2} + x = 0$$

is of the type (1) is above theorem, where P(t) = Q(t) = 1 on every interval $a \leqslant t \leqslant b$. The linearly dependent solutions are A sint and B sint, which have the common zeros $t = \pm n\pi$ ( n= 0, 1, 2…..) and no other zeros.

**The Separation and Comparison Theorems**

**Theorem 6.4 Sturm Separation Theorem**

**Hypothesis** Let f and g be real linearly independent solutions of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q(t)x = 0 \tag{1}$$

on the interval $a \leqslant t \leqslant b$.

**Conclusion**

Between any two consecutive zeros of f (or g) there is precisely one zero of g (or f).

**Proof** Let $t_1$ and $t_2$ be two consecutive zeros of f on [a, b]. Then by Theorem 6.3, Part A, $g(t_1) \neq 0$ and $g(t_2) \neq 0$. Now assume that g has no zero in the open interval $t_1 < t < t_2$. Then since the solutions f and g have continuous derivatives on [a, b], the quotient f/g has a continuous derivative on the interval $t_1 \leqslant t \leqslant t_2$. Further, f(t)/g(t) is zero at the endpoints of this interval. Thus by Rolle's theorem there exists $\xi$, where $t_1 < \xi < t_2$, such that

$$\frac{d}{dt}\left[\frac{f(t)}{g(t)}\right]_{t=\xi} = 0$$

But

$$\frac{d}{dt}\left[\frac{f(t)}{g(t)}\right] = \frac{W(g,f)(t)}{[g(t)]^2};$$

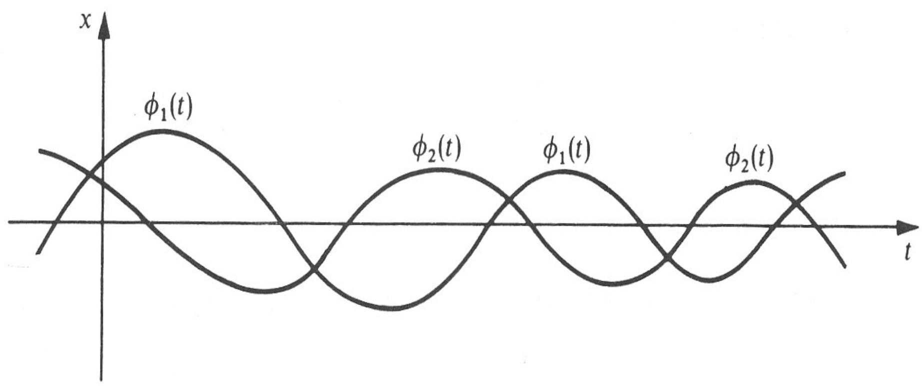
and since f and g are linearly independent on $a \leqslant t \leqslant b$.

$\therefore$ W(f,g) (t) $\neq 0$ on $t_1 < t < t_2$ , i.e.

$$\frac{d}{dt}\left[\frac{f(t)}{g(t)}\right] \neq 0 \qquad \text{on } t_1 < t < t_2.$$

This contradiction shows that g has at least one zero in $t_1 < t < t_2$ .

Now suppose g has more than one zero in $t_1 < t < t_2$, and let $t_3$ and $t_4$ be two such consecutive zeros of g. Then interchanging f and g and using the same arguments we can show that f must have at least one zero $t_5$ in the open interval $t_3 < t < t_4$. Then $t_1 < t_5 < t_2$ and so $t_1$ and $t_2$ would not be consecutive zeros of f, which is a contradiction to our assumption concerning $t_1$ and $t_2$. Thus g has precisely one zero in the open interval $t_1 < t < t_2$.




**Figure**

We may restate Theorem 6.4 in the following form: The zeros of one of two real linearly independent solutions of Equation (1) separate the zeros of the other solution (see figure).

**Example 2**

We have already observed that the equation

$$\frac{d^2x}{dt^2} + x = 0$$

is of the type (1) in above theorem. The functions f and g defined, respectively, by f(t) = sin t and g(t) = cos t are linearly independent solutions of this equation. Between any two consecutive zeros of one of these two linearly independent solutions, there is precisely one zero of the other solution. We know that zeros of f(t) are t = nπ, n = 0, ±1, ±2, …. and zeros of g(t) are t = (2n+1) $\frac{\pi}{2}$, n = 0, ±1, ±2,.... are separated by each other.

**Theorem 6.5 Sturm's Fundamental Comparison Theorem**

**Hypothesis** On the interval a ⩽ t ⩽ b,

1. Let $\phi_1$ be a real solution of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q_1(t)x = 0 \qquad (1)$$

2. Let $\phi_2$ be a real solution of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q_2(t)x = 0 \qquad (2)$$

3. Let P has a continuous derivative and be such that P(t) > 0 and let $Q_1$ and $Q_2$ be continuous and such that $Q_2(t) > Q_1(t)$.

**Conclusion** If $t_1$ and $t_2$ are successive zeros of $\phi_1$ on [a, b], then $\phi_2$ has at least one zero at some point of the open interval $t_1 < t < t_2$.

**Proof**

Assume that $\phi_2$ does not have a zero on the open interval $t_1 < t < t_2$. Then without loss in generality we can assume that $\phi_1(t) > 0$ and $\phi_2(t) > 0$ on $t_1 < t < t_2$. By hypothesis, we have

$$\frac{d}{dt}\left[P(t)\phi_1'(t)\right] + Q_1(t)\phi_1(t) = 0 \qquad (3)$$

$$\frac{d}{dt}\left[P(t)\phi_2'(t)\right]+Q_2(t)\phi_2(t)=0 \tag{4}$$

for all t∈[a, b]. Multiply (3) by $\phi_2(t)$ and (4) by $\phi_1(t)$ and subtract to obtain

$$\phi_2(t)\frac{d}{dt}\left[P(t)\phi_1'(t)\right]-\phi_1(t)\frac{d}{dt}\left[P(t)\phi_2'(t)\right]=[Q_2(t)-Q_1(t)]\ \phi_1(t)\phi_2(t). \tag{5}$$

Since

$$\phi_2(t)\frac{d}{dt}\left[P(t)\phi_1'(t)\right]-\phi_1(t)\frac{d}{dt}\left[P(t)\phi_2'(t)\right]=\frac{d}{dt}\{P(t)[\phi_1'(t)\phi_2(t)-\phi_1(t)\phi_2'(t)]\}.$$

Then (5) reduces to

$$\frac{d}{dt}\{P(t)[\phi_1'(t)\phi_2(t)-\phi_1(t)\phi_2'(t)]\} = [\mathrm{Q}_2(\mathrm{t}) - \mathrm{Q}_1(\mathrm{t})]\phi_1(t)\phi_2(t).$$

Integrating it from $t_1$ to $t_2$, we obtain

$$\int_{t_1}^{t_2}\frac{d}{dt}\{P(t)[\phi_1'(t)\phi_2(t)-\phi_1(t)\phi_2'(t)]\}dt=\int_{t_1}^{t_2}[Q_2(t)-Q_1(t)]\phi_1(t)\phi_2(t)dt$$

or

$$P(t)[\phi_1'(t)\phi_2(t)-\phi_1(t)\phi_2'(t)\Big|_{t_1}^{t_2}=\int_{t_1}^{t_2}[Q_2(t)-Q_1(t)]\phi_1(t)\phi_2(t)dt. \tag{6}$$

Since $\phi_1(t_1)=\phi_1(t_2)=0,$ the equality (6) becomes

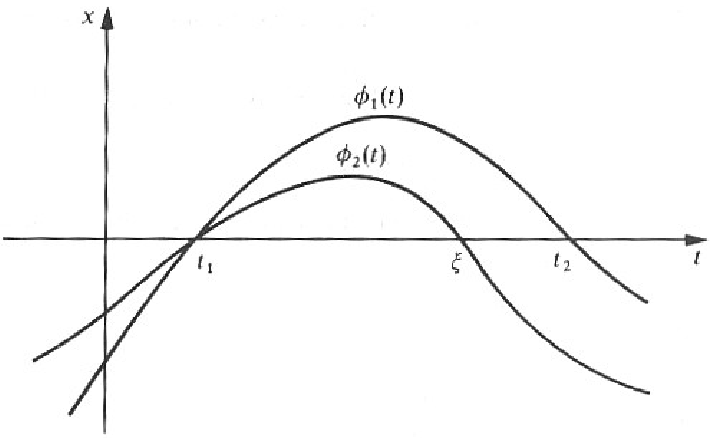
$$P(t_2)\phi_1'(t_2)\phi_2(t_2)-P(t_1)\phi_1'(t_1)\phi_2(t_1)=\int_{t_1}^{t_2}[Q_2(t)-Q_1(t)]\phi_1(t)\phi_2(t)dt. \tag{7}$$

By hypothesis, $P(t_2) > 0$. Since $\phi_1(t_2) = 0$ and $\phi_1(t) > 0$ on $t_1 < t < t_2$, we have $\phi'_1(t_2) < 0$. Since $\phi_2(t) > 0$ on $t_1 < t < t_2$, we have $\phi_2(t_2) \geq 0$. Thus $P(t_2)\ \phi'_1(t_2)\phi_2(t_2) \leqslant 0$. Similarly we have $P(t_1)\ \phi'_1(t_1)\phi_2(t_1) \geq 0$. Thus, the left member of (7) is not positive.

But by hypothesis $Q_2(t) - Q_1(t) > 0$ on $t_1 \leqslant t \leqslant t_2$, and so the right member of (7) is positive. Thus the assumption that $\phi_2$ does not have a zero on the open interval $t_1 < t < t_2$, leads to a contradiction, and so $\phi_2$ has a zero at some point of this open interval.

Hence the proof of the theorem.

As a particular case of importance, suppose that the hypotheses of Theorem 6.5 are satisfied and that $t_1$ is a zero of both $\phi_1$ and $\phi_2$. Then if $t_2$ and $\xi$ are the "next" zeros of $\phi_1$ and $\phi_2$, respectively, we must have $\xi < t_2$.




**Figure**

**Example 3**

Consider the equation

$$\frac{d^2x}{dt^2} + A^2x = 0$$

and

$$\frac{d^2x}{dt^2} + B^2x = 0$$

where A and B are constants such that $B > A > 0$. The functions $\phi_1$ and $\phi_2$ defined respectively by $\phi_1(t) = \sin At$ and $\phi_2(t) = \sin Bt$ are real solutions of these respective equations. Consecutive zeros of sin At are

$$\frac{n\pi}{A} \quad and \quad \frac{(n+1)\pi}{A} \qquad (n = 0, \ \pm 1, \ \pm 2, ....)$$

Then by theorem 6.5, we are assured that sin Bt has at least one zero $\xi_n$ such that

$$\frac{n\pi}{A} < \xi n < \frac{(n+1)\pi}{A} \qquad (n = 0,\ \pm 1,\ \pm 2, ....)$$

In particular, t = 0 is zero of both sin At and sin Bt. The "next" zero of sin At is π/A, while the "next" zero of sin Bt is π/B; and clearly π/B < π/A. This verifies the results of comparison Theorem.

**Non-Oscillatory and Oscillatory function**

A real valued function f(t) defined and continuous in an interval [a, b] is said to be non-oscillatory in [a, b], if f(t) has not more than one zero in [a, b].

If f(t) has at least two zeros in [a, b], then f(t) is said to be oscillatory in [a, b].

**Examples:** (i) Consider the function

$$f(t) = Ae^{-t} + Be^{-t}$$

for $t \geq 0$, A and B are constants, then f(t) is non-oscillatory.

(ii) Let f(t) = sin t, $t \geq 0$.

Then f(t) is oscillatory in [0, 4π].

**Non-Oscillatory and Oscillatory differential equations**

A second order differential equation

$$\frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)y = h(t), \qquad t \geq 0$$

is called "non-oscillatory" if every solution u = u(t) of it, is non-oscillatory. Otherwise, differential equation is called oscillatory.

**Example 1** u" + u = 0 is oscillatory.

Its general solutions is

$$u(t) = A \cos t + B \sin t, \quad t \geq 0.$$

W.l.o.g., we can assume that both A and B are non-zero constants, otherwise, u(t) is trivially oscillatory.

In that case, u(t) has a zero at

$$t = n\pi + \tan^{-1}(A/B), \text{ for } n = 0, 1, 2, 3, .....$$

So, this equation is oscillatory.

**Example 2** Consider the linear equation

$$u'' - u = 0, \quad \text{for } t = 0$$

Its general solution is $u(t) = Ae^{t} + Be^{-t}$, A & B are constants. This solution is non-oscillatory. Hence, this equation is non-oscillatory.

**Definition** Let f(t) and g(t) be two real valued functions defined and continuous in some interval [a, b]. Then f(t) is said to oscillate more rapidly than g(t) if the number of zeros of f(t) in [a, b] exceed the number of zeros of g(t) in [a, b] by more than one.

**Example 3** Let $f(t) = \sin 2t$ in $[0, 4\pi]$,

$g(t) = \sin t$ in $[0, 4\pi]$.

Then zeros of f(t) are only half as far apart as the zeros of g(t). So f(t) oscillates more rapidly than g(t) in the interval $[0, 4\pi]$.

**Riccati Equations**

Before we give the formal definition of **Riccati equations**, a little introduction may be helpful. Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y).$$

If we approximate $f(x,y)$, while $x$ is kept constant, we will get

$$f(x, y) = P(x) + Q(x)y + R(x)y^2 + \ldots\ldots.$$

If we stop at $y$, we will get a linear equation. Riccati looked at the approximation to the second degree: he considered equations of the type

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2.$$

Such type of equations bear his name, **Riccati equations**. They are nonlinear and do not fall under the category of any of the classical equations. In order to solve a Riccati equation, one will need a particular solution. Without knowing at least one solution, there is absolutely no chance to find any solutions to such an equation. Indeed, let $y_1$ be a particular solution of

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2 .$$

Consider the new function *z* defined by

$$z = \frac{1}{y - y_1}$$

Then $$\frac{dz}{dx} = -\,(Q(x) + 2y_1R(x))z - R(x)$$

which is a linear equation satisfied by the new function *z*. Once it is solved, we go back to *y* via the relation

$$y = y_1 + \frac{1}{z}$$

**Example 1** Solve the equation

$$\frac{dy}{dx} = -2 - y + y^2 ,$$

knowing that $y_1 = 2$ is a particular solution.

**Answer** We recognize a Riccati equation. First of all we need to make sure that $y_1$ is indeed a solution.

Consider the new function z defined by

$$z = \frac{1}{y - 2}$$

Then we have

$$y = 2 + \frac{1}{z}$$

which implies

$$y' = -\frac{z'}{z^2}$$

Hence, from the equation satisfied by *y*, we get

$$-\frac{z'}{z^2} = -2 - \left(2 + \frac{1}{z}\right) + \left(2 + \frac{1}{z}\right)^2$$

Then easy calculations give

$$-\frac{z'}{z^2} = \frac{3}{z} + \frac{1}{z^2}.$$

Hence

$$z' = -3z - 1.$$

This is a linear equation. The general solution is given by

$$z = \frac{-1/3e^{3x} + C}{e^{3x}} = -\frac{1}{3} + Ce^{-3x}$$

Therefore, we have

$$y = 2 + \frac{1}{-\frac{1}{3} + Ce^{-3x}}.$$

**Note** If one remembers the equation satisfied by $z$, then the solutions may be found a bit faster. Indeed in this example, we have $P(x) = -2$, $Q(x) = -1$, and $R(x) = 1$. Hence the linear equation satisfied by the new function $z$, is

$$\frac{dz}{dx} = -(Q(x) + 2y_1R(x))z - R(x) = -(-1+4)z - 1 = -3z - 1$$

**Example 2** Check that $y_1 = \sin(x)$ is a solution to

$$\frac{dy}{dx} = \frac{2\cos^2(x) - \sin^2(x) + y^2}{2\cos(x)}$$

Then solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2\cos^2(x) - \sin^2(x) + y^2}{2\cos(x)} \\ y(0) = -1 \end{cases}$$

**Answer** Check that sin(x) is indeed a particular solution of the given differential equation. We also recognize that the equation is of Riccati type. Set

$$z = \frac{1}{y - \sin(x)}$$

which gives

$$y = \sin(x) + \frac{1}{z},$$

Hence

$$y' = \cos(x) - \frac{z'}{z^2}$$

Substituting into the equation gives

$$\cos(x) - \frac{z'}{z^2} = \frac{2\cos^2(x) - \sin^2(x) + \left(\sin(x) + \frac{1}{z}\right)^2}{2\cos(x)}.$$

Easy algebraic manipulations give

$$-\frac{z'}{z^2} = \frac{\left(2\sin(x)\frac{1}{z} + \frac{1}{z^2}\right)}{2\cos(x)} = \frac{\sin(x)}{\cos(x)}\frac{1}{z} + \frac{1}{2\cos(x)}\frac{1}{z^2}$$

Hence

$$z' = -\frac{\sin(x)}{\cos(x)} z - \frac{1}{2\cos(x)}$$

This is the linear equation satisfied by *z*. The integrating factor is

$$u(x) = e^{\int \frac{\sin(x)}{\cos(x)} dx} = e^{-In(\cos(x))} = \frac{1}{\cos(x)} = \sec(x).$$

The general solution is

$$z = \frac{-1/2\int \sec^2(x)dx + C}{u(x)} = \cos(x)\left(-\frac{1}{2}\tan(x) + C\right) = -\frac{1}{2}\sin(x) + C\cos(x).$$

Now go back to the original function *y*. We have

$$y = \sin(x) + \frac{1}{-\frac{1}{2}\sin(x) + C\cos(x)}.$$

The initial condition y(0) = -1 implies $\frac{1}{C} = -1$ or C= -1. Therefore, the solution to the IVP is

$$y = \sin(x) + \frac{1}{-\frac{1}{2}\sin(x) - \cos(x)}.$$

**Relationship between Riccati Equation and linear differential equation of second order**

The importance of Riccati Equation in theory of differential equations is due to the following relationship between it and the general linear differential equation of second order.

Consider the general Riccati Equation

$$\frac{dy}{dx} + Q(x)y + R(x)y^2 = P(x) \tag{1}$$

Let us make the transformation

$$y = \frac{1}{Ru}\frac{du}{dx} = \frac{u'}{Ru} \tag{2}$$

The resulting equation is the following linear differential equation of second order

$$R\frac{d^2u}{dx^2} - (R' - QR)\frac{du}{dx} - PR^2u = 0 \tag{3}$$

Conversely, there corresponds to the general homogenous linear differential equation of second order a Riccati equation.

Given the equation

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0 \tag{4}$$

We make the transformation

$$\frac{du}{dx} = (Ry)u \tag{5}$$

and obtain the following Riccati equation

$$\frac{dy}{dx}+\left(\frac{R'}{R}+\frac{B}{A}\right)y+Ry^2=\frac{-C}{AR}. \tag{6}$$

Comparing (6) with (1), we get

$$Q(x)=\frac{R'(x)}{R(x)}+\frac{B(x)}{A(x)}, \qquad P(x)=\frac{-C(x)}{A(x)R(x)}. \tag{7}$$

Since R(x) is an arbitrary function, we can determine it so that Q(x) is zero. In this case, (6) assumes the simpler form

$$\frac{dy}{dx}+Ry^2=\frac{-C}{AR}, \tag{8}$$

where

$$R(x) = \exp\left[-\int\frac{B(x)}{A(x)}dx\right].$$

**Prüfer Transformation/Polar Co-ordinate Transformation**

This transformation is applicable to linear homogeneous second order differential equations. It yields an equivalent system of two first order differential equations. This transformation changes an equation from Liouville normal form to two successive ordinary differential equations. It is often used to obtain information about the zeros of solutions.

**The Prüfer System**

We will develop a method called phase plane method to seek the solution of Sturm-Liouville equation by using the Prüfer substitution. It yields the phase and the amplitude of the sought after solution to Sturm- Liouville equation.

Let u(x) $\not\equiv 0$ be a real valued solution of the Sturm-Liouville equation.

The method to be developed applies to any differential equation having the form

$$\frac{d}{dx}\left(P(x)\frac{du}{dx}\right)+Q(x)u=0, \quad a < x < b. \tag{1}$$

Here, 0 < P(x), P'(x) and Q(x) are continuous.

We do this by introducing the ``phase" and the ``radius" of a solution u(x). This is done in three steps.

**A)** First apply the Prüfer substitution

$$P(x)u'(x) = r(x)\cos\theta(x); \qquad u(x) = r(x)\sin\theta(x) \tag{2}$$

to the quantities in (1). For this, we introduce the new dependent variables r and θ as defined by the formulae

$$r^2 = u^2 + P^2(u')^2; \qquad \theta = \arctan\frac{u}{Pu'}. \tag{3}$$

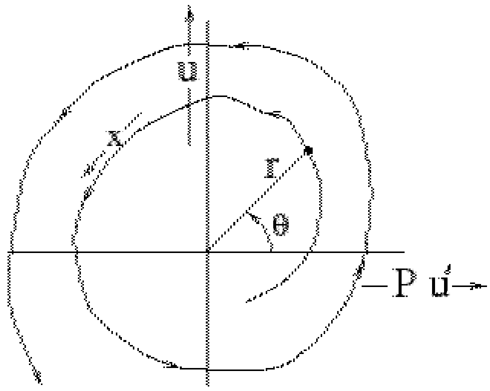



**Figure** The Poincaré phase plane of the second order linear differential equation is spanned by the amplitude u and its derivative u' (multiplied by the positive coefficient P). A solution to the differential equation is represented by an x-parametrized curve. The (Prüfer) phase is the polar angle θ.

(Without loss of generality one may always assume that u(x) is real. Indeed, if u(x) were a complex solution, then it would differ from a real one by a mere complex constant.) A solution u(x) can thus be pictured in this Poincaré plane as a curve parametrized by the independent variable x.

The transformation

$$(Pu', u) \leftrightarrow (r,\theta)$$

is *non-singular* for all r ≠ 0. Furthermore, we always have r > 0 for any non-trivial solutions. Because if r(x) = 0, i.e., u(x) = 0 and u'(x) = 0 for some particular x then by the uniqueness theorem for second order linear o.d.e. u(x) = 0 ∀x, i.e., we have the trivial solution.

**B)** Second, obtain a system of first order o.d.e. which is equivalent to the given differential equation (1).

(i) Differentiate the relation

$$\cot\theta = \frac{Pu'}{u}$$

(If u = 0, then we differentiate $\tan\theta = \frac{u}{Pu'}$ instead. This yields the same result.) One obtains

$$-\operatorname{cosec}^2\theta \, \frac{d\theta}{dx} = \frac{(Pu')'}{u} - \frac{Pu'}{u^2}u'$$

$$= -Q - \frac{1}{P}\frac{\cos^2\theta}{\sin^2\theta}, \text{ (using (1) \& (2)).}$$

or

$$\frac{d\theta}{dx} = Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta \equiv F(x,\theta) \qquad (4)$$

This is Prüfer's differential equation for the phase, the Prüfer phase.

(ii) Differentiate the relation

$$r^2 = u^2 + (Pu')^2$$

and obtain

$$r\frac{dr}{dx} = uu' + (Pu')\,(Pu')'$$

$$= \frac{u}{P}\,Pu' - Pu'\,Qu$$

$$= \frac{r\sin\theta}{P}\,r\cos\theta - r\cos\theta\,Q\,r\sin\theta$$

or

$$\frac{dr}{dx} = \frac{1}{2}\left[\frac{1}{P(x)} - Q(x)\right] r\sin 2\theta\,. \qquad (5)$$

This is Prüfer's differential equation for the amplitude.

**C)** Third, solve the system of Prüfer equations (4) and (5). Doing so is *equivalent* to solving the originally given equation (1). Any solution to the Prüfer system determines a unique solution to the equation (1), and conversely. This system is called the Prüfer system associated with the self adjoint differential equation (1). Of the two Prüfer equations (4) and (5), the one for the phase θ(x) is obviously much more important, it determines the qualitative, e.g. oscillatory, behaviour of u(x). The feature which makes the phase equation so singularly attractive is that it is a first order equation which also is independent of the amplitude r(x). The amplitude r(x) has no influence whatsoever on the phase function θ(x).

Once θ(x) is known from (4), the Prüfer amplitude function r(x) is determined by integrating (5).

One obtains

$$r(x) = K \exp \int_a^x \frac{1}{2}\left[\frac{1}{P(x)} - Q(x)\right] \sin 2\theta(x)\, dx$$

where K = r(a) is the initial amplitude.

**Note** Each solution to the Prüfer system (4) and (5), depends on two constants:

the initial amplitude K = r(a),

the initial phase θ = θ(a).

2. Changing the constant K just multiplies the solution u(x) by a constant factor. Thus the zeros of u(x) can be located by studying only the phase differential equation,

$$\frac{d\theta}{dx} = F(x,\ \theta).$$

Vibrations, oscillations, wiggles, rotations and undulations are all characterized by a changing phase. If the independent variable is the time, then this time, the measure of that aspect of change which permits an enumeration of states, manifests itself physically by the advance of the phase of an oscillating system.

**Summary** The phase of a system is the most direct way of characterizing its oscillatory nature. For a linear $2^{nd}$ order o.d.e., this means the Prüfer phase θ(x), which obeys the first order differential equation.

$$\frac{d\theta}{dx} = Q(x)\ \sin^2\theta + \frac{1}{P(x)}\ \cos^2\theta \equiv F(x,\ \theta)$$

It is obtained from the second order equation

$$\left[\frac{d}{dx}P(x)\frac{d}{dx} + Q(x)\right]u(x) = 0$$

by the Prüfer substitution

$$u(x) = r(x)\sin\theta(x) \qquad Pu'(x) = r(x)\ \cos\theta(x)$$

These equations make it clear that the zeros and the oscillatory behavior of u(x) are controlled by the phase function θ(x).

**Example** Consider the linear second order homogeneous ordinary differential equation

$$x\,u'' - u' + x^3 u = 0. \tag{1}$$

It can be written in Liouville normal form as

$$\frac{d}{dx}\left[\frac{1}{x}\frac{du}{dx}\right] + xu = 0, \tag{2}$$

Here $P(x) = \dfrac{1}{x}$, $Q(x) = x$. (3)

Therefore, the first equation of Prüfer system becomes

$$\frac{d\theta}{dx} = x\cos^2\theta + x\sin^2\theta = x. \tag{4}$$

Solving it, we obtain

$$\theta(x) = \frac{x^2}{2} + C, \tag{5}$$

where C is an arbitrary constant.

Then, the second differential equation of the Prüfer system gives

$$\frac{dr}{dx} = \frac{1}{2}[x - x]r\sin\ 2\theta = 0$$

Integrating

$$r(x) = \text{constant} = r(a) \quad (\text{let}) \tag{6}$$

Thus, we conclude from the Prüfer transformation, that the solution u(x) = r sinθ, now becomes

$$u(x) = r(a)\sin\left[\frac{x^2}{2}+C\right]. \tag{7}$$

Find out r(a) from (7) as

$$r(a) = \frac{u(a)}{\sin\left[\frac{a^2}{2}+C\right]},$$

Then, from (7) we get

$$u(x) = \frac{u(a)\ \sin\left[\frac{x^2}{2}+C\right]}{\sin\left[\frac{a^2}{2}+C\right]}.$$

as the solution of given equation (1).

**Lagrange Identity**

Consider the pair of differential equations.

$$\frac{d}{dt}\left\{p(t)\frac{du}{dt}\right\}+q(t)u(t)=f(t), \tag{1}$$

$$\frac{d}{dt}\left\{p(t)\frac{dv}{dt}\right\}+q(t)\mathrm{v}(t)=g(t), \tag{2}$$

where f = f(t) and g = g(t) are continuous functions on interval I. Multiplying the second relation (2) by u(t), first (1) by v(t) and subtracting the results, we obtain

$$\frac{d}{dt}\left[p(t)\left\{u\frac{dv}{dt}-v\frac{du}{dt}\right\}\right]=gu-fv. \tag{3}$$

The relation (3) is called the Lagrange identity. Its integrated form

$$[p\ (u\,v'-u'v]_a^t=\int_a^t(gu-fv)ds, \tag{4}$$

where [a, t] $\subset$ I, is called Green's formula.

**Summary**

Some basic results of Sturm theory are presented for the self adjoint form of second order equation. Sturm separation and fundamental comparison theorems are proved, with illustrations. Riccati equations are solved and a relationship between the general linear differential equation of second order and Riccati equation is defined. In concluding part of the chapter, phase-plane method is developed to find out the solution of Sturm-Liouville equation, using the Prüffer substitution.

**Keywords**

Sturm theory, Zeros of solutions, Abel's formula, Riccati equation, Prüffer transformation.

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## Chapter-7

# STURM –LIOUVILLE BOUNDARY VALUE PROBLEMS

### Objectives

The main purpose of this chapter is to present the essential concepts and properties of Sturm-Liouville boundary value problems. These concepts are frequently employed in the application of differential equations to physics and engineering.

### Introduction

In this chapter, we shall consider special kind of boundary value problems known as Sturm Liouville problems. These problems arise naturally, for instance, when separation of variables is applied to the wave equation, the potential equation or the diffusion equation. The study of these types of problems will introduce us to several important concepts including characteristic values, characteristic functions, orthogonality and orthonormality of functions, which are very useful in many applied problems.

**Boundary Value Problems** The problems that involve both a differential equation and one or more supplementary conditions, which the solution of given differential equation must satisfy, are called boundary value problems. If all the associated supplementary conditions relate to one x-value, the problem is called an initial-value problem (or one point boundary value problem). If the conditions relate to two different x-values, the problem is called a two-point boundary value problem (or simply a boundary value problem).

### Example 1

Solve the equation

$$\frac{d^2y}{dx^2} + y = 0, \qquad y(1) = 3, \qquad y'(1) = -4 .$$

This problem consists of finding a solution of the given differential equation which assumes the value 3 at x = 1 and whose first derivative assumes the value -4 at x = 1. Both these conditions relate to one x-value, namely x = 1. Thus, this is an initial value problem and has a unique solution.

**Example 2**

Solve the equation

$$\frac{d^2y}{dx^2}+y=0, \qquad y(0)=1, \qquad y\left(\frac{\pi}{2}\right)=5$$

In this problem, we again want a solution of the same differential equation, but this time, the solution must assume the value 1 at x = 0 and 5 at $x=\frac{\pi}{2}$. That is, the conditions relate to the two different x values, 0 and $\frac{\pi}{2}$. This is a (two-point) boundary value problem and also has a unique solution.

**Sturm-Liouville Equation**

In mathematics and its applications, a classical Sturm –Liouville equation, named after Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1882), is a real second-order linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

where y is a function of the free variable x. Here the functions p(x) > 0 has a continuous derivative, q(x) and r(x) > 0 are specified at the outset, and in the simplest of cases are continuous on the finite closed interval [a, b]. In addition, the function y is typically required to satisfy some boundary conditions at a and b. The function r(x), is called the “weight” or “density” function.

The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of (1) satisfying the boundary conditions is part of the problem called the Sturm-Liouville problem (S L).

Such value of λ when they exist are called the eigenvalues of the boundary value problem defined by (1) and the prescribed set of boundary conditions. The corresponding solutions (for such λ) are the eigenfunctions of this problem.

## A. Sturm-Liouville Boundary Value Problems (SLBVP)

We shall consider a special kind of boundary value problem known as Sturm-Liouville problem.

**Definition**

We consider a boundary value problem which consists of

1. A second order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda r(x)]y=0 \qquad (1)$$

where p, q and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \le x \le b$ and λ is a parameter independent of x; and

2. Two supplementary conditions

$$A_1 y(a) + A_2 y'(a) = 0,$$

$$B_1 y(b) + B_2 y'(b) = 0, \qquad (2)$$

where $A_1$, $A_2$, $B_1$ and $B_2$ are real constants such that $A_1$ and $A_2$ are not both zero and $B_1$ and $B_2$ are not both zero.

This type of boundary-value problem is called a Sturm-Liouville problem (or Sturm-Liouville system).

**Note** Two important special cases are those in which the supplementary conditions (2) are either of the form

$$y(a) = 0, \qquad y(b) = 0 \qquad (3)$$

or of the form

$$y'(a) = 0, \qquad y'(b) = 0 \qquad (4)$$

**Example 3**

The boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad (1)$$

$$y(0) = 0, \qquad y(\pi) = 0 \qquad (2)$$

is a Sturm-Liouville problem. The differential equation (1) may be written $\frac{d}{dx}\left[1.\ \frac{dy}{dx}\right] + \left[0 + \lambda.1\right] y = 0$

where p(x) = 1, q(x) = 0, and r(x) = 1. The supplementary conditions (2) are of the special form.

Let us see what is involved in solving Sturm-Liouville problem. We must find a function f which satisfies the given differential equation and the two supplementary conditions. Clearly one solution of any problem of this type is the trivial solution $\phi$ such that $\phi(x) = 0$ for all values of x. But, this trivial solution is not very useful. We shall search for nontrivial solutions of the problem. That is, we shall attempt to find functions, not identically zero, which satisfy the given differential equation and the two conditions. We shall see that the existence of such nontrivial solutions depends upon the value of the parameter $\lambda$ in the differential equation.

**Example 4**

Find nontrivial solutions of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad (1)$$

$$y(0) = 0, \qquad y(\pi) = 0 \qquad (2)$$

**Solution**

Consider separately the three cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$. In each case we shall first find the general solution of the differential equation. We shall then attempt to determine the two arbitrary constants in this solution so that the supplementary conditions (2) are also satisfied.

**Case 1** $\lambda = 0$. In this case the differential equation (1) reduces to

$$\frac{d^2y}{dx^2} = 0,$$

so the general solution is

$$y = c_1 + c_2 x. \quad (3)$$

We now apply the conditions (2) to the solution (3). We find that in order for the solution (3) to satisfy the conditions (2), we must have $c_1 = c_2 = 0$. But then the solution (3) becomes the solution y such that $y(x) = 0$ for all values of x. Thus if the parameter $\lambda = 0$, the only solution of the given problem is the trivial solution.

**Case 2** $\lambda < 0$. The auxiliary equation of the differential equation (1) is $m^2 + \lambda = 0$ and has the roots $\pm\sqrt{-\lambda}$. Since in this case $\lambda < 0$, these roots are real and unequal. Denoting $\sqrt{-\lambda}$ by $\alpha$, we see that for $\lambda < 0$ the general solution of (1) is of the form

$$y = c_1e^{\alpha x} + c_2e^{-\alpha x}. \quad (4)$$

We now apply the conditions (2) to the solution (4). Applying the first condition $y(0) = 0$, we obtain

$$c_1 + c_2 = 0 \quad (5)$$

Applying the second condition $y(\pi) = 0$, we find that

$$c_1e^{\alpha\pi} + c_2e^{-\alpha\pi} = 0 \quad (6)$$

Thus for the solution (4) to satisfy the conditions (2), the constants $c_1$ and $c_2$ must satisfy the system of equations (5) and (6). Obviously $c_1 = c_2 = 0$ is the solution of this system; but these values of $c_1$ and $c_2$ would only give the trivial solution of the given problem. We must therefore seek nonzero values of $c_1$ and $c_2$ which satisfy (5) and (6). This system has nonzero solutions only if the determinant of coefficients is zero. Therefore, we must have

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{vmatrix} = 0$$

which gives $e^{\alpha\pi} = e^{-\alpha\pi}$ and hence that $\alpha = 0$. Since $\alpha = \sqrt{-\lambda}$, we must then have $\lambda = 0$. But $\lambda < 0$ in this case. Thus there are no non-trivial solutions of the given problem in the case $\lambda < 0$.

**Case 3** λ > 0. Since λ > 0 here, the roots $\pm\sqrt{-\lambda}$ of the auxiliary equation of (1) are the conjugate complex numbers $\pm\sqrt{-\lambda}\,\iota$. Thus in this case the general solution of (1) is of the form

$$y = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x. \tag{7}$$

We now apply the condition (2) to this general solution. Applying the first condition y(0) = 0, we obtain

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

and hence $c_2 = 0$. Applying the second condition y(π) = 0, we find that

$$c_1 \sin\sqrt{\lambda}\,\pi + c_2 \cos\sqrt{\lambda}\,\pi = 0.$$

Since $c_2 = 0$, this reduces to

$$c_1 \sin\sqrt{\lambda}\,\pi = 0. \tag{8}$$

We can set $c_1 = 0$ or we can set $\sin\sqrt{\lambda}\,\pi = 0$. However, if we set $c_1 = 0$, then (since $c_2 = 0$ also) the solution (7) reduces immediately to the unwanted trivial solution. Thus to obtain a nontrivial solution we can not set $c_1 = 0$ but rather we must set

$$\sin\sqrt{\lambda}\,\pi = 0. \tag{9}$$

If k > 0, then sin kπ = 0 only if k is a positive integer n = 1, 2, 3…. Thus in order to satisfy (9) we must have $\sqrt{\lambda} = n$, where n = 1, 2, 3, ….. Therefore, in order that the differential equation (9) has a nontrivial solution of the form (7) satisfying the conditions (2), we must have

$$\lambda = n^2, \quad \text{where } n = 1, 2, 3, \ldots. \tag{10}$$

The parameter λ in (2) must be a member of the infinite sequence

$$1, 4, 9, 16, \ldots\ldots, n^2, \ldots\ldots\ldots$$

**Summary** If λ ⩽ 0, the Sturm-Liouville problem consisting of (1) and (2) does not have a nontrivial solution; if λ > 0, a nontrivial solution can exist only if λ is one of the values given by (10). We now note that if λ is one of the values given by (10) , then the problem does have nontrivial solutions. From (7) we see that nontrivial solutions corresponding to $\lambda = n^2$ (n = 1, 2, 3, …. ) are given by

$$y = c_n \sin nx \quad (n = 1, 2, 3, \ldots.) \tag{11}$$

where $c_n$ (n = 1, 2, 3, …. ) are arbitrary nonzero constants. That is, the functions defined by $c_1 \sin x$, $c_2 \sin 2x$, $c_3 \sin 3x$,….., where $c_1, c_2, c_3$,…... are arbitrary nonzero constants, are nontrivial solutions of the given problem.

## B. Characteristic Values and Characteristic Functions

Example 4 shows that the existence of nontrivial solutions of a Sturm-Liouville problem depends upon the value of the parameter $\lambda$ in the differential equation of the problem. Those values of the parameter for which nontrivial solutions do exist, as well as the corresponding nontrivial solutions themselves, are singled out by the following definition:

**Definition**

Consider the Sturm-Liouville problem A, consisting of the differential equation (1) and the supplementary conditions (2). The values of the parameter $\lambda$ for which there exist nontrivial solutions of the problem are called the characteristic values of the problem. The corresponding nontrivial solutions themselves are called the characteristic functions of the problem.

The characteristic values are also called eigenvalues; and the characteristic functions are also called eigenfunctions.

**Example 5**

Consider again the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{1}$$

$$y(0) = 0, \qquad y(\pi) = 0 \tag{2}$$

In example (4) we found that the values of $\lambda$ in (1) for which there exist nontrivial solutions of this problem are the values

$$\lambda = n^2, \qquad \text{where } n = 1, 2, 3\ldots\ldots \tag{3}$$

These are the characteristic values of the problem under consideration. The characteristic functions of the problem are the corresponding nontrivial solutions

$$y = c_n \sin nx \quad (n = 1, 2, 3\ldots), \tag{4}$$

where $c_n$ (n = 1, 2, 3…) are arbitrary nonzero constants.

**Example 6**

Find the characteristic values and characteristic functions of the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \tag{1}$$

$$y'(1) = 0, \qquad y'(e^{2\pi}) = 0 \tag{2}$$

where we assume that the parameter λ in (1) is non-negative.

**Solution** We consider separately the cases λ = 0 and λ > 0. If λ = 0, the differential equation (1) reduces to

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = 0.$$

The general solution of this differential equation is

$$y = C \ln |x| + C_0,$$

where C and $C_0$ are arbitrary constants. If we apply the conditions (2) to this general solution, we find that both of them require that C = 0. Thus for λ = 0 we obtain the solutions $y = C_0$, where $C_0$ is an arbitrary constant. These are nontrivial solutions for all choices of $C_0 \neq 0$. Thus λ = 0 is a characteristic value and the corresponding characteristic functions are given by $y = C_0$, where $C_0$ is an arbitrary nonzero constant.

If λ > 0, we see that for x ≠ 0 this equation is equivalent to the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0. \tag{3}$$

Letting $x = e^t$, Equation (3) transforms into

$$\frac{d^2y}{dt^2} + \lambda y = 0. \tag{4}$$

Since λ > 0, the general solution of (4) is of the form

$$y = c_1 \sin \sqrt{\lambda}\, t + c_2 \cos \sqrt{\lambda} t.$$

Thus for $\lambda > 0$ and $x > 0$, the general solution of (1) may be written

$$y = c_1 \sin(\sqrt{\lambda} \ln x) + c_2 \cos(\sqrt{\lambda} \ln x). \tag{5}$$

We now apply the supplementary conditions (2). From (5) we find that

$$\frac{dy}{dx} = \frac{c_1\sqrt{\lambda}}{x} \cos(\sqrt{\lambda} \ln x) - \frac{c_2\sqrt{\lambda}}{x} \sin(\sqrt{\lambda} \ln x) \tag{6}$$

for $x > 0$. Applying the first condition $y'(1) = 0$ of (2) to (6), we have

$$c_1 \sqrt{\lambda} \cos(\sqrt{\lambda} \ln 1) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} \ln 1) = 0$$

or $c_1 \sqrt{\lambda} = 0$. Thus we must have

$$c_1 = 0 \tag{7}$$

Applying the second condition $y'(e^{2\pi}) = 0$ of (2) to (6) we obtain

$$c_1\sqrt{\lambda}e^{-2\pi} \cos(\sqrt{\lambda} \ln e^{2\pi}) - c_2 \sqrt{\lambda}e^{-2\pi} \sin(\sqrt{\lambda} \ln e^{2\pi}) = 0.$$

Since $c_1 = 0$ by (7) and $\ln e^{2\pi} = 2\pi$, this reduces to

$$c_2\sqrt{\lambda}e^{-2\pi} \sin(2\pi\sqrt{\lambda}) = 0$$

Since $c_1 = 0$, the choice $c_2 = 0$ would lead to the trivial solution.

We must have $\sin\left(2\pi\sqrt{\lambda}\right) = 0$ and hence $2\pi\sqrt{\lambda} = n\pi$, where n = 1, 2, 3. Thus in order to satisfy the second condition (2) nontrivially we must have

$$\lambda = \frac{n^2}{4} (n = 1, 2, 3......) \tag{8}$$

Corresponding to these values of $\lambda$ we obtain for $x > 0$, the nontrivial solutions

$$y = c_n \cos\left(\frac{n \ln x}{2}\right) (n = 1, 2, 3......), \tag{9}$$

where the $c_n (n = 1, 2, 3....)$ are arbitrary nonzero constants.

Thus the values

$$\lambda = 0, \frac{1}{4}, 1, \frac{9}{4}, 4, \frac{25}{4}, ......., \frac{n^2}{4}, ....,$$

given by (8) for $n \geq 0$, are the characteristic values of the given problem. The functions.

$$c_0, c_1 \cos\left(\frac{\ln x}{2}\right), c_2 \cos(\ln x), c_3 \cos\left(\frac{3\ln x}{2}\right), \ldots\ldots,$$

given by (9) for $n \geq 0$, where $c_0, c_1, c_2, c_3\ldots$ are arbitrary nonzero constants, are the corresponding characteristic functions.

We observe that in each of these problems the infinite set of characteristic values can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots$$

such that $\lambda_n \to +\infty$ as $n \to +\infty$. For example, the characteristic values of the problem of Example 4 can be arranged in the monotonic increasing sequence

$$1 < 4 < 9 < 16 < \ldots. \quad (10)$$

such that $\lambda_n = n^2 \to +\infty$ as $n \to +\infty$. We also note that in each problem there is a one-parameter family of characteristic functions corresponding to each characteristic value, and any two characteristic functions corresponding to the same characteristic value are merely nonzero constant multiples of each other. For example, in the problem of Example 4, the one-parameter family of characteristic functions corresponding to the characteristic value $n^2$ is $c_n \sin nx$, where $c_n \neq 0$ is the parameter.

**Theorem 7.1** Prove that eigenvalues of a SLBVP are discrete.

**Proof** Let $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be two linearly independent solutions (for fixed $\lambda$) of a SLBV problem consisting of a differential equation.

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \quad (1)$$

and the boundary conditions $y(a) = y(b) = 0$.
(2)

Then any solution of (1) can be expressed as linear combination of $y_1$ and $y_2$. That is

$$y(x, \lambda) = A\, y_1(x, \lambda) + B\, y_2(x, \lambda). \quad (3)$$

The constants A and B are determined by the fact that y(x, λ) in (3) also satisfies the boundary conditions (2), which leads to

$$A\ y_1(a, \lambda) + B\ y_2(a, \lambda) = 0 \qquad (4)$$

$$A\ y_1(b, \lambda) + B\ y_2(b, \lambda) = 0$$

or in the form of matrix equation

$$\begin{bmatrix} y_1(a,\lambda) & y_2(a,\lambda) \\ y_1(b,\lambda) & y_2(b,\lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (5)$$

Now, condition for the existence of non-trivial solution of (4) is that the determinant of the matrix of coefficients in (5) vanishes. Otherwise, the only solution is A = B =0, which yields the trivial solution y(x) = 0. Thus

$$\begin{vmatrix} y_1(a,\lambda) & y_2(a,\lambda) \\ y_1(b,\lambda) & y_2(b,\lambda) \end{vmatrix} = y_1(a,\lambda)\ y_2(b,\lambda) - y_1(b,\lambda)\ y_2(a,\lambda) = 0. \quad (6)$$

Now, $y_1(x, \lambda)$ and $y_2(x, \lambda)$ being analytic functions of λ, the determinant itself is an analytic function of λ. Therefore, by the theory of complex-valued functions, the zeros of the determinant must be isolated. [The zeros of an analytic function are isolated]. Since the zeros of the determinant correspond to solutions of the SLBVP, we thus conclude that the eigenvalues of (1) and (2) are discrete.

This completes the proof.

**Exercises**

Find the characteristic values and characteristic functions of each of the following Sturm-Liouville problems.

1. $\dfrac{d^2y}{dx^2} + \lambda y = 0,$ y(0) = 0, $y = \left(\dfrac{\pi}{2}\right) = 0.$

2. $\dfrac{d^2y}{dx^2} + \lambda y = 0,$ y(0) = 0, y(L) = 0, where L > 0

3. $\dfrac{d^2y}{dx^2} + \lambda y = 0,$ y(0) = 0, $y(\pi) - y'(\pi) = 0$

4. $\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0,$ $y(1) = 0,$ $y(e^{\pi}) = 0.$

5. $\frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] + \frac{\lambda}{x^2+1}y = 0,$ $y(0) = 0,$ $y(1) = 0.$

[Hind: Let x = tan t.]

**Orthogonality of Characteristic Functions**

**A. Orthogonality**

**Definition** Two functions f and g are called orthogonal with respect to the weight function ω on the interval $a \le x \le b$ if and only if

$$\int_a^b f(x)g(x)\omega(x)dx = 0.$$

**Example 7**

The functions sin x and sin 2x are orthogonal with respect to the weight function having the constant value 1 on the interval $0 \le x \le \pi$, for

$$\int_0^{\pi}(\sin x)(\sin 2x)(1)dx = 0$$

**Definition**

Let $\{\phi_n\}$ n = 1, 2, 3, …. be an infinite set of functions defined on the interval $a \le x \le b$. The set $\{\phi_n\}$ is called an orthogonal system with respect to the weight function ω on $a \le x \le b$, if every two distinct functions of the set are orthogonal with respect to ω on $a \le x \le b$. That is, the set $\{\phi_n\}$ is orthogonal with respect to ω on $a \le x \le b$ if

$$\int_a^b \phi_m(x)\,\phi_n(x)\omega(x)dx = 0 \qquad \text{for } m \ne n.$$

**Example 8**

Consider the infinite set of functions $\{\phi_n\}$, where $\phi_n(x) = \sin nx$ (n = 1, 2, 3, …..) on the interval $0 \le x \le \pi$. The set $\{\phi_n\}$ is an orthogonal system with respect to the weight function having the constant value 1 on the interval $0 \le x \le \pi$, for

$$\int_0^{\pi} (\sin mx)(\sin nx)(1)dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\Bigg|_0^{\pi} = 0 \qquad \text{for } m \neq n.$$

**Note** The weight function ω(x) is not always equal to 1.

**B Orthogonality of Characteristic Functions**

We now state and prove a basic theorem concerning the orthogonality of characteristic functions of a Sturm-Liouville problem, also know as Sturm-Liouville Theorem.

**Theorem 7.2 Sturm - Liouville Theorem**

**Hypothesis** Consider the Sturm-Liouville problem consisting of

1. The differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\ y = 0, \tag{1}$$

where p, q and r are real functions such that p has a continuous derivative, q and r are continuous, and p(x) > 0 and r(x) > 0 for all x on a real interval $a \leq x \leq b$; and λ is a parameter independent of x ; and

2. The conditions

$$A_1\, y(a) + A_2 y'(a) = 0$$

$$B_1\, y(b) + B_2 y'(b) = 0, \tag{2}$$

where $A_1$, $A_2$, $B_1$ and $B_2$ are real constants such that $A_1$ and $A_2$ are not both zero and $B_1$ and $B_2$ are not both zero.

Let $\lambda_m$ and $\lambda_n$ be any two distinct characteristic values of this problem. Let $\phi_m$ be a characteristic function corresponding to $\lambda_m$ and let $\phi_n$ be a characteristic function corresponding to $\lambda_n$.

**Conclusion** The characteristic functions $\phi_m$ and $\phi_n$ are orthogonal with respect to the weight function ω on the interval $a \leq x \leq b$.

**Proof** Since $\phi_m$ is a characteristic function corresponding to $\lambda_m$, the function $\phi_m$ satisfies the differential equation (1) with $\lambda = \lambda_m$; and since $\phi_n$ is a characteristic function corresponding to $\lambda_n$, the function $\phi_n$ satisfies the differential equation (1) with $\lambda = \lambda_n$. Thus, we have

$$\frac{d}{dx}[p(x)\phi_m'(x)] + [q(x) + \lambda_m r(x)]\phi_m(x) = 0, \tag{3}$$

$$\frac{d}{dx}[p(x)\phi_n'(x)] + [q(x) + \lambda_n r(x)]\phi_n(x) = 0, \tag{4}$$

for all x such that $a \le x \le b$. Multiplying both sides of (3) by $\phi_n(x)$ and both sides of (4) by $\phi_m(x)$ and then subtracting the results we obtain

$$\phi_n(x)\frac{d}{dx}[p(x)\phi_m'(x)] + \lambda_m\phi_m(x)\phi_n(x)r(x) - \phi_m(x)\frac{d}{dx}[p(x)\phi_n'(x)]$$

$$-\lambda_n\phi_m(x)\phi_n(x)r(x) = 0$$

and hence

$$(\lambda_m - \lambda_n)\,\phi_m(x)\phi_n(x)r(x) = \phi_m(x)\,\frac{d}{dx}[p(x)\phi_n'(x)] - \phi_n(x)\frac{d}{dx}[p(x)\phi_m'(x)].$$

We now integrate the above identity from a to b and obtain

$$(\lambda_m - \lambda_n)\int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = \int_a^b \phi_m(x)\frac{d}{dx}[p(x)\phi_n'(x)]dx$$

$$-\int_a^b \phi_n(x)\frac{d}{dx}[p(x)\phi_m'(x)]\,dx \tag{5}$$

Integrating by parts, the right member of (5) becomes

$$\phi_m(x)p(x)\phi_n'(x)\Big|_a^b - \int_a^b p(x)\phi_n'(x)\phi_m'(x)\,dx - \phi_n(x)p(x)\phi_n'(x)\Big|_a^b$$

$$+\int_a^b p(x)\phi_m'(x)\phi_n'(x)dx$$

or

$$\left\{p(x)\,\left[\phi_m(x)\phi_n'(x) - \phi_n(x)\phi_m'(x)\right]\right\}\Big|_a^b.$$

Therefore the identity (5) becomes

$$(\lambda_m - \lambda_n)\int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = p(b)[\phi_m(b)\phi_n'(b) - \phi_n(b)\phi_m'(b)]$$

$$- \, p(a)[\phi_m(a)\phi_n'(a) - \phi_n(a)\phi_m'(a)]. \qquad (6)$$

Since $\phi_m$ and $\phi_n$ are characteristic functions of the problem, they satisfy the supplementary conditions (2) of the problem. If $A_2 = B_2 = 0$ in (2), these conditions reduce to $y(a) = 0$, $y(b) = 0$. Then in this case $\phi_m(a) = 0$, $\phi_m(b) = 0$, $\phi_n(a) = 0$, and $\phi_n(b) = 0$, and so the right member of (6) is equal to zero.

If $A_2 = 0$ but $B_2 \neq 0$ in (2), these conditions reduce to $y(a) = 0$, $\beta\, y(b) + y\,'(b) = 0$, where $\beta = B_1/B_2$. Then the second bracket in the right member of (6) is again equal to zero. Also, the first bracket in this member may be written as

$$[\beta\phi_n(b) + \phi_n'(b)]\phi_m(b) - [\beta\phi_m(b) + \phi_m'(b)]\phi_n(b),$$

and so it is also equal to zero. Thus in this case the right member of (6) is equal to zero.

Similarly, if either $A_2 \neq 0$, $B_2 = 0$ or $A_2 \neq 0$, $B_2 \neq 0$ in (2), then the right member of (6) is equal to zero (prove it). Thus in all cases the right member of (6) is equal to zero and so

$$(\lambda_m - \lambda_n) \int_a^b \phi_m(x)\phi_n(x)r(x)\, dx = 0.$$

Since $\lambda_m$ and $\lambda_n$ are distinct characteristic values, so $\lambda_m - \lambda_n \neq 0$. Therefore we must have

$$\int_a^b \phi_m(x)\phi_n(x)r(x)\, dx = 0.$$

This proves that eigen functions $\phi_m$ and $\phi_n$ are orthogonal with respect to r on $a \le x \le b$.

**Remark** Let $\{\lambda_n\}$ be the infinite set of characteristic values of a Sturm Liouville problem, arranged in a monotonic increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \ldots..$ For each n = 1, 2, 3……, let $\phi_n$ be one of the characteristic functions corresponding to the characteristic value $\lambda_n$. Then above Theorem implies at once that the infinite set of characteristic functions $\phi_1$, $\phi_2$, $\phi_3$, ….. is an orthogonal system with respect to the weight function r on $a \le x \le b$.

**Example 9**

Consider again the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{1}$$

$$y(0) = 0, \qquad y(\pi) = 0 \tag{2}$$

which we have already investigated. Corresponding to each characteristic value $\lambda_n = n^2$ (n = 1, 2, 3……), we found the characteristic functions $c_n$ sin nx (n = 1, 2, 3……) where $c_n$(n = 1, 2, 3……) are arbitrary nonzero constants. Let $\{\phi_n\}$ denotes the infinite set of characteristic functions for which $c_n = 1$ (n = 1, 2, 3……). That is

$$\phi_n(x) = \sin nx \qquad (0 \le x \le \pi; \quad n = 1, 2, 3\ldots).$$

Then by the above Theorem, the infinite set $\{\phi_n\}$ is an orthogonal system with respect to the weight function r, where r(x) = 1 for all x, on the interval $0 \le x \le \pi$. That is

$$\int_0^{\pi} (\sin mx)(\sin nx)(1)\, dx = 0 \tag{3}$$

for m = 1, 2, 3, ……; n = 1, 2, 3, ……; $m \ne n$.

**Theorem 7.3** Prove that the eigen values of a SLBVP are real

**Proof** Let $\lambda_n$ be an eigen value corresponding to the eigen function $\phi_n(x)$ of the given SLBVP. Then, by definition

$$\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right] + \left[q(x) + \lambda_n r(x)\right]\phi_n(x) = 0 \tag{1}$$

and $\qquad A_1\, \phi_n(a) + A_2\, \phi'_n(a) = 0$

$$B_1\, \phi_n(b) + B_2\, \phi'_n(b) = 0 \tag{2}$$

We know that p(x), q(x) and r(x) are real valued functions of x over the interval [a, b]. So, taking the complex conjugate of (1) and (2), we obtain

$$\frac{d}{dx}\left[p(x)\frac{d\overline{\phi}_n}{dx}\right] + \left[q(x) + \overline{\lambda}_n r(x)\right]\overline{\phi}_n(x) = 0 \tag{3}$$

and $\qquad A_1\, \overline{\phi}_n(a) + A_2\, \overline{\phi}'_n(a) = 0$

$$B_1\,\bar{\phi}_n(b) + B_2\,\bar{\phi}'_n(b) = 0, \tag{4}$$

where $A_1$, $B_1$, $A_2$, $B_2$ are real constants.

Thus, $\bar{\phi}_n$ is also an eigen function, corresponding to an eigen value $\bar{\lambda}_n$ of the same SLBVP. So from Sturm Liouville Theorem, it follows that

$$(\lambda_n - \bar{\lambda}_n) \int_a^b r(x)\, \phi_n(x)\bar{\phi}_n(x)\, dx = 0$$

or $$(\lambda_n - \bar{\lambda}_n) \int_a^b r(x)\, |\phi_n(x)|^2\, dx = 0. \tag{5}$$

Since, $r(x) > 0$ and $|\phi_n(x)| \neq 0$, being a nontrivial solution, so we must have

$$(\lambda_n - \bar{\lambda}_n) = 0$$

or $$\lambda_n = \bar{\lambda}_n$$

This shows that eigen values are real.

**Exercises**

Verify the validity of the conclusion of S-L Theorem for the characteristic functions of the following Sturm Liouville problems.

1. $\dfrac{d^2y}{dx^2} + \lambda y = 0,\qquad y(0) = 0,\qquad y(\pi/2) = 0.$

2. $\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right] + \dfrac{\lambda}{x}y = 0,\qquad y'(1) = 0,\qquad y'(e^{2\pi}) = 0.$

**C Orthonormal Systems**

**Definition**

A function f is called normalized with respect to the weight function ω on the interval $a \le x \le b$ if and only if

$$\int_a^b [f(x)]^2\, \omega(x)\, dx = 1.$$

**Example 10**

The function $f(x) = \sqrt{2/\pi}\,\sin x$ is normalized with respect to the weight function having the constant value 1 on the interval $0 \le x \le \pi$, for

$$\int_0^{\pi}\left(\sqrt{\frac{2}{\pi}}\ \sin x\right)^2(1)\ dx = \frac{2}{\pi}\int_0^{\pi}\sin^2 x\ dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

**Definition**

Let $\{\phi_n\}$ (n = 1, 2, 3, ......) be an infinite set of functions defined on the interval $a \le x \le b$. The set $\{\phi_n\}$ is called an orthonormal system with respect to the weight function $\omega$ on $a \le x \le b$ if

(1) It is an orthogonal system with respect to $\omega$ on $a \le x \le b$.

(2) Every function of the system is normalized with respect to $\omega$ on $a \le x \le b$. That is, the set $\{\phi_n\}$ is orthonormal with respect to $\omega$ on $a \le x \le b$ if

$$\int_a^b \phi_m(x)\phi_n(x)\omega(x)dx = \begin{cases} 0 & \text{for } m \ne n, \\ 1 & \text{for } m = n. \end{cases}$$

**Example 11**

Consider the infinite set of functions $\{\phi_n\}$, where $\phi_n(x) = \sqrt{2/\pi}\ \sin nx$ (n = 1, 2, 3, ......) on the interval $0 \le x \le \pi$. The set $\{\phi_n\}$ is an orthogonal system with respect to the weight function having the constant value 1 on the interval $0 \le x \le \pi$, for

$$\int_0^{\pi}\left(\sqrt{\frac{2}{\pi}}\ \sin mx\right)\left(\sqrt{\frac{2}{\pi}}\ \sin nx\right)(1)\ dx = 0 \quad \text{for } m \ne n.$$

Further, every function of the system is normalized with respect to this weight function on $0 \le x \le \pi$, for

$$\int_0^{\pi}\left(\sqrt{\frac{2}{\pi}}\ \sin nx\right)^2(1)\ dx = 1.$$

Thus the set $\{\phi_n\}$ is an orthonormal system with respect to the weight function having the constant value 1 on $0 \le x \le \pi$.

**Remark** Consider the Sturm-Liouville problem. Let $\{\lambda_n\}$ be the infinite set of characteristic values of this problem, such that $\lambda_1 < \lambda_2 < \lambda_3 <$ ....... If $\phi_n$ (n = 1, 2, 3, ......) is one of the characteristic functions corresponding to the characteristic value $\lambda_n$, then we know from S-L Theorem that the infinite set of characteristic functions $\phi_1$, $\phi_2$, $\phi_3$, ....... is an orthogonal system with respect to the weight function r on $a \le x \le b$. But this set of characteristic functions is not necessarily orthonormal with respect to r on $a \le x \le b$.

Now, if $\phi_n$ is one of the characteristic functions corresponding to $\lambda_n$, then $k_n\phi_n$, where $k_n$ is an arbitrary nonzero constant, is also a characteristic function corresponding to $\lambda_n$. Thus from the given set of characteristic functions $\phi_1$, $\phi_2$, $\phi_3$, ....... we can form a set of "new" characteristic functions $k_1\phi_1$, $k_2\phi_2$, $k_3\phi_3$, ....... and this "new" set is also orthogonal with respect to r on $a \le x \le b$. Now if we can choose the constants $k_1$, $k_2$, $k_3$, ........ in such a way that every characteristic function of the "new" set is also normalized with respect to r on $a \le x \le b$, then the "new" set of characteristic functions $k_1\phi_1$, $k_2\phi_2$, $k_3\phi_3$, ....... will be an orthonormal system with respect to r on $a \le x \le b$.

The constants $k_1$, $k_2$, $k_3$, ........ can indeed be chosen so that the set $k_1\phi_1$, $k_2\phi_2$, $k_3\phi_3$, ....... is orthonormal. As the function r in the given differential equation is such that $r(x) > 0$ for all x on the interval $a \le x \le b$. Also by definition no characteristic function $\phi_n$ (n = 1, 2. 3,...) is identically zero on $a \le x \le b$. Therefore

$$\int_a^b [\phi_n(x)]^2 \; r(x)\, dx = K_n > 0 \qquad (n = 1, 2, 3, ....),$$

so

$$\int_a^b \left[\frac{1}{\sqrt{K_n}}\,\phi_n(x)\right]^2 r(x)\, dx = 1 \qquad (n = 1, 2, 3, ....).$$

Thus the set

$$\frac{1}{\sqrt{K_1}}\,\phi_1,\ \frac{1}{\sqrt{K_2}}\,\phi_2,\ \frac{1}{\sqrt{K_3}}\,\phi_3, .......$$

is an orthonormal set with respect to r on $a \le x \le b$. Thus, from a given set of orthogonal characteristic functions $\phi_1$, $\phi_2$, $\phi_3$, ......., we can always form the set of orthonormal characteristic functions $k_1\phi_1$, $k_2\phi_2$, $k_3\phi_3$, ....... where

$$k_n = \frac{1}{\sqrt{K_n}} = \frac{1}{\sqrt{\int_a^b [\phi_n(x)]^2 \; r(x)\, dx}} \qquad (n = 1, 2, 3......)$$

**Example 12**

The Sturm-Liouville problem of example 9 has the set of orthogonal characteristic functions $\{\phi_n\}$, where $\phi_n(x) = c_n \sin nx$ (n = 1, 2, 3,...... ; $0 \le x \le \pi$) and $c_n$ (n = 1, 2, 3....) are nonzero constants. We now form the sequence of orthonormal characteristic functions $\{k_n\phi_n\}$, where $k_n$ is defined as above. We have

$$K_n = \int_0^{\pi} \left(c_n \sin nx\right)^2 (1)\, dx = \frac{c_n^2\pi}{2},$$

$$k_n = \frac{1}{\sqrt{K_n}} = \frac{1}{c_n}\sqrt{\frac{2}{\pi}},$$

$$k_n\phi_n(x) = \left(\frac{1}{c_n}\sqrt{\frac{2}{\pi}}\right)\left(c_n \sin nx\right) = \sqrt{\frac{2}{\pi}} \sin nx \qquad (n = 1, 2, 3.....)$$

Thus the Sturm-Liouville problem under consideration has the set of orthonormal characteristic functions $\{\psi_n\}$, where $\psi_n(x) = \sqrt{2/\pi} \sin nx$ (n = 1, 2, 3, ......; $0 \le x \le \pi$). We see that this is the set of orthonormal functions considered in Example 10.

**Summary**

In the course of this chapter, Sturm-Liouville problems are studied and non-trivial solutions of such problems are found. Characteristic values of a SLBVP are found to be real and discrete. A basic theorem known as Sturm-Liouville theorem concerning the orthogonality of characteristic functions is proved and its validity is verified for the characteristic functions of a SLBVP.

**Keywords**

Boundary value problems, Characteristic values, Characteristic functions, Orthogonality, Orthonormality.

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